

# Necessary Optimality Conditions for Non-Smooth Minimax Problems

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Under a suitable assumption necessary optimality conditions are derived for non-smooth minimax problems involving infinitely many functions. The results obtained here generalize some necessary optimality conditions for mathematical programming and minimax problems.

**Key words:** *Clarke tangent cone, contingent cone, Rockafellar derivative, subgradient, non-smooth minimax problems*

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## 1. Introduction

Let  $C$  be a non-empty subset of a normed space  $X$ , and let  $Q$  be a compact topological space. For all  $\alpha \in Q$ , let  $f_\alpha$  be an extended real-valued function on  $X$ . We shall be concerned with the minimax problem

$$(P) \quad \min \{ F(x) := \sup_{\alpha \in Q} f_\alpha(x) \mid x \in C \}.$$

Optimality conditions for minimax problems involving functions that are differentiable in the sense of Fréchet or Gâteaux are given by several authors, but in this paper we are interested in general necessary conditions of the type given in [4, 5, 10]. In recent years, in non-smooth analysis a calculus for various directional derivatives and subgradients of locally Lipschitzian functions and even larger classes of functions has been developed (see, e.g., [3, 8, 11 - 15]). The results obtained in [13, 15] yield necessary optimality conditions for problem (P) of the type mentioned above. The purpose of this paper is to establish various necessary optimality conditions for problem (P) in a rather general setting.

The remainder of the paper is organized as follows. Section 2 is devoted to derive a general necessary optimality condition for problem (P) together with some examples. In Section 3, we give a necessary condition in terms of subgradients and polar cones.

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We also give here examples corresponding to the special cases introduced in Section 2. Finally, in Section 4 we establish necessary optimality conditions for a mathematical program with mixed constraints.

### 2. General necessary optimality conditions

The point  $x_0 \in C$  will be a local minimizer for problem (P). We assume throughout that the function  $\alpha \mapsto f_\alpha(x_0)$  is upper semicontinuous and finite-valued on  $Q$ . This implies in particular that  $F(x_0) = \sup_{\alpha \in Q} f_\alpha(x_0)$  is finite, since  $Q$  is compact.

We recall [3, p.55] that the *contingent cone* to  $C$  at  $x_0$  is the set

$$K_C(x_0) = \left\{ d \in X \left| \begin{array}{l} \text{there exist sequences } \{d_n\} \subset X, \{t_n\} \subset \mathbb{R} \text{ such} \\ \text{that } d_n \rightarrow d, t_n \downarrow 0, x_0 + t_n d_n \in C \text{ for all } n \in \mathbb{N} \end{array} \right. \right\}.$$

Define the set  $Q_0 = \{\alpha \in Q \mid f_\alpha(x_0) = F(x_0)\}$ . Assume that for every  $\alpha \in Q$  we have an extended real-valued function  $\varphi_\alpha$  on  $X$  such that

- (1) for all  $\alpha \in Q$ ,  $\varphi_\alpha$  is convex along rays issuing from the origin, and  $\varphi_\alpha(0) \leq 0$
  - (2) for all  $d \in K_C(x_0)$ ,  $\alpha \mapsto \varphi_\alpha(d)$  is upper semicontinuous on  $Q$  and finite on  $Q \setminus Q_0$ .
- These assumptions are valid throughout.

Let us introduce the following

**Assumption 2.1:** For all  $d \in K_C(x_0)$  and all sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow d$ ,  $t_n \downarrow 0$ ,  $x_0 + t_n d_n \in C$  for all  $n \in \mathbb{N}$  there holds the inequality

$$\varphi_\alpha(d) \geq \limsup_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n}$$

uniformly in  $\alpha \in Q$ .

**Theorem 2.2:** Let  $x_0 \in C$  be a local minimizer for problem (P). Assume that Assumption 2.1 is fulfilled. Then

$$\sup_{\alpha \in Q_0} \varphi_\alpha(d) \geq 0 \quad \text{for all } d \in K_C(x_0). \tag{2.1}$$

**Proof:** Suppose that inequality (2.1) is false. So, there exists  $\bar{d} \in K_C(x_0)$  and  $\mu > 0$  such that

$$\varphi_\alpha(\bar{d}) \leq -\mu < 0 \quad \text{for all } \alpha \in Q_0. \tag{2.2}$$

