

On Spectral Factorization and Generalized Nehari Problems

B. Fritzsche and B. Kirstein

Abstract. The paper is aimed to show that spectral factorization can be reformulated as an appropriately constructed Generalized Nehari Problem for matrix-valued Carathéodory functions.

Key words: *Largest minorants, spectral factorization, Generalized Nehari Problems, matrix-valued Carathéodory functions*

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0. Introduction

In their fundamental papers [1] - [4], Adamjan, Arov and Krein studied carefully the classical Nehari Problem (see [22]) and its matricial generalization. Influenced by the methods and results used there Cotlar and his collaborators initiated a systematic research of various types of Generalized Nehari Problems (see, e.g., [5], [9], [20], [15], [17]). In particular, Cotlar and his group recognized that several important problems of classical analysis can be reformulated as special Generalized Nehari Problems. In this way, short proofs of various well-known theorems could be constructed. In this connection we mention the Helson-Szegő Theorem on the angle between subspaces, the Devinatz-Widom Theorem on invertibility of Toeplitz operators, and some theorems due to Koosis on the Hilbert transform (see [6]). The main goal of our paper is to show that the spectral factorization problem turns out to be intimately related to an appropriately constructed Generalized Nehari Problem for matrix-valued Carathéodory functions. Our method is essentially based on recent results on Weyl matrix balls associated with matrix-valued Carathéodory functions (see [18], [19]).

1. Notation and preliminaries

Throughout this paper, let m, p and q be positive integers. We will use \mathbb{N}_0 and \mathbb{C} to denote the set of all nonnegative integers and the set of all complex numbers, respectively.

B. Fritzsche: Univ. Leipzig, Inst. Math., Augustuspl. 10, D - 04109 Leipzig

B. Kirstein: Univ. Leipzig, Inst. Math., Augustuspl. 10, D - 04109 Leipzig

Further, let $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $T := \{z \in \mathbb{C} : |z| = 1\}$. The linear Lebesgue-Borel measure on T will be designated by $\tilde{\lambda}$. If \mathcal{X} is a nonempty set, then we will write $\mathcal{X}^{p \times q}$ for the set of all $p \times q$ matrices each entry of which belongs to \mathcal{X} . The symbol $O_{p \times q}$ denotes the null matrix that belongs to $\mathbb{C}^{p \times q}$, whereas I_q stands for the $q \times q$ identity matrix. In cases where the size of the null matrix (respectively, the identity matrix) is clear, we will omit the indexes. If A and B are $p \times p$ Hermitian matrices, then $A \geq B$ (respectively, $A > B$) means that $A - B$ is nonnegative Hermitian (respectively, positive Hermitian). Further, let $\mathcal{C}_{\geq}^{p \times p}$ (respectively, $\mathcal{C}_{>}^{p \times p}$) be the set of all $p \times p$ nonnegative Hermitian (respectively, $p \times p$ positive Hermitian) matrices. A $p \times q$ matrix K is called contractive if $I \geq KK^*$. The set of all $p \times q$ contractive matrices will be denoted by $\mathcal{K}_{p \times q}$. If A belongs to $\mathbb{C}^{p \times p}$, then let $\Re A$ and $\Im A$ be the real part of A and the imaginary part of A , respectively, i.e., we set $\Re A := \frac{1}{2}(A + A^*)$ and $\Im A := \frac{1}{2i}(A - A^*)$. If \mathcal{M} stands for the complex linear space of all Borel measurable functions $f : T \rightarrow \mathbb{C}^{p \times q}$, then $\mathcal{Z} := \{f \in \mathcal{M} : \tilde{\lambda}(\{z \in T : f(z) \neq O_{p \times q}\}) = 0\}$ is a linear subspace of \mathcal{M} . If $f \in \mathcal{M}$, then we will use $\langle f \rangle$ to indicate that element of the quotient space \mathcal{M}/\mathcal{Z} which is generated by f . Obviously, $\langle f \rangle = \langle g \rangle$ if and only if $f(z) = g(z)$ for $\tilde{\lambda}$ -almost all $z \in T$.

2. Some facts on various classes of meromorphic matrix functions

First we will summarize some basic facts on particular classes of holomorphic functions. A detailed treatment of this subject can be found, e.g., in [12]. Let $\mathcal{N}(\mathcal{D})$ be the Nevanlinna class of all holomorphic functions $g : \mathcal{D} \rightarrow \mathbb{C}$ which satisfy

$$\sup_{r \in (0,1)} \frac{1}{2\pi} \int_T \log^+ |g(rz)| \tilde{\lambda}(dz) < +\infty$$

where $\log^+ x := \max(\log x, 0)$ for each $x \in [0, \infty)$. If $g \in \mathcal{N}(\mathcal{D})$, then a well-known theorem due to Fatou implies that there exist a Borelian subset \mathcal{B}_0 of the unit circle T with $\tilde{\lambda}(\mathcal{B}_0) = 0$ and a Borel measurable function $g : T \rightarrow \mathbb{C}$ such that

$$\lim_{r \rightarrow 1-0} g(rz) = g(z)$$

for all $z \in T \setminus \mathcal{B}_0$. In the following, we will continue to use the symbol g to denote the boundary function of a function g which belongs to $\mathcal{N}(\mathcal{D})$. For each $g \in \mathcal{N}(\mathcal{D})$, the inequality

$$\frac{1}{2\pi} \int_T \log^+ |g(z)| \tilde{\lambda}(dz) \leq \lim_{r \rightarrow 1-0} \frac{1}{2\pi} \int_T \log^+ |g(rz)| \tilde{\lambda}(dz) \tag{1}$$

is satisfied. By the Smirnov class $\mathcal{N}_+(\mathcal{D})$ we will mean the set of all $g \in \mathcal{N}(\mathcal{D})$ for which equality holds true in (1). The class $\mathcal{N}_+(\mathcal{D})$ proves to be a subalgebra of $\mathcal{N}(\mathcal{D})$. If g is an outer function in $\mathcal{N}(\mathcal{D})$, then g necessarily belongs to $\mathcal{N}_+(\mathcal{D})$. Observe that the Hardy classes $H^t(\mathcal{D})$, $t \in (0, \infty]$, are subsets of $\mathcal{N}_+(\mathcal{D})$. An important subclass of $\{H^\infty(\mathcal{D})\}^{p \times q}$ we will deal with is the Schur class $\mathcal{S}_{p \times q}(\mathcal{D})$ of all holomorphic functions $f : \mathcal{D} \rightarrow \mathbb{C}^{p \times q}$ which satisfy $I - f(z)f^*(z) \geq 0$ for all $z \in \mathcal{D}$.

