# On the Solvability of Nonlinear Singular Integral Equations

P. JUNGHANNS and U. WEBER

Three classes of nonlinear singular integral equations of Cauchy type occuring in the treatment of certain free boundary value problems are investigated. Existence of the solution is proved under weaker conditions than in [13] using the technique which was created in [12, 13] and is based on the application of Schauder's fixed point theorem.

Key words: *Nonlinear Cauchy type singular integral equations* 

AMS subject classification: 45E05, 45G05

# 1 Introduction

In investigating and solving certain free boundary value problems by means of nonlinear singular integral operators there occur equations of the form (cf. [2, 3, 10, 11])<br>  $F(x, u(x)) = \frac{1}{\pi} \int \frac{u(y) dy}{u - x} + c, \qquad -1 \le x \le 1,$ integral operators there occur equations of the form (cf. [2, 3, 10, 11])

$$
F(x,u(x)) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(y) \, dy}{y-x} + c, \qquad -1 \leq x \leq 1,
$$

and

$$
F(x, u(x)) = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{y - x} dx
$$
  
and  

$$
u(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{F(y, u(y)) dy}{y - x} + f(x) + c, \qquad -1 \le x \le 1.
$$
  
Therefore, one of the assumptions, under which in [13] the existence of a solution u with p

summable derivative,  $p > 1$ , was proved, is violated, namely the condition  $F(x, u(x)) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(y) dy}{y - x} + c,$   $-1 \le x \le 1,$ <br>  $u(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{F(y, u(y)) dy}{y - x} + f(x) + c,$   $-1 \le x \le 1.$ <br>
eby, one of the assumptions, under which in [13] the existence of a solution  $u$  with p-<br>
nable derivativ

$$
|f'(x)|, |F_x(x, u)| \le l_0(1 - x^2)^{-\delta}, \tag{1.1}
$$

where  $l_0$  and  $\delta < \frac{1}{2}$  are some nonnegative constants. In the above mentioned situation, the last relation is fulfilled only in case  $\delta = \frac{1}{2}$ . In [10, 11] one can find some remarks that, nevertheless, in the special cases considered there the equations are solvable.

In the present paper we show that the restriction on the constant  $\delta$  in (1.1) can be weakened even in the general situation. In particular, by keeping of all the other assumptions in [13) even  $\delta > \frac{1}{2}$  is permissible. The existence of the solution can be proved in principle with the technique of [12, 13] being based on the application of Schauder's fixed point theorem. In Section 2 this will be done for equations of the first type. Equations of the second type are considered in Section 3. Section 4 is dedicated to a third class of equations, for which an example is given in Section 5.  $\delta > \frac{1}{2}$  is permissible. The existence of the solution can be prof [12, 13] being based on the application of Schauder's fixed<br>be done for equations of the first type. Equations of the se<br>3. Section 4 is dedicated to

P. Junghanns: Techn. Univ. Chemnitz-Zwickau, FB Math., PSF 964, D - 09009 Chemnitz U. Weber: Techn. Univ. Chemnitz - Zwickau, FB Math., PSF 964, D - 09009 Chemnitz

## **2 Equations of the first type**

We consider singular integral equations of the form

P: JUNGHANNS and U. WEBER

\n**Equations of the first type**

\nconsider singular integral equations of the form

\n
$$
F(x, u(x)) = (Su)(x) + c, \quad -1 \leq x \leq 1,
$$
\n(2.1)

\nre S denotes the Cauchy singular integral operator

where S denotes the Cauchy singular integral operator  
\n
$$
(Su)(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(y)}{y - x} dy.
$$

We search for continuous functions *u* and real numbers *c* satisfying (2.1) and the additional **2 Equations of the first type**<br>We consider singular integral equations of the  $F(x, u(x)) = (Su)(x) + c, \t -1 \le x \le$ <br>where S denotes the Cauchy singular integral<br> $(Su)(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(y)}{y - x} dy.$ <br>We search for continuous func

$$
u(-1) = u(1) = 0. \tag{2.2}
$$

More precisely, we will ask for functions u for which a real number  $p > 1$  and a function  $v \in L^p$ *exist such that*  $u(x) = \int_{-1}^{x} v(y) dy$  *and*  $u(1) = 0$ *. Here*  $L^p = L^p(-1, 1)$  *denotes the usual Lebesgue* space of all measurable functions  $v$  for which  $|v(x)|^p$  is summable, and  $||v||_p$  is the usual norm in  $L^p$ . The set of all such functions *u* we will call  $W_0^{1+}$ . In order to prove the existence of a solution  $u \in \mathcal{W}_0^{1+}$  of problem (2.1), (2.2) we assume that the following assumption is fulfilled *hat*  $u(x) = \int_{-1}^{x} v(y) dy$  and  $u(1) = 0$ . Here  $L^p = L^p(-1, 1)$  denotes the usual Lebesgue<br> *l* measurable functions  $v$  for which  $|v(x)|^p$  is summable, and  $||v||_p$  is the usual norm<br>  $e$  set of all such functions  $u$  we wi

*Al:*  $F = F(x, u)$ :  $[-1, 1] \times \mathbb{R} \to \mathbb{R}$  ( $\mathbb{R} := (-\infty, +\infty)$ ) possesses a continuous partial derivative  $F_u$  and a partial derivative  $F_x$  which is continuous with respect to  $u \in \mathbb{R}$  for allmost all *x* and measurable with respect to  $x \in [-1,1]$  *for all*  $u \in \mathbb{R}$  (*Carathéodory condition*). *Furthermore, there exist constants*  $l_0, l_1, l_2, \delta \geq 0$  such that  $l_1 l_2 < 1$  and *F*( $x, u$ ): [-1, 1]  $\times$  **F**( $x$ ): [-22] we assume that the following assumption is fulfilled.<br> *F(x,u)*: [-1, 1]  $\times$  **F** 3 - arctan *l* + *arctan l + arctan <i>l* + *arctan l* + *arctan <i>l* + *arctan decisionstants*  $l_0, l_$ casurable with respect to  $x \in [-1,1]$  for all  $u \in \mathbb{R}$  (Carathéodory condition).<br>
state exist constants  $l_0, l_1, l_2, \delta \ge 0$  such that  $l_1 l_2 < 1$  and<br>  $\le F_u(x, u) \le l_2,$   $(x, u) \in [-1, 1] \times \mathbb{R},$  (2.3)<br>  $[x, u] \le l_0(1 - x^2)^{-\delta$ 

$$
-l_1 \le F_u(x, u) \le l_2, \qquad (x, u) \in [-1, 1] \times \mathbb{R}, \tag{2.3}
$$

$$
|F_x(x,u)| \le l_0(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1] \times \mathbb{R}, \qquad (2.4)
$$

$$
\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}.\tag{2.5}
$$

We remark that in view of  $l_1 l_2 < 1$  the estimation (2.5) is weaker than the condition  $\delta < 1/2$ from [13].  $|F_x(x, u)| \le l_0(1 - x^2)^{-\delta}, \qquad (x, u) \in [-1, 1] \times \mathbb{R},$ <br>  $\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}.$ <br>
We remark that in view of  $l_1 l_2 < 1$  the estimation (2.5) is west<br>
from [13].<br> **2.1** Reduction to a fixed point equation<br>
Let  $u(x) = \$ 

## 2.1 Reduction to a fixed point equation

*z.1 f* **<b>***f f f t*<sup>*t*</sup> *dydy de f f dy<i>f f f f f f f f f f f f f f f f f f f <i>f <i>f <i>f <i>f* Then we can differentiate equation (2.1) (cf. [8], Chapt. II, Lemma6.1). Taking into account We remark that in view of  $l_1 l_2 < 1$ <br>from [13].<br>**2.1** Reduction to a fixed poi<br>Let  $u(x) = \int_{-1}^{x} v(y) dy$  be a solution<br>Then we can differentiate equation<br>condition (2.2) we obtain<br> $a(x)v(x) - (Sv)(x) = g(x)$ , orp<br>(2.1)<br>. **Reduction to a fixed point equation**<br>  $u(x) = \int_{-1}^{x} v(y) dy$  be a solution of problem (2.1), (2.2) with the above required properties.<br>
1 we can differentiate equation (2.1) (cf. [8], Chapt. II, Lemma 6.1). Taking into acco *condition (2.2)* we obtain  $a(x)v(x) - (Sv)(x) = g(x)$ , (2.6)<br>
where  $a(x) = F_u(x, u(x))$  and  $g(x) = -F_x(x, u(x))$ . Since, in view of Assumption A1, the

$$
a(x)v(x) - (Sv)(x) = g(x), \tag{2.6}
$$

function a is continuous there exist continuous functions  $\alpha$ :  $[-1, 1] \rightarrow (0, 1)$  and  $r : [-1, 1] \rightarrow \mathbb{R}$ 

 $\mathcal{L}_{\text{max}}$  and  $\mathcal{L}_{\text{max}}$  and  $\mathcal{L}_{\text{max}}$  $\sqrt{2}$ 

such that  $a(x) + i = r(x) \exp[i\pi\alpha(x)]$ . From (2.2) it follows that the solution v of (2.6) has to satisfy the condition *On the Solvability of Nonlinear Singular Integral Equations* 685<br>
that  $a(x) + i = r(x) \exp[i\pi \alpha(x)]$ . From (2.2) it follows that the solution v of (2.6) has to<br> *J*<sub>1</sub>  $v(y) dy = 0$ . (2.7)<br>
and an explicit expression for the solution

$$
\int_{-1}^{1} v(y) \, dy = 0. \tag{2.7}
$$

To find an explicit expression for the solution  $v$  of problem  $(2.6)$ ,  $(2.7)$  we use the results of  $\lceil l \rceil$ , §9.5] (cf. also [4, 7, 8]). With  $P = \frac{1}{2}(I - iS)$  and  $Q = I - P = \frac{1}{2}(I + iS)$  (I denotes the identity operator) we can write equation (2.6) in the equivalent form ession for the solution<br>
With  $P = \frac{1}{2}(I - I)$ <br>
equation (2.6) is<br>  $Q(a + i)|v = g$ .<br>  $a(x) + i| = \exp\left[-i\pi\alpha(x) + \int_{-1}^{1} \frac{1}{2} \right]$ 

$$
\mathbf{B}\mathbf{v} := [\mathbf{P}(a-i) + \mathbf{Q}(a+i)]\mathbf{v} = g.
$$

Put  $c(x) = [a(x) - i]/[a(x) + i] = \exp[-2\pi i \alpha(x)]$ . Then, for all sufficiently small  $p > 1$ , the function *c* admits a generalized **L**<sup>p</sup>-factorization  $c(x) = c_-(x)(x - i)^{-1}c_+(x)$ , where )]. Then, for all sufficiently small<br>  $c(x) = c_-(x)(x - i)^{-1}c_+(x)$ , where<br>  $= \frac{(x - i)[a(x) - i]}{(1 - x)r(x)} exp[\pi(S\alpha)(x)],$ 

1 that 
$$
a(x) + i = r(x) \exp[i\pi\alpha(x)]
$$
. From (2.2) it follows that the solution v of (2.6) has to  
\n
$$
\int_{-1}^{1} v(y) dy = 0.
$$
\n(2.7)  
\nfind an explicit expression for the solution v of problem (2.6), (2.7) we use the results of [1,  
\n| (cf. also [4, 7, 8]). With  $P = \frac{1}{2}(I - iS)$  and  $Q = I - P = \frac{1}{2}(I + iS)$  (I denotes the identity  
\nratio) we can write equation (2.6) in the equivalent form  
\n
$$
Bv := [P(a - i) + Q(a + i)]v = g.
$$
\n
$$
c(x) = [a(x) - i]/[a(x) + i] = \exp[-2\pi i\alpha(x)].
$$
 Then, for all sufficiently small  $p > 1$ , the  
\nition c admits a generalized  $L^p$ -factorization  $c(x) = c_-(x)(x - i)^{-1}c_+(x)$ , where  
\n
$$
c_-(x) = \frac{x - i}{1 - x} \exp\left[-i\pi\alpha(x) + \int_{-1}^{1} \frac{\alpha(y)}{y - x} dy\right] = \frac{(x - i)[a(x) - i]}{(1 - x)r(x)} \exp[\pi(S\alpha)(x)],
$$
\n
$$
c_+(x) = (1 - x) \exp\left[-i\pi\alpha(x) - \int_{-1}^{1} \frac{\alpha(y)}{y - x} dy\right] = \frac{(1 - x)[a(x) - i]}{r(x)} \exp[-\pi(S\alpha)(x)].
$$
\n
$$
B^{(-1)} = \frac{c_+^{-1}}{a + i}[P(a + i) + Q(a - i)]\frac{c_+}{a - i}I = z^{-1}(aI + S)\frac{z}{r^2}I,
$$
\n(2.8)  
\n
$$
B(z) = (1 - x)r(x) \exp[-\pi(S\alpha)(x)].
$$
 Because of (2.4) and (2.5) we have  $g \in L^p$  for all  
\nciently small  $p > 1$ . We show that  $v = B^{(-1)}g$  is the solution of problem (2.6), (2.7). For

