Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 13 (1994), No. 1, 89–96

General Approximation-Solvability of Nonlinear Equations Involving $\mathcal{A}$-regular Operators

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Abstract. Here we consider a general approximation-solvability scheme involving $\mathcal{A}$-regular operators – a generalization of $\mathcal{A}$-proper operators – introduced and studied by Petryshyn [4] and further studied by Milojevic' [3], Petryshyn [6] and others. Among significant applications, an $\mathcal{A}$-regular version of the Petryshyn theorem is given.

Keywords: *$\mathcal{A}$-regular operators, discrete convergence, consistency, stability*

AMS subject classification: Primary 65J15, secondary 47H17

0. Introduction

We consider an approximation scheme $\pi_0 = \{X_n, Y_n, E_n, R_n, Q_n\}$ represented by the diagram

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ R_n \downarrow\uparrow E_n & & \downarrow Q_n \\ X_n & \xrightarrow{A_n} & Y_n \end{array}$$

Diagram 1

Here X and Y are infinite-dimensional normed spaces over $\mathbf{K}$ (the real or complex field), and X_n and Y_n are finite-dimensional normed spaces over $\mathbf{K}$ for all n. For all n, $E_n : X_n \to X$, $Q_n : Y \to Y_n$ and $R_n : X \to X_n$ are connection operators. The operator $A : X \to Y$ is defined by $A_n = Q_n A E_n$, that is, Diagram 1 is commutative. Petryshyn [4 - 6], in a series of papers, studied the following problem:

Construct a solution x of the equation

$$Ax = b \qquad (x \in X; b \in Y) \tag{1}$$

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ISSN 0232-2064 / $ 2.50

as a strong limit of solutions x_n of the simpler finite-dimensional equations (the so-called *approximate equations*)

$$A_n x_n = Q_n b \qquad (x_n \in X_n, n = 1, 2, \ldots) \tag{2}$$

with respect to the approximation scheme π_0.

In order to solve this problem, Petryshyn developed the notion of *$\mathcal{A}$-properness* of an operator $A : X \to Y$, which is not only closely connected with the approximation-solvability of the equation $Ax = b$, but it extends and unifies investigations concerning Galerkin type methods for linear and nonlinear operator equations with other newer results in the theory of strongly ϕ-monotone and accretive operators, operators of the type (S), ball-condensing and other mappings. The $\mathcal{A}$-properness is equally applicable to the approximation-solvability of abstract semilinear equations of the form

$$Ax + Nx = f, \tag{3}$$

where A is a Fredholm mapping of index zero and N a quasibounded nonlinear mapping such that $A + N$ is $\mathcal{A}$-proper. Among the significant applications of these results is the approximation-solvability of semilinear elliptic equations of the type

$$Ax + F(x, u, Du, \ldots, D^{2m}u) = f \tag{4}$$

exhibiting double resonance with F having a linear growth. It is known that many boundary value problems for ordinary and partial differential equations can be formulated as abstract operator equations of the type (3) if the operators are chosen in suitable spaces. While the $\mathcal{A}$-properness has the tremendous power of unifying several classes of $\mathcal{A}$-proper mappings, it is not strong enough to give any information about the existence of solutions of the finite-dimensional approximate equations, and even if they exist and are uniformly bounded, there exists just a subsequence converging to some solution. This leads to expect that the numerical applications of $\mathcal{A}$-proper mappings are feasible under extra arguments. For more selected details on the approximation-solvability involving $\mathcal{A}$-proper mappings, we refer to [3 - 8] and [10].

In this paper, we first consider the approximation-solvability of the equation $Ax = b$ in the context of $\mathcal{A}$-regularity – a generalization of the $\mathcal{A}$-properness – in general normed spaces. Second, we give the $\mathcal{A}$-regular version of Petryshyn's theorem [4]. Finally, we apply the obtained results to approximation schemes in Banach spaces.

Note that the approximation schemes in the context of $\mathcal{A}$-regular operators have definite edge over the approximation schemes for $\mathcal{A}$-proper operators in the sense that these require minimal hypotheses.

A word of caution: Here and in what follows, the symbols $\to$ and $\xrightarrow{w}$ shall denote strong convergence and weak convergence, respectively.