We will essentially use the following maximum modulus principle for the Smirnov class. It follows immediately from the corresponding scalar version (see, e.g., [12, Theorem 2.11]) and a simple property of contractive matrices (see, e.g., [13, Lemma 0.8] or [10, Lemma 1.1.11]).

Lemma 1: *If $f \in [\mathcal{N}_+(\mathcal{D})]^{p \times q}$ is such that $f(z)$ is contractive for $\bar{\lambda}$ -almost all $z \in T$, then f belongs to $S_{p \times q}(\mathcal{D})$.*

For the convenience of the reader, now we are going to recall some facts on outer functions which belong to the matricial Smirnov class. A function $\Phi \in [\mathcal{N}_+(\mathcal{D})]^{m \times m}$ is called *outer* (in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$) if $\det \Phi$ is outer in $\mathcal{N}(\mathcal{D})$. An outer function $\Phi \in [\mathcal{N}_+(\mathcal{D})]^{m \times m}$ is called *normalized* if $\Phi(0)$ is nonnegative Hermitian. The following useful properties of matrix-valued outer functions can be taken, e.g., from [7].

Remark 1: $\Phi \in [\mathcal{N}_+(\mathcal{D})]^{m \times m}$ is outer if and only if Φ admits the representation $\Phi = \frac{1}{\varphi} \Phi_1$ with some outer functions $\Phi_1 \in [H^\infty(\mathcal{D})]^{m \times m}$ and $\varphi \in H^\infty(\mathcal{D})$.

Remark 2: (a) If Φ is an outer function in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$, then $\det \Phi(z) \neq 0$ for all $z \in \mathcal{D}$, and Φ^{-1} is an outer function in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$.

(b) If $\Phi \in [\mathcal{N}_+(\mathcal{D})]^{m \times m}$ satisfies $\det \Phi(z) \neq 0$ for all $z \in \mathcal{D}$ and if Φ^{-1} belongs to $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$, then both Φ and Φ^{-1} are outer functions in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$.

Remark 3: If both Φ and Ψ are outer functions in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$, then the product $\Phi\Psi$ is also an outer function in $[\mathcal{N}_+(\mathcal{D})]^{m \times m}$.

Remark 4: $\Phi \in [H^2(\mathcal{D})]^{m \times m}$ is outer if and only if $\det \Phi$ is outer in $H^{2/m}(\mathcal{D})$.

A function $\Phi \in [H^2(\mathcal{D})]^{m \times m}$ is called *left maximal* if Φ has the following property: For each $\Sigma \in [H^2(\mathcal{D})]^{m \times m}$ with $\langle \Sigma \Sigma^* \rangle = \langle \Phi \Phi^* \rangle$, the inequality $\Sigma(0)\Sigma^*(0) \leq \Phi(0)\Phi^*(0)$ holds true. A function $\Psi \in [H^2(\mathcal{D})]^{m \times m}$ is said to be *right maximal* if Ψ fulfills the analogous condition: If $\Sigma \in [H^2(\mathcal{D})]^{m \times m}$ is a function such that $\langle \Sigma^* \Sigma \rangle = \langle \Psi^* \Psi \rangle$ is satisfied, then $\Sigma^*(0)\Sigma(0) \leq \Psi^*(0)\Psi(0)$. A left (respectively, right) maximal function is called *normalized* if it has a nonnegative Hermitian value at $z = 0$.

Let $W : T \rightarrow \mathcal{C}_\Sigma^{m \times m}$ be Lebesgue integrable. A function $\Sigma \in [H^2(\mathcal{D})]^{m \times m}$ is called a *left* (respectively, *right*) *minorant* of $\langle W \rangle$ if

$$\underline{\Sigma} \underline{\Sigma}^* \leq W \quad \bar{\lambda}\text{-a.e. on } T \quad (\text{respectively, } \Sigma^* \Sigma \leq W \quad \bar{\lambda}\text{-a.e. on } T). \quad (2)$$

A left (respectively, right) minorant Σ of $\langle W \rangle$ is said to be a *largest left minorant* (respectively, a *largest right minorant*) of $\langle W \rangle$ if the following condition is satisfied: If Ξ is an arbitrary left (respectively, right) minorant of $\langle W \rangle$, then $\Xi(0)\Xi^*(0) \leq \Sigma(0)\Sigma^*(0)$ (respectively, $\Xi^*(0)\Xi(0) \leq \Sigma^*(0)\Sigma(0)$).

In the context of prediction theory of stationary sequences in Hilbert space, a particular subclass of minorants plays a distinguished role (see, e.g., [26], [27]), namely the class of spectral factors. A function $\Sigma \in [H^2(\mathbb{D})]^{m \times m}$ is called a *left* (respectively, *right*) *spectral factor* of $\langle W \rangle$ if $\underline{\Sigma} \underline{\Sigma}^* = W$ λ -a.e. on T (respectively, $\underline{\Sigma}^* \underline{\Sigma} = W$ λ -a.e. on T). If a Lebesgue integrable matrix-valued function $W : T \rightarrow \mathcal{C}_{\Sigma}^{m \times m}$ is given, then the Left and Right Spectral Factorization Problem Associated with $\langle W \rangle$ consists of the description of the set $LS(\langle W \rangle)$ of all left spectral factors of $\langle W \rangle$ and the set $RS(\langle W \rangle)$ of all right spectral factors of $\langle W \rangle$. If $LS(\langle W \rangle)$ (respectively, $RS(\langle W \rangle)$) is nonempty, then $LS(\langle W \rangle)$ (respectively, $RS(\langle W \rangle)$) contains a unique normalized left (respectively, right) maximal function (which provides all information for calculating the best linear prediction in the framework of stationary sequences in Hilbert space (see [26], [27])).

The following theorem, which shows the existence of largest minorants, was proved by Masani [21], Rozanov [24] and Smuljan [25] in the context of prediction theory.

