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Behaviour of Solutions of a Third Order Differential Equation

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Properties of proper and regular non-oscillatory solutions of the equation $x''' + f(t)g(x, x', x'') = 0$, and some sufficient conditions for regular solutions to be oscillatory or non-oscillatory are derived.

Key words: *Proper solutions, regular solutions, oscillatory and non-oscillatory solutions, third order differential equations*

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1. We will deal with the differential equation

$$x''' + f(t)g(x, x', x'') = 0, \tag{1}$$

where

(i) f is continuous on $(a, +\infty), a > -\infty$, and $f(t) > 0$ for all $t \in (a, +\infty)$
(ii) g is continuous on R^3 and $g(y_0, y_1, y_2) > 0$ for all $(y_0, y_1, y_2) \in R^3$ with $y_0 \neq 0$.

Equation (1) is a generalization of equations examined in [1, 2] and a special case of equations examined in [4, 5].

2. The aim of this paper is to investigate so-called proper and regular solutions of equation (1).

Definition 1: A function $x \in C^3\langle T_x, +\infty)$ $(T_x \geq a)$ being a solution of equation (1) is said to be a

a) *proper sulution* if $\sup\{|x(t)| : t \geq T\} > 0$ for every $T > T_x$;

b) *regular solution* if it is proper and if for some $T \geq T_x$ it does not have a triple null point in $\langle T, +\infty)$, i.e. there is no $t_0 \geq T$ such that $x(t_0) = x'(t_0) = x''(t_0) = 0$;

c) *oscillatory solution* if it is proper or regular and if there exists a sequence $t_i \uparrow +\infty$ such that $x(t_i) = 0$ for all i;

d) *non-oscillatory solution* if it is not oscillatory, i.e. if there exists a $T \geq T_x$ such that $x(t) > 0$ for all $t > T$ or $x(t) < 0$ for all $t > T$.

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Theorem 1: *Let x be a proper solution of equation* (1) *on $\langle T_x, +\infty)$ and let $\rho > T_x$ be a triple null point of x. Then there exists an $\epsilon > 0$ such that on $(\rho, \rho+\epsilon)$ either $x = 0$ or $x \neq 0$ and x has infinite many null points with ρ as limiting point.*

Proof: It is sufficient to prove that there does not exist an $\epsilon > 0$ such that $x(t) \neq 0$ for all $t \in (\rho, \rho+\epsilon)$. Suppose $x(t) > 0$ for all $t \in (\rho, \rho+\epsilon)$. Then $x'''(t) < 0$ for all $t \in (\rho, \rho+\epsilon)$ and x'' is decreasing. But $x''(\rho) = 0$ and therefore $x''(t) < 0$ for all $t \in (\rho, \rho+\epsilon)$. In the same way we get $x'(t) < 0$ and $x(t) < 0$ for all $t \in (\rho, \rho+\epsilon)$ which leads to a contradiction. In the case $x(t) < 0$ for all $t \in (\rho, \rho+\epsilon)$ the proof is similar ∎

Theorem 2: *Let f and g be functions satisfying conditions* (i) *and* (ii), *respectively. Moreover, let there exists for any $(y_0, y_1, y_2) \in R^3$ with $y_0 \neq 0$*

$$(iii) \quad g_1(y_1, y_2) := \lim_{y_0 \to 0} \frac{g(y_0, y_1, y_2)}{y_0}$$

and let the function g_1 be continuous on R^2. Then the trivial solution $x = 0$ of equation (1) *fulfilling the initial conditions $x(t_0) = x'(t_0) = x''(t_0) = 0$ $(t_0 > a)$ is unique on $\langle t_0, +\infty)$.*

Proof: Let x be a solution of equation (1) with t_0 as triple null point and $x(t) \neq 0$ in $(t_0, t_0+\epsilon)$ for some $\epsilon > 0$. Then x fulfils also the equation

$$u'''(t) + f(t)G(t)u = 0 \tag{2}$$

on $\langle t_0, t_0+\epsilon)$, where

$$G(t) = \begin{cases} g(x(t), x'(t), x''(t))/x(t) & \text{for} \quad x(t) \neq 0 \\ g_1(x'(t), x''(t)) & \text{for} \quad x(t) = 0. \end{cases} \tag{3}$$