Definition 0.1 (*Discrete convergence*). A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to *converge discretely* to an element $x \in X$, denoted $x_n \xrightarrow{d} x$, if $\lim_{n\to\infty} \|x_n - R_n x\|_{X_n} = 0$. Similarly, a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$ is said to *converge discretely* to an element $b \in Y$, denoted $y_n \xrightarrow{d} b$, if $\lim_{n\to\infty} \|y_n - Q_n b\|_{Y_n} = 0$.

Definition 0.2 (*Discrete equality*). An element $x \in X$ is said to *equal discretely* another element $u \in X$, denoted $x \stackrel{d}{=} u$, if $\lim_{n\to\infty} \|R_n x - R_n u\|_{X_n} = 0$.

Definition 0.3 (*Discrete Unique Approximation-Solvability*). The equation $Ax = b$ has *discretely unique approximation-solvability* if, for each $b \in Y$, the following hold:

(i) For each $n \geq n_0$, the approximate equation $A_n x_n = Q_n b$ $(x_n \in X_n)$ has a unique solution.

(ii) The equation $Ax = b$ has a discretely unique solution.

(iii) The sequence $\{x_n\}$ converges discretely to the solution x of the equation $Ax = b$ in the sense that $\lim_{n\to\infty} \|x_n - R_n x\|_{X_n} = 0$.

Definition 0.4 ($\mathcal{A}$-*Regularity*). The operator $A : X \to Y$ is $\mathcal{A}$-*regular* with respect to the approximation scheme π_0 if, for given any bounded sequence $\{x_n\}$ with $A_n x_n \stackrel{d}{\to} b$ (that is, $\lim_{n\to\infty} \|A_n x_n - Q_n b\|_{Y_n} = 0$), there exists a subsequence $\{x_{n'}\}$ and an $x \in X$ such that $x_{n'} \stackrel{d}{\to} x$ and $Ax = b$.

To this end, we are just about to consider the general approximation-solvability of equation (1) involving $\mathcal{A}$-regular operators relating to the approximation scheme π_0.

1. General approximation-solvability

In this section we give an upgrade of the approximation-solvability based on the concept of $\mathcal{A}$-regularity – a generalization of $\mathcal{A}$-properness.

Theorem 1.1. *Let $\pi_0 = \{X_n, Y_n, E_n, R_n, Q_n\}$ be an approximation scheme represented by Diagram 1 with the following assumptions:*

(A1) *Admissible Approximation Scheme. π_0 is an approximation scheme, that is, the following hold:*

(i) *X and Y are infinite-dimensional normed spaces over* $\mathbf{K}$.

(ii) *X_n and Y_n are finite-dimensional normed spaces over* $\mathbf{K}$ *with* $\dim X_n = \dim Y_n$.

(iii) *$Q_n : Y \to Y_n$ are not necessarily linear connection operators with $\|Q_n b\| \leq \nu(b)$ for each $b \in Y$, where $\nu(b) > 0$, and is independent of n. The other connection operators $R_n : X \to X_n$ and $E_n : X_n \to X$ are not necessarily linear with $E_n(0) = 0$ for all n. All operators $A_n = Q_n A E_n$ are nonlinear and continuous, where the operator $A : X \to Y$ is also nonlinear.*

(A2) *Consistency. π_0 is consistend, that is, for all $x \in X$, $\lim_{n\to\infty} \|Q_n Ax - A_n R_n x\|_{Y_n} = 0$.*

(A3) *Stability. π_0 is stable, that is, for all $u, \nu \in X_n$, there is an n_0 such that $\|A_n u - A_n \nu\|_{Y_n} \geq \mu(\|u - \nu\|_{X_n})$ for all $n \geq n_0$. Here $\mu : \mathbf{R}^+ \to \mathbf{R}^+$ is a gauge function, that is, μ is continuous and strictly monotone increasing with $\mu(0) = 0$ and $\mu(t) \to \infty$ as $t \to \infty$.*

Then the following conditions are equivalent:

(C1) *Solvability. For each $b \in Y$, the equation $Ax = b$ has a solution.*

(C2) *Discretely Unique Approximation-Solvability. The equation $Ax = b$ has a discretely unique approximation-solvability.*

(C3) *$\mathcal{A}$-Regularity. The operator $A : X \to Y$ is $\mathcal{A}$-regular with respect to the approximation scheme π_0.*

More precisely, Theorem 1.1 can be stated as: If π_0 is an admissible approximation scheme with consistency and stability, then the equation $Ax = b$ has a discretely unique approximation-solvability if and only if the operator A is $\mathcal{A}$-regular.