Theorem 1: *Let $W : T \rightarrow \mathcal{C}_{\Sigma}^{m \times m}$ be Lebesgue-integrable. Then:*

(a) *There exists a unique largest normalized left minorant Φ_0 of $\langle W \rangle$. This function Φ_0 is left maximal.*

(b) *If Φ is a largest left minorant of $\langle W \rangle$, then $\Phi = \Phi_0 U$ with some unitary matrix U . In particular, Φ is left maximal.*

(c) *There exists a unique largest normalized right minorant Ψ_0 of $\langle W \rangle$. This function Ψ_0 is right maximal.*

(d) *If Ψ is a largest right minorant of $\langle W \rangle$, then $\Psi = V \Psi_0$ with some unitary matrix V . In particular, Ψ is right maximal.*

(e) *The set $L(\langle W \rangle)$ of all left minorants of $\langle W \rangle$ and the set $R(\langle W \rangle)$ of all right minorants of $\langle W \rangle$ admit the representations $L(\langle W \rangle) = \{\Phi_0 S : S \in \mathcal{S}_{q \times q}(\mathbb{D})\}$ and $R(\langle W \rangle) = \{S \Psi_0 : s \in \mathcal{S}_{q \times q}(\mathbb{D})\}$.*

(f) *Suppose $\int_T \log(\det W) d\lambda > -\infty$. Then Φ_0 and Ψ_0 are an outer left spectral factor of $\langle W \rangle$ and an outer right spectral factor of $\langle W \rangle$, respectively. In particular, the sets $LS(\langle W \rangle)$ and $RS(\langle W \rangle)$ are nonempty. Moreover, the set of all largest left (respectively, largest right) minorants coincides with the set of all outer left (respectively, outer right) spectral factors of $\langle W \rangle$.*

3. On matrix-valued functions belonging to the classes of Carathéodory and Schur

A function $\Omega : \mathbb{D} \rightarrow \mathcal{C}^{m \times m}$ is called *$m \times m$ Carathéodory function* if Ω is both holomorphic in \mathbb{D} and has nonnegative Hermitian real part $\Re \Omega(z)$ for every choice of z in \mathbb{D} . We will use $\mathcal{C}_m(\mathbb{D})$ to denote the set of all $m \times m$ Carathéodory functions. One can show that $\mathcal{C}_m(\mathbb{D}) \subseteq [N_+(\mathbb{D})]^{m \times m}$ (see, e.g., [19, Corollary 2]). In particular, every $m \times m$ Carathéodory function has boundary values λ -almost everywhere on T . There are several interrelations between matrix-valued Carathéodory functions and the class $\mathcal{S}_{p \times q}(\mathbb{D})$ of all

$p \times q$ Schur functions.

Let $\tau \in \mathbb{N}$ or $\tau = \infty$. A sequence $(\Gamma_k)_{k=0}^\tau$ of $m \times m$ complex matrices is called $m \times m$ Carathéodory sequence (respectively, nondegenerate $m \times m$ Carathéodory sequence) if, for every integer n with $0 \leq n \leq \tau$, the block Toeplitz matrix

$$T_n := \begin{pmatrix} \Re\Gamma_0 & \frac{1}{2}\Gamma_1^* & \frac{1}{2}\Gamma_2^* & \dots & \frac{1}{2}\Gamma_n^* \\ \frac{1}{2}\Gamma_1 & \Re\Gamma_0 & \frac{1}{2}\Gamma_1^* & \dots & \frac{1}{2}\Gamma_{n-1}^* \\ \frac{1}{2}\Gamma_2 & \frac{1}{2}\Gamma_1 & \Re\Gamma_0 & \dots & \frac{1}{2}\Gamma_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\Gamma_n & \frac{1}{2}\Gamma_{n-1} & \frac{1}{2}\Gamma_{n-2} & \dots & \Re\Gamma_0 \end{pmatrix}$$

is nonnegative Hermitian (respectively, positive Hermitian). If $(\Gamma_n)_{k=0}^\infty$ is a given sequence of $m \times m$ complex matrices, then the power series

$$\Omega(z) := \sum_{k=0}^\infty \Gamma_k z^k, \quad z \in \mathbb{D},$$

defines an $m \times m$ Carathéodory function if and only if $(\Gamma_n)_{k=0}^\infty$ is an $m \times m$ Carathéodory sequence (see, e.g., [10, Theorems 2.2.1 and 2.2.2]). An $m \times m$ Carathéodory function Ω is said to be nondegenerate if the sequence of the Taylor coefficients of Ω (in the Taylor series representation around the origin) is a nondegenerate $m \times m$ Carathéodory sequence.

Let $n \in \mathbb{N}_0$, and let $(\Gamma_n)_{k=0}^n$ be a sequence of $m \times m$ complex matrices. Then the set $\mathcal{C}_m[\Gamma_0, \Gamma_1, \dots, \Gamma_n]$ of all $m \times m$ Carathéodory functions Ω with first $n + 1$ Taylor coefficients $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ (in the Taylor series representation of Ω around the origin) is nonempty if and only if $(\Gamma_n)_{k=0}^n$ is an $m \times m$ Carathéodory sequence (see, e.g., [14, Section 4]). If $(\Gamma_n)_{k=0}^n$ is a nondegenerate Carathéodory sequence, then $\mathcal{C}_m[\Gamma_0, \Gamma_1, \dots, \Gamma_n]$ can be described by certain linear fractional transformations (see, e.g., [16, Theorem 28]). Furthermore, one can show that, for each $z \in \mathbb{D}$, the set $\mathcal{F}(z) := \{\Omega(z) : \Omega \in \mathcal{C}_m[\Gamma_0, \Gamma_1, \dots, \Gamma_n]\}$ admits a representation as a so-called matrix ball. To be more precise, there are functions $\mathcal{M}_n : \mathbb{D} \rightarrow \mathcal{O}^{m \times m}$, $\mathcal{L}_n^\# : \mathbb{D} \rightarrow \mathcal{O}_>^{m \times m}$ and $\mathcal{R}_n : \mathbb{D} \rightarrow \mathcal{O}_>^{m \times m}$, which are explicitly constructed from $\Gamma_0, \Gamma_1, \dots, \Gamma_n$, such that

$$\mathcal{F}(z) = \left\{ X \in \mathcal{O}^{m \times m} \mid X = \mathcal{M}_n(z) + |z|^{n+1} \sqrt{2\mathcal{L}_n^\#(z)} K \sqrt{2\mathcal{R}_n(z)}, \quad K \in \mathcal{K}_{m \times m} \right\}$$

(see [16, Theorem 29]). The matrix-valued functions $\mathcal{M}_n, \mathcal{L}_n^\#$ and \mathcal{R}_n are said to be the Weyl-Carathéodory center function, the canonical normalized left Weyl-Carathéodory semi-radius function and the canonical right Weyl-Carathéodory semi-radius function, respectively, associated with $(\Gamma_n)_{k=0}^n$.