Equation (2) is linear and therefore it has only one solution with t_0 as triple null point, namely $u = 0$. But this is in contradiction with the supposition of x ∎

Corollary 1: *Let x be a proper solution of equation* (1) *on $\langle T_x, +\infty)$. Then the following statements are true.*

a) *The non-trivial solution x has at most one double or triple null point.*
b) *If $x(t) = c$ $(c \in R)$ on some subinterval I, then $c = 0$.*
c) *If t_1, t_2 are triple null points of x with $t_1 < t_2$, then $x = 0$ on the interval $\langle t_1, t_2\rangle$.*
d) *The solution x cannot be zero on any different disjoint subintervals I_1 and I_2.*

Proof: a) Let $t_0 > T_x$ be a double null point of x and $t_1 > t_0$ its next double null point. Denote $F(x) = xx'' - x'^2/2$. Then multiplying equation (1) by x and integrating from t_0 to t we obtain

$$F(x(t)) = \int_{t_0}^{t} f(\tau)g(x(\tau), x'(\tau), x''(\tau))x(\tau)\,d\tau = F(x(t_0)). \tag{4}$$

Because t_0, t_1 are double null points the function $F(x(\cdot))$ is non-increasing for $t \geq t_0$ and $F(x(t_0)) = F(x(t_1)) = 0$ which contradicts that x is a non-trivial solution of equation (1). The case of a solution x with triple null point can be proved by analogous arguments.

b) Let $t_0 \in I$ and $c \neq 0$. Then from (4) we get $F(x(t)) = -\int_{t_0}^{t} f(\tau)g(c,0,0)c\,d\tau < 0$ for $t > t_0$ and $F(x(\cdot))$ is non-increasing. But $F(x(t)) = F(c) = 0$ which and is possible for $c = 0$ only.

c) - d) The proof of these suppositions follows immediately from the monotonicity of the function $F(x(\cdot))$ ∎

3. In this section we will be interested in non-oscillatory solutions of equation (1).

Lemma 1. *Let x be a solution of equation (1) on $\langle T_x, +\infty)$ without null points and with $F(x(t)) > 0$ $(t \geq t_1)$ for the function $F(x(\cdot))$ defined by (4). Then there exists a point $T \geq T_x$ such that, for $t > T$, the functions x, x', x'' are monotone, have no null point, and either*

$$\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t) \quad \text{or} \quad \operatorname{sgn} x(t) = \operatorname{sgn} x''(t) = \operatorname{sgn} x'(t).$$

Proof: From (4) it follows that $x''(t) \neq 0$ and $\operatorname{sgn} x(t) = \operatorname{sgn} x''(t)$ for $t \geq T_x$. From the properties of x and x'' it follows that x' is monotone for $t \geq T_x$ and that it can have at most one null point. Therefore there exists a point $T \geq T_x$ such that either $\operatorname{sgn} x'(t) = \operatorname{sgn} x(t)$ or $\operatorname{sgn} x'(t) \neq \operatorname{sgn} x(t)$ for $t \geq T$. It remains to prove that x'' is monotone for $t \geq T$. From $x(t) \neq 0$ it follows that $x'''(t) = -f(t)g(x(t), x'(t), x''(t)) \neq 0$ for $t \geq T_x$, that is $x'''(t)$ does not change the sign and therefore x'' is monotone ∎

Theorem 3: *Let x be a solution of equation (1) with $x(t) \neq 0$ and $F(x(t)) > 0$ for all $t \in \langle T_x, +\infty)$. Let further $\int_{t_0}^{+\infty} f(t)\, dt = +\infty$ for $t_0 > a$ and*

$$\liminf_{t \to \infty} \varphi(t) g(\varphi(t), \varphi'(t), \varphi''(t)) > 0 \tag{5}$$

for any function $\varphi \neq 0$ with $\lim_{t \to +\infty} \varphi(t) = \pm\infty$ and which is monotone with φ' and φ'' on $(a, +\infty)$. Then there exists a $T \geq T_x$ such that, for $t > T$,

$$x(t)x'(t)x''(t) \neq 0 \qquad \text{and} \qquad \operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t)$$

and $x(t), x'(t), x''(t) \to 0$ as $t \to +\infty$.