Define  $\psi_\alpha(d) = f_\alpha(x_0) + \varphi_\alpha(d)$ . It follows from (2.2) that, for all  $\alpha \in Q_0$ ,  $\psi_\alpha(\bar{d}) \leq \hat{m} - \mu$ , where  $\hat{m} = F(x_0)$ . Note that  $\hat{m}$  is finite. We shall begin with showing that there is  $\hat{d} = \lambda \bar{d} \in K_C(x_0)$  such that

$$\psi_\alpha(\hat{d}) < \hat{m} \quad \text{for all } \alpha \in Q. \tag{2.3}$$

To do this, we set  $U = \{\alpha \in Q \mid \varphi_\alpha(\bar{d}) < -\mu/2\}$ . In view of (2.2) one has  $Q_0 \subset U$ . By virtue of the upper semicontinuity of the mapping  $\alpha \mapsto \varphi_\alpha(\bar{d})$ ,  $Q \setminus U$  is compact. Hence, by the upper semicontinuity of the mapping  $\alpha \mapsto f_\alpha(x_0)$ , we can find a constant  $l > 0$  such that, for all  $\beta \in Q \setminus U$ ,  $f_\beta(x_0) \leq \hat{m} - l$ , and therefore also

$$\psi_\beta(0) = f_\beta(x_0) + \varphi_\beta(0) \leq \hat{m} - l. \tag{2.4}$$

Since the set  $Q \setminus U$  is compact and the mapping  $\alpha \mapsto \varphi_\alpha(\bar{d})$  is upper semicontinuous and finite on  $Q \setminus U$ , we can find a constant  $\gamma \in \mathbb{R}$  such that  $\varphi_\beta(\bar{d}) \leq \gamma$  for all  $\beta \in Q \setminus U$ , whence

$$\psi_\beta(\bar{d}) = f_\beta(x_0) + \varphi_\beta(\bar{d}) \leq \hat{m} + \gamma. \tag{2.5}$$

For  $\lambda \in (0, 1]$ ,  $d_\lambda := \lambda\bar{d} = \lambda\bar{d} + (1 - \lambda)0 \in K_C(x_0)$ . Then by virtue of the convexity along rays of  $\psi_\alpha$  and the definition of  $U$  we get that, for all  $\alpha \in U$ ,

$$\psi_\alpha(d_\lambda) \leq \lambda\psi_\alpha(\bar{d}) + (1 - \lambda)\psi_\alpha(0) \leq \lambda(\hat{m} - \frac{\mu}{2}) + (1 - \lambda)\hat{m} = \hat{m} - \frac{1}{2}\lambda\mu < \hat{m}. \tag{2.6}$$

For  $\beta \in Q \setminus U$ , it follows from (2.4) and (2.5) that

$$\psi_\beta(d_\lambda) \leq \lambda(\hat{m} + \gamma) + (1 - \lambda)(\hat{m} - l) = \hat{m} - l + \lambda(\gamma + l).$$

For  $\lambda$  small enough,  $-l + \lambda(\gamma + l) < 0$ , which implies  $\psi_\beta(d_\lambda) < \hat{m}$ . This together with (2.6) gives (2.3), whence  $\sup_{\alpha \in Q}(f_\alpha(x_0) + \varphi_\alpha(\hat{d})) < \hat{m}$ . Then for some number  $\hat{\mu} > 0$  we obtain the inequality

$$f_\alpha(x_0) + \varphi_\alpha(\hat{d}) \leq \hat{m} - \hat{\mu} \quad \text{for all } \alpha \in Q. \tag{2.7}$$

Since  $\hat{d} \in K_C(x_0)$ , there exist sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow \hat{d}$ ,  $t_n \downarrow 0$  such that  $x_0 + t_n d_n \in C$ . Taking account of Assumption 2.1, we get

$$\limsup_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \varphi_\alpha(\hat{d}) \tag{2.8}$$

uniformly in  $\alpha$ . Combining (2.7) and (2.8) yields that

$$\limsup_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \hat{m} - \hat{\mu} - f_\alpha(x_0)$$

uniformly in  $\alpha$ . Consequently, for  $\varepsilon > 0$  there is a natural number  $N$  (not depending on  $\alpha$ ) such that, for all  $n \geq N$ ,

$$\frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \leq \hat{m} - \hat{\mu} - f_\alpha(x_0) + \varepsilon. \tag{2.9}$$

We can assume that  $t_n \leq 1$ , as  $t_n \downarrow 0$ . So, observing that  $\hat{m} - f_\alpha(x_0) \geq 0$ , from (2.9) it follows that

$$\begin{aligned}
 f_\alpha(x_0 + t_n d_n) &\leq f_\alpha(x_0) + t_n(\widehat{m} - \widehat{\mu} - f_\alpha(x_0) + \varepsilon) \\
 &= f_\alpha(x_0) + t_n(\widehat{m} - f_\alpha(x_0)) - t_n(\widehat{\mu} - \varepsilon) \\
 &\leq f_\alpha(x_0) + \widehat{m} - f_\alpha(x_0) - t_n(\widehat{\mu} - \varepsilon) \\
 &= \widehat{m} - t_n(\widehat{\mu} - \varepsilon).
 \end{aligned}$$

In particular, for  $\varepsilon < \widehat{\mu}$  we obtain  $F(x_0 + t_n d_n) < \widehat{m} = F(x_0)$  for all  $n \geq N$ , which conflicts with the hypothesis that  $x_0$  is a local minimizer for (P) ■

In the sequel we apply Theorem 2.2 to various forms of derivatives of the functions  $f_\alpha : X \rightarrow \mathbb{R}$ . In all these examples, property (1) of the function  $\varphi_\alpha$  will be satisfied automatically. It remains to enforce property (2) and Assumption 2.1.