Now let Ω be a nondegenerate $m \times m$ Carathéodory function, and let

$$\Omega(z) = \sum_{k=0}^{\infty} \Gamma_k z^k, \quad z \in \mathcal{D},$$

be the Taylor series representation of Ω . For each $n \in \mathbb{N}_0$, let $\mathcal{M}_n, \mathcal{L}_n^\#$ and \mathcal{R}_n be the Weyl-Carathéodory center function, the canonical normalized left Weyl-Carathéodory semi-radius function and the canonical right Weyl-Carathéodory semi-radius function, respectively, associated with $(\Gamma_k)_{k=0}^n$. Then one can show (see [18, Theorem 5]) that there are (unique) functions $\mathcal{L}_{[\Omega]}^\# : \mathcal{D} \rightarrow \mathcal{O}_{\mathbb{C}}^{m \times m}$ and $\mathcal{R}_{[\Omega]}^\# : \mathcal{D} \rightarrow \mathcal{O}_{\mathbb{C}}^{m \times m}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{L}_n^\#(z) = \mathcal{L}_{[\Omega]}^\#(z) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{R}_n(z) = \mathcal{R}_{[\Omega]}(z)$$

for all $z \in \mathcal{D}$. Moreover, observe that

$$\lim_{n \rightarrow \infty} \mathcal{M}_n^\#(z) = \Omega(z), \quad z \in \mathcal{D}.$$

We call $\mathcal{L}_{[\Omega]}^\#$ and $\mathcal{R}_{[\Omega]}$ the *canonical normalized left Weyl-Carathéodory limit semi-radius function* and the *canonical right Weyl-Carathéodory limit semi-radius function*, respectively, associated with (the nondegenerate $m \times m$ Carathéodory function) Ω .

Theorem 2: *Let Ω be an $m \times m$ Carathéodory function which satisfies*

$$\frac{1}{2\pi} \int_{\Gamma} \log \det[\Re \Omega] d\tilde{\lambda} > -\infty.$$

Then Ω is nondegenerate. Let $\mathcal{L}_{[\Omega]}^\#$ and $\mathcal{R}_{[\Omega]}$ be the canonical normalized left Weyl-Carathéodory limit semi-radius function and the canonical right Weyl-Carathéodory limit semi-radius function, respectively, associated with Ω .

- (a) *If Φ is an arbitrary outer left spectral factor of $\langle \Re \Omega \rangle$, then $\Phi \Phi^* = \mathcal{L}_{[\Omega]}^\#$.*
- (b) *If ψ is an arbitrary outer right spectral factor of $\langle \Re \Omega \rangle$, then $\Psi^* \Psi = \mathcal{R}_{[\Omega]}$.*

Proof: Use Lemma 5, Theorem 7 in [19] and Theorem 1 ■

At the end of this section, we will recall the notion of nondegenerate $p \times q$ Schur functions. Assume that $f : \mathcal{D} \rightarrow \mathcal{O}^{p \times q}$ is holomorphic in \mathcal{D} . Let

$$f(z) = \sum_{k=0}^{\infty} A_k z^k, \quad z \in \mathcal{D},$$

be the Taylor series representation of f around the origin. Then one can show (see, e.g., [10, Theorem 3.1.1]) that f belongs to the Schur class $\mathcal{S}_{p \times q}(\mathcal{D})$ if and only if, for each $n \in \mathbb{N}_0$, the block Toeplitz matrix

$$S_n := \begin{pmatrix} A_0 & 0 & 0 & \dots & 0 \\ A_1 & A_0 & & \dots & 0 \\ A_2 & A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & A_{n-2} & \dots & A_0 \end{pmatrix}$$

is contractive. If S_n is strictly contractive for each $n \in \mathbb{N}_0$, then the $p \times q$ Schur function f is said to be *nondegenerate*.

4. Spectral factorization as a generalized Nehari problem

In this section, we will indicate that spectral factorization can be conceived as an appropriately constructed generalized Nehari problem. For this reason, we are going to start with a special case which sheds much light to the general situation. First we recall the following Generalized Nehari Problem (SNP) for matrix-valued Schur functions which was studied in [17]. (Observe that Bakonyi [8] and the authors [11] treated particular Nehari Problems of this type with pseudocontinuable original data.)

(SNP) : Let $f : \mathbb{D} \rightarrow \mathbb{C}^{p \times r}$, $g : \mathbb{D} \rightarrow \mathbb{C}^{r \times s}$ and $h : \mathbb{D} \rightarrow \mathbb{C}^{q \times s}$ be matrix-valued function which are holomorphic in \mathbb{D} . Describe the set $\mathcal{M}(f, g, h)$ of all functions $e : \mathbb{D} \rightarrow \mathbb{C}^{q \times r}$ such that

$$S := \begin{pmatrix} f & g \\ e & h \end{pmatrix} \tag{3}$$

belongs to $\mathcal{S}_{(p+q) \times (r+s)}(\mathbb{D})$.

Obviously, if $\mathcal{M}(f, g, h)$ is nonempty, then f, g and h are matrix-valued Schur functions. We will turn our attention to the situation that the block e is quadratic and that some of the blocks f, g and h identically vanish.

