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On the K-Theory Groups of Certain Crossed Product C^*-Algebras

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Abstract. Given α, β two irrational numbers in (0,1) we define an analogue of the irrational rotation C^*-algebra on $T^2 = S^1 \times S^1$. Hence we compute explicitly the K-theory groups associated to that crossed product C^*-algebra.

Keywords: *C^*-algebras, crossed product C^*-algebras, K-theory*

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Let $C(T^2)$ be the C^*-algebra of continuous functions on $T^2 = S^1 \times S^1$ and let $\rho_{(\alpha,\beta)}$: $T^2 \longrightarrow T^2$ be the automorphism corresponding to a rotation of angles $2\pi\alpha, 2\pi\beta$ where α, β are irrational numbers, i.e., $(z_1, z_2) \longrightarrow (e^{2\pi i\alpha} z_1, e^{2\pi i\beta} z_2)$. Then the group $\boldsymbol{Z}$ acts as a transformation group on

$$T^2 : (z_1, z_2) \longrightarrow (e^{2\pi in\alpha} z_1, e^{2\pi in\beta} z_2) \ (n \in \boldsymbol{Z})$$

by powers of the rotation automorphism. We can extend this action to the C^*-algebra of continuous functions on T^2 so it gives an action of $\boldsymbol{Z}$ on $C(T^2)$. We define $A_{(\alpha,\beta)}$ to be the crossed product C^*-algebra of $C(T^2)$ under the automorphism $\rho_{(\alpha,\beta)} : n \longrightarrow \rho_{(\alpha,\beta)}(n)$ $(n \in \boldsymbol{Z})$. By [4] we have that $A_{(\alpha,\beta)} = C(T^2) \times_{\rho_{(\alpha,\beta)}} \boldsymbol{Z}$ is a simple C^*-algebra.

Denote by U_1 the unitary operator in $A_{(\alpha,\beta)}$ corresponding to the element $n = \mathbf{1} \in \boldsymbol{Z}$ and defined by

$$(U_1 f)(z_1, z_2) = f((e^{2\pi i\alpha})z_1, z_2) \qquad (f \in L^2(T^2))$$

and similarly denote

$$(V_1 f)(z_1, z_2) = f(z_1, (e^{2\pi i\beta})z_2) \qquad (f \in L^2(T^2)).$$

Let W_1 be the multiplication operator on $L^2(T^2)$. Therefore we have

$$U_1 V_1 = V_1 U_1, \qquad U_1 W_1 = (e^{2\pi i\alpha}) W_1 U_1, \qquad V_1 W_1 = (e^{2\pi i\beta}) W_1 V_1. \qquad (1)$$

Let $C^*(U, V, W)$ be the C^*-algebra generated by three unitary operators U, V, W such that they satisfy relations (1). Then by [3: Formula (7.6.6)] and by the fact that $A_{(\alpha,\beta)}$

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is simple we can conclude that $A_{(\alpha,\beta)}$ is isomorphic to the C^*-algebra $C^*(U,V,W)$, where the isomorphism maps $U_1 \longrightarrow U, V_1 \longrightarrow V, W_1 \longrightarrow W$. Let $A_\alpha = C(T) \times_{\rho_\alpha} \boldsymbol{Z}$ be the irrotational rotation C^*-algebra, i.e., the crossed product of the C^*-algebra of the continuous functions on the unit circle by the automorphism corresponding to a rotation of angle $2\pi\alpha$. This C^*-algebra is simple (see [4]). It is generated by two unitaries u, w such that $u\,w = (e^{2\pi i\alpha})w\,u$ (see [5]).

By the same argument we can conclude that A_α is contained in $A_{(\alpha,\beta)}$.

Let γ_β be the automorphism of A_α defined by $\gamma_\beta = a\,d\,V$, with V the unitary operator in $A_{(\alpha,\beta)}$. Then we form the C^*-algebra crossed product $A_\alpha \times_{\gamma_\beta} \boldsymbol{Z}$. We observe that

$$\gamma_\beta(u) = V\,u\,V^* = u \quad \text{and} \quad \gamma_\beta(w) = V\,w\,V^* = (e^{-2\pi i\beta})w$$

so that the automorphism γ_β leaves the operator u fixed. Therefore we have $u\,V = V\,u, u\,w = (e^{2\pi i\alpha})w\,u$ and $V\,w = (e^{2\pi i\beta})w\,V$. By using [1: Prop. 5.1] we have $\Gamma(\alpha) = T$ and by [2: Formula (8.11.12)] we get that the C^*-algebra $A_\alpha \times_{\gamma_\beta} \boldsymbol{Z}$ is simple.

