Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1993), 723–728

A Higher Order Uniform Convergence Result for a Turning Point Problem

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We describe a new fitted scheme, of k-th order uniform accuracy with arbitrary k, for a turning point problem of cusp type. The scheme is constructed by applying an iterative technique to an auxiliary problem obtained after replacing coefficient functions in the original problem by piecewise polynomials. The analysis of the scheme is based on an improved stability result.

Key words: *Singular perturbations, boundary value problems, turning points, higher order schemes*

AMS Subject Classifications: 34B15, 65L10

1 Introduction

Let us consider a turning point problem of the form

$$\begin{aligned} Lu := -\epsilon u'' - xa(x)u' + b(x)u &= f(x), \quad x \in (-1,1), \\ u(-1) = u(1) &= 0, \end{aligned} \tag{1}$$

where ϵ is a small positive parameter and the functions a, b and f are sufficiently smooth. We assume that the coefficient of u' has a single simple zero and exclude any resonance phenomena. Thus, our additional assumptions are:

$$(i) \quad a(x) \geq \alpha > 0 \qquad (ii) \quad b(x) \geq 0, \quad b(0) > 0. \tag{2}$$

Under these conditions, problem (1) admits a unique solution which satisfies the maximum principle and has a single, isolated turning point of *cusp* type at $x = 0$.

Abrahamsson [1] has derived the asymptotic behaviour of the solution as $\epsilon \to 0$. The unique solution of (1) converges to the solution of the reduced equation which satisfies *both* boundary conditions. The general solution of the reduced equation admits the following representation:

$$v(x) = w(x) + \begin{cases} c_1|x|^\lambda \exp(\int_0^x \psi(t)dt) & \text{if } x < 0, \\ c_2\, x^\lambda \exp(\int_0^x \psi(t)dt) & \text{if } x > 0, \end{cases} \tag{3}$$

where $\lambda = b(0)/a(0)$, and w and ψ are some smooth functions (see Lemma 3.1 in [1]). The constants c_1 and c_2 are determined by the boundary conditions in the limit of u as $\epsilon \to 0$. A priori estimates of the derivatives of the exact solution have been obtained in [2]:

$$|u^{(l)}(x)| \leq C(x^2 + \epsilon)^{\frac{\lambda - l}{2}}, \qquad l = 0, 1, \ldots. \tag{4}$$

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ISSN 0232-2064 / $ 2.50 © Heldermann Verlag Berlin

The representation (3) and the sharp estimates (4) are the main reason that the standard discretization methods fail at least in the most complicated situation $0 < \lambda < 1$. Let us generally assume that

$$0 < \lambda < 1, \tag{5}$$

even though our method works for *every* $\lambda > 0$. Under the assumption (5), the well known El-Mistikawy-Werle scheme has been shown to be of order $O(h^\lambda)$, uniformly with respect to the parameter ϵ (see section 3 in [2]). Farrell [4] has given sufficient conditions for a class of finite-difference schemes (including the standard upwinding) to have the same uniform rate of convergence in the maximum norm.

The accuracy $O(h^\lambda)$ is somewhat unsatisfactory, but Farrell and Gartland [5] were able to construct a uniform $O(h)$-scheme. In this paper we shall show that their approach can be regarded as the first step of an iterative process which yields $O(h^k)$-accuracy in the maximum norm for arbitrary k. This result is based on a new stability estimate. The iterative process is used for handling an auxiliary problem with piecewise polynomial coefficients. This is similar to the technique from [3], where singular perturbation problems without turning points have been treated. Another approach for numerical treatment of singular perturbation problems is to use the classical finite-difference schemes on special discretization meshes which are dense in the layers. The first result by this approach for the problem (1) has been given by Liseikin [6]. It was later improved in [10], [7] and finally in [11]. The papers [6], [7] and [10] give uniform methods of first order, while a uniform second order result has been proved in [11]. Thus, our higher order uniform convergence result for turning point problems is new.

Since the main purpose of this paper is to introduce a technique by which the method from [5] can be improved, we decided to present no numerical results.