Hence,  $B: L^p \longrightarrow L^p$  is a Fredholm operator with index 1, and a right inverse of B is given by

$$
\mathbf{B}^{(-1)} = \frac{c_{+}^{-1}}{a+i} [\mathbf{P}(a+i) + \mathbf{Q}(a-i)] \frac{c_{+}}{a-i} \mathbf{I} = z^{-1} (a\mathbf{I} + \mathbf{S}) \frac{z}{r^{2}} \mathbf{I}, \qquad (2.8)
$$

where  $z(x) = (1-x)r(x) \exp[-\pi(S\alpha)(x)]$ . Because of (2.4) and (2.5) we have  $q \in L^p$  for all sufficiently small  $p > 1$ . We show that  $v = B^{(-1)}q$  is the solution of problem (2.6), (2.7). For this end we have to prove that  $v = B^{(-1)}q$  satisfies condition (2.7). Indeed, taking into account relation (2.8) we conclude that  $(a + i)c_{+}v$  lies in the image of the operator  $P(a + i)I + Q(a - i)I$ , which implies (cf. [1, §9.5]) *Let*  $\exp[-\pi(S\alpha)(x)]$ . Because of (2.4) and (2.5) we have  $g \in L^p$  for all  $e$  show that  $v = B^{(-1)}g$  is the solution of problem (2.6), (2.7). For and  $v = B^{(-1)}g$  satisfies condition (2.7). Indeed, taking into account  $(\alpha + i)c_+$ 

$$
0=\int_{-1}^{1}(a+i)c_{+}v[c_{-}-c_{+}^{-1}(x-i)](x-i)^{-1}dx=\int_{-1}^{1}(a+i)(c-1)v\,dx=-2i\int_{-1}^{1}v(x)\,dx.
$$

Integrating the obtained expression for *v,* we can summarize our investigations to the result that each solution  $u \in W_0^{1+}$  of problem  $(2.1), (2.2)$  is a solution of the fixed point equation

$$
u(x) = (\mathbf{T}u)(x) := \int_{-1}^{x} L(y, u(y)) dy, \qquad u \in \mathbf{C}_0,
$$
\n(2.9)

where

\n (2.8) we conclude that 
$$
(a + i)c_+v
$$
 lies in the image of the operator  $P(a + i)I + Q(a - i)I$ ,  $h$  implies (cf.  $[1, \S 9.5]$ ).\n

\n\n
$$
0 = \int_{-1}^{1} (a + i)c_+v[c_--c_+^{-1}(x - i)](x - i)^{-1}dx = \int_{-1}^{1} (a + i)(c - 1)v \, dx = -2i \int_{-1}^{1} v(x) \, dx
$$
.\n (rating the obtained expression for  $v$ , we can summarize our investigations to the result that solution  $u \in \mathcal{W}_0^{1+}$  of problem (2.1), (2.2) is a solution of the fixed point equation\n

\n\n
$$
u(x) = (\mathbf{T}u)(x) := \int_{-1}^{x} L(y, u(y)) \, dy, \quad u \in \mathbf{C}_0,
$$
\n

\n\n
$$
L(x, u(x)) = \frac{a(x)}{r^2(x)}g(x) + \frac{z^{-1}(x)}{\pi} \int_{-1}^{1} \frac{z(y)g(y)}{r^2(y)} \, dy - x
$$
\n

\n\n (2.10) 
$$
C_0 = C_0[-1, 1]
$$
 denotes the Banach space of all continuous functions on  $[-1, 1]$  satisfying\n

and  $C_0 = C_0[-1,1]$  denotes the Banach space of all continuous functions on  $[-1,1]$  satisfying the boundary conditions (2.2). In the sequel it will be proved that the kernel  $L(x, u(x))$  is an element of  $L^p$  for some  $p > 1$  if  $u \in C_0$ . Consequently, each solution of equation (2.9) is also a solution of problem  $(2.1)$ ,  $(2.2)$ .

The expression of the kernel of the integral operator  $T$  considered in  $[13]$  is a little bit more complicated than expression (2.10). It is possible to choose this simpler version since  $v = B^{(-1)}g$ fulfils condition (2.7).

### 2.2 Investigation of the kernel  $L(x,u(x))$

The aim of this subsection is to obtain some preliminary results which give us the possibility to handle the fixed point equation (2.9) with the help of Schauder's fixed point theorem. By 2.2 Investigation of the kernel  $L(x,u(x))$ <br>The aim of this subsection is to obtain some preliminary results which give us the possibility<br>to handle the fixed point equation (2.9) with the help of Schauder's fixed point theor  $a(x) = F_u(x, u(x))$ ,  $a_n(x) = F_u(x, u_n(x))$ , and analogous notations for functions depending on *u* or u,. Obviously we have P. JUNGHANNS and U. WEBER<br>
Investigation of the kernel  $L(x,u)$ <br>
aim of this subsection is to obtain some  $|$ <br>
andle the fixed point equation (2.9) with<br>  $\sigma = \sup\{|u(x)| : x \in [-1,1]\}$  we denote<br>
ws we assume  $\{u_n\} \subset C_0$ ,  $u \in C_0$ aim of this subsection is<br>
andle the fixed point equ<br>  $\infty = \sup\{|u(x)| : x \in [-\pi, u(x)) : a_n(x) =$ <br>  $F_u(x, u(x)), a_n(x) =$ <br>  $u_n$ . Obviously we have<br>  $||a_n - a||_{\infty} \longrightarrow 0$  (*n* -<br>  $\therefore$  the norms  $||u_n||_{\infty}$  are un<br>  $g_n \longrightarrow g$  a.e.<br>
Assumption A1). Wi

$$
\|\mathbf{a}_n - \mathbf{a}\|_{\infty} \longrightarrow 0 \qquad (n \to \infty), \tag{2.11}
$$

since the norms  $||u_n||_{\infty}$  are uniformly bounded. Furthermore, it holds

$$
g_n \longrightarrow g \quad a.e. \tag{2.12}
$$

(cf. Assumption A1). With  $\gamma(x) := \arctan a(x) = \pi/2 - \pi\alpha(x)$  we can write

$$
z(x) = \sqrt{1-x^2} \, r(x) \exp[(S\gamma)(x)].
$$

From (2.10) we conclude

$$
u_n. Obviously we have
$$
\n
$$
||a_n - a||_{\infty} \longrightarrow 0 \qquad (n \to \infty),
$$
\n
$$
||a_n||_{\infty} \text{ are uniformly bounded. Furthermore, it holds}
$$
\n
$$
g_n \longrightarrow g \quad a.e.
$$
\n
$$
g_n \longrightarrow g \quad a.e.
$$
\n
$$
(2.12)
$$
\nAssumption A1). With  $\gamma(x) := \arctan a(x) = \pi/2 - \pi \alpha(x)$  we can write\n
$$
z(x) = \sqrt{1 - x^2} \tau(x) \exp[(S\gamma)(x)].
$$
\n
$$
u(2.10) \text{ we conclude}
$$
\n
$$
L(x, u(x)) = \sin[\gamma(x)]h(x) + \cos[\gamma(x)]M(x, u(x)),
$$
\n
$$
u(x) = g(x)/r(x) = \cos[\gamma(x)]g(x) \text{ and}
$$
\n
$$
M(x, u(x)) = (1 - x^2)^{-1/2} \sin\left(\frac{(S\gamma(x))}{2}\right)g(x) \text{ and}
$$

where  $h(x) = g(x)/r(x) = \cos[\gamma(x)]g(x)$  and

$$
L(x, u(x)) = \sin[\gamma(x)] h(x) + \cos[\gamma(x)] M(x, u(x)),
$$
  
we  $h(x) = g(x)/r(x) = \cos[\gamma(x)] g(x)$  and  

$$
M(x, u(x)) := (1 - x^2)^{-1/2} \exp[-(S\gamma)(x)] \Big( S\Big\{ (1 - y^2)^{1/2} h(y) \exp[(S\gamma)(y)] \Big\} \Big)(x).
$$

**Lemma 2.1:** Let  $1 < p < \delta^{-1}$ . Then  $h \in L^p$  and  $h_n \sin \gamma_n \longrightarrow h \sin \gamma$  in  $L^p$ . Moreover, the *estimation*  $||h||_p \le c(\delta, p)$  *is valid, where the constant*  $c(\delta, p)$  *does not depend on*  $u \in \mathbb{C}_0$ *.* 

**Proof:** In view of the estimate (2.4) we have

$$
\int_{-1}^1 |h(x)|^p dx \leq l_0^p \int_{-1}^1 (1-x^2)^{-\delta p} dx =: c(\delta, p)^p < \infty.
$$

From (2.11) and (2.12) it follows  $h_n(x) \sin \gamma_n(x) \longrightarrow h(x) \sin \gamma(x)$  a.e., and (2.4) implies

 $|h_n(x) \sin \gamma_n(x) - h(x) \sin \gamma(x)|^p \leq 2^p l_0^p (1 - x^2)^{-\delta p}$ .

Now, from the Lebesgue theorem we conclude the assertion  $\blacksquare$ 

Let us introduce the notations  $2\omega_- := \arctan l_2 - \arctan l_1$ ,  $2\omega_+ := \arctan l_2 + \arctan l_1$ . Because of  $2|\omega_-| < \pi/2$  and  $2\omega_+ = \arctan((l_1 + l_2)/(1 - l_1 l_2)] \in (0, \frac{\pi}{2})$  we have  $3/(1+2|\omega_-|/\pi) > 2$ and  $\pi/2\omega_+ > 2$ . Thus, taking into account (2.5) there exists a  $\kappa > 2$  such that  $0 < 3 - 2\delta\kappa$ ,  $x \leq l_0^p \int_{-1}^1 (1-x^2)^{-\delta p} dx =: c(\delta, p)^p < \infty.$ <br>
(2.12) it follows  $h_n(x) \sin \gamma_n(x) \longrightarrow h(x)$ ;<br>  $((x) - h(x) \sin \gamma(x)) \longrightarrow (x^p)^{\delta p}$ <br>  $(\delta x) = (x^p - \delta x)^{\delta p}$ .<br>  $(\delta x) = (x^p - \delta x)^{\delta p}$ <br>  $(\delta x) = (x^p - \delta x)^{\delta p}$ .<br>  $(\delta x) = (x^p - \delta x)^{\delta p}$ .<br>  $(\delta x) = (x^p - \delta x)^{\$  $\begin{aligned}\n\mathbf{a}x &\leq t_0' \int_{-1}^{\infty} (1-x^2)^{-\nu} dx =: c(\delta, p)^{\nu} < \infty. \\
\mathbf{d} \left(2.12\right) \text{ it follows } h_n(x) \sin \gamma_n(x) \longrightarrow h(x) \sin \gamma(x) \\
\gamma_n(x) - h(x) \sin \gamma(x) |^p &\leq 2^p l_0^p (1-x^2)^{-\delta p}. \\
\text{Lebesgue theorem we conclude the assertion } \blacksquare \\
\text{roduce the notations } 2\omega_- := \arctan l_2 - \arctan l_1, \\
\lfloor \langle \pi/2 \text{ and } 2\omega_+$  $\begin{aligned}\n\gamma_n(x) - h(x) \sin \gamma(x) &= 2^p l_0^p (1 - x^2)^{-\delta p}.\n\end{aligned}$ <br>
e Lebesgue theorem we conclude the assertion  $\blacksquare$ <br>  $\begin{aligned}\n\text{troduce the notations } 2\omega_- &:= \arctan l_2 - \arctan l_1, 2\omega_+ &:= \arctan l_2 + \arctan l_1.\n\end{aligned}$ <br>  $\begin{aligned}\n\gamma_n(z) = \frac{z}{\pi} \arctan \left[ (l_$ 

$$
\kappa < \min\left\{\frac{\pi}{2\omega_+}, \frac{2}{\delta}, \frac{3}{1+2|\omega_-|/\pi}\right\} \quad \text{and} \quad \frac{\kappa}{3-2\delta\kappa} < \frac{\pi}{2\omega_+}.
$$