Theorem 3: *The following statements hold true:*

(a) Let $h \in \mathcal{S}_{q \times s}(\mathbb{D})$. Then $\mathcal{M}(O_{p \times q}, O_{p \times s}, h)$ is exactly the set $L(h)$ of all left minorants of $(I - \underline{h} \underline{h}^*)$, i.e.,

$$\mathcal{M}(O_{p \times q}, O_{p \times s}, h) = \{\varphi S : S \in \mathcal{S}_{q \times q}(\mathbb{D})\}$$

where φ is an arbitrary fixed largest left minorant of $(I - \underline{h} \underline{h}^*)$.

(b) Let $f \in \mathcal{S}_{p \times r}(\mathbb{D})$. Then $\mathcal{M}(f, O_{p \times s}, O_{r \times s})$ is exactly the set $R(f)$ of all right minorants of $(I - \underline{f}^* \underline{f})$, i.e.,

$$\mathcal{M}(f, O_{p \times s}, O_{r \times s}) = \{S \psi : S \in \mathcal{S}_{r \times r}(\mathbb{D})\}$$

where ψ is an arbitrary fixed largest right minorant of $(I - \underline{f}^* \underline{f})$.

Proof: First assume that $e \in \mathcal{M}(O_{p \times q}, O_{p \times s}, h)$. Then $e \in \mathcal{S}_{q \times q}(\mathbb{D}) \subseteq [H^2(\mathbb{D})]^{q \times q}$ and $(e, h) \in \mathcal{S}_{q \times (q+s)}(\mathbb{D})$. Hence, $I - h(z)h^*(z) \succeq e(z)e^*(z)$ for all $z \in \mathbb{D}$. This implies

$$I - \underline{h} \underline{h}^* \succeq \underline{e} \underline{e}^* \tag{4}$$

$\bar{\lambda}$ -a.e. on T . Thus, e belongs to $L(h)$.

Conversely, now suppose that e is an arbitrary element of $L(h)$. Then e is a function which belongs to $[H^2(\mathbb{D})]^{q \times q} \subseteq [\mathcal{N}_+(\mathbb{D})]^{q \times q}$ and satisfies (4) $\bar{\lambda}$ -a.e. on T . Therefore, $I - \underline{(e, h)} (e, h)^* \geq 0$ $\bar{\lambda}$ -a.e. on T . Because of $(e, h) \in [\mathcal{N}_+(\mathbb{D})]^{q \times (q+s)}$ and Lemma 1, we then get that $(e, h) \in \mathcal{S}_{q \times (q+s)}(\mathbb{D})$. Consequently, $e \in \mathcal{M}(O_{p \times q}, O_{p \times s}, h)$. The application of parts (b) and (e) of Theorem 1 completes the proof of part (a). Part (b) can be shown analogously ■

The Generalized Nehari Problem for matrix-valued Schur functions can be considered as special case of the following Generalized Nehari Problem (CNP) for matrix-valued Carathéodory functions:

(CNP): Let $\alpha : \mathbb{D} \rightarrow \mathbb{C}^{p \times p}$, $\beta : \mathbb{D} \rightarrow \mathbb{C}^{p \times q}$ and $\delta : \mathbb{D} \rightarrow \mathbb{C}^{q \times q}$ be matrix-valued functions which are holomorphic in \mathbb{D} . Describe the set $\mathcal{N}(\alpha, \beta, \delta)$ of all functions $\xi : \mathbb{D} \rightarrow \mathbb{C}^{q \times p}$ such that

$$\Omega := \begin{pmatrix} \alpha & \beta \\ \xi & \delta \end{pmatrix}$$

belongs to $\mathcal{C}_{p+q}(\mathbb{D})$.

This problem was posed by V.E. Katsnelson [20] and studied by the authors in [15]. In the following, we will consider the particular case that β identically vanishes. However, first we will show how the Generalized Nehari Problem (SNP) for matrix-valued Schur functions formulated above can be expressed as Generalized Nehari Problem for matrix-valued Carathéodory functions.

Lemma 2: Let $A \in \mathbb{C}^{p \times q}$. Then the matrix

$$G := \begin{pmatrix} I & 0 \\ 2A & I \end{pmatrix}$$

has nonnegative (respectively, positive) real part if and only if A is contractive (respectively, strictly contractive).

Proof: Obviously, $\Re G = \begin{pmatrix} I & A^* \\ A & I \end{pmatrix}$. Thus, $\Re G \geq 0$ if and only if $I - A^*A \geq 0$ (see, e.g., [10, Lemma 1.1.12]) ■

Remark 5: Let $V \in \mathbb{C}^{q \times q}$ be nonsingular, and let $\Gamma \in \mathbb{C}^{q \times q}$. Then $\Re \Gamma \geq 0$ if and only if $\Re(V^* \Gamma V) \geq 0$.

If e is a $q \times r$ matrix-valued function, then we set

$$e^\square := \begin{pmatrix} O_{p \times r} & O_{p \times s} \\ e & O_{q \times s} \end{pmatrix} \tag{5}$$

We will use the symbol $Z_{p,q,r,s}$ to denote the set of all such $(p+q) \times (r+s)$ matrix-valued functions e^\square given by (5) where e is some $q \times r$ matrix-valued function.

Lemma 3: Let $f : \mathbb{D} \rightarrow \mathbb{C}^{p \times r}$, $g : \mathbb{D} \rightarrow \mathbb{C}^{p \times s}$ and $h : \mathbb{D} \rightarrow \mathbb{C}^{q \times s}$ be matrix-valued functions, and let

$$F := \begin{pmatrix} I & 0 \\ 2f & I \end{pmatrix}, \quad G := \begin{pmatrix} 0 & 0 \\ 2g & 0 \end{pmatrix} \quad \text{and} \quad H := \begin{pmatrix} I & 0 \\ 2h & I \end{pmatrix}. \tag{6}$$

(a) The set $\mathcal{M}(f, g, h)$ is nonempty if and only if the sets $\mathcal{N}(F, G, H)$ and $Z_{p,q,r,s}$ have a nonempty intersection.

(b) Suppose $\mathcal{M}(f, g, h) \neq \emptyset$. A function $e : \mathbb{D} \rightarrow \mathbb{C}^{q \times r}$ belongs to $\mathcal{M}(f, g, h)$ if and only if the function e^\square given by (5) belongs to $\mathcal{N}(F, G, H)$.