**Example 1.** Let the functions  $f_\alpha$  be Fréchet differentiable at  $x_0$ . Then, for all sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow d$ ,  $t_n \downarrow 0$ , there holds

$$\lim_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} = \langle f'_\alpha(x_0), d \rangle. \quad (2.10)$$

We set  $\varphi_\alpha(d) = \langle f'_\alpha(x_0), d \rangle$ . We require for all  $d \in K_C(x_0)$  that the mapping  $\alpha \mapsto \langle f'_\alpha(x_0), d \rangle$  is upper semicontinuous, and that (2.10) holds uniformly in  $\alpha$ . Then Assumption 2.1 is satisfied, and from Theorem 2.2 we obtain

$$\sup_{\alpha \in Q_0} \langle f'_\alpha(x_0), d \rangle \geq 0 \quad \text{for all } d \in K_C(x_0).$$

**Example 2.** Let the functions  $f_\alpha$  be Hadamard differentiable at  $x_0$ . This means that for each  $\alpha \in Q$  there exists  $\nabla f_\alpha(x_0) \in X^*$  such that, for all  $d \in X$ ,

$$\lim_{d' \rightarrow d, t \downarrow 0} \frac{f_\alpha(x_0 + t d') - f_\alpha(x_0)}{t} = \langle \nabla f_\alpha(x_0), d \rangle. \quad (2.11)$$

We set  $\varphi_\alpha(d) = \langle \nabla f_\alpha(x_0), d \rangle$ . We require for all  $d \in K_C(x_0)$  that the mapping  $\alpha \mapsto \langle \nabla f_\alpha(x_0), d \rangle$  is upper semicontinuous, and that the limit (2.11) is uniform in  $\alpha$ . Then Assumption 2.1 is satisfied, and from Theorem 2.2 we obtain

$$\sup_{\alpha \in Q_0} \langle \nabla f_\alpha(x_0), d \rangle \geq 0 \quad \text{for all } d \in K_C(x_0).$$

This is a generalization of Theorem 1 in [7].

**Example 3.** Let the functions  $f_\alpha$  be Lipschitz in a neighborhood of  $x_0$  with a Lipschitz constant  $L$  being independent of  $\alpha$ . Clarke's directional derivative of  $f_\alpha$  at  $x_0$  with respect to  $d$  is defined as

$$f_\alpha^0(x_0; d) := \limsup_{x \rightarrow x_0, t \downarrow 0} \frac{f_\alpha(x + t d) - f_\alpha(x)}{t}. \quad (2.12)$$

We set  $\varphi_\alpha(d) = f_\alpha^0(x_0; d)$ . We require for all  $d \in K_C(x_0)$  that the mapping  $\alpha \mapsto f_\alpha^0(x_0; d)$  is upper semicontinuous and that (2.12) holds uniformly in  $\alpha$ . Then Assumption 2.1 is satisfied, since

$$\begin{aligned} & \limsup_{d_n \rightarrow d, t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \\ & \leq \limsup_{d_n \rightarrow d, t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d) - f_\alpha(x_0) + t_n L \|d_n - d\|}{t_n} \\ & = \limsup_{t_n \downarrow 0} \frac{f_\alpha(x_0 + t_n d) - f_\alpha(x_0)}{t_n} \\ & \leq \limsup_{x \rightarrow x_0, t \downarrow 0} \frac{f_\alpha(x + td) - f_\alpha(x)}{t} \\ & = f_\alpha^0(x_0; d) \end{aligned}$$

uniformly in  $\alpha$ . Hence from Theorem 2.2 we obtain

$$\sup_{\alpha \in Q_0} f_\alpha^0(x_0; d) \geq 0 \quad \text{for all } d \in K_C(x_0).$$

**Example 4.** Let the functions  $f_\alpha$  be directionally Lipschitzian at  $x_0$  in the sense that

$$f_\alpha^+(x_0; d) < \infty \quad \text{for all } d \in K_C(x_0),$$

where

$$f_\alpha^+(x_0; d) := \limsup_{\substack{(x, \gamma) \rightarrow (x_0, f_\alpha(x_0)), \\ (x, \gamma) \in \text{epi} f_\alpha, d' \rightarrow d, t \downarrow 0}} \frac{f_\alpha(x + td') - \gamma}{t}. \tag{2.13}$$

Here  $\text{epi} f_\alpha := \{(x, \gamma) \in X \times \mathbb{R} \mid \gamma \geq f_\alpha(x)\}$  denotes the *epigraph* of the function  $f_\alpha$ . We set  $\varphi_\alpha(d) = f_\alpha^+(x_0; d)$ . We require for all  $d \in K_C(x_0)$  that the mapping  $\alpha \mapsto f_\alpha^+(x_0; d)$  is upper semicontinuous, and that the limit (2.13) is uniform in  $\alpha$ . Then Assumption 2.1 is satisfied, since

$$f_\alpha^+(x_0; d) \geq \limsup_{d' \rightarrow d, t \downarrow 0} \frac{f_\alpha(x_0 + td') - f_\alpha(x_0)}{t} \geq \limsup_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n}$$

uniformly in  $\alpha$ . Hence from Theorem 2.2 we obtain

$$\sup_{\alpha \in Q_0} f_\alpha^+(x_0; d) \geq 0 \quad \text{for all } d \in K_C(x_0).$$

