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On the Solvability of Nonlinear Singular Integral Equations

P. JUNGHANNS and U. WEBER

Three classes of nonlinear singular integral equations of Cauchy type occuring in the treatment of certain free boundary value problems are investigated. Existence of the solution is proved under weaker conditions than in [13] using the technique which was created in [12, 13] and is based on the application of Schauder's fixed point theorem.

Key words: *Nonlinear Cauchy type singular integral equations*

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1 Introduction

In investigating and solving certain free boundary value problems by means of nonlinear singular integral operators there occur equations of the form (cf. [2, 3, 10, 11])

$$F(x, u(x)) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(y)\, dy}{y - x} + c, \qquad -1 \le x \le 1,$$

and

$$u(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{F(y, u(y))\, dy}{y - x} + f(x) + c, \qquad -1 \le x \le 1.$$

Thereby, one of the assumptions, under which in [13] the existence of a solution u with p-summable derivative, $p > 1$, was proved, is violated, namely the condition

$$|f'(x)|, |F_x(x, u)| \le l_0 (1 - x^2)^{-\delta}, \tag{1.1}$$

where l_0 and $\delta < \frac{1}{2}$ are some nonnegative constants. In the above mentioned situation, the last relation is fulfilled only in case $\delta = \frac{1}{2}$. In [10, 11] one can find some remarks that, nevertheless, in the special cases considered there the equations are solvable.

In the present paper we show that the restriction on the constant δ in (1.1) can be weakened even in the general situation. In particular, by keeping of all the other assumptions in [13] even $\delta > \frac{1}{2}$ is permissible. The existence of the solution can be proved in principle with the technique of [12, 13] being based on the application of Schauder's fixed point theorem. In Section 2 this will be done for equations of the first type. Equations of the second type are considered in Section 3. Section 4 is dedicated to a third class of equations, for which an example is given in Section 5.

P. Junghanns: Techn. Univ. Chemnitz - Zwickau, FB Math., PSF 964, D - 09009 Chemnitz
U. Weber: Techn. Univ. Chemnitz - Zwickau, FB Math., PSF 964, D - 09009 Chemnitz

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2 Equations of the first type

We consider singular integral equations of the form

$$F(x,u(x)) = (\mathbf{S}u)(x) + c, \qquad -1 \le x \le 1, \tag{2.1}$$

where $\mathbf{S}$ denotes the Cauchy singular integral operator

$$(\mathbf{S}u)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)}{y-x}dy.$$

We search for continuous functions u and real numbers c satisfying (2.1) and the additional conditions

$$u(-1) = u(1) = 0. \tag{2.2}$$

More precisely, we will ask for functions u for which a real number $p > 1$ and a function $v \in \mathbf{L}^p$ exist such that $u(x) = \int_{-1}^{x} v(y)\,dy$ and $u(1) = 0$. Here $\mathbf{L}^p = \mathbf{L}^p(-1,1)$ denotes the usual Lebesgue space of all measurable functions v for which $|v(x)|^p$ is summable, and $\|v\|_p$ is the usual norm in $\mathbf{L}^p$. The set of all such functions u we will call $\mathcal{W}_0^{1+}$. In order to prove the existence of a solution $u \in \mathcal{W}_0^{1+}$ of problem (2.1),(2.2) we assume that the following assumption is fulfilled.

A1: $F = F(x,u): [-1,1]\times\mathbb{R} \to \mathbb{R}$ ($\mathbb{R} := (-\infty,+\infty)$) *possesses a continuous partial derivative* F_u *and a partial derivative* F_x *which is continuous with respect to* $u \in \mathbb{R}$ *for allmost all* x *and measurable with respect to* $x \in [-1,1]$ *for all* $u \in \mathbb{R}$ *(Carathéodory condition). Furthermore, there exist constants* $l_0, l_1, l_2, \delta \ge 0$ *such that* $l_1 l_2 < 1$ *and*

$$-l_1 \le F_u(x,u) \le l_2, \qquad (x,u) \in [-1,1]\times\mathbb{R}, \tag{2.3}$$

$$|F_x(x,u)| \le l_0(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1]\times\mathbb{R}, \tag{2.4}$$

$$\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}. \tag{2.5}$$

We remark that in view of $l_1 l_2 < 1$ the estimation (2.5) is weaker than the condition $\delta < 1/2$ from [13].

2.1 Reduction to a fixed point equation

Let $u(x) = \int_{-1}^{x} v(y)\,dy$ be a solution of problem (2.1),(2.2) with the above required properties. Then we can differentiate equation (2.1) (cf. [8], Chapt. II, Lemma 6.1). Taking into account condition (2.2) we obtain

$$a(x)v(x) - (\mathbf{S}v)(x) = g(x), \tag{2.6}$$

where $a(x) = F_u(x,u(x))$ and $g(x) = -F_x(x,u(x))$. Since, in view of Assumption A1, the function a is continuous there exist continuous functions $\alpha : [-1,1] \to (0,1)$ and $r : [-1,1] \to \mathbb{R}$

such that $a(x) + i = r(x)\exp[i\pi\alpha(x)]$. From (2.2) it follows that the solution v of (2.6) has to satisfy the condition

$$\int_{-1}^{1} v(y)\,dy = 0. \tag{2.7}$$

To find an explicit expression for the solution v of problem (2.6), (2.7) we use the results of [1, §9.5] (cf. also [4, 7, 8]). With $\mathbf{P} = \frac{1}{2}(\mathbf{I} - i\mathbf{S})$ and $\mathbf{Q} = \mathbf{I} - \mathbf{P} = \frac{1}{2}(\mathbf{I} + i\mathbf{S})$ ($\mathbf{I}$ denotes the identity operator) we can write equation (2.6) in the equivalent form

$$\mathbf{B}v := [\mathbf{P}(a - i) + \mathbf{Q}(a + i)]v = g.$$

Put $c(x) = [a(x) - i]/[a(x) + i] = \exp[-2\pi i\alpha(x)]$. Then, for all sufficiently small $p > 1$, the function c admits a generalized $\mathbf{L}^p$-factorization $c(x) = c_-(x)(x - i)^{-1}c_+(x)$, where

$$c_-(x) = \frac{x - i}{1 - x}\exp\left[-i\pi\alpha(x) + \int_{-1}^{1}\frac{\alpha(y)}{y - x}dy\right] = \frac{(x - i)[a(x) - i]}{(1 - x)r(x)}\exp[\pi(\mathbf{S}\alpha)(x)],$$

$$c_+(x) = (1 - x)\exp\left[-i\pi\alpha(x) - \int_{-1}^{1}\frac{\alpha(y)}{y - x}dy\right] = \frac{(1 - x)[a(x) - i]}{r(x)}\exp[-\pi(\mathbf{S}\alpha)(x)].$$