Proof: First suppose that $\mathcal{M}(f, g, h) \neq \emptyset$. Let $e \in \mathcal{M}(f, g, h)$, i.e., the function S given by (3) belongs to $\mathcal{S}_{(p+q) \times (r+s)}(\mathbb{D})$. Hence, Lemma 2 shows that

$$\Gamma := \begin{pmatrix} I & 0 \\ 2S & I \end{pmatrix} \tag{7}$$

satisfies $\Re e \Gamma \geq 0$, i.e., Γ belongs to $\mathcal{C}_{p+q+r+s}(\mathbb{D})$. Setting

$$V := \begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix} \quad \text{and} \quad \Omega := \begin{pmatrix} F & G \\ e^\square & H \end{pmatrix}, \tag{8}$$

we obtain $V^* \Gamma V = \Omega$. By virtue of Remark 5, we get $\Re e \Omega \geq 0$, i.e., Ω belongs to $\mathcal{C}_{p+q+r+s}(\mathbb{D})$. Consequently, $e^\square \in \mathcal{N}(F, G, H)$. From (5) we see that e^\square belongs obviously to $Z_{p,q,r,s}$. Conversely, now suppose that $\mathcal{N}(F, G, H) \cap Z_{p,q,r,s} \neq \emptyset$. Let $E \in \mathcal{N}(F, G, H) \cap Z_{p,q,r,s}$. Then there is a (unique) holomorphic function $e : \mathbb{D} \rightarrow \mathbb{C}^{q \times r}$ such that $E = e^\square$, and Ω defined by (8) belongs to $\mathcal{C}_{p+q+r+s}(\mathbb{D})$. If V is given by (8), then we obtain $V \Omega V^* = \Gamma$ where Γ is defined by (3) and (7). Remark 5 yields $\Re e \Gamma \geq 0$. Because of $\Omega \in \mathcal{C}_{p+q+r+s}(\mathbb{D})$, the matrix-valued functions f, g and h are holomorphic in \mathbb{D} . Hence, $\Gamma \in \mathcal{C}_{p+q+r+s}(\mathbb{D})$. Applying Lemma 2 we obtain $S \in \mathcal{S}_{p \times q}(\mathbb{D})$, i.e., e belongs to $\mathcal{N}(f, g, h)$ ■

Now we will study a structured Generalized Nehari Problem for matrix-valued Carathéodory functions. For this reason, we need the following algebraic result.

Lemma 4: Let $A \in \mathbb{C}^{p \times q}$, $B \in \mathbb{C}^{q \times q}$, $K \in \mathbb{C}^{p \times q}$, and let

$$G := \begin{pmatrix} A^* A & A^* K^* B^* \\ B K A & B B^* \end{pmatrix}.$$

Suppose $\det A \neq 0$ and $\det B \neq 0$. Then :

- (a) $G \geq 0$ if and only if K is contractive.
- (b) $G > 0$ if and only if K is strictly contractive.

Proof: Use [10: Lemma 1.1.12] and $G = \text{diag}(A^*, B) \cdot \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \cdot [\text{diag}(A^*, B)]^* \blacksquare$

The following theorem is the main result of this paper. Its proof is based on four cornerstones, namely, the maximum modulus principle for the Smirnov class, Theorem 2, the algebraic Lemma 4, and a nonobvious inequality for matrix-valued Carathéodory functions which was recently found (see [18, Theorem 5]).

Theorem 4: Let $\alpha \in C_p(\mathbb{D})$ and $\delta \in C_q(\mathbb{D})$ be such that

$$\frac{1}{2\pi} \int_T \det[\Re \underline{\alpha}] d\bar{\lambda} > -\infty \tag{9}$$

and

$$\frac{1}{2\pi} \int_T \det[\Re \underline{\delta}] d\bar{\lambda} > -\infty. \tag{10}$$

Further, let Ψ be an outer right spectral factor associated with $\langle \underline{\alpha} + \underline{\alpha}^* \rangle$, and let Φ be an outer left spectral factor associated with $\langle \underline{\delta} + \underline{\delta}^* \rangle$. Then $\mathcal{N}(\alpha, O_{p \times q}, \delta) = \{\Phi S \Psi : S \in \mathcal{S}_{q \times p}(\mathbb{D})\}$. In particular, $\mathcal{N}(\alpha, O_{p \times q}, \delta)$ is a nonempty subset of $[H^1(\mathbb{D})]^{q \times p}$.

Proof: We know from Lemma 5 in [19] that both α and δ are nondegenerate matrix-valued Carathéodory functions. Suppose $S \in \mathcal{S}_{q \times p}(\mathbb{D})$. Because of $\Phi \in [H^2(\mathbb{D})]^{q \times q}$ and $\Psi \in [H^2(\mathbb{D})]^{p \times p}$, the function $Y := \Phi S \Psi$ belongs to $[H^1(\mathbb{D})]^{q \times p}$. In particular,

$$\Omega := \begin{pmatrix} \alpha & 0 \\ Y & \delta \end{pmatrix} \tag{11}$$

is holomorphic with

$$\Re \Omega(z) = \begin{pmatrix} \Re \alpha(z) & \frac{1}{2} Y^*(z) \\ \frac{1}{2} Y(z) & \Re \delta(z) \end{pmatrix} \tag{12}$$

for all $z \in \mathbb{D}$. Using Theorem 2 (and the notations given there) we get

$$\Psi^*(z)\Psi(z) = \mathcal{R}_{[2\alpha]}(z) \tag{13}$$

and

$$\Phi(z)\Phi^*(z) = \mathcal{L}_{[2\delta]}^{\#}(z) \tag{14}$$

for all $z \in \mathbb{D}$. By virtue of Lemma 9 in [18], $f := (I - 2\alpha)(I + 2\alpha)^{-1}$ and $g := (I - 2\delta)(I + 2\delta)^{-1}$ are nondegenerate matrix-valued Schur functions. From Theorem 5 in [18] and Theorem 5.6.4 in [10] we then see that

$$\mathcal{R}_{[2\alpha]}(z) \leq ([I + f(z)]^{-1})^* [I - f^*(z)f(z)] [I + f(z)]^{-1} \tag{15}$$

and

$$\mathcal{L}_{[2\delta]}^{\#}(z) \leq [I + f(z)]^{-1} [I - f(z)f^*(z)] ([I + f(z)]^{-1})^* \tag{16}$$