Defining  $\mu(x) := \gamma(x) - \omega$  we obtain (cf. (2.3))

$$
\kappa < \min\left\{\frac{\pi}{2\omega_{+}}, \frac{2}{\delta}, \frac{3}{1+2|\omega_{-}|/\pi}\right\} \quad \text{and} \quad \frac{\kappa}{3-2\delta\kappa} < \frac{\pi}{2\omega_{+}}.
$$
\n
$$
\min g \ \mu(x) := \gamma(x) - \omega_{-} \text{ we obtain (cf. (2.3))}
$$
\n
$$
|\mu(x)| \leq \omega_{+} \ (x \in [-1,1]) \quad \text{and} \quad (\text{S}\gamma)(x) = (\text{S}\mu)(x) + \frac{\omega_{-}}{\pi}\ln\frac{1-x}{1+x}, \tag{2.14}
$$

and, consequently,

On the Solvability of Nonlinear Singular Integral Equations 687  
consequently,  

$$
M(x, u(x)) = \tilde{R}(x)(\tilde{S}\phi)(x),
$$
 (2.15)

where

On the Solvability of Nonlinear Singular Integral Equations 687  
\nconsequently,  
\n
$$
M(x, u(x)) = \bar{R}(x)(\bar{S}\phi)(x),
$$
\n
$$
(\hat{S}\phi)(x) = \bar{N}(x)(1-x^2)^{1/2\kappa} \Big(S[\tilde{N}^{-1}(y)(1-y^2)^{-1/2\kappa}\phi(y)]\Big)(x),
$$
\n
$$
\bar{N}(x) = (1-x)^{-1/2-\omega_{-}/\kappa}(1+x)^{-1/2+\omega_{-}/\kappa},
$$
\n
$$
\phi(x) = (1-x^2)^{1/2\kappa}h(x) \exp[(S\mu)(x)],
$$
\n
$$
\bar{R}(x) = (1-x^2)^{-1/2\kappa} \exp[-(S\mu)(x)].
$$
\n
$$
\tilde{R}(x) = (1-x^2)^{-1/2\kappa} \exp[-(S\mu)(x)].
$$
\n
$$
\text{We obtain}
$$
\n
$$
||\tilde{R}||_{\kappa} \leq \left(\frac{\pi}{\cos(\kappa\omega_{+})}\right)^{1/\kappa} =: d(\omega_{+}, \kappa) \quad \forall u \in \mathbb{C}_{0}.
$$
\n
$$
\text{times 2: } \text{It holds } \tilde{R}_{n} \longrightarrow \tilde{R} \text{ in } \mathbb{L}^{\kappa}.
$$
\n(2.17)

In view of (2.14) and a lemma from [13, Appendix] we obtain  
\n
$$
\|\tilde{R}\|_{\kappa} \leq \left(\frac{\pi}{\cos(\kappa\omega_{+})}\right)^{1/\kappa} =: d(\omega_{+}, \kappa) \qquad \forall u \in \mathbf{C}_{0}.
$$
\n(2.17)

Lemma 2.2: It holds  $\tilde{R}_n \longrightarrow \tilde{R}$  in  $L^{\kappa}$ .

 $\Phi(u) = u^{\vartheta}$  and  $f_n(x) = |\tilde{R}_n(x)|^{\kappa}$  we can estimate

$$
ln \ln 2 - \cos(\kappa \omega_+)/
$$
\nLemma 2.2: *It holds*  $\hat{R}_n \longrightarrow \hat{R}$  in L<sup>κ</sup>.  
\nProof: Let  $\varepsilon > 0$ ,  $0 < \vartheta < \varepsilon/(1+\varepsilon)$ , and  $\kappa(1+\varepsilon)(1+\vartheta) < \pi/2\omega_+$ . Defining the functions  
\n
$$
\int_{-1}^{1} |f_n(x)| \Phi(|f_n(x)|) dx
$$
\n
$$
= \int_{-1}^{1} (1 - x^2)^{-(1+\vartheta)/2} \exp[-\kappa(1+\vartheta)(S\mu_n)(x)] dx
$$
\n
$$
= \int_{-1}^{1} (1 - x^2)^{-(2(1+\varepsilon)-\vartheta/2)} (1 - x^2)^{-1/2(1+\varepsilon)} \exp[-\kappa(1+\vartheta)(S\mu_n)(x)] dx
$$
\n
$$
\leq \left\{ \int_{-1}^{1} (1 - x^2)^{-(\varepsilon/2)(1+\varepsilon)+\vartheta/2} (1 - x^2)^{-1/2(1+\varepsilon)} \left\{ \int_{-1}^{1} (1 - x^2)^{-1/2} \exp[-\nu(S\mu_n)(x)] dx \right\}^{1/(1+\varepsilon)} \right\}
$$
\n
$$
= \left\{ \int_{-1}^{1} (1 - x^2)^{-1/2 - \vartheta(1+\varepsilon)/2\varepsilon} dx \right\}^{\varepsilon/(1+\varepsilon)} \left\| (1 - x^2)^{-1/2\nu} \exp[-(S\mu_n)(x)] \right\|_{\nu}^{\nu/(1+\varepsilon)}
$$
\n
$$
\leq const \left( \frac{\pi}{\cos(\nu \omega_+)} \right)^{1/(1+\varepsilon)},
$$

where we used the lemma from [13, Appendix] and the notation  $\nu := \kappa(1 + \varepsilon)(1 + \vartheta)$ . Since where we used the lemma from [13, Appendix] and the notation  $\nu := \kappa (1 + \varepsilon)(1 + \vartheta)$ . Since  $\|\mu_n - \mu\|_{\infty} \longrightarrow 0$  and since the operator S is continuous in L<sup>9</sup> for all  $q > 1$ , it follows  $S\mu_n \longrightarrow S\mu$  in measure, which impli in measure, which implies, in view of the monotonicity of  $e^u$ ,  $f_n \longrightarrow f$  in measure. From the lemmas of Vallée-Poussin and Vitali (cf. [9, Chapt. VI, §3]) we obtain

$$
\int_{-1}^x \tilde{R}_n(y) dy \longrightarrow \int_{-1}^x \tilde{R}(y) dy, \qquad \forall x \in [-1,1],
$$

as well as  $\|\tilde{R}_n\|_{\kappa} \longrightarrow \|\tilde{R}\|_{\kappa}$ . This yields the assertion **I** 

**Lemma 2.3:** For the above defined functions  $\phi_n$  and  $\phi$  (cf. (2.16)) we have  $\|\phi_n - \phi\|_{\kappa} \longrightarrow 0$ .

**Proof:** Let  $\max\{\kappa, \kappa/(3 - 2\delta\kappa)\} < \varepsilon < \pi/2\omega_+$ , and  $\phi = \psi\chi$ , where

$$
\psi(x) = h(x)(1-x^2)^{(\epsilon+\kappa)/2\epsilon\kappa}, \quad \chi(x) = (1-x^2)^{-1/2\epsilon} \exp[(S\mu)(x)].
$$

Froof: Let  $\max\{\kappa, \kappa/(3-2\delta\kappa)\} < \varepsilon < \pi/2\omega_+$ , and  $\phi = \psi\chi$ , where<br>  $\psi(x) = h(x)(1-x^2)^{(\varepsilon+\kappa)/2\varepsilon\kappa}$ ,  $\chi(x) = (1-x^2)^{-1/2\varepsilon} \exp[(S\mu)(x)].$ <br>
Further, let  $\tilde{p} := \varepsilon$  and  $\tilde{q} := \varepsilon\kappa/(\varepsilon-\kappa)$ . Then  $\tilde{p}^{-1} + \tilde{q}^{-1}$ where  $||\chi||_{\bar{p}} \leq d(\omega_{+}, \varepsilon)$  and  $\tilde{p} = h(x)(1 - x^2)^{(\epsilon + \kappa)/2\epsilon\kappa}, \quad \chi(x) = (1 - x^2)^{-1/2\epsilon} e$ <br> *it*  $\tilde{p} := \varepsilon$  and  $\tilde{q} := \varepsilon\kappa/(\varepsilon - \kappa)$ . Then  $\tilde{p}^{-1} + \tilde{q}^{-1} =$ <br>  $\tilde{p} \leq d(\omega_+, \varepsilon)$  and<br>  $\leq l_0^{\tilde{q}} \int_{-1}^{1} (1 - x^2)^{(\varepsilon + \kappa - 2\delta\epsilon\kappa)/2(\varepsilon - \$ 

$$
\|\psi\|_{\tilde{q}}^{\tilde{q}}\leq l_0^{\tilde{q}}\int_{-1}^1(1-x^2)^{(\epsilon+\kappa-2\delta\epsilon\kappa)/2(\epsilon-\kappa)}dx=:e(\kappa,\varepsilon)^{\tilde{q}}<\infty
$$

because of  $(\epsilon + \kappa - 2\delta\epsilon\kappa)/2(\epsilon - \kappa) > -1$ . In the same manner as in the proof of Lemma 2.2 we can show that  $\chi_n \longrightarrow \chi$  in  $L^{\tilde{p}}$ . Since  $\psi_n \longrightarrow \psi$  a.e. and

 $|\psi_n(x)-\psi(x)|^{\bar{q}} \leq (2l_0)^{\bar{q}}(1-x^2)^{(\epsilon+\kappa-2\delta\epsilon\kappa)/2(\epsilon-\kappa)}.$ 

the Lebesgue theorem implies  $\psi_n \longrightarrow \psi$  in  $L^{\tilde{g}}$ . Hence,  $\phi_n \longrightarrow \phi$  in  $L^{\kappa}$ 

**Lemma 2.4:** The operator  $\bar{S}$  (cf. (2.16)) is continuous in the space  $L^k$ , and

 $\|\bar{\mathbf{S}}\phi_n - \bar{\mathbf{S}}\phi\|_{\epsilon} \longrightarrow 0.$ 

Proof: The assertions follow from [1, *§* 1.4, Lemma 4.2] and Lemma 2.3 **<sup>U</sup>**

#### 2.3 Existence proof

Now, we are going to prove some assertions about the image and the continuity of the operator **T** defined in (2.9). For constants *R*,  $R_0 \ge 0$  and  $\lambda \in (0, 1)$  we define

**1.** if ined in (2.9). For constants  $R$ ,  $R_0 \ge 0$  and  $\lambda \in (0, 1)$  we define<br>  $K_{R, R_0, \lambda}^0 = \{ u \in \mathbf{C}_0 : ||u||_{\infty} \le R$ ,  $|u(x_1) - u(x_2)| \le R_0 |x_1 - x_2|^\lambda$ ,  $\forall x_1, x_2 \in [-1, 1] \}.$ 

**Proposition 2.5:** *There exist constants R, R<sub>0</sub>, and*  $\lambda$  *such that for the operator T defined by (2.9) and (2.10) the inclusion*  $\mathbf{T}(\mathbf{C_0}) \subset \mathcal{K}_{R,R_0,\lambda}^0$  holds.