Proof: From Lemma 1 it follows that x, x', x'' are monotone on $\langle T_x, +\infty)$ and therefore having proper or improper limits at infinity. Let e.g. $\lim_{t \to +\infty} x(t) \neq 0$. Then clearly by (5), or directly by (ii), if $\lim_{t \to +\infty} x(t)$ is finite, then $\liminf_{t \to +\infty} x(t) g(x(t), x'(t), x''(t)) \geq c > 0$. From (4) for x there follows the existence of a $T \geq T_x$ such that, for $t > T$, $F(x(t)) \uparrow$ and

$$F(x(t)) - F(x(T)) = -\int_T^t f(s)x(s)g(x(s), x'(s), x''(s))\, ds < 0.$$

On the other hand we have

$$F(x(t)) - F(x(T)) = -\int_T^t f(s)x(s)g(x(s), x'(s), x''(s))\, ds \leq -c \int_T^t f(s)\, ds.$$

From here for $t \to +\infty$ we obtain a contradiction to $F(x(t)) > 0$ for $t > T_x$ and therefore $x(t) \to 0$ for $t \to +\infty$. From the properties of x there follows $\operatorname{sgn} x(t) \neq \operatorname{sgn} x'(t)$ for $t \geq T' \geq T$, where T' is sufficiently large. From $F(x(t)) > 0$ there follows $\operatorname{sgn} x(t) = \operatorname{sgn} x''(t)$ and from the properties of x there follows $x'(t), x''(t) \to 0$ as $t \to +\infty$ ∎

Example 1: Let us have the differential equation

$$x''' + x\frac{\arctan\sqrt{1 + x^2 + x'^2 + x''^2}}{\arctan\sqrt{1 + 3e^{-2t}}} = 0$$

for $t \in (0, +\infty)$. This equation fulfils the suppositions of Theorem 3. Its every solution x defined on $\langle T_x, +\infty)$ for some $T_x > 0$, with $x(t) \neq 0$ and $F(x(t)) > 0$ for all $t \in \langle T_x, +\infty)$ has the property $x(t)x'(t)x''(t) \neq 0, \operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t)$ for $t \in \langle T, +\infty)$ $(T \geq T_x)$ and $x(t), x'(t), x''(t) \to 0$ as $t \to +\infty$. The function $x = e^{-t}$ is clearly a solution of this equation.

Lemma 2 (see [6]): *Let $h \in C(a, +\infty)$ and $\int_{t_0}^{+\infty} t^2 h(t)\, dt < +\infty, t_0 > a$. Then the differential equation*

$$y''' + hy = 0 \tag{6}$$

is non-oscillatory on $(a, +\infty)$, i.e. every solution of equation (6) has a finite set of null points on $(a, +\infty)$.

Theorem 4: *Let f and g be functions satisfying conditions (i) – (iii) and let x be a solution of equation (1) on $\langle T_x, +\infty)$ such that*

$$|g(x(t), x'(t), x''(t))| \leq |x(t)|\varphi(t) \quad (t > t_0) \tag{7}$$

for some positive function $\varphi \in C\langle t_0, +\infty)$ $(t_0 > T_x)$ satisfying the condition

$$\int_{t_0}^{+\infty} t^2 f(t)\varphi(t)\, dt < +\infty. \tag{8}$$

Then x is non-oscillatory on $\langle T_0, +\infty)$.