2 The Farrell–Gartland Approach

Let some grid $-1 = x_0 < x_1 < \cdots < x_N = 1$ be given with $h_i = x_{i+1} - x_i$ and the mesh width $h = \max h_i$. We define $\bar{a}$ as a piecewise constant approximation to a:

$$\bar{a}(x) = \bar{a}_i \quad \text{on} \quad (x_i, x_{i+1}) \quad \text{with} \quad \|a - \bar{a}\|_\infty \leq Ch.$$

Here and throughout the paper C denotes a generic constant independent of ϵ and h. The functions b and f are approximated in the same way. Then, Farrell and Gartland's basic idea is to define the approximate solution u_h as the solution to the boundary value problem

$$\begin{aligned} \bar{L}u_h := -\epsilon u_h'' - x\bar{a}(x)u_h' + \bar{b}(x)u_h &= \bar{f}(x), \quad x \in (-1,1), \\ u_h(-1) = u_h(1) &= 0. \end{aligned} \tag{6}$$

Farrell and Gartland use the stability result for $\bar{L}$

$$\|v\|_\infty \leq C\|\bar{L}v\|_\infty \tag{7}$$

to estimate the error $u - u_h$. Since

$$\bar{L}(u - u_h) = f - \bar{f} + (\bar{b} - b)u - (\bar{a} - a)xu',$$

the stability estimate (7) yields

$$\|u - u_h\|_\infty \leq C(\|f - \bar{f}\|_\infty + \|b - \bar{b}\|_\infty + \|\bar{a} - a\|_\infty \|xu'\|_\infty). \tag{8}$$

Then (4) implies that

$$\|xu'\|_\infty \leq C \max_{x\in[-1,1]} \{|x|(x^2 + \epsilon)^{\frac{\lambda-1}{2}}\} \leq C,$$

and the desired first order uniform convergence result follows from (8):

$$\|u - u_h\|_\infty \le Ch. \tag{9}$$

For our approach we need some improvement of the stability result (7) as well as the error estimate (9). First, we announce the improved stability result.

Lemma 1. *Under the conditions* (2), *the stability estimate* $\|v\|_\infty + \|xv'\|_\infty \le C\,\|\bar{L}v\|_\infty$ *holds for* $h \le h_0$, *where* h_0 *is sufficiently small but independent of* ϵ.

Further on, we shall always assume that $h \le h_0$. We will prove Lemma 1 in the appendix. Applying Lemma 1 we get a sharpened version of the corresponding theorem in [5]:

Theorem 1. *Let the approximate problem* (6) *be constructed by replacing the functions* a, b, f *from* (1) *by functions* $\bar{a}, \bar{b}, \bar{f}$ *satisfying* $\|\bar{a} - a\|_\infty + \|b - \bar{b}\|_\infty + \|f - \bar{f}\|_\infty \le Ch$. *Then the error between the solutions to the original problem* (1) *and to the approximate problem* (6) *satisfies*

$$\|u - u_h\|_\infty + \|x(u - u_h)'\|_\infty \le Ch.$$

3 The New Higher Order Scheme

On the given grid we approximate a by $\bar{a}$, a piecewise polynomial of degree k, such that

$$\bar{a} = \sum_{\mu=0}^{k} a_\mu, \quad \|a - \sum_{\mu=0}^{l} a_\mu\|_\infty \le Ch^{l+1} \quad \text{for} \quad l = 0(1)k,$$

$$\|a_\mu\|_\infty \le Ch^\mu \quad \text{for} \quad \mu = 0(1)k.$$

We shall use analogous approximations to b and f as well. Further, we introduce the abbreviations

$$\bar{L}_0 v := -\epsilon v'' - x a_0 v' + b_0 v \quad \text{and} \quad \bar{L}_l v := -x a_l v' + b_l v.$$

Theoretically, we could introduce an approximation u_h^* which solves the problem

$$\begin{aligned} -\epsilon w'' - x\bar{a}(x)w' + \bar{b}(x)w &= \bar{f}(x), \quad x \in (-1, 1), \\ w(-1) = w(1) &= 0. \end{aligned}$$

In the same way as in Section 2, we are able to prove $\|u - u_h^*\|_\infty \le Ch^k$, but this result is practically worthless because it is impossible to handle equations with piecewise polynomial coefficients of higher degree.