Proof: Using (2.13), (2.15), Lemma 2.4, (2.17), Lemma 2.1, and the proof of Lemma 2.3, one can estimate  $||L(\cdot, u)||_p$ ,  $p = \kappa/2$ , by some constant  $R_0$ . If we define  $R = 2^{1/q} R_0$ ,  $q^{-1} = 1 - p^{-1}$ , and  $\lambda = q^{-1}$  we conclude  $||\mathbf{T}u||_{\infty} \leq R$  and  $|(\mathbf{T}u)(x_1) - (\mathbf{T}u)(x_2)| \leq R_0|x_1 - x_2|^{\lambda}$ **T** defined in (2.9). For constants *R*,<br>  $\mathcal{K}_{R,R_0,\lambda}^0 = \{u \in \mathbf{C}_0 : ||u||_{\infty} \leq R$ <br> **Proposition 2.5:** There exist c<br>
by (2.9) and (2.10) the inclusion **T**(<br> **Proof:** Using (2.13), (2.15), Lem<br>
can estimate  $||L(\cdot, u)||_p$ 

**Proposition 2.6 (cf. [12]):** *The operator* **T** :  $C_0 \longrightarrow C_0$  *defined by (2.9) and (2.8) is continuous.*  $\lambda = q^{-1}$  we conclude  $||\mathbf{T}u||_{\infty} \leq R$  and  $|(\mathbf{T}u)(x_1) - \theta$ <br>**Proposition 2.6 (cf. [12]):** The operator  $\mathbf{T}$  : C<sub>0</sub><br>*tinuous.*<br>**Proof:** One can show that  $||L(\cdot, u_n) - L(\cdot, u)||_p \longrightarrow$ 

0 **<sup>1</sup>**

As a consequence of Proposition 2.5 we obtain  $T(\mathcal{K}_{R,R_0,\lambda}^0) \subset \mathcal{K}_{R,R_0,\lambda}^0$ . Since  $\mathcal{K}_{R,R_0,\lambda}^0$  is a convex and compact subset of  $C_0$  we are able to apply Schauder's fixed point theorem to equation (2.9) in accordance to Propositions 2.5 and 2.6. We terminate at the following theorem.

**Theorem 2.7:** Let *A1* be fulfilled. Then problem (2.1), (2.2) possesses a solution  $u \in W_0^{1+}$ .

Let  $u \in C_0$  be a solution of the fixed point equation (2.9). Then the functions  $a(x)$  =  $F_u(x,u(x))$  and  $\alpha(x) = \frac{1}{2} - \frac{1}{\pi} \arctan a(x)$  are continuous. Consequently, the lemma from [13, Appendix] implies the representation

$$
z(x)=(1-x)\sqrt{1+[a(x)]^2}\exp[-\pi(S\alpha)(x)]=(1+x)^{\beta_1}(1-x)^{\beta_2}w(x),
$$

where  $w, w^{-1} \in L^r$  for  $r \in (1, \infty), \beta_1 = \alpha(-1), \beta_2 = 1 - \alpha(1), \text{ and } \beta_j \in (0, 1), j = 1, 2.$  Hence,  $a g/(1+a^2) \in L^s$  for  $1 < s < \delta^{-1}$ . Since the operator  $(1+x)^{-\beta_1}(1-x)^{-\beta_2}S(1+y)^{\beta_1}(1-y)^{\beta_2}$ is continuous in  $\mathbf{L}^s$  for  $1 < s < \min\left\{\beta_1^{-1},\beta_2^{-1}\right\}$ , it follows, for  $1 < p < \min\left\{\beta_1^{-1},\beta_2^{-1},\delta^{-1}\right\}$ ,  $L(\cdot, u) \in L^p$  (cf. (2.10)). Thus, we have proved the following (cf. [13, Part I, Theorem 2]). blies the representation<br>  $- x \sqrt{1 + [a(x)]^2} \exp[-\pi(S\alpha)(x)] = (1 + x)^{\beta_1}(1 - x)^{\beta_2}w(x)$ ,<br>  $\in L^r$  for  $r \in (1, \infty)$ ,  $\beta_1 = \alpha(-1)$ ,  $\beta_2 = 1 - \alpha(1)$ , and  $\beta_j \in (0, 1)$ <br>  $L^s$  for  $1 < s < \delta^{-1}$ . Since the operator  $(1 + x)^{-\beta_1}(1 - x)^{-\beta_2}S$ **13**  $\sqrt{1 + [a(x)]^2} \exp[-\pi(S\alpha)(x)] = (1 + x)^{\beta_1}(1 - x)^{\beta_2}w(x)$ ,<br>
L' for  $r \in (1, \infty), \beta_1 = \alpha(-1), \beta_2 = 1 - \alpha(1),$  and  $\beta_j \in (0, 1)$ <br>
<sup>1</sup> for  $1 < s < \delta^{-1}$ . Since the operator  $(1 + x)^{-\beta_1}(1 - x)^{-\beta_2}S$ <br>
L<sup>3</sup> for  $1 < s < \min\left\{\beta_1^{-1}, \beta_2^{-1}\right\}$ 

**Corollary 2.8:** *Let Assumption Al be fulfilled. Then problem (2.1), (2.2) possesses a solu-***COPOILATY 2.6:** Let Assumption  $AT$  be jujit<br>
tion  $u(x) = \int_{-1}^{x} v(t) dt$ , where  $v \in \bigcap_{1 \le p \le p_0} L^p$  and

$$
p_0 = \min\left\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\delta}\right\}, \qquad \beta_1 = \frac{1}{2} - \frac{1}{\pi} \arctan F_u(-1, 0), \qquad \beta_2 = \frac{1}{2} + \frac{1}{\pi} \arctan F_u(1, 0).
$$

**Remark 2.9:** *The assertions of Theorem 2.7 and Corollary 2.8 remain valid if instead of condition (2.4) we assume that there exist constants*  $b_1, b_2 \ge 0$  and  $\nu \in [0, 1)$  such that  $p_0 = \min \left\{ \frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\delta} \right\}, \qquad \beta_1 = \frac{1}{2} - \frac{1}{\pi} \arctan F_u(-1, 0), \qquad \beta_2 = \frac{1}{2}$ <br> **Lemark 2.9:** The assertions of Theorem 2.7 and Corollary 2.8 remation (2.4) we assume that there exist constants  $b_1, b_2 \ge$ 

$$
|F_x(x,u)| \leq l_0(K) (1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1] \times [-K,K], \quad K > 0,
$$

*with*  $l_0(K) = b_1 + b_2 K^{\nu}$  and  $\delta \ge 0$  satisfying (2.5) *(cf. [13, Part I, §4.1, Remark]*).

**Proof:** Let  $u \in \mathbb{C}_0$  and  $||u||_{\infty} \leq K$ . From the proofs of Lemma 2.1, Lemma 2.3, and **Proposition** 2.5 we conclude

$$
||h||_p \le l_0(K) c_1, ||\phi||_{\kappa} \le l_0(K) c_2, ||L(\cdot, u)||_p \le l_0(K) c_3,
$$

where

ere  
\n
$$
c_{1} = \left\{ \int_{-1}^{1} (1 - x^{2})^{-\delta p} dx \right\}^{1/p}, \quad c_{3} = c_{1} + d(\omega_{+}, \kappa) ||\tilde{S}||_{\kappa} c_{2},
$$
\n
$$
c_{2} = \left\{ \int_{-1}^{1} (1 - x^{2})^{(\epsilon + \kappa - 2\delta \epsilon \kappa)/2(\epsilon - \kappa)} dx \right\}^{(\epsilon - \kappa)/\epsilon \kappa} \left( \frac{\pi}{\cos(\epsilon \omega_{+})} \right)^{1/\epsilon},
$$

If we choose  $K > 0$  such that  $2^{1/q} c_3 l_0(K) \leq K_{\tau_q} q^{-1} = 1 - p^{-1}$ , we obtain  $\mathbf{T}(\mathcal{K}_{R,R_0,\lambda}^0) \subset$  $c_2 = \left\{ \int_{-1}^1 (1 - x^2)^{(\epsilon + \kappa - 2\delta\epsilon\kappa)/2(\epsilon - \kappa)} dx \right\}^{(\epsilon - \kappa)/\epsilon\kappa} \left( \frac{\pi}{\cos(\epsilon \omega_+)} \right)^{1/\epsilon},$ <br>If we choose  $K > 0$  such that  $2^{1/q} c_3 l_0(K) \le K$ ,  $q^{-1} = 1 - p^{-1}$ , we obtain  $\mathbf{T}(\mathcal{K}_{R,R_0,\lambda}^0) \subset \mathcal{K}_{R,R_0,\lambda}^0$ <br>for  $R =$ Schauder's fixed point theorem  $\blacksquare$  $c_2 = \left\{ \int_{-1}^{1} (1 - x^2)^{(e+\kappa-2\delta\varepsilon\kappa)/2(e-\kappa)} dx \right\}^{(e-\kappa)/\varepsilon\kappa} \left( \frac{\pi}{\cos(\varepsilon \omega_+)} \right)^{1/\varepsilon},$ <br>
choose  $K > 0$  such that  $2^{1/q} c_3 l_0(K) \le K, q^{-1} = 1 - p^{-1}$ , we obtain  $\mathbf{T}(K_{R,R_0,\lambda}^0) \subset K_{R,R_0,\lambda}^0$ <br>  $k = 2^{1/q} R_0$ ,  $R_0 = l$ 

### **3 Equations of the second type**

Now, consider equations of the kind

$$
u(x) = (S[F(\cdot, u)])(x) + f(x) + c, \qquad -1 \le x \le 1,
$$
\n(3.1)

with the conditions

$$
F(-1, u(-1)) = F(1, u(1)) = 0.
$$
\n(3.2)

P. JUNGHANNS and U. WEBER<br>
the conditions<br>  $F(-1, u(-1)) = F(1, u(1)) = 0.$  (3.2)<br>
set of functions u for which a real number  $p > 1$  and a function  $v \in L^p$  exist such that<br>  $= u(-1) + \int_{-1}^{x} v(y) dy$  and for which the conditions (3.2) The set of functions *u* for which a real number  $p > 1$  and a function  $v \in L^p$  exist such that  $u(x) = u(-1) + \int_{-1}^{x} v(y) dy$  and for which the conditions (3.2) are fulfilled will be denoted by  $W_F^{1+}$ . We distinguish two cases and make the following assumption.

**A2:**  $F = F(x, u) : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  possesses a continuous partial derivative  $F_u$ , where

$$
F_{u}(-1, u) \ge 0, \ F(1, u) = 0, \qquad \forall u \in \mathbb{R},
$$
\n(3.3)

$$
F(x, u) : [-1, 1] \times \mathbb{R} \to \mathbb{R} \text{ possesses a continuous partial derivative } F_u, \text{ where}
$$
  
\n
$$
F_u(-1, u) \ge 0, F(1, u) = 0, \quad \forall u \in \mathbb{R},
$$
  
\n
$$
\lim_{u \to -\infty} F(-1, u) < 0, \lim_{u \to +\infty} F(-1, u) > 0,
$$
  
\n
$$
-l_1 \le F_u(x, u) \le l_2, (x, u) \in [-1, 1] \times \mathbb{R}, l_1, l_2 \ge 0, l_1 l_2 < 1
$$
  