Hence, $\mathbf{B} : \mathbf{L}^p \longrightarrow \mathbf{L}^p$ is a Fredholm operator with index 1, and a right inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{(-1)} = \frac{c_+^{-1}}{a + i}[\mathbf{P}(a + i) + \mathbf{Q}(a - i)]\frac{c_+}{a - i}\mathbf{I} = z^{-1}(a\mathbf{I} + \mathbf{S})\frac{z}{r^2}\mathbf{I}, \tag{2.8}$$

where $z(x) = (1 - x)r(x)\exp[-\pi(\mathbf{S}\alpha)(x)]$. Because of (2.4) and (2.5) we have $g \in \mathbf{L}^p$ for all sufficiently small $p > 1$. We show that $v = \mathbf{B}^{(-1)}g$ is the solution of problem (2.6), (2.7). For this end we have to prove that $v = \mathbf{B}^{(-1)}g$ satisfies condition (2.7). Indeed, taking into account relation (2.8) we conclude that $(a + i)c_+v$ lies in the image of the operator $\mathbf{P}(a + i)\mathbf{I} + \mathbf{Q}(a - i)\mathbf{I}$, which implies (cf. [1, §9.5])

$$0 = \int_{-1}^{1}(a + i)c_+v[c_- - c_+^{-1}(x - i)](x - i)^{-1}dx = \int_{-1}^{1}(a + i)(c - 1)v\,dx = -2i\int_{-1}^{1}v(x)\,dx.$$

Integrating the obtained expression for v, we can summarize our investigations to the result that each solution $u \in \mathcal{W}_0^{1+}$ of problem (2.1), (2.2) is a solution of the fixed point equation

$$u(x) = (\mathbf{T}u)(x) := \int_{-1}^{x} L(y, u(y))\,dy, \qquad u \in \mathbf{C}_0, \tag{2.9}$$

where

$$L(x, u(x)) = \frac{a(x)}{r^2(x)}g(x) + \frac{z^{-1}(x)}{\pi}\int_{-1}^{1}\frac{z(y)g(y)}{r^2(y)}\frac{dy}{y - x} \tag{2.10}$$

and $\mathbf{C}_0 = \mathbf{C}_0[-1, 1]$ denotes the Banach space of all continuous functions on $[-1, 1]$ satisfying the boundary conditions (2.2). In the sequel it will be proved that the kernel $L(x, u(x))$ is an element of $\mathbf{L}^p$ for some $p > 1$ if $u \in \mathbf{C}_0$. Consequently, each solution of equation (2.9) is also a solution of problem (2.1), (2.2).

The expression of the kernel of the integral operator $\mathbf{T}$ considered in [13] is a little bit more complicated than expression (2.10). It is possible to choose this simpler version since $v = \mathbf{B}^{(-1)}g$ fulfils condition (2.7).

2.2 Investigation of the kernel L(x,u(x))

The aim of this subsection is to obtain some preliminary results which give us the possibility to handle the fixed point equation (2.9) with the help of Schauder's fixed point theorem. By $\|u\|_\infty = \sup\{|u(x)| : x \in [-1,1]\}$ we denote the norm in the Banach space $\mathbf{C}_0$. In all what follows we assume $\{u_n\} \subset \mathbf{C}_0$, $u \in \mathbf{C}_0$ and $\|u_n - u\|_\infty \longrightarrow 0$ $(n \longrightarrow \infty)$ and use the notations $a(x) = F_u(x, u(x))$, $a_n(x) = F_u(x, u_n(x))$, and analogous notations for functions depending on u or u_n. Obviously we have

$$\|a_n - a\|_\infty \longrightarrow 0 \qquad (n \to \infty), \tag{2.11}$$

since the norms $\|u_n\|_\infty$ are uniformly bounded. Furthermore, it holds

$$g_n \longrightarrow g \ \ a.e. \tag{2.12}$$

(cf. Assumption A1). With $\gamma(x) := \arctan a(x) = \pi/2 - \pi\alpha(x)$ we can write

$$z(x) = \sqrt{1-x^2}\, r(x) \exp[(S\gamma)(x)].$$

From (2.10) we conclude

$$L(x, u(x)) = \sin[\gamma(x)]\, h(x) + \cos[\gamma(x)]\, M(x, u(x)), \tag{2.13}$$

where $h(x) = g(x)/r(x) = \cos[\gamma(x)]\, g(x)$ and

$$M(x, u(x)) := (1-x^2)^{-1/2} \exp[-(S\gamma)(x)] \Big(S\Big\{(1-y^2)^{1/2} h(y) \exp[(S\gamma)(y)]\Big\}\Big)(x).$$

Lemma 2.1: *Let $1 < p < \delta^{-1}$. Then $h \in \mathbf{L}^p$ and $h_n \sin\gamma_n \longrightarrow h \sin\gamma$ in $\mathbf{L}^p$. Moreover, the estimation $\|h\|_p \le c(\delta, p)$ is valid, where the constant $c(\delta, p)$ does not depend on $u \in \mathbf{C}_0$.*

Proof: In view of the estimate (2.4) we have

$$\int_{-1}^{1} |h(x)|^p dx \le l_0^p \int_{-1}^{1} (1-x^2)^{-\delta p} dx =: c(\delta, p)^p < \infty.$$

From (2.11) and (2.12) it follows $h_n(x) \sin\gamma_n(x) \longrightarrow h(x) \sin\gamma(x)$ a.e., and (2.4) implies

$$|h_n(x)\sin\gamma_n(x) - h(x)\sin\gamma(x)|^p \le 2^p l_0^p (1-x^2)^{-\delta p}.$$

Now, from the Lebesgue theorem we conclude the assertion ∎

Let us introduce the notations $2\omega_- := \arctan l_2 - \arctan l_1$, $2\omega_+ := \arctan l_2 + \arctan l_1$. Because of $2|\omega_-| < \pi/2$ and $2\omega_+ = \arctan[(l_1+l_2)/(1-l_1l_2)] \in (0, \frac{\pi}{2})$ we have $3/(1+2|\omega_-|/\pi) > 2$ and $\pi/2\omega_+ > 2$. Thus, taking into account (2.5) there exists a $\kappa > 2$ such that $0 < 3 - 2\delta\kappa$,

$$\kappa < \min\left\{\frac{\pi}{2\omega_+}, \frac{2}{\delta}, \frac{3}{1+2|\omega_-|/\pi}\right\} \qquad \text{and} \qquad \frac{\kappa}{3-2\delta\kappa} < \frac{\pi}{2\omega_+}.$$