Corollary 1.2. *Let π_0 be an admissible approximation scheme satisfying the stability condition. If $A : X \to Y$ is $\mathcal{A}$-regular, then the following hold:*

(a) *For each $b \in Y$, the equation $Ax = b$ $(x \in X)$ has a solution.*

(b) *For each $b \in Y$ and each $n \geq n_0$, the approximate equation $A_n x_n = Q_n b$ $(x_n \in X_n)$ has a unique solution.*

(c) *The sequence $\{x_n\}$ has a subsequence $\{x_{n'}\}$ such that $x_{n'} \xrightarrow{d} x$ in X.*

(d) *If, for each $b \in Y$, the equation $Ax = b$ has exactly one solution x, then the whole sequence $\{x_n\}$ converges discretely to x, that is, $x_n \xrightarrow{d} x$.*

Corollary 1.3. *Let π_0 represent an admissible approximation scheme, and let $A : X \to Y$ be $\mathcal{A}$-regular. If, for some $b \in Y$ and all $n \geq n_0$, the approximate equation $A_n x_n = Q_n b$ $(x_n \in X_n)$ has a unique solution, and if we have the a-priori estimate $\sup_n \|x_n\| < \infty$, then the following implications hold:*

(a) *There exists a subsequence $\{x_{n'}\}$ such that $x_{n'} \xrightarrow{d} x$ and $Ax = b$.*

(b) *If the equation $Ax = b$ $(x \in X)$ has exactly one solution x, then $x_n \xrightarrow{d} x$.*

Proof of Theorem 1.1. The method of the proof follows the sequence *(C3)* $\Rightarrow$ *(C2)* $\Rightarrow$ *(C1)* $\Rightarrow$ *(C3)*.

Step (C3) $\Rightarrow$ *(C2)*. We first show that, for fixed $b \in Y$ and each $n \geq n_0$, the approximate equation $A_n x_n = Q_n b$ $(x_n \in X_n)$ has a unique solution. To achieve this, we show that the set $A_n X_n$ is both open and closed. The set $A_n X_n$ is open. Indeed, the operator $A_n : X_n \to Y_n$ is injective by the stability condition (A3). The set $A_n X_n$ is open by the invariance of the Domain Theorem [9: Theorem 16.C]. The set $A_n X_n$ is closed. To show this, assume $A_n x_k \to z$ as $k \to \infty$. Then $\{A_n x_k\}$ is a Cauchy sequence. By the stability condition (A3), $\{x_k\}$ is also a Cauchy sequence, and hence $x_k \to x$ as $k \to \infty$. Since the operator $A_n : X_n \to Y_n$ is continuous this implies $A_n x = z$, that is, $z \in A_n X_n$. Since the non-empty set $A_n X_n$ is both open and closed in Y_n, we get $A_n X_n = Y_n$.

Second, we show that, for fixed $b \in Y$, the equation $Ax = b$ $(x \in X)$ has at most one discrete solution. To prove this, assume $Ax = Au$ for $x, u \in X$. Then the consistency and stability conditions imply that, as $n \to \infty$,

$$\begin{aligned}\mu(\|R_n x - R_n u\|) &\leq \|A_n R_n x - A_n R_n u\| \\ &\leq \|A_n R_n x - Q_n A x\| + \|Q_n A u - A_n R_n u\| \to 0.\end{aligned}$$

Hence, $\|R_n x - R_n u\| \to 0$ as $n \to \infty$, that is, $x \overset{d}{=} u$.

Third, we prove that, for each $b \in Y$, the equation $Ax = b$ has exactly one solution $x \in X$. Let us choose $x_n \in X_n$ with $A_n x_n = Q_n b$. From $A_n(0) = Q_n A E_n(0) = Q_n A(0)$ and Assumption (A1)/(iii), it follows that

$$\nu(b) \geq \|Q_n b\| = \|A_n x_n\| = \|A_n x_n - A_n(0)\| - \|A_n(0)\| \geq \mu(\|x_n\|) - \nu(A(0)),$$

that is, $\mu(\|x_n\|) \leq \nu(b) + \nu(A(0))$ for all n. This implies that $\sup_n \mu(\|x_n\|) < \infty$, and hence $\sup_n \|x_n\| < \infty$. It easily follows from the hypothesis that $A_n x_n \xrightarrow{d} b$ as $n \to \infty$.