\n
$$
0 < l_1 \le F_u(x, u) \le l_2 < \infty
$$
  
\n(3.6)  
\n
$$
0 < l_1 \le F_u(x, u) \le l_2 < \infty
$$
  
\n(3.6)  
\n(3.6)  
\n(3.7)

$$
-l_1 \le F_u(x, u) \le l_2, \quad (x, u) \in [-1, 1] \times \mathbb{R}, \quad l_1, l_2 \ge 0, \quad l_1 l_2 < 1 \tag{3.5}
$$

in Case 1, *and* 

$$
0 < l_1 \le F_u(x, u) \le l_2 < \infty \tag{3.6}
$$

*in* Case 2. The partial derivative  $F_x$  satisfies the Carathéodory condition. The function  $f$  is absolutely continuous and possesses a measurable derivative  $f'$ . Furthermore, there exist constants  $\delta$ ,  $l_0 \ge 0$  such *f is absolutely continuous and* possesses *a measurable derivative 1' . Furthermore, there exist constants*  $\delta$ *,*  $l_0 \geq 0$  *such that (3.4)*<br>  $\times \mathbb{R}, l_1, l_2 \ge 0, l_1 l_2 < 1$  (3.5)<br> *(3.6)*<br> *sfies the Carathéodory condition. The function*<br> *u measurable derivative f'. Furthermore, there*<br>  $(x, u) \in [-1, 1] \times \mathbb{R},$  (3.7)  $H_1 \leq F_u(x, u) \leq$ <br>
2. The partial<br>
blutely continuor<br>
stants  $\delta, l_0 \geq 0$ <br>  $x, u$ ,  $|f'(x)| \leq$ <br>  $\frac{1}{4}$   $\frac{1}{2\pi}$ <br>  $\frac{3}{4}$   $\frac{1}{2\pi}$ <br>  $\frac{1}{2\pi}$ <br>  $\frac{1}{2\pi}$ 1, and<br>  $l_1 \leq F_u(x, u) \leq l_2 < \infty$ <br>
2. The partial derivative  $F_x$  satisficatively continuous and possesses a r<br>
stants  $\delta$ ,  $l_0 \geq 0$  such that<br>  $|x, u)|$ ,  $|f'(x)| \leq l_0(1 - x^2)^{-\delta}$ , (x<br>  $x, u$ ,  $|f'(x)| \leq l_0(1 - x^2)^{-\delta}$ , (x<br> bsolutely continuous and possenstants  $\delta$ ,  $l_0 \geq 0$  such that<br>  $\int_{x}^{1} (x, u) |, |f'(x)| \leq l_0 (1 - x^2)^{-\delta}$ <br>
in Case 1,<br>  $\leq \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}$ <br>
a Case 2,<br>  $\leq \frac{1}{2} + \frac{1}{2\pi} \arctan l_1$ .<br>
that in Case 1 the con

$$
|F_x(x, u)|, |f'(x)| \le l_0(1 - x^2)^{-\delta}, \qquad (x, u) \in [-1, 1] \times \mathbb{R}, \tag{3.7}
$$

*where, in* **Case 1,** 

$$
\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi} \tag{3.8}
$$

and, in Case 2,  
\n
$$
\delta < \frac{1}{2} + \frac{1}{2\pi} \arctan l_1.
$$
\n(3.9)

We remark that in Case 1 the condition  $F(1, u(1)) = 0$  is automatically satisfied.

#### **3.1 Reduction to a fixed point equation**

Differentiation of (3.1) with **regard to (3.2)** yields

**Reduction to a fixed point equation**  
erentiation of (3.1) with regard to (3.2) yields  

$$
v(x) - (Sav)(x) = g(x),
$$
 (3.10)

where  $a(x) = F_u(x, u(x))$  and  $g(x) = f'(x) + (S[F_x(\cdot, u)])(x)$ . We define  $\gamma(x) = \arctan a(x)$ (principal branch),  $r(x) = \sqrt{1 + a^2(x)}$ , and  $z_j(x) = (1 - x)^{\delta_j x} r(x) \exp[-(\mathsf{S}\gamma)(x)]$ , where  $\delta_{jk}$ 

denotes the Kronecker delta. Using again the results of  $[1, \S 9.5]$ , we obtain the fixed point equation

On the Solvability of Nonlinear Singular Integral Equations 691  
ates the Kronecker delta. Using again the results of [1, §9.5], we obtain the fixed point  
ation  

$$
u(x) = (\mathbf{T}_j u)(x) := k_j + \int_{-1}^x [L_j(y, u(y)) + c_j z_j^{-1}(y)] dy, \quad u \in \mathbf{C}
$$
(3.11)  
ase j, where

in Case j, where

the Kronecker delta. Using again the results of [1, §9.5], we obtain the fixed point

\ntion

\n
$$
u(x) = (\mathbf{T}_j u)(x) := k_j + \int_{-1}^x [L_j(y, u(y)) + c_j z_j^{-1}(y)] dy, \quad u \in \mathbf{C}
$$
\nase j, where

\n
$$
L_j(x, u(x)) = \frac{g(x)}{r^2(x)} + \frac{z_j^{-1}(x)}{\pi} \int_{-1}^1 \frac{a(y)z_j(y)g(y)}{r^2(y)} \frac{dy}{y-x},
$$
\n(3.12)

\nthe constants  $k_j$  and  $c_j$  ( $j = 1, 2$ ) have to fulfill the equations  $c_1 = 0$ ,

\n
$$
F(-1, k_j) = 0,
$$
\n
$$
F(1, k_2 + \int_{-1}^1 L_2(x, u(x)) dx + c_2 \int_{-1}^1 z_2^{-1}(x) dx = 0.
$$
\n(3.13)

\nthe same arguments as in Subsection 2.1 one can show that the problem (3.1), (3.2) for

and the constants  $k_j$  and  $c_j$  ( $j = 1, 2$ ) have to fulfil the equations  $c_1 = 0$ ,

$$
F(-1,k_j) = 0, \t\t(3.13)
$$

$$
F\left(1, k_2 + \int_{-1}^1 L_2(x, u(x)) dx + c_2 \int_{-1}^1 z_2^{-1}(x) dx\right) = 0.
$$
 (3.14)

With the same arguments as in Subsection 2.1 one can show that the problem (3.1), (3.2) for  $u \in \mathcal{W}_F^{1+}$  is equivalent to the fixed point problem (3.11) in the respective Case j.

#### **3.2** Investigation of the kernels  $L_i(x, u(x))$

We shall investigate the operators  $T_j$  in the Banach space  $C = C[-1, 1]$ . To prove the contiwe shall investigate the operators  $\mathbf{1}_j$  in the banach space  $\mathbf{C} = \mathbf{C}[-1,1]$ . To prove the conti-<br>nuity of  $\mathbf{T}_j$  we consider an arbitrary sequence  $\{u_n\} \subset \mathbf{C}$  with  $||u_n - u||_{\infty} \longrightarrow 0$  and use analogous notations as in Subsection 2.2. Obviously, relation (2.11) remains valid, which implies shall in<br>y of  $\mathbf{T}_j$ <br>us notat<br> $-\gamma \rVert_{\infty}$  $\|\gamma_n-\gamma\|_{\infty}\longrightarrow 0.$  Analogously to (2.13) we write *L*,( $x_2 + \int_{-1} L_2(x, u(x)) dx + c_2 \int_{-1}^{2} (x) dx = 0$ . (3.14)<br> *L*, the same arguments as in Subsection 2.1 one can show that the problem (3.1), (3.2) for<br> *W*<sup>1</sup><sup>+</sup> is equivalent to the fixed point problem (3.11) in the respect

$$
L_j(x, u(x)) = \cos^2(\gamma(x))g(x) + \cos(\gamma(x))M_j(x, u(x)), \qquad (3.15)
$$

where  $h(x) = a(x)g(x)/r(x) = \sin[\gamma(x)]g(x)$  and

$$
M_j(x, u(x)) := (1-x)^{-\delta_{j2}} \exp[(S\gamma)(x)] \Big(S\Big\{ (1-y)^{\delta_{j2}} h(y) \exp[-(S\gamma)(y)] \Big\}\Big)(x).
$$

Lemma 3.1: *If*  $1 < p < \delta^{-1}$ , *then*  $g_n \longrightarrow g$  in  $L^p$ . Moreover, we have

emma 3.1: *If* 1 < *p* < *δ*<sup>−1</sup>,<br>||*g*||<sub>*p* ≤ (1 + ||S||<sub>*p*</sub>) c(*δ*, *p*)</sub>  $||g||_p \leq (1+||S||_p) c(\delta, p) \quad \forall u \in \mathbf{C}.$ 

**Proof:** Compare the proof of Lemma 2.1

In Case 1, let  $2\omega_{-} := \arctan l_2 - \arctan l_1$ ,  $2\omega_{+} := \arctan l_2 + \arctan l_1$ , and, in Case 2,  $2\omega_- := \frac{\pi}{2}$  + arctan  $l_1$ ,  $2\omega_+ := \frac{\pi}{2}$  – arctan  $l_1$ . Furthermore, let  $\mu(x) := \gamma(x) - \omega_-$ . We remark that in both cases we have  $\omega_+ \in (0, \pi/4)$ ,  $\delta < 3/4 - \omega_+/\pi$ ,  $|\mu(x)| \leq \omega_+$ . Furthermore, the relations  $|\omega_-| < \pi/4$  in **Case 1** and  $\pi/4 < \omega_- < \pi/2$  in **Case 2** are fulfilled. Again we write **emma 3.1:**  $\int f \mathbf{1} < p < \delta^{-1}$ , then  $g_n \longrightarrow g$  in  $\mathbf{L}^p$ . Moreover, we have<br>  $\|g\|_p \leq (1 + \|\mathbf{S}\|_p) c(\delta, p)$   $\forall u \in \mathbf{C}$ .<br> **'roof:** Compare the proof of Lemma 2.1 <br> **a** Case 1, let  $2\omega$ . := arctan  $l_2$  - arcta

$$
M_j(x, u(x)) = \tilde{R}(x)(\tilde{S}_j \phi)(x). \tag{3.16}
$$

 $\bar{\mathbf{S}}_j \phi$  is defined as in (2.16) (with  $\tilde{N}_j$  instead of  $\tilde{N}$ ), where

P. JUNGHANNS and U. WEBER  
\nis defined as in (2.16) (with 
$$
\tilde{N}_j
$$
 instead of  $\tilde{N}$ ), where  
\n
$$
\tilde{N}_j(x) = (1+x)^{-\omega/\pi}(1-x)^{-\delta_{j2}+\omega_{-}/\pi},
$$
\n
$$
\phi(x) = (1-x^2)^{1/2\kappa}h(x) \exp[-(\mathbf{S}\mu)(x)],
$$
\n
$$
\tilde{R}(x) = (1-x^2)^{-1/2\kappa} \exp[(\mathbf{S}\mu)(x)],
$$
\nthe number  $\kappa$  can be chosen so that  $3 - 2\delta\kappa > 0$  and  
\n $2 < \kappa < \min \left\{\frac{\pi}{2\omega_{+}}, 3, \frac{2}{\delta}\right\}, \frac{\kappa}{3 - 2\delta\kappa} < \frac{\pi}{2\omega_{+}}.$   
\n $\kappa < \frac{\pi}{2\omega_{+}}$  and  $|\mu(x)| \leq \omega_{+}$ , the estimation (2.17) for all  $u \in \mathbb{C}$  and Lemma 2.2 remain  
\n. In Case 2 we additionally require

and the number  $\kappa$  can be chosen so that  $3 - 2\delta \kappa > 0$  and

$$
2 < \kappa < \min\left\{\frac{\pi}{2\omega_+}, 3, \frac{2}{\delta}\right\}, \qquad \frac{\kappa}{3 - 2\delta\kappa} < \frac{\pi}{2\omega_+}.
$$

and the number  $\kappa$  can be chosen so that  $3 - 2\delta\kappa > 0$  and<br>  $2 < \kappa < \min\left\{\frac{\pi}{2\omega_+}, 3, \frac{2}{\delta}\right\}, \qquad \frac{\kappa}{3 - 2\delta\kappa} < \frac{\pi}{2\omega_+}.$ <br>
Since  $\kappa < \frac{\pi}{2\omega_+}$  and  $|\mu(x)| \leq \omega_+$ , the estimation (2.17) for all  $u \in \mathbb{C}$  an valid. In **Case 2** we additionally require

\n The image shows a function of the equation 
$$
(2.17)
$$
 for all  $u \in \mathbb{C}$  and Lemma 2.2 remain. The equation  $K < \frac{3}{2(1 - \omega_{-}/\pi)}$ .\n

\n\n The formula  $3.2$ : The relation  $\|\phi_n - \phi\|_{\kappa} \longrightarrow 0$  holds true.\n