**Example 5.** We recall [3, p.53] that *Clarke's tangent cone to a set A at  $x_0$*  is defined as

$$T_A(x_0) = \left\{ d \in X \left| \begin{array}{l} \text{for all sequences } \{x_n\} \subset A, \{t_n\} \subset \mathbb{R} \text{ with} \\ x_n \rightarrow x_0, t_n \downarrow 0 \text{ there exists a sequence } \{d_n\} \subset X \\ \text{with } d_n \rightarrow d, x_n + t_n d_n \in A \text{ for all } n \in \mathbb{N} \end{array} \right. \right\}.$$

For a function  $f$  which is finite at  $x_0$  the *Rockafellar directional derivative of  $f$  at  $x_0$  in the direction  $d$*  is then defined as

$$f^1(x_0; d) = \inf \{r \mid (d, r) \in T_{\text{epi}f}(x_0, f(x_0))\}.$$

The extended real-valued function  $f^1(x_0; \cdot)$  has the following properties:

$$\text{epi}f^1(x_0; \cdot) = T_{\text{epi}f}(x_0, f(x_0))$$

$f^1(x_0; \cdot)$  is lower semicontinuous and sublinear.

If  $f^1(x_0; \cdot)$  is proper, then  $f^1(x_0; 0) = 0$ , else  $f^1(x_0; 0) = -\infty$ . If  $f^+(x_0; d) < \infty$ , then  $f^1(x_0; d) = f^+(x_0; d)$ . For further discussions on the Rockafellar derivative we refer to [3, 8, 11, 13]. We set now  $\varphi_\alpha(d) = f_\alpha^1(x_0; d)$ . We require for all  $d \in K_C(x_0)$  that the mapping  $\alpha \mapsto f_\alpha^1(x_0; d)$  is upper semicontinuous on  $Q$  and finite on  $Q \setminus Q_0$ , and that for all sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow d, t_n \downarrow 0$  there holds

$$f_\alpha^1(x_0; d) \geq \limsup_{n \rightarrow \infty} \frac{f_\alpha(x_0 + t_n d_n) - f_\alpha(x_0)}{t_n} \tag{2.14}$$

uniformly in  $\alpha$ . Then Assumption 2.1 is satisfied, and from Theorem 2.2 we obtain

$$\sup_{\alpha \in Q_0} f_\alpha^1(x_0; d) \geq 0 \quad \text{for all } d \in K_C(x_0).$$

We give now another optimality condition, which also involves  $f_\alpha^1$ , but is independent of Theorem 2.2.

**Assumption 2.3:** For all  $d \in K_C(x_0)$ , the following is true:

$$(i) \liminf_{d' \rightarrow d, d'' \rightarrow d, t \downarrow 0} \left( \frac{f_\alpha(x_0 + td') - f_\alpha(x_0 + td'')}{t} \right) = 0$$

uniformly in  $\alpha$ ; for all  $\alpha \in Q$  and all  $d \in K_C(x_0)$ ,  $f_\alpha(x_0 + td')$  and  $f_\alpha(x_0 + td'')$  are finite for all  $t > 0$  in a neighborhood of 0 and all  $(d', d'')$  in a neighborhood of  $(d, d)$ .

$$(ii) \sup_{\alpha \in Q_0} f_\alpha^1(x_0; d) \geq F^1(x_0; d) \text{ for all } d \in K_C(x_0).$$

**Proposition 2.4:** Let  $x_0 \in C$  be a local minimizer for problem (P). Assume that Assumption 2.3 is fulfilled. Then

$$\sup_{\alpha \in Q_0} f_\alpha^1(x_0; d) \geq 0 \quad \text{for all } d \in K_C(x_0). \tag{2.15}$$

**Proof:** Suppose that (2.15) is false. This means that there exist  $\bar{d} \in K_C(x_0)$  and  $\mu > 0$  such that

$$\sup_{\alpha \in Q_0} f_\alpha^1(x_0; \bar{d}) \leq -\mu < 0. \tag{2.16}$$

It follows from Assumption 2.3/(ii) and inequality (2.16) that  $F^\uparrow(x_0; \bar{d}) \leq -\mu$ , that is  $(\bar{d}, -\mu) \in \text{epi}F^\uparrow(x_0; \cdot)$ . Moreover,