To this end, the $\mathcal{A}$-regularity of the operator A ensures the existence of a subsequence $\{x_{n'}\}$ such that $x_{n'} \xrightarrow{d} x$ and $Ax = b$.

Finally, we show that $x_n \xrightarrow{d} x$ and $Ax = b$. Let us choose, for fixed $b \in Y$ and $n \geq n_0$, $x_n \in X_n$ with $A_n x_n = Q_n b$. Then we have $\sup_n \|x_n\| < \infty$. It is obvious that $\|A_n x_n - Q_n b\| \to 0$ as $n \to \infty$. The $\mathcal{A}$- regularity of the operator A ensures the existence of a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} \xrightarrow{d} x$ and $Ax = b$. It follows that each subsequence $\{x_{n'}\}$ of $\{x_n\}$ has another subsequence $\{x_{n''}\}$ with $x_{n''} \xrightarrow{d} x$ and $Ax = b$. The limit element x is the same for all subsequences since the equation$Ax = b$ has exactly one solution x. It follows that the entire sequence converges discretely to x, that is $x_n \xrightarrow{d} x$. As a matter of fact, this follows from an analogous argument as in the proof of the convergence principle [9: Proposition 10.13 (1)].

Step (C2) ⇒ (C1). The proof of this implication is trivial.

Step (C1) ⇒ (C3). For given $b \in Y$, we solve the equation $Ax = b$ $(x \in X)$. To show that $A : X \to Y$ is $\mathcal{A}$- regular, assume $\sup_n \|x_n\| < \infty$ and $A_n x_n \xrightarrow{d} b$, that is, $\|A_n x_n - Q_n b\| \to 0$ as $n \to \infty$. For the sake of brevity, we shall write n for n'. To this end, by the stability and consistency conditions, we obtain, as $n \to \infty$,

$$\mu(\|x_n - R_n x\|) \leq \|A_n x_n - A_n R_n x\| \leq \|A_n x_n - Q_n b\| + \|Q_n A x - A_n R_n x\| \to 0.$$

This implies that $\|x_n - R_n x\| \to 0$ as $n \to \infty$, that is, the operator A is $\mathcal{A}$-regular ∎

Remark 1.4. *Suppose that we replace the assumptions for the connection operators E_n and Q_n in Theorem 1.1 by the following:*

(A4) *$E_n : X_n \to X$ and $Q_n : Y \to Y_n$ are linear and continuous with* $\sup_n \|E_n\| < \infty$ *and* $\sup_n \|Q_n\| < \infty$.

(A5) *π_0 is compatible, that is,* $\lim_{n\to\infty} \|E_n R_n x - x\|_X = 0$.

Then $\mathcal{A}$-regularity implies $\mathcal{A}$-properness.

Proof. Since $A : X \to Y$ is $\mathcal{A}$-regular, this implies that there exists a subsequence, again denoted by $\{x_n\}$, such that $x_n \xrightarrow{d} x$ in X, that is, $\|x_n - R_n x\| \to 0$ as $n \to \infty$. Since $\sup_n \|E_n\| < \infty$, we obtain $\|E_n x_n - E_n R_n x\| \to 0$ as $n \to \infty$. By the compatibility condition (A5), it follows that $\|E_n x_n - x\| \to 0$ as $n \to \infty$, that is, A is $\mathcal{A}$-proper.

Now, in the light of Remark 1.4, we give the $\mathcal{A}$-regular version of Petryshyn's theorem [4].

Theorem 1.5. *Let $\pi_1 = \{X_n, Y_n, E_n, R_n, Q_n\}$ be a consistend and stable approximation scheme represented by Diagram 1 with the following assumptions:*

(A) *Admissible Inner Approximation Scheme. π_1 is an admissible approximation scheme, that is, we have the following:*

(i) *X and Y are infinite-dimensional normed spaces over* **K**.

(ii) *X_n and Y_n are finite-dimensional normed spaces over* **K** *with* $\dim X_n = \dim Y_n$.