Similarly we can consider the irrational rotation C^*-algebra A_β generated by v, w and form the crossed product C^*-algebra $A_\beta \times_{\gamma_\beta} \boldsymbol{Z}$ under the automorphism γ_α given by $a\,d\,U$. Let (π, V) be a covariant representation of $(A_\alpha, \gamma_\beta \boldsymbol{Z})$. Let $\tilde{\pi} : A_\alpha \times_{\gamma_\beta} \boldsymbol{Z} \longrightarrow A_{(\alpha,\beta)}$ be defined by $u \longrightarrow U_1, w \longrightarrow W_1, v \longrightarrow V_1$. Then by [3: Formula (7.6.6)], since both C^*-algebras are simple $\tilde{\pi}$ gives an isomorphism. Similarly $A_{(\alpha,\beta)}$ and $A_\beta \times_{\gamma_\alpha} \boldsymbol{Z}$ are isomorphic. Therefore the C^*-algebras $A_{(\alpha,\beta)}, A_\alpha \times_{\gamma_\beta} \boldsymbol{Z}$ and $A_\beta \times_{\gamma_\alpha} \boldsymbol{Z}$ are all isomorphic.

Let $A_\alpha = C(T) \times_{\rho_\alpha} \boldsymbol{Z}$ and $A_\beta \times_{\gamma_\alpha} \boldsymbol{Z}$ be the irrational rotation C^*-algebras and let p and $\tilde{p}$ be the Rieffel projections in A_α and A_β, respectively (see [5]). There exists a unitary operator $\mathcal{U}$ in $M_n(A_{(\alpha,\beta)})$ such that for the boundary map ∂ we have $\partial[\mathcal{U}] = [p]$, where M_n stands for the $n \times n$-matrices. As in [6: Section 8.5], let us call $[Q]$ the *Bott generator* of $K_0(C(T^2))$.

Our main result is the following

Theorem. *The following two statements are true.*

(a) The group $K_0(C(T^2) \times_{\rho_{(\alpha,\beta)}} \boldsymbol{Z}$ *is isomorphic to* $\boldsymbol{Z}^4$ *and it is generated by* $[1], [p], [\tilde{p}], [Q]$.

(b) The group $K_1(C(T^2) \times_{\rho(\alpha,\beta)} \boldsymbol{Z})$ *is isomorphic to* $\boldsymbol{Z}^4$ *and it is generated by* $[U], [V], [W], [\mathcal{U}]$.

Proof. (a) Let $C(T^2) \times_{\rho_{(\alpha,\beta)}} \boldsymbol{Z}$ be the C^*-algebra generated by three unitary operators U, V, W satisfying relations (1). Let the C^*-algebra $C^*(U,V)$ be generated by U, V, $C^*(U,W)$ by U, W and $C^*(V,W)$ by V, W. Observe the identities

$$C^*(U,V) = C(T^2), \quad C^*(U,W) = A_\alpha, \quad C^*(V,W) = A_\beta$$

and that all of these C^*-algebras are contained in $A_{(\alpha,\beta)}$. By [3] we have the following exact sequence:

$$\begin{array}{ccccc} K_0(A_\alpha) & \xrightarrow{(id)^*-(\gamma_\beta)^*} & K_0(A_\alpha) & \xrightarrow{i^*} & K_0(A_\alpha \times_{\gamma_\beta} \mathbf{Z}) \\ {\scriptstyle \partial}\uparrow & & & & \downarrow{\scriptstyle \tilde{\partial}} \\ K_1(A_\alpha \times_{\gamma_\beta} \mathbf{Z}) & \xleftarrow{i^*} & K_1(A_\alpha) & \xleftarrow{(id)^*-(\gamma_\beta)^*} & K_1(A_\alpha) \end{array} \tag{2}$$

where $i : A_\alpha \longrightarrow A_\alpha \times_{\gamma_\beta} \boldsymbol{Z}$ is the inclusion map and ∂ is the index map coming from the Toepliz extension for (A_α, γ_β). Starting from the left upper side of the sequence we see that, since $K_0(A_\alpha)$ is generated by $[\mathbf{1}]$ and $[p]$, $(id)^* - (\gamma_\beta)^*$ is the zero map on $K_0(A_\alpha)$, then $[\mathbf{1}]$ and $[p]$ generate a copy of $\boldsymbol{Z}^2$ in $K_0(A_\alpha \times_{\gamma_\beta} \boldsymbol{Z})$. On $K_1(A_\alpha)$ the map $(id)^* - (\gamma_\beta)^*$ is the zero map and because of $K_1(A_\alpha)$ is generated by $[U], [W]$ we have a copy of $\boldsymbol{Z}^2$ in $K_1(A_\alpha \times_{\gamma_\beta} \boldsymbol{Z})$. Note that since