hold true for all $z \in \mathbb{D}$. Parts (f) and (g) of Lemma 1.3.12 in [10] then imply that the right-hand sides of (15) and (16) coincide with $\Re[2\alpha(z)]$ and $\Re[2\delta(z)]$, respectively. In view of (13) and (14), it follows then

$$\Psi^*(z)\Psi(z) \leq \Re[2\alpha(z)] = \alpha(z) + \alpha^*(z) \tag{17}$$

and

$$\Phi(z)\Phi^*(z) \leq \Re e[2\delta(z)] = \delta(z) + \delta^*(z) \tag{18}$$

for each $z \in \mathbb{D}$. Hence, in view of (12), we thus obtain

$$\begin{aligned} \Re e \Omega(z) &\geq \frac{1}{2} \begin{pmatrix} \Psi^*(z)\Psi(z) & Y^*(z) \\ Y(z) & \Phi(z)\Phi^*(z) \end{pmatrix} \\ &= \frac{1}{2} \text{diag}[\Psi^*(z), \Phi(z)] \cdot \begin{pmatrix} I_p & S^*(z) \\ S(z) & I \end{pmatrix} \cdot \text{diag}[\Psi(z), \Phi^*(z)] \geq 0 \end{aligned}$$

for all $z \in \mathbb{D}$. Therefore, $\Omega \in \mathcal{C}_{p+q}(\mathbb{D})$. This implies $Y \in \mathcal{N}(\alpha, O_{p \times q}, \delta)$.

Conversely, now assume that Y is an arbitrary function which belongs to $\mathcal{N}(\alpha, O_{p \times q}, \delta)$, i.e., Ω defined by (11) is a $(p + q) \times (p + q)$ Carathéodory function. In particular, Ω belongs to the Smirnov class $[\mathcal{N}_+(\mathbb{D})]^{(p+q) \times (p+q)}$. Hence, $Y \in [\mathcal{N}_+(\mathbb{D})]^{p \times q}$. Since Ψ (respectively, Φ) is an outer right (respectively, left) spectral factor associated with $(\underline{\alpha} + \underline{\alpha}^*)$ (respectively, $(\underline{\delta} + \underline{\delta}^*)$), we see from (12) that

$$\begin{pmatrix} \underline{\Psi}^*(\xi)\underline{\Psi}(\xi) & \underline{Y}^*(\xi) \\ \underline{Y}(\xi) & \underline{\Phi}(\xi)\underline{\Phi}^*(\xi) \end{pmatrix} \geq 0 \tag{19}$$

holds true for $\bar{\lambda}$ -almost all $\xi \in T$. Because Φ and Ψ are outer functions in $[H^2(\mathbb{D})]^{p \times q}$ and $[H^2(\mathbb{D})]^{q \times p}$, respectively, Φ^{-1} and Ψ^{-1} belong to $[H^2(\mathbb{D})]^{q \times q}$ and $[H^2(\mathbb{D})]^{p \times p}$, respectively. Therefore, $S := \Phi^{-1}Y\Psi^{-1}$ is a member of $[H^2(\mathbb{D})]^{q \times p}$ with $(\underline{Y}) = (\underline{\Phi} \underline{S} \underline{\Psi})$. From inequality (19), Lemma 4, Remarks 2 and 4 it follows that $\underline{S}(\xi)$ is contractive for $\bar{\lambda}$ -almost all $\xi \in T$. Then Lemma 1 provides finally $S \in \mathcal{S}_{q \times p}(\mathbb{D})$ ■

Observe that, for the particular Generalized Nehari Problem considered in Theorem 4, we were able to obtain an analytical description of $\mathcal{N}(\alpha, O_{p \times q}, \delta)$, whereas in the general case studied in [15] we got a purely algebraic description of this set (in terms of the Taylor coefficients of α, β and δ).

In the situation of Theorem 4, we will call a function X a *canonical element* of $\mathcal{N}(\alpha, O_{p \times q}, \delta)$ if there exists an isometric or coisometric $q \times p$ matrix U such that $X = \Phi U \Psi$.

Corollary 1: *Let $\alpha \in \mathcal{C}_q(\mathbb{D})$ and $\delta \in \mathcal{C}_q(\mathbb{D})$ be such that (9) and (10) are satisfied. Then every canonical element of $\mathcal{N}(\alpha, O_{q \times q}, \delta)$ is an outer function in $[H^1(\mathbb{D})]^{q \times q}$. (In particular, the set $\mathcal{N}(\alpha, O_{q \times q}, \delta)$ contains outer functions belonging to $[H^1(\mathbb{D})]^{q \times q}$.)*

Proof: Let $U \in \mathcal{C}^{q \times q}$ be unitary. Let Ψ be a right outer spectral factor associated with $(\underline{\alpha} + \underline{\alpha}^*)$, and let Φ be a left outer spectral factor associated with $(\underline{\delta} + \underline{\delta}^*)$. Since Ψ and Φ belong to $[H^2(\mathbb{D})]^{q \times q}$ it follows $\Phi U \Psi \in [H^1(\mathbb{D})]^{q \times q}$. The application of Theorem 4 yields the rest of the assertion ■

Note that, in view of part (f) of Theorem 1, the following theorem characterizes the set of all outer spectral factors as canonical elements in $\mathcal{N}(\alpha, O_{q \times q}, \delta)$.