$$\text{epi}F^\uparrow(x_0; \cdot) = T_{\text{epi}F}(x_0, F(x_0)). \tag{2.17}$$

Since  $\bar{d} \in K_C(x_0)$ , there exist sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow \bar{d}, t_n \downarrow 0$  such that  $x_0 + t_n d_n \in C$ . It follows from (2.17) that there exist sequences  $\{d'_n\} \subset X$ ,  $\{\mu_n\} \subset \mathbb{R}$  with  $d'_n \rightarrow \bar{d}, \mu_n \rightarrow -\mu$  such that  $(x_0, F(x_0)) + t_n(d'_n, \mu_n) \in \text{epi}F$ , which implies that

$$\frac{F(x_0 + t_n d'_n) - F(x_0)}{t_n} \leq \mu_n. \tag{2.18}$$

By Assumption 2.3/(i), for  $\varepsilon > 0$  there exists a subsequence  $\{n_k\} \subset \mathbb{N}$  (not depending on  $\alpha$ ) such that

$$\frac{f_\alpha(x_0 + t_{n_k} d_{n_k}) - f_\alpha(x_0 + t_{n_k} d'_{n_k})}{t_{n_k}} < \varepsilon \quad \text{for all } \alpha \in Q. \tag{2.19}$$

Combining (2.18) and (2.19) yields that

$$\frac{F(x_0 + t_{n_k} d_{n_k}) - F(x_0)}{t_{n_k}} < \mu_{n_k} + \varepsilon.$$

Since  $\mu_{n_k} \rightarrow -\mu$ , there is an  $N \in \mathbb{N}$  (not depending on  $\alpha$ ) such that, for all  $k \geq N$ ,

$$\frac{F(x_0 + t_{n_k} d_{n_k}) - F(x_0)}{t_{n_k}} < -\mu + 2\varepsilon,$$

which implies that  $F(x_0 + t_{n_k} d_{n_k}) - F(x_0) < t_{n_k}(2\varepsilon - \mu)$ . For  $\varepsilon$  small enough,  $2\varepsilon - \mu < 0$ . Then for all  $k \geq N$  we have  $F(x_0 + t_{n_k} d_{n_k}) - F(x_0) < 0$ , which contradicts the hypothesis that  $x_0$  is a local minimizer for (P) ■

**Remark 2.5:** Proposition 2.4 includes Theorem 6 of [8] as a special case.

### 3. Necessary conditions in terms of subgradients

In this section we assume that the function  $\varphi_\alpha$  is sublinear for all  $\alpha \in Q$ . Let  $X^*$  be the topological dual of  $X$ . Let

$$\partial\varphi_\alpha(0) := \{x^* \in X^* \mid \langle x^*, d \rangle \leq \varphi_\alpha(d) \quad \text{for all } d \in X\}.$$

In what follows  $M$  is a closed convex subcone of  $K_C(x_0)$  with vertex at the origin,  $M^*$  denotes the polar cone of  $M$ , i.e.,

$$M^* = \{x^* \in X^* \mid \langle x^*, d \rangle \geq 0 \quad \text{for all } d \in M\}.$$

We write  $X_\sigma^*$  to indicate that  $X^*$  is endowed with the weak\* topology.

**Theorem 3.1:** Let  $x_0$  be a local minimizer for problem (P). Assume that the hypotheses of Theorem 2.2 hold;  $\partial\varphi_\alpha(0) \neq \emptyset$  and  $\varphi_\alpha(d) = \sup_{x^* \in \partial\varphi_\alpha(0)} \langle x^*, d \rangle$  for all  $\alpha \in Q_0$ . Then

$$0 \in \text{cl}(\text{co}(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*) \quad (3.1)$$

where *co* and *cl* denote convex hull and weak\* closure, respectively.

**Proof:** Taking account of Theorem 2.2 we get

$$\sup_{\alpha \in Q_0} \varphi_\alpha(d) \geq 0 \quad \text{for all } d \in M. \quad (3.2)$$

We now assume that the inclusion (3.1) is false, i.e.,

$$0 \notin \text{cl}(\text{co}(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*). \quad (3.3)$$

The right-hand side of (3.3) is weak\* closed convex. So from a standard separation theorem for convex sets (see, e.g., Theorem 3.6 in [6]) there exist  $d_0 \in (X_\sigma^*)^* = X$  and  $\gamma \in \mathbb{R}$  such that

$$0 > \gamma \geq \langle \xi, d_0 \rangle \quad \text{for all } \xi \in \text{co}(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*.$$

Since  $M^*$  is a cone containing the origin this implies

$$0 \geq \langle \xi, d_0 \rangle \quad \text{for all } \xi \in -M^* \quad (3.4)$$

$$0 > \gamma \geq \langle \xi, d_0 \rangle \quad \text{for all } \xi \in \cup_{\alpha \in Q_0} \partial\varphi_\alpha(0). \quad (3.5)$$

It follows from (3.4) that  $d_0 \in M^{**} = M$ . It follows from (3.5) that

$$0 > \gamma \geq \sup_{\xi \in \partial\varphi_\alpha(0)} \langle \xi, d_0 \rangle = \varphi_\alpha(d_0) \quad \text{for all } \alpha \in Q_0$$

whence  $0 > \sup_{\alpha \in Q_0} \varphi_\alpha(d_0)$ . This contradicts inequality (3.2) ■