Therefore we introduce the following iterative process:

$$\begin{aligned} &(i) \qquad \bar{L}_0 u_h^0 = f_0, \quad u_h^0(-1) = u_h^0(1) = 0, \\ &(ii) \qquad \bar{L}_0 u_h^{i+1} = \sum_{l=0}^{i+1} f_l - \sum_{l=1}^{i+1} \bar{L}_l u_h^i, \quad u_h^{i+1}(-1) = u_h^{i+1}(1) = 0, \quad i = 0(1)k-1. \end{aligned} \tag{10}$$

Thus u_h^0, u_h^1, ..., u_h^k solve differential equations with the same left hand side as in the Farrell–Gartland approach.

We analyse the method by mathematical induction starting with u_h^0 which is the same as u_h from Theorem 1, and thus satisfies

$$\|u - u_h^0\|_\infty + \|x(u - u_h^0)'\|_\infty \le Ch.$$

Then for $i \geq 0$, we have

$$\begin{aligned} \bar{L}_0(u_h^{i+1} - u) &= \sum_{l=0}^{i+1} f_l - \sum_{l=1}^{i+1} \bar{L}_l u_h^i - Lu + (L - L_0)u \\ &= (\sum_{l=0}^{i+1} f_l - f) - \sum_{l=1}^{i+1} \bar{L}_l u_h^i + (b - \bar{b})u - (a - \bar{a})xu' + \sum_{l=1}^{k} \bar{L}_l u, \end{aligned}$$

thus

$$\bar{L}_0(u_h^{i+1} - u) = (\sum_{l=0}^{i+1} f_l - f) + \sum_{l=1}^{i+1} \bar{L}_l(u - u_h^i) + (b - \bar{b})u - (a - \bar{a})xu' + \sum_{l=i+2}^{k} \bar{L}_l u.$$

By applying our stability result from Lemma 1 and the uniform boundedness of $\|u\|_\infty + \|xu'\|_\infty$, we obtain

$$\|u - u_h^{i+1}\|_\infty + \|x(u - u_h^{i+1})'\|_\infty \leq C\left\{h^{i+2} + h(\|u - u_h^i\|_\infty + \|x(u - u_h^i)'\|_\infty) + h^{k+1}\right\}.$$

So, we have proved the following theorem by mathematical induction.

Theorem 2. *The error between the solutions to the original problem* (1) *and to the i-th iteration of the process* (10) *satisfies*

$$\|u - u_h^i\|_\infty + \|x(u - u_h^i)'\|_\infty \leq Ch^{i+1}, \quad i = 0(1)k,$$

thus u_h^i is an $O(h^{i+1})$ uniform approximation to u.

Finally, let us remark that there exist well known procedures to handle the problem

$$\bar{L}_0 u_h = g, \quad u_h(-1) = u_h(1) = 0,$$

in an effective way (using patched spline functions [8], local Green's functions, or Marchuk type integral relations [3], [5]). They all lead to tridiagonal linear systems which have nice properties (their matrices are M–matrices).

4 Appendix: Proof of the Stability Estimate

We consider the boundary value problem (using the notation of Section 2 again)

$$\begin{aligned} \bar{L}v = -\epsilon v'' - x\bar{a}(x)v' + \bar{b}(x)v &= f(x), \quad x \in (-1, 1), \\ v(-1) = v(1) &= 0, \end{aligned}$$

and try to estimate xv' in terms of F, where $\|v\|_\infty \leq CF$, $F = \|f\|_\infty = \|\bar{L}v\|_\infty$. Introducing the integrating factor $e^{\phi(x)}$ with $\epsilon\phi(x) = \int_0^x \bar{a}(t)t\,dt$, we obtain

$$\epsilon(e^\phi v')' = (\bar{b}v - f)e^\phi, \quad \text{or} \quad xv'(x) = S_1 + S_2,$$

where

$$S_1 = xv'(0)e^{-\phi(x)} \quad \text{and} \quad S_2 = \frac{x}{\epsilon}\int_0^x (\bar{b}(t)v(t) - f(t))e^{\phi(t)-\phi(x)}dt.$$