Proof: Suppose that x is oscillatory. Then x is also a solution of equation (2) and from our suppositions it follows that $\int_{t_0}^{+\infty} t^2 f(t)G(t)\, dt \leq \int_{t_0}^{+\infty} t^2 f(t)\varphi(t)\, dt < +\infty$. Therefore from Lemma 2 it follows that equation (2) is non-oscillatory. This is a contradiction with the supposition that x is oscillatory ▮

Example 2: Let us have the differential equation

$$x''' + \frac{1}{t^4}\frac{\arctan x}{1 + x^2 + x'^2 + x''^2} = 0$$

for $t \in (0, +\infty)$. This equation fulfils the suppositions of Theorem 4 for $\varphi(t) = M > \pi/2$ $(t \in (0, +\infty))$. Therefore any of its solution on $\langle t_0, +\infty)$ $(t_0 > 0)$ is non-oscillatory.

Corollary 2: *Let f and g be functions satisfying (i) – (iii) and such that (8) holds for $\varphi = 1$. Then equation (1) has no oscillatory solution x bounded with x', x'' on $\langle t_0, +\infty), t_0 > a$.*

Proof: Let x be a solution of equation (1) on $\langle T_x, +\infty)$ and let x, x', x'' be bounded on this interval. Then from the continuity of g and from supposition (iii) there follows the existence of a constant $M > 0$ such that $0 \leq G(t) \leq M$ for all $t \in \langle T_x, +\infty)$ where G is given by (3). Then the assertion follows from Theorem 4 for $\varphi = M$ ▮

Remark: Theorem 4 and Corollary 2 generalize the results of Theorem 3 and of Corollary 2 of [1].

4. We deduce in this section sufficient conditions for the oscillatory and non-oscillatory nature of solutions of equation (1) by means of the so-called *generalized Kneser criterion*

(for second order linear differential equations) generalized in [3] for the linear equation (6).

Lemma 3: *Let* $0 \leq h \in C(0, +\infty)$ *and* $t_0 > 0$.

a) *If* $h \geq 0, h(t) \leq \frac{2}{3\sqrt{3}} \frac{1}{t^3}$ *for* $t \in (t_0, +\infty)$ *and* $h \neq 0$ *on any subinterval of* $(t_0, +\infty)$, *then equation* (6) *is not oscillatory on* $(t_0, +\infty)$.

b) *If there is a* $\delta > 0$ *such that* $h(t) \geq \frac{1}{3}(1+\delta)\sqrt{\frac{1}{3}(4+\delta)}\frac{1}{t^3}$ *for* $t \in (t_0, +\infty)$, *then equation* (6) *is oscillatory on* $(t_0, +\infty)$, *i.e. every solution of equation* (6) *with a null point in* $(t_0, +\infty)$ *oscillates on this interval.*

Proof: The proof is given in [3] as proof of Corollary 2.7 ∎

Theorem 5: *Let f and g be functions satisfying conditions* (i) – (iii) *on* $(0, +\infty)$ *and let* $\delta, t_0 > 0$.

a) *If for all* $(y_0, y_1, y_2) \in R^3$ *and* $t \in (t_0, +\infty)$

$$f(t)|g(y_0, y_1, y_2)| \leq \frac{2}{3\sqrt{3}} \frac{1}{t^3}|y_0|, \tag{9}$$

then equation (1) *does not have an oscillatory solution defined on* $(t_0, +\infty)$.

b) *If for all* $(y_0, y_1, y_2) \in R^3$ *and* $t \in (t_0, +\infty)$

$$f(t)|g(y_0, y_1, y_2)| \geq \frac{1}{3}(1+\delta)\sqrt{\frac{1}{3}(4+\delta)}\frac{1}{t^3}|y_0|, \tag{10}$$

then any solution x of equation (1) *defined on* $(t_0, +\infty)$ *and having a null point is oscillatory on the right of this null point.*

Proof: a) Under our conditions x is a solution of equation (2) on $(t_0, +\infty)$ and by (9) satisfies the inequality $f(t)G(t) \leq \frac{2}{3\sqrt{3}}\frac{1}{t^3}$ for all $t \in (t_0, +\infty)$, where G is given by (3). Now from Lemma 3 it follows that equation (2) is non-oscillatory, x is a non-oscillatory solution of equation (2) and of equation (1), too.The assertion b) can be proved with the same arguments as in the foregoing assertion ∎

The proof of the next lemma is given in [3: Theorem 3.6].