(iii) *E_n and Q_n are linear and continuous operators with* $\sup_n \|E_n\| < \infty$ *and* $\sup_n \|Q_n\| < \infty$. *All operators $A_n = Q_n A E_n$ are nonlinear and continuous, and R_n are also nonlinear.*

(iv) *Compatibility Condition (A5) is satisfied.*

Then equation $Ax = b$, for each $b \in Y$, is solvable and uniquely approximation-solvable and the operator $A : X \to Y$ is $\mathcal{A}$-regular with respect to π_1.

Corollary 1.6. *Let π_1 be an admissible inner approximation scheme. If the operator $A : X \to Y$ is $\mathcal{A}$-regular with respect to π_1, and if $C : X \to Y$ is compact, then $A + C : X \to Y$ is also $\mathcal{A}$-regular.*

Proof. For brevity, we shall write n for n'. Let $\sup_n \|x_n\| < \infty$, and let $A_n x_n \xrightarrow{d} b$, that is, $\lim_{n\to\infty} \|A_n x_n - Q_n b\| = 0$ with $A_n = Q_n(A + C)E_n$. Since $\sup_n \|E_n\| < \infty$, the sequence $\{E_n x_n\}$ is bounded. Given that the operator C is compact, there exists a subsequence of $\{x_n\}$, again denoted $\{x_n\}$, such that

$$CE_n x_n \to z \qquad \text{in } Y \text{ as } n \to \infty. \tag{5}$$

Since $\sup_n \|Q_n\| < \infty$, this implies that $\|Q_n CE_n x_n - Q_n z\| \to 0$ as $n \to \infty$. It follows that $\lim_{n\to\infty} \|Q_m AE_n x_n - Q_n(b - z)\| = 0$. Given that the operator A is $\mathcal{A}$-regular, there is a subsequence, again denoted by $\{x_n\}$, such that $\lim_{n\to\infty} \|x_n - R_n x\| = 0$ and $Ax = b - z$. Since $\sup_n \|E_n\| < \infty$, this implies that $\|E_n x_n - E_n R_n x\| \to 0$ as $n \to \infty$. By the compatibility condition (A5), we get

$$\|E_n x_n - x\| \to 0 \qquad \text{as } n \to \infty. \tag{6}$$

Since C is compact, by (5) and (6), we obtain $Cx = z$. Hence, $x_n \xrightarrow{d} x$ and $(A+C)x = b$, that is, $A + C$ is $\mathcal{A}$-regular ∎

Here we consider an example, in which we compare the $\mathcal{A}$-properness with $\mathcal{A}$-regularity.

Example 1.7 [2: Theorem 17.1]. Let X be a Banach space, $T \in L(X)$, X_n a closed subspace of X and $P_n : D(P_n) \subset X \to X_n$ a (possibly) unbounded projection onto X_n such that $R(T) \subset D(P_n)$ and $P_n T \in L(X)$. Consider

$$x = Tx + y \quad \text{with } y \in D(P_n) \qquad \text{and} \qquad x = T_n x + y_n \quad \text{with } T_n \in L(X_n)$$

for $y_n \in X_n$. Suppose that $(I - T)^{-1} \in L(X)$ and

$$\|T_n - P_n T_{|X_n}\| \to 0, \quad \|t - P_n T\| \to 0, \quad \|y_n - P_n y\| \to 0, \quad \|P_n y - y\| \to 0.$$

Then $x = T_n x + y_n$ has a unique solution x_n for all large n, and $x_n \to x_0 = (I - T)^{-1} y$.

Note that if we try to place this result in an $\mathcal{A}$-proper setting of Theorem 1.1 under the approximation scheme represented by Diagram 1, it is difficult, while it is easier to handle with an $\mathcal{A}$-regular setting.