**Remark 3.2:** We remark that, conversely, (3.1) implies (3.2). Indeed, assume now that (3.1) is true. Observe that if  $A$  is any subset of  $X^*$ , then  $\bar{a}$  being an element of the weak\* closure of  $A$  implies that for all  $d \in X$  and  $\varepsilon > 0$  there exists  $a \in A$  such that  $|\langle a - \bar{a}, d \rangle| \leq \varepsilon$ . Hence for every  $d \in X$  and  $\varepsilon > 0$  there exist finitely many  $\alpha_i \in Q_0$ ,  $\xi_i \in \partial\varphi_{\alpha_i}(0)$ ,  $\lambda_i \geq 0$  satisfying  $\sum_i \lambda_i = 1$ , and  $m^* \in M^*$  such that

$$-\varepsilon \leq \sum_i \lambda_i \langle \xi_i, d \rangle - \langle m^*, d \rangle \leq \sum_i \lambda_i \varphi_{\alpha_i}(d) - \langle m^*, d \rangle.$$



Choosing  $d \in M$  we get

$$-\varepsilon \leq \sum_i \lambda_i \varphi_{\alpha_i}(d) \leq \sup_{\alpha \in Q_0} \varphi_{\alpha}(d).$$

Since  $\varepsilon$  is arbitrary this implies  $\sup_{\alpha \in Q_0} \varphi_{\alpha}(d) \geq 0$ . Hence (3.1) implies (3.2).

**Remark 3.3:** If  $\text{cl}(\text{co } \cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0))$  is weak\* compact, then (3.1) becomes

$$0 \in \text{cl} \left( \text{co } \bigcup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0) \right) - M^*. \tag{3.6}$$

Indeed, from the general fact that the equality  $\text{cl}(A + B) = \text{cl}(\text{cl}A + B)$  is true, if we take  $A = \text{co } \cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0)$  and  $B = -M^*$ , then  $\text{cl}A + B$  is weak\* closed. So the assertion is proved.

For the remaining part of this section we assume that  $X$  is a Banach space.

**Corollary 3.4:** Let  $x_0$  be a local minimizer for problem (P). Assume that the hypotheses of Theorem 2.2 hold,  $\varphi_{\alpha}$  is lower semicontinuous, proper and upper-bounded in a neighborhood of 0 for all  $\alpha \in Q_0$ , and that the mapping  $\alpha \mapsto \partial\varphi_{\alpha}(0)$  is upper semicontinuous from  $Q$  into  $X_{\sigma}^*$ . Then  $0 \in \text{cl}(\text{co } \cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0)) - M^*$ .

**Proof:** From Theorem 3.1 and Remark 3.3 we need only to show that the set  $\text{cl}(\text{co } \cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0))$  is weak\* compact. By virtue of Proposition 2.1.4 in [3],  $\partial\varphi_{\alpha}(0)$  for all  $\alpha \in Q$  are non-empty weak\* compact subsets of  $X^*$ , and the equality  $\varphi_{\alpha}(d) = \max_{x^* \in \partial\varphi_{\alpha}(0)} \langle x^*, d \rangle$  is true. Making use of the compactness of  $Q_0$  and the upper semicontinuity of the mapping  $\alpha \mapsto \partial\varphi_{\alpha}(0)$  we get that  $\cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0)$  is weak\* compact (see, e.g., [2, p.116]). Since  $X$  is a Banach space, the set  $\text{cl}(\text{co } \cup_{\alpha \in Q_0} \partial\varphi_{\alpha}(0))$  is weak\* compact ■

**Examples.** We will apply Corollary 3.4 to some of the examples considered already in Section 2.

1. The functions  $f_{\alpha}$  are Fréchet differentiable at  $x_0$ , and  $\varphi_{\alpha}(d) := \langle f'_{\alpha}(x_0), d \rangle$ . We require that the mapping  $\alpha \mapsto f'_{\alpha}(x_0)$  is continuous from  $Q$  into  $X_{\sigma}^*$ . Then from the inequality  $\sup_{\alpha \in Q_0} \langle f'_{\alpha}(x_0), d \rangle \geq 0$  for all  $d \in K_C(x_0)$  it follows that  $0 \in \text{cl } \text{co}(\cup_{\alpha \in Q_0} f'_{\alpha}(x_0)) - M^*$ . This is a generalization of a necessary condition in [5, p.59].