Defining $\mu(x) := \gamma(x) - \omega_-$ we obtain (cf. (2.3))

$$|\mu(x)| \le \omega_+ \ \ (x \in [-1,1]) \qquad \text{and} \qquad (S\gamma)(x) = (S\mu)(x) + \frac{\omega_-}{\pi} \ln\frac{1-x}{1+x}, \tag{2.14}$$

and, consequently,

$$M(x,u(x)) = \tilde{R}(x)(\tilde{S}\phi)(x), \tag{2.15}$$

where

$$\begin{aligned}
(\tilde{S}\phi)(x) &= \tilde{N}(x)(1-x^2)^{1/2\kappa}\left(S\left[\tilde{N}^{-1}(y)(1-y^2)^{-1/2\kappa}\phi(y)\right]\right)(x),\\
\tilde{N}(x) &= (1-x)^{-1/2-\omega_-/\pi}(1+x)^{-1/2+\omega_-/\pi},\\
\phi(x) &= (1-x^2)^{1/2\kappa}h(x)\exp[(S\mu)(x)],\\
\tilde{R}(x) &= (1-x^2)^{-1/2\kappa}\exp[-(S\mu)(x)].
\end{aligned} \tag{2.16}$$

In view of (2.14) and a lemma from [13, Appendix] we obtain

$$\|\tilde{R}\|_\kappa \le \left(\frac{\pi}{\cos(\kappa\omega_+)}\right)^{1/\kappa} =: d(\omega_+,\kappa) \qquad \forall u \in \mathbf{C}_0. \tag{2.17}$$

Lemma 2.2: *It holds $\tilde{R}_n \longrightarrow \tilde{R}$ in $\mathbf{L}^\kappa$.*

Proof: Let $\varepsilon > 0$, $0 < \vartheta < \varepsilon/(1+\varepsilon)$, and $\kappa(1+\varepsilon)(1+\vartheta) < \pi/2\omega_+$. Defining the functions $\Phi(u) = u^\vartheta$ and $f_n(x) = |\tilde{R}_n(x)|^\kappa$ we can estimate

$$\begin{aligned}
&\int_{-1}^{1} |f_n(x)|\,\Phi(|f_n(x)|)\,dx\\
&= \int_{-1}^{1}(1-x^2)^{-(1+\vartheta)/2}\exp[-\kappa(1+\vartheta)(S\mu_n)(x)]\,dx\\
&= \int_{-1}^{1}(1-x^2)^{-\varepsilon/2(1+\varepsilon)-\vartheta/2}(1-x^2)^{-1/2(1+\varepsilon)}\exp[-\kappa(1+\vartheta)(S\mu_n)(x)]\,dx\\
&\le \left\{\int_{-1}^{1}(1-x^2)^{-[\varepsilon/2(1+\varepsilon)+\vartheta/2](1+\varepsilon)/\varepsilon}dx\right\}^{\varepsilon(1+\varepsilon)}\left\{\int_{-1}^{1}(1-x^2)^{-1/2}\exp[-\nu(S\mu_n)(x)]\,dx\right\}^{1/(1+\varepsilon)}\\
&= \left\{\int_{-1}^{1}(1-x^2)^{-1/2-\vartheta(1+\varepsilon)/2\varepsilon}dx\right\}^{\varepsilon/(1+\varepsilon)}\left\|(1-x^2)^{-1/2\nu}\exp[-(S\mu_n)(x)]\right\|_\nu^{\nu/(1+\varepsilon)}\\
&\le const\left(\frac{\pi}{\cos(\nu\omega_+)}\right)^{1/(1+\varepsilon)},
\end{aligned}$$

where we used the lemma from [13, Appendix] and the notation $\nu := \kappa(1+\varepsilon)(1+\vartheta)$. Since $\|\mu_n-\mu\|_\infty \longrightarrow 0$ and since the operator S is continuous in $\mathbf{L}^q$ for all $q > 1$, it follows $S\mu_n \longrightarrow S\mu$ in measure, which implies, in view of the monotonicity of e^u, $f_n \longrightarrow f$ in measure. From the lemmas of Vallée-Poussin and Vitali (cf. [9, Chapt. VI, § 3]) we obtain

$$\int_{-1}^{x}\tilde{R}_n(y)\,dy \longrightarrow \int_{-1}^{x}\tilde{R}(y)\,dy, \qquad \forall x \in [-1,1],$$

as well as $\|\tilde{R}_n\|_\kappa \longrightarrow \|\tilde{R}\|_\kappa$. This yields the assertion ∎

Lemma 2.3: *For the above defined functions ϕ_n and ϕ (cf. (2.16)) we have $\|\phi_n - \phi\|_\kappa \longrightarrow 0$.*

Proof: Let $\max\{\kappa, \kappa/(3-2\delta\kappa)\} < \varepsilon < \pi/2\omega_+$, and $\phi = \psi\chi$, where

$$\psi(x) = h(x)(1-x^2)^{(\varepsilon+\kappa)/2\varepsilon\kappa}, \quad \chi(x) = (1-x^2)^{-1/2\varepsilon}\exp[(S\mu)(x)].$$

Further, let $\tilde{p} := \varepsilon$ and $\tilde{q} := \varepsilon\kappa/(\varepsilon-\kappa)$. Then $\tilde{p}^{-1} + \tilde{q}^{-1} = \kappa^{-1}$, which implies $\|\phi\|_\kappa \le \|\psi\|_{\tilde{q}}\|\chi\|_{\tilde{p}}$, where $\|\chi\|_{\tilde{p}} \le d(\omega_+, \varepsilon)$ and

$$\|\psi\|_{\tilde{q}}^{\tilde{q}} \le l_0^{\tilde{q}} \int_{-1}^{1} (1-x^2)^{(\varepsilon+\kappa-2\delta\varepsilon\kappa)/2(\varepsilon-\kappa)} dx =: e(\kappa, \varepsilon)^{\tilde{q}} < \infty$$

because of $(\varepsilon + \kappa - 2\delta\varepsilon\kappa)/2(\varepsilon-\kappa) > -1$. In the same manner as in the proof of Lemma 2.2 we can show that $\chi_n \longrightarrow \chi$ in $\mathbf{L}^{\tilde{p}}$. Since $\psi_n \longrightarrow \psi$ a.e. and

$$|\psi_n(x) - \psi(x)|^{\tilde{q}} \le (2l_0)^{\tilde{q}}(1-x^2)^{(\varepsilon+\kappa-2\delta\varepsilon\kappa)/2(\varepsilon-\kappa)},$$

the Lebesgue theorem implies $\psi_n \longrightarrow \psi$ in $\mathbf{L}^{\tilde{q}}$. Hence, $\phi_n \longrightarrow \phi$ in $\mathbf{L}^\kappa$ ∎