2. $\mathcal{A}$-Regularity in Banach spaces

Let $\pi_2 = \{X_n, X_n^*, E_n, R_n, E_n^*\}$ be an approximation scheme represented by the diagram

$$\begin{array}{ccc} X & \overset{A}{\rightarrow} & X^* \\ R_n \downarrow\uparrow E_n & & \downarrow E_n^* \\ X_n & \overset{A_n}{\rightarrow} & X_n^* \end{array}$$

Diagram 2

Here X is a real reflexive separable infinite-dimensional Banach space. Let $\{X_n\}$ be a Galerkin scheme in X with $X_n = \text{span}\{e_{1n}, ..., e_{n'n}\}, n \in \mathbb{N}$. Let $E_n : X_n \rightarrow X$ be the embedding operator with $X_n \subseteq X$. The connection operator $R_n : X \rightarrow X_n$ is constructed in a manner that, for each $x \in X$, there exists at least one element $R_n x$ such that $\|x - R_n x\| = \text{dist}(x, X_n)$. Let all operators $A_n = E_n^* A E_n$ be continuous.

Consider the operator equation

$$Ax = b \qquad (x \in X, b \in X^*) \tag{7}$$

along with the approximate equations

$$A_n x_n = E_n^* b \qquad (x_n \in X_n; n = 1, 2, ...) \tag{8}$$

with respect to approximation scheme π_2. For $n = 1, 2, ...,$ the approximate equations above are equivalent to Galerkin equations

$$[Ax_n, e_{jn}] = [b, e_{jn}] \qquad (x_n \in X_n; j = 1, ..., n') \tag{9}$$

where $[\cdot,\cdot]$ is the dual pairing between X^* and X.

Theorem 2.1. *Let $\pi_2 = \{X_n, X_n^*, E_n, R_n, E_n^*\}$ be an approximation scheme represented by Diagram 2. If the operator $A : X \rightarrow X^*$ is continuous and uniformly monotone, then A is $\mathcal{A}$-regular and, for each $b \in X^*$, the equation $Ax = b$ $(x \in X)$ is uniquely approximation-solvable. If $C : X \rightarrow X^*$ is compact, then the operator $A + C : X \rightarrow X^*$ is $\mathcal{A}$-regular.*

Proof. Under the assumptions, it follows easily that π_2 is an admissible approximation scheme with the consistency and stability ensured by the uniform monotonicity of the operator $A : X \rightarrow X^*$. To this end, it suffices to show that A is $\mathcal{A}$-regular. Let $\sup_n \|x_n\| < \infty$ and $A_n x_n \overset{d}{\rightarrow} b$ in X^* for $x_n \in X_n$. Since X is reflexive, there exists a subsequence, again denoted by $\{x_n\}$, such that $x_n \overset{w}{\rightarrow} x$ in X. Since $\{X_n\}$ is a Galerkin scheme, $\text{dist}(x, X_n) \rightarrow 0$ for all $x \in X$ and, hence, it follows from the construction of the operator $R_n : X \to X_n$ that $\|x - R_nx\| \to 0$. This implies that $b - AR_nx \to b - Ax$ as $n \to \infty$. Thus, $x_n - R_nx \xrightarrow{w} 0$. Now, by the stability condition, we have, as $n \to \infty$,

$$\begin{aligned}\mu(\|x_n - R_nx\|) &\le [A_nx_n - A_nR_nx, x_n - R_nx] \\ &= [A_nx_n - E_n^*b + E_n^*b - A_nR_nx, x_n - R_nx] \\ &= [A_nx_n - E_n^*b, x_n - R_nx] + [E_n^*b - A_nR_nx, x_n - R_nx] \\ &= [A_nx_n - E_n^*b, x_n - R_nx] + [E_n^*b - E_n^*AR_nx, x_n - R_nx] \\ &= [A_nx_n - E_n^*b, x_n - R_nx] + [b - A_nR_nx, E_nx_n - E_nR_nx] \\ &= [A_nx_n - E_n^*b, x_n - R_nx] + [b - AR_nx, x_n - R_nx] \to 0.\end{aligned}$$

It follows that $\|x_n - R_nx\| \to 0$, that is, $x_n \xrightarrow{d} x$. Since A is continuous and $\|x_n - R_nx\| \to 0$, we get $Ax = b$. Hence, A is $\mathcal{A}$-regular, and Theorem 2.1 follows from an application of Theorem 1.1 ∎

Remark 2.2. *Under the assumptions of Theorem 2.1, $x_n \xrightarrow{d} x$ impies $x_n \to x$, and, for $x, u \in X$, we have that $x \overset{d}{=} u$ implies $x = u$.*

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Received 06.11.1992; in revised form 21.09.1993