2. The functions  $f_{\alpha}$  are Lipschitz in a neighborhood of  $x_0$ , and  $\varphi_{\alpha}(d) := f_{\alpha}^0(x_0; d)$ . Then  $\varphi_{\alpha}$  is lower semicontinuous, proper, and upper bounded in a neighborhood of 0. Let

$$\partial^0 f_{\alpha}(x_0) := \{x^* \in X^* \mid \langle x^*, d \rangle \leq f_{\alpha}^0(x_0; d) \text{ for all } d \in X\}.$$

We require that the mapping  $\alpha \mapsto \partial^0 f_\alpha(x_0)$  is upper semicontinuous from  $Q$  into  $X_\sigma^*$ . Then from the inequality  $\sup_{\alpha \in Q_0} f_\alpha^0(x_0; d) \geq 0$  for all  $d \in K_C(x_0)$  it follows that  $0 \in \text{cl co}(\cup_{\alpha \in Q_0} \partial^0 f_\alpha(x_0)) - M^*$ .

3. Let  $\varphi_\alpha(d) := f_\alpha^1(x_0; d)$ . Then  $\varphi_\alpha$  is lower semicontinuous. Let

$$\partial^1 f_\alpha(x_0) := \{x^* \in X^* \mid \langle x^*, d \rangle \leq f_\alpha^1(x_0; d) \text{ for all } d \in X\}.$$

We require that the mapping  $\alpha \mapsto \partial^1 f_\alpha(x_0)$  is upper semicontinuous from  $Q$  into  $X_\sigma^*$ , and that  $f_\alpha^1(x_0; \cdot)$  is proper and upper bounded in a neighborhood of 0. Then from the inequality  $\sup_{\alpha \in Q_0} f_\alpha^1(x_0; d) \geq 0$  for all  $d \in K_C(x_0)$  it follows that  $0 \in \text{cl co}(\cup_{\alpha \in Q_0} \partial^1 f_\alpha(x_0)) - M^*$ .

#### 4. A constrained mathematical program

Let us consider the problem

$$(MP) \quad \min\{f(x) \mid x \in C, F(x) \leq 0\}$$

where  $F(x) = \sup_{\alpha \in Q} f_\alpha(x)$ . Here  $f$  is an extended real-valued function on  $X$ ,  $f_\alpha$  ( $\alpha \in Q$ ) and  $C$  are as in problem (P),  $x_0 \in C$  will be a local minimizer of problem (MP). The functions  $f_\alpha$  and  $\varphi_\alpha$  have the same properties as requested at the beginning of Section 2. The function  $f$  is supposed to be finite in  $x_0$ . The function  $\varphi$  is upper semicontinuous and positively homogeneous on  $X$ . A further requirement for  $\varphi$  is given in the following

**Assumption 4.1:** Let the following conditions be true.

(i) For all  $d \in K_C(x_0)$  and all sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow d$ ,  $t_n \downarrow 0$ ,  $x_0 + t_n d_n \in C$  there holds

$$\varphi(d) \geq \limsup_{n \rightarrow \infty} \frac{f(x_0 + t_n d_n) - f(x_0)}{t_n}.$$

(ii)  $\text{cl}\{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\} \supset \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0\}$ , where cl indicates the norm closure.

**Theorem 4.2:** Let  $x_0 \in C$  be a local minimizer for problem (MP). Assume that Assumptions 2.1 and 4.1 are fulfilled. Then

$$\varphi(d) \geq 0 \text{ for all } d \in K_C(x_0) \text{ satisfying } \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0. \tag{4.1}$$

**Proof:** We first prove that  $\varphi(d) \geq 0$  for all  $d \in K_C(x_0)$  satisfying  $\sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0$ . Suppose that this is false. So there is a  $\bar{d} \in K_C(x_0)$  satisfying  $\sup_{\alpha \in Q_0} \varphi_\alpha(\bar{d}) < 0$

and  $\varphi(\bar{d}) < 0$ . Hence, for some  $\mu > 0$ ,  $\varphi(\bar{d}) \leq -\mu < 0$ . Define  $\psi_\alpha(d) = f_\alpha(x_0) + \varphi_\alpha(d)$ . In the same way as in the proof of Theorem 2.2 we can find a  $\hat{d} = \lambda\bar{d} \in K_C(x_0)$  such that

$$\sup_{\alpha \in Q} \psi_\alpha(\hat{d}) < \hat{m}, \quad \text{where } \hat{m} = F(x_0).$$

Since  $\hat{d} \in K_C(x_0)$ , there exist sequences  $\{d_n\} \subset X$ ,  $\{t_n\} \subset \mathbb{R}$  with  $d_n \rightarrow \hat{d}$ ,  $t_n \downarrow 0$  such that  $x_0 + t_n d_n \in C$ . Making use of Assumption 2.1, by an argument analogous to that used for the proof of Theorem 2.2 we can find an  $N_1 \in \mathbb{N}$  such that, for all  $n \geq N_1$ ,  $F(x_0 + t_n d_n) - F(x_0) < 0$ . Hence  $x_0 + t_n d_n$  is a feasible point of problem (MP). On the other hand, since  $\varphi(\bar{d}) \leq -\mu$  it follows from the positive homogeneity of  $\varphi$  that  $\varphi(\hat{d}) = \lambda\varphi(\bar{d}) \leq -\tilde{\mu} < 0$ , where  $\tilde{\mu} = \lambda\mu$ . By Assumption 4.1/(i), for  $\varepsilon > 0$  there is an  $N_2 \in \mathbb{N}$ ,  $N_2 \geq N_1$  such that, for all  $n \geq N_2$ ,