Lemma 2.4: *The operator $\bar{\mathrm{S}}$ (cf. (2.16)) is continuous in the space $\mathbf{L}^\kappa$, and*

$$\|\bar{\mathrm{S}}\phi_n - \bar{\mathrm{S}}\phi\|_\kappa \longrightarrow 0.$$

Proof: The assertions follow from [1, §1.4, Lemma 4.2] and Lemma 2.3 ∎

2.3 Existence proof

Now, we are going to prove some assertions about the image and the continuity of the operator $\mathbf{T}$ defined in (2.9). For constants R, $R_0 \ge 0$ and $\lambda \in (0,1)$ we define

$$\mathcal{K}^0_{R,R_0,\lambda} = \{u \in \mathbf{C}_0 : \|u\|_\infty \le R, \ |u(x_1) - u(x_2)| \le R_0|x_1 - x_2|^\lambda, \ \forall x_1, x_2 \in [-1,1]\}.$$

Proposition 2.5: *There exist constants R, R_0, and λ such that for the operator $\mathbf{T}$ defined by (2.9) and (2.10) the inclusion $\mathbf{T}(\mathbf{C}_0) \subset \mathcal{K}^0_{R,R_0,\lambda}$ holds.*

Proof: Using (2.13), (2.15), Lemma 2.4, (2.17), Lemma 2.1, and the proof of Lemma 2.3, one can estimate $\|L(\cdot,u)\|_p$, $p = \kappa/2$, by some constant R_0. If we define $R = 2^{1/q}R_0$, $q^{-1} = 1 - p^{-1}$, and $\lambda = q^{-1}$ we conclude $\|\mathbf{T}u\|_\infty \le R$ and $|(\mathbf{T}u)(x_1) - (\mathbf{T}u)(x_2)| \le R_0|x_1 - x_2|^\lambda$ ∎

Proposition 2.6 (cf. [12]): *The operator $\mathbf{T} : \mathbf{C}_0 \longrightarrow \mathbf{C}_0$ defined by (2.9) and (2.8) is continuous.*

Proof: One can show that $\|L(\cdot,u_n) - L(\cdot,u)\|_p \longrightarrow 0$ ∎

As a consequence of Proposition 2.5 we obtain $\mathbf{T}(\mathcal{K}^0_{R,R_0,\lambda}) \subset \mathcal{K}^0_{R,R_0,\lambda}$. Since $\mathcal{K}^0_{R,R_0,\lambda}$ is a convex and compact subset of $\mathbf{C}_0$ we are able to apply Schauder's fixed point theorem to equation (2.9) in accordance to Propositions 2.5 and 2.6. We terminate at the following theorem.

Theorem 2.7: *Let A1 be fulfilled. Then problem (2.1), (2.2) possesses a solution $u \in \mathcal{W}_0^{1+}$.*

Let $u \in \mathbf{C}_0$ be a solution of the fixed point equation (2.9). Then the functions $a(x) = F_u(x, u(x))$ and $\alpha(x) = \frac{1}{2} - \frac{1}{\pi}\arctan a(x)$ are continuous. Consequently, the lemma from [13, Appendix] implies the representation

$$z(x) = (1-x)\sqrt{1+[a(x)]^2}\exp[-\pi(\mathrm{S}\alpha)(x)] = (1+x)^{\beta_1}(1-x)^{\beta_2}w(x),$$

where $w, w^{-1} \in \mathbf{L}^r$ for $r \in (1,\infty)$, $\beta_1 = \alpha(-1)$, $\beta_2 = 1-\alpha(1)$, and $\beta_j \in (0,1)$, $j = 1,2$. Hence, $a\,g/(1+a^2) \in \mathbf{L}^s$ for $1 < s < \delta^{-1}$. Since the operator $(1+x)^{-\beta_1}(1-x)^{-\beta_2}\mathrm{S}(1+y)^{\beta_1}(1-y)^{\beta_2}$ is continuous in $\mathbf{L}^s$ for $1 < s < \min\left\{\beta_1^{-1}, \beta_2^{-1}\right\}$, it follows, for $1 < p < \min\left\{\beta_1^{-1}, \beta_2^{-1}, \delta^{-1}\right\}$, $L(\cdot,u) \in \mathbf{L}^p$ (cf. (2.10)). Thus, we have proved the following (cf. [13, Part I, Theorem 2]).

Corollary 2.8: *Let Assumption A1 be fulfilled. Then problem (2.1), (2.2) possesses a solution $u(x) = \int_{-1}^{x} v(t)\,dt$, where $v \in \bigcap_{1<p<p_0} \mathbf{L}^p$ and*

$$p_0 = \min\left\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\delta}\right\}, \qquad \beta_1 = \frac{1}{2} - \frac{1}{\pi}\arctan F_u(-1,0), \qquad \beta_2 = \frac{1}{2} + \frac{1}{\pi}\arctan F_u(1,0).$$

Remark 2.9: *The assertions of Theorem 2.7 and Corollary 2.8 remain valid if instead of condition (2.4) we assume that there exist constants $b_1, b_2 \geq 0$ and $\nu \in [0,1)$ such that*

$$|F_x(x,u)| \leq l_0(K)(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1]\times[-K,K],\ K > 0,$$

with $l_0(K) = b_1 + b_2K^\nu$ and $\delta \geq 0$ satisfying (2.5) (cf. [13, Part I, §4.1, Remark]).

Proof: Let $u \in \mathbf{C}_0$ and $\|u\|_\infty \leq K$. From the proofs of Lemma 2.1, Lemma 2.3, and Proposition 2.5 we conclude

$$\|h\|_p \leq l_0(K)\,c_1, \quad \|\phi\|_\kappa \leq l_0(K)\,c_2, \quad \|L(\cdot,u)\|_p \leq l_0(K)\,c_3,$$

where

$$c_1 = \left\{\int_{-1}^{1}(1-x^2)^{-\delta p}dx\right\}^{1/p}, \quad c_3 = c_1 + d(\omega_+,\kappa)\|\bar{\mathrm{S}}\|_\kappa c_2,$$

$$c_2 = \left\{\int_{-1}^{1}(1-x^2)^{(\varepsilon+\kappa-2\delta\varepsilon\kappa)/2(\varepsilon-\kappa)}dx\right\}^{(\varepsilon-\kappa)/\varepsilon\kappa}\left(\frac{\pi}{\cos(\varepsilon\,\omega_+)}\right)^{1/\varepsilon},$$

If we choose $K > 0$ such that $2^{1/q}c_3 l_0(K) \leq K$, $q^{-1} = 1 - p^{-1}$, we obtain $\mathbf{T}(\mathcal{K}^0_{R,R_0,\lambda}) \subset \mathcal{K}^0_{R,R_0,\lambda}$ for $R = 2^{1/q}R_0$, $R_0 = l_0(K)\,c_3$, and $\lambda = q^{-1}$. Since Proposition 2.6 remains valid we can apply Schauder's fixed point theorem ∎

3 Equations of the second type

Now, consider equations of the kind

$$u(x) = (\mathrm{S}[F(\cdot,u)])(x) + f(x) + c, \qquad -1 \leq x \leq 1, \tag{3.1}$$

with the conditions

$$F(-1,u(-1)) = F(1,u(1)) = 0. \tag{3.2}$$

The set of functions u for which a real number $p > 1$ and a function $v \in \mathbf{L}^p$ exist such that $u(x) = u(-1) + \int_{-1}^{x} v(y)\,dy$ and for which the conditions (3.2) are fulfilled will be denoted by $\mathcal{W}_F^{1+}$. We distinguish two cases and make the following assumption.