Theorem 5: *The following statements hold true:*

(a) *Let the function $\alpha \in C_q(\mathbb{D})$ be such that condition (9) is satisfied. Then $\mathcal{N}(\alpha, O_{q \times q}, \delta)$ is exactly the set of all right minorants of $(\underline{\alpha} + \underline{\alpha}^*)$. The set of all largest right minorants of $(\underline{\alpha} + \underline{\alpha}^*)$ coincides with the set of all canonical elements in $\mathcal{N}(\alpha, O_{q \times q}, \delta)$.*

(b) *Let $\delta \in C_q(\mathbb{D})$ be such that (10) is fulfilled. Then $\mathcal{N}(\alpha, O_{q \times q}, \delta)$ is exactly the set of all left minorants of $(\underline{\delta} + \underline{\delta}^*)$. The set of all largest left minorants of $(\underline{\delta} + \underline{\delta}^*)$ coincides with the set of all canonical elements in $\mathcal{N}(\alpha, O_{q \times q}, \delta)$.*

Proof: Let Ψ be an arbitrary outer right spectral factor associated with $(\underline{\alpha} + \underline{\alpha}^*)$. By virtue of Theorem 4, we get then $\mathcal{N}(\alpha, O_{q \times q}, \delta) = \{S\Psi : S \in \mathcal{S}_{q \times q}(\mathbb{D})\}$. Thus, Theorems 1 and 4 yield the assertion stated in (a). Part (b) can be proved analogously to part (a) ■

References

- [1] ADAMJAN, V.M., AROV, D.Z. and M.G. KREIN: *On infinite Hankel matrices and generalized Carathéodory-Fejér and F. Riesz problems* (in Russian). Funk. Anal. i Prilo. 2 (1968) No. 1, 1 - 19.
- [2] ADAMJAN, V.M., AROV, D.Z. and M.G. KREIN: *Infinite Hankel matrices and generalized Carathéodory-Fejér and Schur problems* (in Russian). Funkt. Anal. i Prilo. 2 (1968) No.4, 1 - 17.
- [3] ADAMJAN, V.M., AROV, D.Z. and M.G. KREIN: *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem* (in Russian). Math. USSR-Sbornik, 15 (1971), 34 - 75.
- [4] ADAMJAN, V.M., AROV, D.Z. and M.G. KREIN: *Infinite block Hankel matrices and related extension problems* (in Russian). Izv. Akad. Nauk Armjan. SSSR, Ser. Mat. 6 (1971), 87 - 112.
- [5] AROCENA, R., and M. COTLAR: *Generalized Toeplitz kernels and Adamjan-Arov-Krein moment problems*. Oper. Theory: Adv. Appl. 4 (1982), 37 - 55.
- [6] AROCENA, R. and M. COTLAR: *Dilation of generalized Toeplitz kernels and some vectorial moment and weighted problem*. Lecture Notes Math. 908 (1982), 169 - 188.
- [7] AROV, D.Z.: *Regular J-inner matrix functions and related continuation problems*. Oper. Theory: Adv. Appl. 43 (1990), 63 - 87.
- [8] BAKONYI, M.: *Spectral factors and analytic completion*, Int. Equ. Oper. Theory 13 (1990), 149 - 164.
- [9] COTLAR, M. and C. SADOSKY: *On the Helson-Szegő theorem and a related class of modified Toeplitz kernels*. In: Harmonic Analysis in Euclidean Spaces. Proc. Symp. Pure Math.: Vol.35, Part I (eds.: G. Weiss and S. Wainger). Providence, Rhode Island: Amer. Math. Soc. 1979, pp. 383 - 407.
- [10] DUBOVOJ, V.K., FRITZSCHE, B. and B. KIRSTEIN: *Matricial Version of the Classical Schur Problem* (Teubner-Texte zur Mathematik: Vol. 129). Leipzig - Stuttgart: B.G. Teubner 1992.
- [11] DUBOVOJ, V.K., FRITZSCHE, B. and B. KIRSTEIN: *On spectrally associated Schur functions, Arov-inner functions and a Nehari-type completion problem for Schur functions*. Int. Equ. Oper. Theory 17 (1993), 247 - 276.
- [12] DUREN, P.L.: *Theory of H^p Spaces*. New York: Academic Press 1970.

- [13] DYM, H.: *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation* (CBMS Lecture Notes: Vol. 71). Providence, Rhode Island: Amer. Math. Soc. 1989.
- [14] FRITZSCHE, B. and B. KIRSTEIN: *An extension problem for non-negative Hermitian block Toeplitz matrices*. Math. Nachr. 130 (1987), 121 - 135.
- [15] FRITZSCHE, B. and B. KIRSTEIN: *On generalized Nehari problems*. Math. Nachr. 138 (1988), 217 - 237.
- [16] FRITZSCHE, B. and B. KIRSTEIN: *An extension problem for non-negative Hermitian block Toeplitz matrices*. Part V. Math. Nachr. 144 (1989), 283 - 308.
- [17] FRITZSCHE, B. and B. KIRSTEIN: *A Nehari type problem for matricial Schur functions*. Z. Anal. Anw. 9 (1990), 73 - 84.
- [18] FRITZSCHE, B. and B. KIRSTEIN: *On the Weyl matrix balls associated with non-degenerate matrix-valued Carathéodory functions*. Z. Anal. Anw. 12 (1993), 239 - 261.
- [19] FRITZSCHE, B. and B. KIRSTEIN: *On the largest minorants associated with a matrix-valued Carathéodory function*. Z. Anal. Anw. 12 (1993), 471 - 490.
- [20] KATSNELSON, V.E.: *Integral representation of Hermitian positive kernels of mixed type and the generalized Nehari problem* (in Russian). Teor. Funkcii. Funkcional Anal. i Priložen. (Kharkov) 43 (1985), 54 - 70.
- [21] MASANI, P.R.: *Cramér's theorem on monotone matrix-valued functions and the Wold decomposition*. In: Probability and Statistics - The Harald Cramér Volume (ed.: U. Grenander). Stockholm: Almqvist & Wiksell 1959, pp. 175 - 189.
- [22] MASANI, P.R.: *Shift invariant spaces and prediction theory*. Acta Math. 107 (1962), 275 - 290.
- [23] NEHARI, Z.: *On bounded bilinear forms*. Ann. Math. 65 (1957), 153 - 162.
- [24] ROZANOV, J.A.: *Spectral properties of multivariate stationary processes and boundary properties of holomorphic matrix functions* (in Russian). Theory of Probability and Its Applications 5 (1960), 399-414.
- [25] SMULJAN, J.L.: *Optimal factorization of nonnegative matrix functions* (in Russian). Theory Prob. and Its Appl. 9 (1964), 382 - 386.
- [26] WIENER, N. and P. MASANI: *The prediction theory of multivariate stochastic processes*. Part I: *The regularity condition*. Acta Math. 98 (1957), 111 - 150.
- [27] WIENER, N. and P. MASANI: *The prediction theory of multivariate stochastic processes*. Part II: *The linear predictor*. Acta Math. 99 (1958), 93 - 137.

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