Let us start estimating S_2. We get

$$|S_2| \leq \frac{CF}{\epsilon}\left|x\int_0^x e^{\phi(t)-\phi(x)}dt\right|.$$

Let us assume $x > 0$ (the case $x < 0$ can be handled analogously). By taking into account that $\bar{a}(x) \geq \alpha > 0$ for $h \leq h_0$, we get, for $0 \leq t \leq x$,

$$\phi(t) - \phi(x) = \frac{1}{\epsilon}\int_x^t \bar{a}(s)sds \leq \frac{\alpha}{2\epsilon}(t^2 - x^2) = \frac{\alpha}{2\epsilon}(t-x)(t+x) \leq \frac{\alpha}{2\epsilon}x(t-x)$$

and obtain

$$|S_2| \leq CF\frac{x}{\epsilon}\int_0^x e^{\frac{\alpha x}{2\epsilon}(t-x)}dt \leq CF.$$

Let us now estimate S_1. We only have to prove

$$|v'(0)| \leq CF/\sqrt{\epsilon}, \tag{11}$$

since this inequality and $e^{-\phi(x)} \leq e^{-\frac{\alpha x^2}{2\epsilon}}$ imply

$$|S_1| \leq CF\frac{1}{\sqrt{\epsilon}}\left|xe^{-\frac{\alpha x^2}{2\epsilon}}\right| \leq CF.$$

Let us prove (11). We choose $x^* \in (0, \sqrt{\epsilon})$ such that $v'(x^*) = (v(\sqrt{\epsilon}) - v(0))/\sqrt{\epsilon}$. Then from (7) we conclude

$$|v'(x^*)| \leq CF/\sqrt{\epsilon}. \tag{12}$$

Integrating the differential equation from 0 to x^* we get

$$-\epsilon v'(x^*) + \epsilon v'(0) - \int_0^{x^*} \bar{a}(t)tv'(t)dt = \int_0^{x^*}(f(t) - \bar{b}(t)v(t))dt,$$

wherefrom we obtain the inequality

$$|v'(0)| \leq |v'(x^*)| + \frac{1}{\epsilon}\left|\int_0^{x^*}\bar{a}(t)tv'(t)dt\right| + \frac{1}{\epsilon}\left|\int_0^{x^*}(f(t) - \bar{b}(t)v(t))dt\right|.$$

Due to (12) and $x^* \in (0, \sqrt{\epsilon})$ we get

$$|v'(0)| \leq \frac{CF}{\sqrt{\epsilon}} + \frac{S_3}{\epsilon}, \quad \text{where} \quad S_3 = \left|\int_0^{x^*}\bar{a}(t)tv'(t)dt\right|. \tag{13}$$

Let us assume $x^* \in (x_k, x_{k+1}]$. Then we have

$$S_3 = \left|\sum_{i=0}^{k}\bar{a}_i\int_{x_i}^{\hat{x}_{i+1}}\left[(xv)' - v\right]dx\right| \quad \text{with} \quad \hat{x}_l = \begin{cases} x_l & \text{if} \quad l \leq k, \\ x^* & \text{if} \quad l = k+1. \end{cases} \tag{14}$$

This representation leads to

$$S_3 \leq \sum_{i=0}^{k}\bar{a}_i\left|\int_{x_i}^{\hat{x}_{i+1}} v\,dx\right| + \left|\sum_{i=0}^{k}\bar{a}_i[\hat{x}_{i+1}v(\hat{x}_{i+1}) - x_iv(x_i)]\right|.$$

The first term is immediately bounded by $CF\sqrt{\epsilon}$. The second term can be written in the form

$$\left|\sum_{i=1}^{k}(\bar{a}_{i+1} - \bar{a}_i)x_iv(x_i) + \bar{a}_k\hat{x}_{k+1}v(\hat{x}_{k+1})\right|,$$

thus, $CF\sqrt{\epsilon}$ is an upper bound again. Therefore we obtain $S_3 \leq CF\sqrt{\epsilon}$, and finally from the inequality (13) $|v'(0)| \leq CF/\sqrt{\epsilon}$. Thus, Lemma 1 is proved.

The technique used here is essentially the one introduced in [6]. We made some necessary modifications to handle the piecewise constant coefficient $\bar{a}$.

Acknowledgements: A part of the first author's work and the second author's entire work on this paper were done during their visit to the Department of Mathematics and Computer Science at Kent State University, Kent, Ohio, U.S.A. The authors would like to use this opportunity to thank Professor O. P. Stackelberg, Chairman, and Professors P. A. Farrell and E. C. Gartland for their hospitality.

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Received 13.10.1992