A2: $F = F(x,u) : [-1,1] \times \mathbb{R} \to \mathbb{R}$ *possesses a continuous partial derivative* F_u, *where*

$$F_u(-1,u) \geq 0, \ F(1,u) = 0, \qquad \forall u \in \mathbb{R}, \tag{3.3}$$

$$\lim_{u \to -\infty} F(-1,u) < 0, \ \lim_{u \to +\infty} F(-1,u) > 0, \tag{3.4}$$

$$-l_1 \leq F_u(x,u) \leq l_2, \ \ (x,u) \in [-1,1] \times \mathbb{R}, \ \ l_1, l_2 \geq 0, \ \ l_1 l_2 < 1 \tag{3.5}$$

in **Case 1**, *and*

$$0 < l_1 \leq F_u(x,u) \leq l_2 < \infty \tag{3.6}$$

in **Case 2**. *The partial derivative* F_x *satisfies the Carathéodory condition. The function* f *is absolutely continuous and possesses a measurable derivative* f'. *Furthermore, there exist constants* δ, $l_0 \geq 0$ *such that*

$$|F_x(x,u)|, |f'(x)| \leq l_0(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1] \times \mathbb{R}, \tag{3.7}$$

where, in **Case 1**,

$$\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi} \tag{3.8}$$

and, in **Case 2**,

$$\delta < \frac{1}{2} + \frac{1}{2\pi} \arctan l_1. \tag{3.9}$$

We remark that in **Case 1** the condition $F(1,u(1)) = 0$ is automatically satisfied.

3.1 Reduction to a fixed point equation

Differentiation of (3.1) with regard to (3.2) yields

$$v(x) - (\mathrm{S}av)(x) = g(x), \tag{3.10}$$

where $a(x) = F_u(x,u(x))$ and $g(x) = f'(x) + (\mathrm{S}[F_x(\cdot,u)])(x)$. We define $\gamma(x) = \arctan a(x)$ (principal branch), $r(x) = \sqrt{1+a^2(x)}$, and $z_j(x) = (1-x)^{\delta_{j2}} r(x) \exp[-(\mathrm{S}\gamma)(x)]$, where δ_{jk}

denotes the Kronecker delta. Using again the results of [1, §9.5], we obtain the fixed point equation

$$u(x) = (\mathbf{T}_j u)(x) := k_j + \int_{-1}^{x} [L_j(y, u(y)) + c_j z_j^{-1}(y)]dy, \quad u \in \mathbf{C} \tag{3.11}$$

in **Case j**, where

$$L_j(x, u(x)) = \frac{g(x)}{r^2(x)} + \frac{z_j^{-1}(x)}{\pi} \int_{-1}^{1} \frac{a(y)z_j(y)g(y)}{r^2(y)} \frac{dy}{y-x}, \tag{3.12}$$

and the constants k_j and c_j ($j = 1, 2$) have to fulfil the equations $c_1 = 0$,

$$F(-1, k_j) = 0, \tag{3.13}$$

$$F\Big(1, k_2 + \int_{-1}^{1} L_2(x, u(x))\, dx + c_2 \int_{-1}^{1} z_2^{-1}(x)\, dx\Big) = 0. \tag{3.14}$$

With the same arguments as in Subsection 2.1 one can show that the problem (3.1),(3.2) for $u \in \mathcal{W}_F^{1+}$ is equivalent to the fixed point problem (3.11) in the respective **Case j**.

3.2 Investigation of the kernels $\mathrm{L_j(x, u(x))}$

We shall investigate the operators $\mathbf{T}_j$ in the Banach space $\mathbf{C} = \mathbf{C}[-1, 1]$. To prove the continuity of $\mathbf{T}_j$ we consider an arbitrary sequence $\{u_n\} \subset \mathbf{C}$ with $\|u_n - u\|_\infty \longrightarrow 0$ and use analogous notations as in Subsection 2.2. Obviously, relation (2.11) remains valid, which implies $\|\gamma_n - \gamma\|_\infty \longrightarrow 0$. Analogously to (2.13) we write

$$L_j(x, u(x)) = \cos^2[\gamma(x)]g(x) + \cos[\gamma(x)]M_j(x, u(x)), \tag{3.15}$$

where $h(x) = a(x)g(x)/r(x) = \sin[\gamma(x)]\, g(x)$ and

$$M_j(x, u(x)) := (1-x)^{-\delta_{j2}} \exp[(\mathbf{S}\gamma)(x)]\Big(\mathbf{S}\Big\{(1-y)^{\delta_{j2}} h(y) \exp[-(\mathbf{S}\gamma)(y)]\Big\}\Big)(x).$$

Lemma 3.1: *If* $1 < p < \delta^{-1}$, *then* $g_n \longrightarrow g$ *in* $\mathbf{L}^p$. *Moreover, we have*

$$\|g\|_p \le (1 + \|\mathbf{S}\|_p)\, c(\delta, p) \qquad \forall u \in \mathbf{C}.$$

Proof: Compare the proof of Lemma 2.1 ∎

In **Case 1**, let $2\omega_- := \arctan l_2 - \arctan l_1$, $2\omega_+ := \arctan l_2 + \arctan l_1$, and, in **Case 2**, $2\omega_- := \frac{\pi}{2} + \arctan l_1$, $2\omega_+ := \frac{\pi}{2} - \arctan l_1$. Furthermore, let $\mu(x) := \gamma(x) - \omega_-$. We remark that in both cases we have $\omega_+ \in (0, \pi/4)$, $\delta < 3/4 - \omega_+/\pi$, $|\mu(x)| \le \omega_+$. Furthermore, the relations $|\omega_-| < \pi/4$ in **Case 1** and $\pi/4 < \omega_- < \pi/2$ in **Case 2** are fulfilled. Again we write

$$M_j(x, u(x)) = \bar{R}(x)(\bar{\mathbf{S}}_j \phi)(x). \tag{3.16}$$

$\bar{\mathbf{S}}_j\phi$ is defined as in (2.16) (with $\tilde{N}_j$ instead of $\tilde{N}$), where

$$\begin{aligned}\tilde{N}_j(x) &= (1+x)^{-\omega_-/\pi}(1-x)^{-\delta_{j2}+\omega_-/\pi},\\ \phi(x) &= (1-x^2)^{1/2\kappa}h(x)\exp[-(\mathbf{S}\mu)(x)],\\ \tilde{R}(x) &= (1-x^2)^{-1/2\kappa}\exp[(\mathbf{S}\mu)(x)],\end{aligned} \tag{3.17}$$

and the number κ can be chosen so that $3-2\delta\kappa>0$ and

$$2<\kappa<\min\left\{\frac{\pi}{2\omega_+},3,\frac{2}{\delta}\right\},\qquad \frac{\kappa}{3-2\delta\kappa}<\frac{\pi}{2\omega_+}.$$