\n\n The formula  $2.2 \cdot 10^{-10} \, \text{m} \cdot \$ 

**Lemma 3.2:** The relation  $\|\phi_n - \phi\|_{\kappa} \longrightarrow 0$  holds true.

**Proof:** We proceed in the same way as in the proof of Lemma 2.3. Hence, we put  $\phi = (\psi + \omega)\chi$ with

$$
\psi(x) = \sin[\gamma(x)](1-x^2)^{\eta} f'(x),
$$
  
\n
$$
\omega(x) = \sin[\gamma(x)](1-x^2)^{\eta} \Big( S\Big\{(1-y^2)^{-\eta}[(1-y^2)^{\eta} F_x(y,u(y))] \Big\}\Big)(x),
$$
  
\n
$$
\chi(x) = (1-x^2)^{-1/2\epsilon} \exp[-(S\mu)(x)],
$$
  
\n
$$
\psi(x) = (1-x^2)^{-1/2\epsilon} \exp[-(S\mu)(x)],
$$
  
\n
$$
\psi(x) = \frac{\pi}{2\epsilon},
$$
  
\n
$$
\
$$

where  $\eta = (\varepsilon + \kappa)/2\varepsilon\kappa$  and  $\varepsilon$  is a fixed number such that

$$
\max\left\{\kappa,\frac{\kappa}{3-2\delta\kappa}\right\}<\varepsilon<\frac{\pi}{2\omega_+}.
$$

Define  $\tilde{p} = \varepsilon$  and  $\tilde{q} = \varepsilon \kappa/(\varepsilon - \kappa)$ . Then  $\tilde{p}^{-1} + \tilde{q}^{-1} = \kappa^{-1}$  and  $||\phi||_{\kappa} \leq (||\psi||_{\tilde{q}} + ||\omega||_{\tilde{q}})||\chi||_{\tilde{p}}$ . where  $\eta = (\varepsilon + \kappa)/2\varepsilon\kappa$  and  $\varepsilon$  is a fixed number such that<br>  $\max\left\{\kappa, \frac{\kappa}{3-2\delta\kappa}\right\} < \varepsilon < \frac{\pi}{2\omega_{+}}.$ <br>
Define  $\tilde{p} = \varepsilon$  and  $\tilde{q} = \varepsilon\kappa/(\varepsilon - \kappa)$ . Then  $\tilde{p}^{-1} + \tilde{q}^{-1} = \kappa^{-1}$  and  $||\phi||_{\kappa} \leq ($  $\rho(x) = (1 - x^2)^{-\eta}$ . Consequently, in the same manner as in the proof of Lemma 2.3 it follows Since  $3\varepsilon/2\varepsilon\kappa < 3/2 < \kappa$ , we obtain  $\tilde{q}^-$ <br> $\rho(x) = (1 - x^2)^{-\eta}$ . Consequently, in the<br>the assertion and  $||\phi||_{\kappa} \leq e(\kappa, \varepsilon)[1 + ||\rho^{-1}]$ the assertion and  $\|\phi\|_{\kappa} \leq e(\kappa,\varepsilon)[1 + \|\rho^{-1}S\rho I\|_{\tilde{q}}]d(\omega_{+},\varepsilon)$  $\|\phi\|_{\kappa} \leq (\|\psi\|_{\tilde{q}} + \|\omega\|_{\tilde{q}}) \|\chi\|_{\tilde{p}}.$ <br>
us,  $\rho^{-1}S\rho I \in \mathcal{L}(L^{\tilde{q}})$ , where<br>
roof of Lemma 2.3 it follows<br>
re continuous in L<sup>k</sup>.<br>  $\rho_j^{-1}S\rho_j I$ , where<br>  $-\frac{1}{2\kappa}$ . (3.19)<br>  $\omega_{-}/\pi < 3/2\kappa$ , i.e.  $\$ on  $\tilde{p}^{-1} + \tilde{q}^{-1} = \kappa^{-1}$  and  $||\phi||_{\kappa}$ <br>  $e^{-1} - 1 < -\eta < \tilde{q}^{-1}$ . Thus,  $\mu$ <br>
same manner as in the proof<br>  ${}^{1}S\rho I||_{\tilde{q}}]d(\omega_{+}, \varepsilon)$  <br>
1, 2 (cf. (2.16), (3.17)), are consider the value of  $\rho_{j}^{-1}S$ <br>  $\frac{\partial f}{$ 

**Lemma 3.3:** *The operators*  $\bar{S}_j$ ,  $j = 1,2$  *(cf. (2.16), (3.17)), are continuous in*  $L^2$ .

**Proof:** As in the proof of Lemma 2.4 we write  $\bar{\mathbf{S}}_j$  in the form  $\rho_j^{-1} \mathbf{S} \rho_j \mathbf{I}$ , where

Lemma 3.3: The operators 
$$
\bar{S}_j
$$
,  $j = 1, 2$  (cf. (2.16), (3.17)), are continuous in L<sup>2</sup>.  
\nProof: As in the proof of Lemma 2.4 we write  $\bar{S}_j$  in the form  $\rho_j^{-1} S \rho_j I$ , where  
\n
$$
\rho_j(x) = (1+x)^{\alpha_1} (1-x)^{\alpha_2}, \quad \alpha_1 = \frac{\omega_-}{\pi} - \frac{1}{2\kappa}, \quad \alpha_2 = \delta_{j2} - \frac{\omega_-}{\pi} - \frac{1}{2\kappa}.
$$
\n(3.19)

Because of  $2 < \kappa < 3$  and  $-1/4 < \omega_-/\pi < 1/2$  we have  $3/2\kappa - 1 < \omega_-/\pi < 3/2\kappa$ , i.e.  $\kappa^{-1} - 1 <$  $\alpha_1<\kappa^{-1}.$  Since in  $\bf Case~1$  we have  $|\omega_-|/\pi< 1/4,$  it also holds  $3/2\kappa-1<-\omega_-/\pi< 3/2\kappa,$  which implies  $\kappa^{-1} - 1 < \alpha_2 < \kappa^{-1}$ . In Case 2 we obtain from (3.18) that  $3/2\kappa - 1 < 1 - \omega_-/\pi < 3/2\kappa$ , which yields  $\kappa^{-1} - 1 < \alpha_2 < \kappa^{-1}$ , too

**Corollary 3.4:** In both cases  $j = 1,2$  there holds  $\|\bar{S}_j\phi_n - \bar{S}_j\phi\|_{\kappa} \longrightarrow 0$ .

 $(3.20)$ 

**Lemma 3.5:** In Case 2 the  $L^{\kappa/2}$ -norm of  $z_2^{-1}$  is uniformly bounded with respect to  $u \in \mathbb{C}$ . *Furthermore, it holds*  $z_{2,n}^{-1} \longrightarrow z_2^{-1}$  in  $L^{\kappa/2}$ .

**Proof:** Write  $z_2^{-1}(x) = \cos[\gamma(x)]\rho_2^{-1}(x)\bar{R}(x)$ , where  $\rho_2$  and  $\bar{R}$  are defined in (3.19) and (3.17). Since  $\alpha_k < \kappa^{-1}$ ,  $k = 1,2$  (cf. the proof of Lemma 3.3), it holds  $\rho_2^{-1} \in L^*$ . Thus, in view of (2.17),

 $||z_2^{-1}||_n \leq ||\rho_2^{-1}||_2 d(\omega_{\pm}, \kappa).$ 

The second assertion follows from Lemma 2.2 **<sup>U</sup>**

#### 3.3 Existence proof

Because of (3.4) there exists a solution  $k_1$  of equation (3.13) in Case 1. which we will fix for the sequel. In Case 2 the solutions  $k_2$  of (3.13) and  $\tilde{k}_2$  of  $F(1, \tilde{k}_2) = 0$  are uniquely determined. use of (3.4) there exists a soluti<br> *2* I. In Case 2 the solutions  $k_2$ .<br> **2** emma 3.6 ([13], Part II, §3<br>  $\int_{-1}^{1} z_2^{-1}(x) dx \ge D \quad \forall u \in \mathbb{C}$ .

**Lemma 3.6 ([13], Part II, §3.2):** *There exists a constant*  $D > 0$  *such that* 

For constants  $R$ ,  $R_0 \ge 0$  and  $\lambda \in (0, 1)$  we define

$$
\mathcal{K}_{R,R_0,\lambda}^0 = \{u \in \mathbf{C} : ||u||_{\infty} \leq R, |u(x_1) - u(x_2)| \leq R_0 |x_1 - x_2|^{\lambda}, \forall x_1, x_2 \in [-1,1]\}.
$$

**Proposition 3.7:** *It holds*  $T_j(C) \subset K_{R,R_0,\lambda}$  *for some constants R.*  $R_0 \geq 0$  *and*  $\lambda \in (0,1)$ .

**Proof:** We estimate  $||L_j(\cdot, u)||_p$  (cf. (3.12)). Having regard to (3.20). Lemma 3.6. and

have of (3.4) there exists a solution 
$$
k_1
$$
 of equation (3.13) in Case 1, which we will fix for the el. In Case 2 the solutions  $k_2$  of (3.13) and  $k_2$  of  $F(1, k_2) = 0$  are uniquely determined.

\nLemma 3.6 ([13], Part II, §3.2): There exists a constant  $D > 0$  such that

\n
$$
\int_{-1}^{1} z_2^{-1}(x) \, dx \geq D \qquad \forall u \in \mathbb{C}.
$$
\nFor constants  $R, R_0 \geq 0$  and  $\lambda \in (0, 1)$  we define

\n
$$
\mathcal{K}_{R, R_0, \lambda}^0 = \{u \in \mathbb{C} : ||u||_{\infty} \leq R, |u(x_1) - u(x_2)| \leq R_0 |x_1 - x_2|^\lambda, \forall x_1, x_2 \in [-1, 1]\}.
$$
\nProposition 3.7: It holds

\n
$$
T_j(C) \subset K_{R, R_0, \lambda}
$$
\nfor some constants  $R, R_0 \geq 0$  and  $\lambda \in (0, 1)$ .

\nProof: We estimate  $||L_j(\cdot, u)||_p$  (cf. (3.12)). Having regard to (3.20). Lemma 3.6, and

\n
$$
c_2 = \left\{ \int_{-1}^{1} z_2^{-1}(x) \, dx \right\}^{-1} \left\{ k_2 - k_2 - \int_{-1}^{1} L_2(x, u(x)) \, dx \right\}
$$
\nproof can be completed in the same way as the proof of Proposition 2.5

\nProposition 3.8 (cf. [12]):

\nThe operators

\n
$$
T_j: C \to C
$$
\ndefined by (3.11) are continuous.

the proof can be completed in the same way as the proof of Proposition 2.5 **<sup>U</sup>**

**Proposition 3.8 (cf. [12]):** The operators  $T_j: C \to C$  defined by (3.11) are continuous.

Froposition 3.8 (cf. [12]): The operators  $T_j : C \to C$  defined by (3.11) are continuous.<br>In accordance to Propositions 3.7 and 3.8 the operators  $T_j : K_{R,R_0,\lambda} \to K_{R,R_0,\lambda}$ ,  $j = 1, 2$ .<br>In accordance to Propositions 3.7 and 3.8 satisfy the conditions of Schauder's fixed point theorem. Thus, the following theorem is proved.

**Theorem 3.9:** *Let Assumption Al for the* Case 1 *or* Case 2 *be fulfilled. Then probknm*   $(3.1), (3.2)$  possesses a solution  $u \in \mathcal{W}_{F}^{1+}$ .

In the same manner as Corollary 2.8 and Remark 2.9 we can prove the following assertions.