Lemma 4: *Let* $h \in C(a, +\infty)$ *and* $d > 0$ *be such that* $h(t) \geq d$ *for all* $t \in (a, +\infty)$. *Then every solution of equation* (6) *is oscillatory in* $(a, +\infty)$ *except solutions y (unique up to linear dependence) satisfying the conditions*

$$\int_\alpha^{+\infty} y^2(t)\,dt < +\infty \quad (a < \alpha < +\infty) \qquad \text{and} \qquad y, y' \to 0 \text{ as } t \to +\infty.$$

Theorem 6: *Let x be a solution of equation* (1) *on* $(T_x, +\infty)$, *let the functions f and g satisfy the conditions* (i) – (iii) *and*

$$f(t)|g(y_0, y_1, y_2)| \geq M|y_0| \quad (M > 0) \tag{11}$$

for arbitrary $(y_0, y_1, y_2) \in R^3$ *and* $t \in (a, +\infty)$.

a) *If* $x \neq 0$, *then* $\int_{T_x}^{+\infty} x^2(t)\,dt < +\infty$ *and* $x, x' \to 0$ *as* $t \to +\infty$.

b) *If x has a null point, then x is oscillatory.*

Proof: a) The function x fulfils equation (2) and by (11) we have the inequality $f(t)|g(x(t), x'(t), x''(t))| \geq M|x(t)|$ which yields $f(t)G(t) \geq M$. By Lemma 4 equation (2) is oscillatory on $\langle T_x, +\infty)$ and x is its solution without null points. Again by Lemma 4 it must have the properties stated. Assertion b) follows immediately by Theorem 5 ∎

Lemma 5: *Let $h \in C(a, +\infty)$ and $h(t) > 0$ for all $t \in (a, +\infty)$.*

a) *If equation (6) has an oscillatory solution, then every solution but one (up to linear dependence) is oscillatory and the non-oscillatory solution y has the properties $y(t) \neq 0, \operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t)$ for all $t \in (a, +\infty)$, y, y', y'' are monotone on $(a, +\infty)$ and $y'(t), y''(t) \to 0$ as $t \to +\infty$.*

b) *If y is a monotone solution of (6) and $y(t) \neq 0, \operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t)$ for all $t \in (a, +\infty)$, then $y(t) \to 0$ as $t \to +\infty$ if and only if $\int_{t_0}^{+\infty} t^2 h(t)\, dt = +\infty$.*

Proof: The proof of statement a) is given in [3: Theorem 2.13] and that of statement b) is given in [7] ∎

Theorem 7: *Let f and g be functions satisfying the properties (i) – (iii) and (10). Then every solution x of equation (1) with the property $x(t) \neq 0$ in $\langle T_x, +\infty)$ tends to zero as $t \to +\infty$.*

Proof: The function x fulfils equation (2). By Theorem 5 this equation is oscillatory (i.e. every solution with a null point in $\langle T_x, +\infty)$ is oscillatory). By Lemma 5/a) equation (2) has only one solution without null points in $\langle T_x, +\infty)$. Therefore this solution must be x and it has the properties as in Lemma 5/a). It follows from (10) that $\int_{T_x}^{+\infty} t^2 f(t)G(t)\, dt = +\infty$ and therefore by Lemma 5/b) $x(t) \to 0$ as $t \to +\infty$ ∎

The equation in Example 1 fulfils the suppositions of Theorem 7 on $\langle t_0, +\infty)$ for sufficiently large $t_0 > 0$ and $x = e^{-t}$ is its solution with $x(t) \neq 0$ for $t > t_0$.

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