Since $\kappa<\frac{\pi}{2\omega_+}$ and $|\mu(x)|\le\omega_+$, the estimation (2.17) for all $u\in\mathbf{C}$ and Lemma 2.2 remain valid. In **Case 2** we additionally require

$$\kappa<\frac{3}{2(1-\omega_-/\pi)}. \tag{3.18}$$

Lemma 3.2: *The relation $\|\phi_n-\phi\|_\kappa\longrightarrow 0$ holds true.*

Proof: We proceed in the same way as in the proof of Lemma 2.3. Hence, we put $\phi=(\psi+\omega)\chi$ with

$$\begin{aligned}\psi(x) &= \sin[\gamma(x)](1-x^2)^\eta f'(x),\\ \omega(x) &= \sin[\gamma(x)](1-x^2)^\eta\Big(\mathbf{S}\Big\{(1-y^2)^{-\eta}[(1-y^2)^\eta F_x(y,u(y))]\Big\}\Big)(x),\\ \chi(x) &= (1-x^2)^{-1/2\varepsilon}\exp[-(\mathbf{S}\mu)(x)],\end{aligned}$$

where $\eta=(\varepsilon+\kappa)/2\varepsilon\kappa$ and ε is a fixed number such that

$$\max\left\{\kappa,\frac{\kappa}{3-2\delta\kappa}\right\}<\varepsilon<\frac{\pi}{2\omega_+}.$$

Define $\tilde{p}=\varepsilon$ and $\tilde{q}=\varepsilon\kappa/(\varepsilon-\kappa)$. Then $\tilde{p}^{-1}+\tilde{q}^{-1}=\kappa^{-1}$ and $\|\phi\|_\kappa\le(\|\psi\|_{\tilde{q}}+\|\omega\|_{\tilde{q}})\|\chi\|_{\tilde{p}}$. Since $3\varepsilon/2\varepsilon\kappa<3/2<\kappa$, we obtain $\tilde{q}^{-1}-1<-\eta<\tilde{q}^{-1}$. Thus, $\rho^{-1}\mathbf{S}\rho\mathbf{I}\in\mathcal{L}(\mathbf{L}^{\tilde{q}})$, where $\rho(x)=(1-x^2)^{-\eta}$. Consequently, in the same manner as in the proof of Lemma 2.3 it follows the assertion and $\|\phi\|_\kappa\le e(\kappa,\varepsilon)[1+\|\rho^{-1}\mathbf{S}\rho\mathbf{I}\|_{\tilde{q}}]d(\omega_+,\varepsilon)$ ∎

Lemma 3.3: *The operators $\bar{\mathbf{S}}_j$, $j=1,2$ (cf. (2.16), (3.17)), are continuous in $\mathbf{L}^\kappa$.*

Proof: As in the proof of Lemma 2.4 we write $\bar{\mathbf{S}}_j$ in the form $\rho_j^{-1}\mathbf{S}\rho_j\mathbf{I}$, where

$$\rho_j(x)=(1+x)^{\alpha_1}(1-x)^{\alpha_2},\quad \alpha_1=\frac{\omega_-}{\pi}-\frac{1}{2\kappa},\quad \alpha_2=\delta_{j2}-\frac{\omega_-}{\pi}-\frac{1}{2\kappa}. \tag{3.19}$$

Because of $2<\kappa<3$ and $-1/4<\omega_-/\pi<1/2$ we have $3/2\kappa-1<\omega_-/\pi<3/2\kappa$, i.e. $\kappa^{-1}-1<\alpha_1<\kappa^{-1}$. Since in **Case 1** we have $|\omega_-|/\pi<1/4$, it also holds $3/2\kappa-1<-\omega_-/\pi<3/2\kappa$, which implies $\kappa^{-1}-1<\alpha_2<\kappa^{-1}$. In **Case 2** we obtain from (3.18) that $3/2\kappa-1<1-\omega_-/\pi<3/2\kappa$, which yields $\kappa^{-1}-1<\alpha_2<\kappa^{-1}$, too ∎

Corollary 3.4: *In both cases $j=1,2$ there holds $\|\bar{\mathbf{S}}_j\phi_n-\bar{\mathbf{S}}_j\phi\|_\kappa\longrightarrow 0$.*

Lemma 3.5: *In* **Case 2** *the* $\mathbf{L}^{\kappa/2}$*-norm of* z_2^{-1} *is uniformly bounded with respect to* $u \in \mathbf{C}$. *Furthermore, it holds* $z_{2,n}^{-1} \longrightarrow z_2^{-1}$ *in* $\mathbf{L}^{\kappa/2}$.

Proof: Write $z_2^{-1}(x) = \cos[\gamma(x)]\rho_2^{-1}(x)\tilde{R}(x)$, where ρ_2 and $\tilde{R}$ are defined in (3.19) and (3.17). Since $\alpha_k < \kappa^{-1}$, $k = 1,2$ (cf. the proof of Lemma 3.3), it holds $\rho_2^{-1} \in \mathbf{L}^{\kappa}$. Thus, in view of (2.17),

$$\|z_2^{-1}\|_p \leq \|\rho_2^{-1}\|_\kappa d(\omega_+,\kappa). \tag{3.20}$$

The second assertion follows from Lemma 2.2 ∎

3.3 Existence proof

Because of (3.4) there exists a solution k_1 of equation (3.13) in **Case 1**, which we will fix for the sequel. In **Case 2** the solutions k_2 of (3.13) and $\tilde{k}_2$ of $F(1,\tilde{k}_2) = 0$ are uniquely determined.

Lemma 3.6 ([13], Part II, §3.2): *There exists a constant* $D > 0$ *such that*

$$\int_{-1}^{1} z_2^{-1}(x)\,dx \geq D \qquad \forall u \in \mathbf{C}.$$

For constants R, $R_0 \geq 0$ and $\lambda \in (0,1)$ we define

$$\mathcal{K}^0_{R,R_0,\lambda} = \{u \in \mathbf{C} : \|u\|_\infty \leq R,\ |u(x_1) - u(x_2)| \leq R_0|x_1 - x_2|^\lambda,\ \forall x_1, x_2 \in [-1,1]\}.$$

Proposition 3.7: *It holds* $\mathbf{T}_j(\mathbf{C}) \subset \mathcal{K}_{R,R_0,\lambda}$ *for some constants* R, $R_0 \geq 0$ *and* $\lambda \in (0,1)$.