**Corollary 3.10 (cf. [13], Part** II, Theorem 2): *Under the conditions of l'heore7n* **3.9 the problem (3.1), (3.2) possesses a solution**  $u(x) = u(-1) + \int_{-1}^{x} v(t) dt$ **, where**  $v \in \bigcap_{1 \leq p \leq p_0}$ *form 3.9*<br> *form 3.9*<br>  $\bigcap_{1 \leq p \leq p_0} L^p$ *and in*  $(f_{-1}^x v(t))$ <br>+  $f_{-1}^x v(t)$ **consition 3.8 (cf. [12]):** The operators  $T_j$ :  $\theta$ <br>
i accordance to Propositions 3.7 and 3.8 the op<br>
iy the conditions of Schauder's fixed point theore<br> **heorem 3.9:** Let Assumption A1 for the Case<br>  $\theta$ , (3.2) possesse **13], Part II, Theorem 2):** Under the consesses a solution  $u(x) = u(-1) + \int_{-1}^{x} v(t) dt$ <br> $\frac{1}{\mu_1}, \frac{1}{\delta}$ , Case 2:  $p_0 = \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\delta}\right\}$ 

*where* 

**Case 1:**  $p_0 = \min \left\{ \frac{1}{\mu_1}, \frac{1}{\delta} \right\},$  **Case 2:**  $p_0 = \min \left\{ \frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\delta} \right\}.$ <br> **P**<br>  $\mu_1 = \frac{1}{\pi} \sup \{ \arctan F_u(-1, u) : u \in \mathbb{R} \}, \quad \mu_2 = 1 - \frac{1}{\pi} \inf \{ \arctan F_u(1, u) : u \in \mathbb{R} \}.$ 

**Remark 3.11 (cf. [13], Part II, §4.1, Remark 1):** *Condition (3.7)* **(with** *respect to*  $F_x$ *) can be replaced by the weaker conditions of Remark 2.9 with*  $\delta \geq 0$  *satisfying (3.8) resp. (3.9).* 

#### **3.4** The case  $F(\pm 1, u) \equiv 0$

**Let** us consider the equation

P. JUNGHANNS and U. WEBER  
\nThe case 
$$
F(\pm 1, u) \equiv 0
$$
  
\nus consider the equation  
\n
$$
u(x) = (S[F(\cdot, u)])(x) + f(x), \quad -1 \le x \le 1, \quad u \in W_F^{\perp +},
$$
\n
$$
B = F(x, u) : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R} \text{ possesses a continuous partial derivative } F \text{, satisfying}
$$

under the following assumption.

A3:  $F = F(x, u) : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$  possesses a continuous partial derivative  $F_u$  satisfying

UNGHANNS and U. WEBER

\ne case 
$$
F(\pm 1, u) \equiv 0
$$

\nside  $F(\pm 1, u) \equiv 0$ 

\nside  $F(\pm 1, u) \equiv 0$ 

\nfollowing assumption.

\n $F(x, u) : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$  possesses a continuous partial derivative  $F_u$  satisfying

\n $-l_1 \leq F_u(x, u) \leq l_2, \quad (x, u) \in [-1, 1] \times \mathbb{R}, \quad l_1, l_2 \geq 0, \quad l_1 l_2 < 1,$ 

\n(3.23)

\n $F(\pm 1, u) = 0 \qquad \forall u \in \mathbb{R}.$ 

\n(3.24)

\nrespect to  $F_x$  and  $f'$  let Assumption A2 be fulfilled with relations (3.7) and (3.8).

*and*

$$
F(\pm 1, u) = 0 \qquad \forall u \in \mathbb{R}.\tag{3.24}
$$

*With respect to*  $F_x$  *and f' let Assumption A2 be fulfilled with relations (3.7) and (3.8).* 

As in Subsection 3.1 we obtain a fixed point equation

$$
-l_1 \le F_u(x, u) \le l_2, \quad (x, u) \in [-1, 1] \times \mathbb{R}, \quad l_1, l_2 \ge 0, \quad l_1l_2 < 1,\tag{3.23}
$$
\nand

\n
$$
F(\pm 1, u) = 0 \qquad \forall u \in \mathbb{R}.
$$
\nWith respect to  $F_x$  and  $f'$  let Assumption A2 be fulfilled with relations (3.7) and (3.8).

\nAs in Subsection 3.1 we obtain a fixed point equation

\n
$$
u(x) = (\mathbf{T}_0 u)(x) := k_0 + \int_{-1}^x L_0(y, u(y)) dy, \quad u \in \mathbf{C},\tag{3.25}
$$
\nthe equivalent to problem (3.22). The Kernel  $L_0(x, u(x))$  is given by equation (3.12) and

which is equivalent to problem (3.22). The kernel  $L_0(x, u(x))$  is given by equation (3.12) and  $z_0(x) = r(x) \exp[-(S\gamma)(x)]$ . The constant  $k_0$  is determined by

and  
\n
$$
F(\pm 1, u) = 0 \quad \forall u \in \mathbb{R}.
$$
\n(3.24)  
\nWith respect to  $F_x$  and  $f'$  let Assumption A2 be fulfilled with relations (3.7) and (3.8).  
\nAs in Subsection 3.1 we obtain a fixed point equation  
\n
$$
u(x) = (T_0u)(x) := k_0 + \int_{-1}^{x} L_0(y, u(y)) dy, \quad u \in C,
$$
\n(3.25)  
\nch is equivalent to problem (3.22). The Kernel  $L_0(x, u(x))$  is given by equation (3.12) and  
\n
$$
y(x) = r(x) \exp[-(\mathbf{S}\gamma)(x)].
$$
\nThe constant  $k_0$  is determined by  
\n
$$
k_0 = F_0 - \frac{1}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} \int_{-1}^{x} L_0(y, u(y)) dy dx,
$$
\n(3.26)  
\n
$$
x^2)^{-1/2}
$$
\nand integrating over [-1, 1]. Thus, all the results obtained in Subsection 3.2 with

where  $F_0 = \int_{-1}^{1} (1 - x^2)^{-1/2} f(x) dx$ . Relation (3.26) follows by multiplying equation (3.22) by  $(1 - x^2)^{-1/2}$  and integrating over  $[-1,1]$ . Thus, all the results obtained in Subsection 3.2 with respect to the kernel  $L_1(x, u(x))$  remain valid for the kernel  $L_0(x, u(x))$ . Taking into account  $(3.26)$ , which yields  $|k_0| \leq |F_0| + ||L_0(\cdot,u)||_p$ , it is easily seen that Propositions 3.7 and 3.8 hold true also for  $j = 0$ . Thus, the following theorem is in force (cf. [13, Part II, Theorem 2]).

**Theorem 3.12:** *If Assumption A3 is fulfilled, problem (3.22) possesses a solution*  $u \in W_F^{1+}$ *. Furthermore,*  $u(x) = u(-1) + \int_{-1}^{x} v(t) dt$ , *heorem is in force (cf.*<br> *s fulfilled, problem (3.2)*<br> *where*  $v \in \bigcap_{1 \leq p \leq 1/\delta}$  **LP**. also for  $j = 0$ . Thus, the following theorem is in force (cf. [13, Part II, Theorem 2]).<br> **Theorem 3.12:** If Assumption A3 is fulfilled, problem (3.22) possesses a solution  $u \in W_f^{1+}$ .<br>
thermore,  $u(x) = u(-1) + \int_{-1}^{x} v(t) dt$ 

The second part of the theorem can be proved in the same way as Corollary 2.8. Remark 3.11 also remains valid (with  $\delta \ge 0$  satisfying (3.8), cf. also [13, Part II, §4.1, Remark 1]).

### **4 Equations of the third type**

We consider the equation

$$
u(x) = (S[F(\cdot, u)])(x) + f(x) + c + d x, \qquad -1 \le x \le 1,
$$
\n(4.1)

with the conditions

$$
u(-1) = u(1) = 0 \tag{4.2}
$$

and make the following assumption.

A4:  $F = F(x, u) : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$  possesses a continuous partial derivative  $F_u$ , where

On the Solvability of Nonlinear Singular Integral Equations 695  
\n
$$
F(x, u) : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R} \text{ possesses a continuous partial derivative } F_u. \text{ where}
$$
\n
$$
-l_1 \le F_u(x, u) \le l_2, \quad (x, u) \in [-1, 1] \times \mathbb{R}, \quad l_1, l_2 \ge 0, \quad l_1 l_2 < 1,
$$
\n
$$
F(\pm 1, 0) = 0,
$$
\n
$$
(4.3)
$$
\n
$$
F_u(-1, 0) \ge 0, \qquad F_u(1, 0) \le 0.
$$
\n
$$
(4.5)
$$

$$
F(\pm 1,0) = 0,\t(4.4)
$$

$$
F_{u}(-1,0) \ge 0, \qquad F_{u}(1,0) \le 0.
$$
\n(4.5)

(4.5)<br>
(4.5) Continuous partial derivative  $F_u$ , where<br>  $\begin{aligned}\n & -1, 1] \times \mathbb{R}, \quad l_1, l_2 \geq 0, \quad l_1 l_2 < 1, \\
 & (4.3)\n \end{aligned}$ <br>
(4.5)<br>
(4.5)<br>
(Carathéodory condition. The function f is absolutely<br>
(4.5) The partial derivative  $F_x$  satisfies the Carathéodory condition. The function f is absolutely continuous *and possesses a measurable drivative 1'. Furthermore, there exist constants*  vartial derivative  $F_x$  satisfies the Caral<br>nuous and possesses a measurable de<br> $\geq 0$  such that<br> $F_x(x, u)|, |f'(x)| \leq l_0(1 - x^2)^{-\delta}.$ *(a continuous partial derivative*  $F_u$ *, where*<br>  $\times \mathbb{R}$ ,  $l_1, l_2 \ge 0$ ,  $l_1 l_2 < 1$ , (4.3)<br>
(4.4)<br>
(4.5)<br> *(a.b)*<br> *(a.b)*<br> *(a.b)*<br> *(a.e.g)*  $\in$  *[-1,1]*  $\times \mathbb{R}$ . (4.6)<br>
(a.g) derivative  $F_x$  satisfies the Carathéodory con<br>
and possesses a measurable derivative f'.<br>
ch that<br>
w)|,  $|f'(x)| \le l_0(1 - x^2)^{-\delta}$ .  $(x, u) \in [-1,$ <br>  $-\frac{\arctan l_1 + \arctan l_2}{2\pi}$ .<br>
tions  $u \in W_0^{1+}$  and real numbers c, d satisfy al derivative  $F_x$ <br>us and possesse<br>such that<br> $[x, u]$ ,  $|f'(x)| \le$ <br> $\frac{3}{4} - \frac{\arctan l_1 +}{2\pi}$ <br>nctions  $u \in \mathcal{W}_0^{1}$ 

$$
|F_x(x,u)|, |f'(x)| \le l_0(1-x^2)^{-\delta}, \qquad (x,u) \in [-1,1] \times \mathbb{R}.
$$
 (4.6)

*where* 

$$
\delta, l_0 \ge 0 \text{ such that}
$$
\n
$$
|F_x(x, u)|, |f'(x)| \le l_0(1 - x^2)^{-\delta}, \qquad (x, u) \in [-1, 1] \times \mathbb{R}.
$$
\n
$$
\text{where}
$$
\n
$$
\delta < \frac{3}{4} - \frac{\arctan l_1 + \arctan l_2}{2\pi}.
$$
\n
$$
(4.7)
$$

We seek functions  $u \in \mathcal{W}^{1+}_0$  and real numbers  $c,$   $d$  satisfying (4.1) and (4.2). Having regard to (4.2) and (4.4), for the derivative *v* of *u*, we again obtain equation (3.10) with  $g + d$  instead of q. The index of this equation in  $L^p$  for all sufficiently small  $p > 1$  is equal to zero because of

(4.5). Taking into account  

$$
\frac{z(x)}{r^2(x)} + \frac{1}{\pi} \int_{-1}^{1} \frac{a(y)z(y)}{r^2(y)} \frac{dy}{y-x} = 1
$$

(cf. [4, §1.1]) this leads to the fixed point equation

$$
\frac{z(x)}{r^2(x)} + \frac{1}{\pi} \int_{-1}^1 \frac{a(y)z(y)}{r^2(y)} \frac{dy}{y-x} = 1
$$
\n[4, §1.1]) this leads to the fixed point equation\n
$$
u(x) = \int_{-1}^1 [L(y, u(y)) + dz^{-1}(y)] dy, \quad u \in \mathbb{C},
$$

where

Taking into account  
\n
$$
\frac{z(x)}{r^2(x)} + \frac{1}{\pi} \int_{-1}^{1} \frac{a(y)z(y)}{r^2(y)} \frac{dy}{y-x} = 1
$$
\n4, §1.1]) this leads to the fixed point equation  
\n
$$
u(x) = \int_{-1}^{1} [L(y, u(y)) + dz^{-1}(y)] dy, \qquad u \in \mathbb{C},
$$
\ne  
\n
$$
L(x, u(x)) = \frac{g(x)}{r^2(x)} + \frac{z^{-1}(x)}{\pi} \int_{-1}^{1} \frac{a(y)z(y)g(y)}{r^2(y)} \frac{dy}{y-x}, \quad z(x) = r(x) \exp[-(S\gamma)(x)].
$$

 $r(x)$ ,  $\gamma(x)$  as in Section 3.1, and  $d = -\int_{-1}^{1} L(x, u(x)) dx / \int_{-1}^{1} z^{-1}(x) dx$ . Now, the proof of the following theorem can be given in the same manner as that of Theorem 3.9 for the Case I in Subsections 3.2 and 3.3.