$$\frac{f(x_0 + t_n d_n) - f(x_0)}{t_n} \leq -\tilde{\mu} + \varepsilon,$$

whence  $f(x_0 + t_n d_n) - f(x_0) \leq t_n(-\tilde{\mu} + \varepsilon)$ . Consequently, for  $\varepsilon < \tilde{\mu}$  we get  $f(x_0 + t_n d_n) - f(x_0) < 0$ , which contradicts the hypothesis that  $x_0$  is a local minimizer for problem (MP). So, we have proved that

$$\varphi(d) \geq 0 \quad \text{on } \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\}.$$

Since  $\varphi$  is upper semicontinuous, it follows that

$$\varphi(d) \geq 0 \quad \text{on } \text{cl} \{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) < 0\}.$$

By Assumption 4.1/(ii), we get  $\varphi(d) \geq 0$  on  $\{d \in K_C(x_0) \mid \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0\}$  ■

To derive a necessary optimality condition for problem (MP) in terms of subgradients, we now assume that  $\varphi$  and  $\varphi_\alpha$  are sublinear for all  $\alpha \in Q$ . Let

$$\partial\varphi(0) := \{x^* \in X^* \mid \langle x^*, d \rangle \leq \varphi(d) \quad \text{for all } d \in X\}.$$

As before,  $M$  is a closed convex subcone of  $K_C(x_0)$  with vertex at the origin.

**Theorem 4.3:** *Let  $x_0 \in C$  be a local minimizer for problem (MP). Assume that Assumptions 2.1 and 4.1 are fulfilled. Suppose, furthermore, that  $\partial\varphi(0)$  is non-empty, weak\* compact and  $\varphi(d) = \sup_{x^* \in \partial\varphi(0)} \langle x^*, d \rangle$ ; for each  $\alpha \in Q_0$ ,  $\partial\varphi_\alpha(0)$  is non-empty and  $\varphi_\alpha(d) = \sup_{x^* \in \partial\varphi_\alpha(0)} \langle x^*, d \rangle$ . Then*

$$0 \in \partial\varphi(0) + \text{cl} \left( \text{cc}(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^* \right) \tag{4.2}$$

where *cc* and *cl* denote convex conical hull and weak\* closure, respectively.

**Proof:** By Theorem 4.2 we get

$$\varphi(d) \geq 0 \text{ for all } d \in M \text{ satisfying } \sup_{\alpha \in Q_0} \varphi_\alpha(d) \leq 0. \quad (4.3)$$

Assume now that (4.2) is not true. So 0 does not belong to the set on the right-hand side. The latter is weak\* closed, since  $\partial\varphi(0)$  is weak\* compact. Moreover it is convex. So from a standard separation theorem (see, e.g., Theorem 3.6 in [6]) there exist  $d_0 \in (X_\sigma^*)^* = X$  and  $\gamma \in \mathbb{R}$  such that

$$0 > \gamma \geq \langle \xi, d_0 \rangle \quad \forall \xi \in \partial\varphi(0) + cc(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0)) - M^*.$$

Since  $cc(\cup_{\alpha \in Q_0} \partial\varphi_\alpha(0))$  and  $M^*$  are cones it follows from this that

$$\begin{aligned} 0 > \gamma &\geq \langle \xi, d_0 \rangle && \text{for all } \xi \in \partial\varphi(0), \\ 0 &\geq \langle \xi, d_0 \rangle && \text{for all } \xi \in \cup_{\alpha \in Q_0} \partial\varphi_\alpha(0), \\ 0 &\geq \langle \xi, d_0 \rangle && \text{for all } \xi \in -M^*. \end{aligned}$$

The first of these inequalities implies  $\varphi(d_0) < 0$ . The second implies  $\varphi_\alpha(d_0) \leq 0$  for all  $\alpha \in Q_0$ . The third implies  $d_0 \in M^{**} = M$ , which is a contradiction with (4.3) ■

**Remark 4.4:** We remark that, conversely, (4.2) implies (4.3). The proof is similar to the one given in Remark 3.2.

**Remark 4.5:** If we assume that the function  $\varphi$  is lower semicontinuous, proper, sublinear, then it can be expressed by equality  $\varphi(d) = \sup_{x^* \in \partial\varphi(0)} \langle x^*, d \rangle$  where  $\partial\varphi(0)$  is non-empty, weak\* closed (see, e.g., [3, p.29]). If we suppose, in addition, that  $\varphi$  is upper-bounded in a neighborhood of 0, or that  $X$  is a Banach space and  $\varphi$  is a finite function on  $X$ , then  $\partial\varphi(0)$  is weak\* compact (see, e.g., Theorem 5 in [8] and Proposition 2.1.4 in [3]).

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