Proof: We estimate $\|L_j(\cdot,u)\|_p$ (cf. (3.12)). Having regard to (3.20), Lemma 3.6, and

$$c_2 = \left\{\int_{-1}^{1} z_2^{-1}(x)\,dx\right\}^{-1}\left\{\tilde{k}_2 - k_2 - \int_{-1}^{1} L_2(x,u(x))\,dx\right\} \tag{3.21}$$

the proof can be completed in the same way as the proof of Proposition 2.5 ∎

Proposition 3.8 (cf. [12]): *The operators* $\mathbf{T}_j : \mathbf{C} \to \mathbf{C}$ *defined by (3.11) are continuous.*

In accordance to Propositions 3.7 and 3.8 the operators $\mathbf{T}_j : \mathcal{K}_{R,R_0,\lambda} \to \mathcal{K}_{R,R_0,\lambda}$, $j = 1,2$, satisfy the conditions of Schauder's fixed point theorem. Thus, the following theorem is proved.

Theorem 3.9: *Let Assumption A1 for the* **Case 1** *or* **Case 2** *be fulfilled. Then problem (3.1), (3.2) possesses a solution* $u \in \mathcal{W}_F^{1+}$.

In the same manner as Corollary 2.8 and Remark 2.9 we can prove the following assertions.

Corollary 3.10 (cf. [13], Part II, Theorem 2): *Under the conditions of Theorem 3.9 the problem (3.1), (3.2) possesses a solution* $u(x) = u(-1) + \int_{-1}^{x} v(t)\,dt$, *where* $v \in \bigcap_{1<p<p_0} \mathbf{L}^p$ *and in*

$$\textbf{Case 1:}\quad p_0 = \min\left\{\frac{1}{\mu_1}, \frac{1}{\delta}\right\}, \qquad \textbf{Case 2:}\quad p_0 = \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\delta}\right\},$$

where

$$\mu_1 = \frac{1}{\pi}\sup\{\arctan F_u(-1,u) : u \in \mathbb{R}\},\quad \mu_2 = 1 - \frac{1}{\pi}\inf\{\arctan F_u(1,u) : u \in \mathbb{R}\}.$$

Remark 3.11 (cf. [13], Part II, §4.1, Remark 1): *Condition (3.7) (with respect to* F_x*) can be replaced by the weaker conditions of Remark 2.9 with* $\delta \geq 0$ *satisfying (3.8) resp. (3.9).*

3.4 The case $\mathbf{F(\pm 1, u) \equiv 0}$

Let us consider the equation

$$u(x) = (\mathbf{S}[F(\cdot,u)])(x) + f(x), \qquad -1 \le x \le 1, \quad u \in \mathcal{W}_F^{1+}, \tag{3.22}$$

under the following assumption.

A3: *$F = F(x,u) : [-1,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ possesses a continuous partial derivative F_u satisfying*

$$-l_1 \le F_u(x,u) \le l_2, \quad (x,u) \in [-1,1] \times \mathbb{R}, \quad l_1, l_2 \ge 0, \quad l_1 l_2 < 1, \tag{3.23}$$

and

$$F(\pm 1, u) = 0 \qquad \forall u \in \mathbb{R}. \tag{3.24}$$

With respect to F_x and f' let Assumption A2 be fulfilled with relations (3.7) and (3.8).

As in Subsection 3.1 we obtain a fixed point equation

$$u(x) = (\mathbf{T}_0 u)(x) := k_0 + \int_{-1}^{x} L_0(y, u(y))\, dy, \quad u \in \mathbf{C}, \tag{3.25}$$

which is equivalent to problem (3.22). The kernel $L_0(x,u(x))$ is given by equation (3.12) and $z_0(x) = r(x)\exp[-(\mathbf{S}\gamma)(x)]$. The constant k_0 is determined by

$$k_0 = F_0 - \frac{1}{\pi}\int_{-1}^{1} (1-x^2)^{-1/2} \int_{-1}^{x} L_0(y,u(y))\, dy\, dx, \tag{3.26}$$

where $F_0 = \int_{-1}^{1}(1-x^2)^{-1/2} f(x)\, dx$. Relation (3.26) follows by multiplying equation (3.22) by $(1-x^2)^{-1/2}$ and integrating over $[-1,1]$. Thus, all the results obtained in Subsection 3.2 with respect to the kernel $L_1(x,u(x))$ remain valid for the kernel $L_0(x,u(x))$. Taking into account (3.26), which yields $|k_0| \le |F_0| + \|L_0(\cdot,u)\|_p$, it is easily seen that Propositions 3.7 and 3.8 hold true also for $j = 0$. Thus, the following theorem is in force (cf. [13, Part II, Theorem 2]).

Theorem 3.12: *If Assumption A3 is fulfilled, problem (3.22) possesses a solution $u \in \mathcal{W}_F^{1+}$. Furthermore, $u(x) = u(-1) + \int_{-1}^{x} v(t)\, dt$, where $v \in \bigcap\limits_{1<p<1/\delta} \mathbf{L}^p$.*

The second part of the theorem can be proved in the same way as Corollary 2.8. Remark 3.11 also remains valid (with $\delta \ge 0$ satisfying (3.8), cf. also [13, Part II, §4.1, Remark 1]).

4 Equations of the third type

We consider the equation

$$u(x) = (\mathbf{S}[F(\cdot,u)])(x) + f(x) + c + d\,x, \qquad -1 \le x \le 1, \tag{4.1}$$

with the conditions

$$u(-1) = u(1) = 0 \tag{4.2}$$

and make the following assumption.

A4: *$F = F(x,u) : [-1,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ possesses a continuous partial derivative F_u, where*

$$-l_1 \le F_u(x,u) \le l_2, \quad (x,u) \in [-1,1] \times \mathbb{R}, \quad l_1, l_2 \ge 0, \quad l_1 l_2 < 1, \tag{4.3}$$

$$F(\pm 1, 0) = 0, \tag{4.4}$$

$$F_u(-1,0) \ge 0, \qquad F_u(1,0) \le 0. \tag{4.5}$$

The partial derivative F_x satisfies the Carathéodory condition. The function f is absolutely continuous and possesses a measurable derivative f'. Furthermore, there exist constants δ, $l_0 \ge 0$ such that

$$|F_x(x,u)|, |f'(x)| \le l_0(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1] \times \mathbb{R}, \tag{4.6}$$

where

$$\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}. \tag{4.7}$$

We seek functions $u \in \mathcal{W}_0^{1+}$ and real numbers c, d satisfying (4.1) and (4.2). Having regard to (4.2) and (4.4), for the derivative v of u, we again obtain equation (3.10) with $g + d$ instead of g. The index of this equation in $\mathbf{L}^p$ for all sufficiently small $p > 1$ is equal to zero because of (4.5). Taking into account

$$\frac{z(x)}{r^2(x)} + \frac{1}{\pi} \int_{-1}^{1} \frac{a(y)z(y)}{r^2(y)} \frac{dy}{y-x} = 1$$

(cf. [4, §1.1]) this leads to the fixed point equation

$$u(x) = \int_{-1}^{1} [L(y, u(y)) + d\, z^{-1}(y)]\, dy, \qquad u \in \mathbf{C},$$

where

$$L(x, u(x)) = \frac{g(x)}{r^2(x)} + \frac{z^{-1}(x)}{\pi} \int_{-1}^{1} \frac{a(y)z(y)g(y)}{r^2(y)} \frac{dy}{y-x}, \quad z(x) = r(x)\exp[-(\mathbf{S}\gamma)(x)],$$

$r(x)$, $\gamma(x)$ as in Section 3.1, and $d = -\int_{-1}^{1} L(x, u(x))\, dx / \int_{-1}^{1} z^{-1}(x)\, dx$. Now, the proof of the following theorem can be given in the same manner as that of Theorem 3.9 for the Case 1 in Subsections 3.2 and 3.3.