**Theorem 4.1:** Let Assumption A4 be fulfilled. Then, for problem (4.1). (4.2). then exists *a solution u* $\in W_0^{1+}$ **. Moreover,**  $u(x) = \int_{-1}^x v(t) dt$ **, where**  $v \in \bigcap_{1 \leq p \leq p_0} L^p$  **and**  $v \in \bigcap_{1 \leq p \leq p_0} L^p$  **and** 

$$
L(x, u(x)) = \frac{g(x)}{r^2(x)} + \frac{z^{-1}(x)}{\pi} \int_{-1}^{1} \frac{a(y)z(y)g(y)}{r^2(y)} \frac{dy}{y-x}, \quad z(x) = r(x) \exp[-(S\gamma \cdot \tau(x)), \gamma(x) \text{ as in Section 3.1, and } d = -\int_{-1}^{1} L(x, u(x)) dx / \int_{-1}^{1} z^{-1}(x) dx.
$$
 Now.  
following theorem can be given in the same manner as that of Theorem 3.9 for  
Subsections 3.2 and 3.3.  
**Theorem 4.1:** Let Assumption A4 be fulfilled. Then, for problem (4.1). (4  
a solution  $u \in W_0^{1+}$ . Moreover,  $u(x) = \int_{-1}^{x} v(t) dt$ , where  $v \in \bigcap_{1 \leq p \leq p_0} L^p$  and  
 $p_0 = \min \left\{ \frac{1}{\beta_1}, \frac{1}{\beta_2}, \frac{1}{\delta} \right\}, \quad \beta_1 = \frac{1}{\pi} \arctan F_u(-1, 0), \quad \beta_2 = -\frac{1}{\pi} \arctan F_u(1, 0)$   
 $(\beta_j^{-1} = \infty \text{ if } \beta_j = 0).$ 

**Remark 4.2:** *The condition (4.6) On F.* **in** *Assumption A4 can be replaced by the wuikr conditions of Remark 2.9 with*  $\delta \geq 0$  satisfying  $(4.7)$ .

## 5 An example

In [11] there is considered the problem of two-dimensional free surface seepage flow from a nonlinear channel (see the picture). The problem is transformed into a nonlinear singular integral equation **g(z(s))** ds<br>  $g(x(s))ds - D = R(t),$  <br>  $\Rightarrow$  <br>  $f(x) = f(t)$ , <br>  $\Rightarrow$  <br>  $f(x)$ 

$$
x(t) - \frac{1}{\pi} \int_{-1}^{1} \frac{g(x(s))ds}{s - t} - D = R(t), \qquad -1 < t < 1,
$$
\n(5.1)

where  $g(x)$ ,  $0 \le x \le b$ , describes the shape of the channel and the right-hand side is given by

An example  
\n11] there is considered the problem of two-dimensional free surface seepage flow for  
\nlinear channel (see the picture). The problem is transformed into a nonlinear singular inte  
\nation  
\n
$$
z(t) - \frac{1}{\pi} \int_{-1}^{1} \frac{g(x(s)) ds}{s - t} - D = R(t), \qquad -1 < t < 1,
$$
\n(10)  
\n11)  
\n12
$$
z(t) - \frac{1}{\pi} \int_{-1}^{1} \frac{g(x(s)) ds}{s - t} - D = R(t), \qquad -1 < t < 1,
$$
\n(21)  
\n22
$$
z(t) = \frac{1}{\pi} \int_{1}^{1/k} \left[ \frac{r(k, \sigma)}{\sigma - t} - \frac{r(k, -\sigma)}{\sigma + t} \right] d\sigma, \quad r(k, t) = H - \frac{H}{K'} \int_{t}^{-1} \frac{d\sigma}{\sqrt{(\sigma^2 - 1)(1 - k^2 \sigma^2)}},
$$
\n
$$
K' = F(\sqrt{1 - k^2}, 1), \qquad F(k, \zeta) = \int_{0}^{\zeta} \frac{d\tau}{\sqrt{1 - \tau^2} \sqrt{1 - k^2 \tau^2}}.
$$
\nfunction x and the parameters  $k \in (0, 1)$  and  $D \in \mathbb{R}$  are unknown, where x has to sat  
\nboundary conditions  
\n
$$
x(-1) = b, x(1) = 0.
$$

The function x and the parameters  $k \in (0, 1)$  and  $D \in \mathbb{R}$  are unknown, where x has to satisfy the boundary conditions

$$
x(-1) = b, x(1) = 0.
$$
 (5.2)



Substituting  $x(t) = u(t) + \frac{b}{2}(1 - t)$ , from (5.1) we obtain

$$
u(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{F(s, u(s)) ds}{s - t} - D - E t = f(t), \qquad -1 < t < 1,
$$
 (5.3)

where

$$
F(t, u) = g(\frac{b}{2}(1-t) + u) - H,
$$
\n(5.4)

On the Solvability of Nonlinear Singular Integral Equations 697  
\n
$$
f(t) = R(t) - \frac{b}{2}(1-t) + \frac{H}{\pi} \ln \frac{1-t}{1+t}.
$$
\n(5.5)  
\nconditions (5.2) are transformed into the homogeneous conditions  
\n $u(-1) = u(1) = 0.$   
\nAssume that *g* is continuously differentiable with  $g'(0) < 0$ ,  $g'(b) > 0$  and put

The conditions (5.2) are transformed into the homogeneous conditions

$$
u(-1) = u(1) = 0. \tag{5.6}
$$

We assume that  $g$  is continuously differentiable with  $g'(0) < 0$  ,  $g'(b) > 0$  and put

$$
u(-1) = u(1) = 0.
$$
  
assume that  $g$  is continuously differenti  

$$
g(x) = \begin{cases} g'(0) x + H & , x < 0, \\ g'(b)(x - b) + H & , x > b, \end{cases}
$$

such that *F* in (5.4) is defined for all  $(t, u) \in [-1, 1] \times \mathbb{R}$ . The parameter *E* in (5.3) is an additional unknown, which is to be considered as a function  $E = E(k)$  of  $k \in (0, 1)$ . So we obtain the equation  $E(k) = 0$  for the determination of k. We ask wether, for each fixed parameter  $k$ , Theorem 4.1 is applicable to equation (5.3). If we assume that  $g'$  is monotonously increasing, relation (4.3) is satisfied under the condition  $|g'(0)g'(b)| < 1$ . Relations (4.4) and (4.5) follow from  $g(0) = g(b) = H$  and  $F_u(-1,0) = g'(b)$ ,  $F_u(1,0) = g'(0)$ . It remains to show the existence *of constants*  $l_0 > 0$  and  $\delta \geq 0$  such that  $y(x) = \int g'(b)(x - b) + H$ ,  $x > b$ ,<br>that *F* in (5.4) is defined for all  $(t, u) \in [-1, 1] \times \mathbb{R}$ . The parameter *E* in (5.3) is an<br>ional unknown, which is to be considered as a function  $E = E(k)$  of  $k \in (0, 1)$ . So we<br>in the equation EXERECT: The determinal text of the determinal<br>
1.1 is applicable to equation (5.3).<br>
is satisfied under the condition  $|n(b) = H$  and  $F_u(-1, 0) = g'(b)$ ,  $h$ <br>  $h_0 > 0$  and  $\delta \ge 0$  such that<br>  $h_0(1-t^2)^{-\delta}$ ,  $-1 < t < 1$ ,<br>
satis

$$
|f'(t)| \le l_0(1-t^2)^{-\delta}, \qquad -1 < t < 1,
$$
\n<sup>(5.7)</sup>

and (4.7) are satisfied. By partial integration we obtain from (5.5)

$$
g(0) = g(b) = H \text{ and } F_u(-1,0) = g'(b), F_u(1,0) =
$$
  
nstants  $l_0 > 0$  and  $\delta \ge 0$  such that  

$$
|f'(t)| \le l_0(1-t^2)^{-\delta}, \qquad -1 < t < 1,
$$
  
(4.7) are satisfied. By partial integration we obtain  

$$
f(t) = \frac{H}{\pi K'} \int_1^{1/k} \frac{\ln \frac{s-t}{s+t} ds}{\sqrt{(s^2-1)(1-k^2s^2)}} - \frac{b}{2}(1-t),
$$
  
h implies  

$$
f'(t) = \frac{H}{\pi K'} \int_1^{1/k^2} \frac{ds}{(t^2-s)\sqrt{(s-1)(1-k^2s)}} + \frac{b}{2}.
$$
  
ng into account [7, §22] we conclude (5.7) for  $\delta =$ 

which implies

$$
f'(t) = \frac{H}{\pi K'} \int_1^{1/k^2} \frac{ds}{(t^2 - s)\sqrt{(s - 1)(1 - k^2 s)}} + \frac{b}{2}.
$$

Taking into account [7, §22] we conclude (5.7) for  $\delta = \frac{1}{2}$ . Thus, with the help of Theorem 4.1 it is possible to ensure the existence *of* a continuous solution *u of (5.3), (5.4)* for each fixed parameter  $k \in (0, 1)$  under natural conditions on the shape of the channel.

### **References**

- [1] GOHBERG, 1. and N. KRUPNIK: *One-Dimensional Linear Singular Integral Equations.* Vol. I and II, Basel, Boston, Berlin: Birkhäuser Verlag 1992.
- [2] JUNGHANN5, P.: *Numerical solution* of *free* surface *seepage problems from nonlinear channels.* To appear.
- [3] JUNGHANNS, P. and D. OESTREICH: *Numerische Lösung des Staudammproblcms mit Drainage.*  ZAMM 69 (1989),  $83 - 92$ .
- [4] JUNGHANNS, P. and B. SILBERMANN: *Numerical Analysis of the Quadrature Method for Solving Linear and Nonlinear Singular Integral Equations.* Wiss. Schriftenreihe Techn. Univ. Chemnilz, Nt. 10, 1988.
- (5) KANTOROWITSCH, L. W. and C. P. ANILov: *Funktionalanalysis* in *nor;nierten Ràu,nen,* Berlin: Akademie-Verlag 1964.
- [6] LJIJSTERNIK, L. A. and W. I. SOBOLEW: *Elemente der Funktionalanalysis.* Berlin: Akademie-Verlag 1975.
- [7] MUSKHELISHVILI. N. I,: *Singular Integral Equations.* Groningen: Nordhoff 1953.
- [8] MIKHLIN, *S. C.* and S. PROSSDORF; *Singular Integral Operators.* Berlin: Akaderriie-Verlag, 1986.
- [9] NATANSON, I. *P.: Theorre der Funktionen rifler reellen Verdnderlichen.* Berlin: Akademie-Verlag 1975.
- [10] OESTREICH. D.: *Am •Stauda,nmprobleni i*<sup>n</sup> *d Drainage.* ZAMM 67 (1987), *293* 300.
- [ii] OESTRECH, D,: *Singular integral equations applied to free surface seepage* from *nonlinear channels.*  Math. Meth. AppI. Sci. 12 (1990), 209- 219.
- [12] WOLFERSDORF, L. V.: *A class of nonlinear Riemann-Hulbert problems for holomorphic functions.*  Math. Nachr. 116 (1984), 89 - 107.
- [13] WOLFERSDORF, L. V.:. On *the theory of nonlinear singular integral equations of Cauchy type.* Math. Meth. Appl. Sci. 7 (1985), 493 - 517.

Received 22. 12. 1992

 $\mathcal{P}_{\mathcal{A}}$