$$A_\beta \longrightarrow A_\beta \times_{\gamma_\alpha} \boldsymbol{Z} \quad \text{and} \quad K_0(A_\alpha \times_{\gamma_\beta} \boldsymbol{Z}) \simeq K_0(A_\beta \times_{\gamma_\alpha} \boldsymbol{Z}), \;\; C^*(W) \longrightarrow A_\alpha$$

we have the following diagram:

$$\begin{array}{ccccccccc} \longrightarrow & K_0(A_\alpha \times_{\gamma_\beta} \mathbf{Z}) & \xrightarrow{\tilde{\partial}} & K_1(A_\alpha) & \xrightarrow{(id)^*-(\gamma_\beta)^*} & K_1(A_\alpha) & \longrightarrow \\ & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & K_1(A_\beta) & \xrightarrow{\partial} & K_1(C^*(W)) & \xrightarrow{0} & K_1(C^*(W)) & \longrightarrow \end{array} \tag{3}$$

where the vertical rows are the inclusion maps. The above diagram is obtained by considering A_β as the C^*-algebra crossed product $C(T) \times_{\rho_\beta} \boldsymbol{Z}$ where $C^*(W) = C(T)$. By the exactness of the bottom line of the diagram we get $\partial[\tilde{p}] = [W]$. Now the rectangles in the diagram commute so also on the top line we have $\tilde{\partial}[\tilde{p}] = [W]$. Let us show that $\tilde{\partial}[Q] = [V]$. So considering $C^*(U, V) = C(T^2)$, then we have the following diagram:

$$\begin{array}{ccccc} K_0(A_{(\alpha,\beta)}) & \xrightarrow{\tilde{\partial}} & K_1(C(T^2)) & \longrightarrow & K_1(C(T^2)) \\ \uparrow & & \uparrow & & \uparrow \\ K_0(C(T^2)) & \longrightarrow & K_1(C^*(V)) & \xrightarrow{0} & K_1(C^*(V)) \end{array} \tag{4}$$

where $C^*(V) = C(T)$. The bottom part is the Pimsner-Voiculescu six term exact sequence for the C^*-algebra $C(T^2)$ by seeing it as the crossed product of $C^*(V)$ by the trivial action of $\boldsymbol{Z}$. As in [6: Section 8.5] let $[Q]$ be the generator of $K_0(C(T^2))$. Then we have $\partial[Q] = [V]$ and $\partial[\mathbf{1}] = 0$, since the diagram commute this holds also for the top part so $\tilde{\partial}[Q] = [V]$. Hence we can conclude that $K_0(A_{(\alpha,\beta)}) = \boldsymbol{Z}^4$ with generators $[1], [p], [\tilde{p}], [Q]$.

(b) The map $(id)^* - (\gamma_\beta)^*$ on $K_1(A_\alpha)$ is the zero map and because of $K_1(A_\alpha) = \boldsymbol{Z}^2$ we have a copy of $\boldsymbol{Z}^2$ in $K_1(A_{(\alpha,\beta)})$ with generators $[U], [W]$. Also at the $K_0(A_\alpha)$ level the map $(id)^* - (\gamma_\beta)^*$ is the zero map. Thus we get another copy of $\boldsymbol{Z}^2$ in $K_1(A_{(\alpha,\beta)})$. Let $\mathcal{U}$ be the unitary operator in $M_n(A_{(\alpha,\beta)})$ such that $\partial[\mathcal{U}] = [p]$. As in (3) we consider the diagram

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
K_1(C^*(W)) & \longrightarrow & K_1(A_\alpha) \\
\downarrow & & \downarrow {\scriptstyle (id)^* - (\gamma_\beta)^*} \\
K_1(C^*(W)) & \longrightarrow & K_1(A_\alpha) \\
\downarrow & & \downarrow \\
K_1(A_\beta) & \longrightarrow & K_1(A_{(\alpha,\beta)}) \\
\downarrow & & \downarrow {\scriptstyle \tilde{\partial}} \\
K_0(C^*(W)) & \longrightarrow & K_0(A_\alpha) \\
\downarrow & & \downarrow
\end{array}
$$

By the exactness of the bottom line of the diagram we get $\partial[W] = [1]$. Now the rectangles in the diagram commute. Since this holds also for the top part, we have $\tilde{\partial}[W] = [1]$. Thus statement (b) is proved ∎

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