Theorem 4.1: *Let Assumption A4 be fulfilled. Then, for problem (4.1), (4.2), there exists a solution $u \in \mathcal{W}_0^{1+}$. Moreover, $u(x) = \int_{-1}^{x} v(t)\, dt$, where $v \in \bigcap_{1<p<p_0} \mathbf{L}^p$ and*

$$p_0 = \min\left\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\delta}\right\}, \quad \beta_1 = \frac{1}{\pi} \arctan F_u(-1,0), \quad \beta_2 = -\frac{1}{\pi} \arctan F_u(1,0)$$

($\beta_j^{-1} = \infty$ if $\beta_j = 0$).

Remark 4.2: *The condition (4.6) on F_x in Assumption A4 can be replaced by the weaker conditions of Remark 2.9 with $\delta \ge 0$ satisfying (4.7).*

5 An example

In [11] there is considered the problem of two-dimensional free surface seepage flow from a nonlinear channel (see the picture). The problem is transformed into a nonlinear singular integral equation

$$x(t) - \frac{1}{\pi}\int_{-1}^{1}\frac{g(x(s))\,ds}{s-t} - D = R(t)\,, \qquad -1 < t < 1\,, \tag{5.1}$$

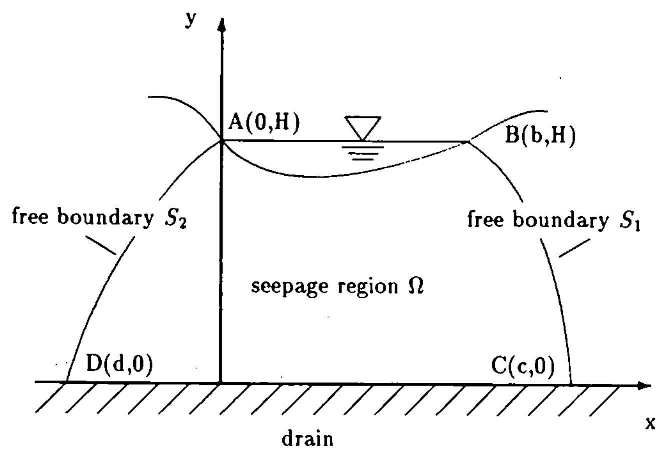
where $g(x)$, $0 \le x \le b$, describes the shape of the channel and the right-hand side is given by

$$R(t) = \frac{1}{\pi}\int_{1}^{1/k}\left[\frac{r(k,\sigma)}{\sigma - t} - \frac{r(k,-\sigma)}{\sigma + t}\right]d\sigma\,, \quad r(k,t) = H - \frac{H}{K'}\int_{t}^{-1}\frac{d\sigma}{\sqrt{(\sigma^2-1)(1-k^2\sigma^2)}}\,,$$

$$K' = F(\sqrt{1-k^2},1)\,, \qquad F(k,\zeta) = \int_0^{\zeta}\frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}}\,.$$

The function x and the parameters $k \in (0,1)$ and $D \in \mathbb{R}$ are unknown, where x has to satisfy the boundary conditions

$$x(-1) = b\,,\ x(1) = 0\,. \tag{5.2}$$



Substituting $x(t) = u(t) + \frac{b}{2}(1-t)$, from (5.1) we obtain

$$u(t) = \frac{1}{\pi}\int_{-1}^{1}\frac{F(s,u(s))\,ds}{s-t} - D - E\,t = f(t)\,, \qquad -1 < t < 1\,, \tag{5.3}$$

where

$$F(t,u) = g(\frac{b}{2}(1-t) + u) - H\,, \tag{5.4}$$

$$f(t) = R(t) - \frac{b}{2}(1-t) + \frac{H}{\pi}\ln\frac{1-t}{1+t}. \tag{5.5}$$

The conditions (5.2) are transformed into the homogeneous conditions

$$u(-1) = u(1) = 0. \tag{5.6}$$

We assume that g is continuously differentiable with $g'(0) < 0$, $g'(b) > 0$ and put

$$g(x) = \begin{cases} g'(0)\,x + H & , \quad x < 0, \\ g'(b)(x-b) + H & , \quad x > b, \end{cases}$$

such that F in (5.4) is defined for all $(t,u) \in [-1,1] \times \mathbb{R}$. The parameter E in (5.3) is an additional unknown, which is to be considered as a function $E = E(k)$ of $k \in (0,1)$. So we obtain the equation $E(k) = 0$ for the determination of k. We ask wether, for each fixed parameter k, Theorem 4.1 is applicable to equation (5.3). If we assume that g' is monotonously increasing, relation (4.3) is satisfied under the condition $|g'(0)\,g'(b)| < 1$. Relations (4.4) and (4.5) follow from $g(0) = g(b) = H$ and $F_u(-1,0) = g'(b)$, $F_u(1,0) = g'(0)$. It remains to show the existence of constants $l_0 > 0$ and $\delta \geq 0$ such that

$$|f'(t)| \leq l_0(1-t^2)^{-\delta}, \qquad -1 < t < 1, \tag{5.7}$$

and (4.7) are satisfied. By partial integration we obtain from (5.5)

$$f(t) = \frac{H}{\pi K'}\int_1^{1/k} \frac{\ln\frac{s-t}{s+t}\,ds}{\sqrt{(s^2-1)(1-k^2s^2)}} - \frac{b}{2}(1-t),$$

which implies

$$f'(t) = \frac{H}{\pi K'}\int_1^{1/k^2} \frac{ds}{(t^2-s)\sqrt{(s-1)(1-k^2s)}} + \frac{b}{2}.$$

Taking into account [7, §22] we conclude (5.7) for $\delta = \frac{1}{2}$. Thus, with the help of Theorem 4.1 it is possible to ensure the existence of a continuous solution u of (5.3), (5.4) for each fixed parameter $k \in (0,1)$ under natural conditions on the shape of the channel.

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