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Tangential Carathéodory-Fejer Interpolation for Stieltjes Functions at Real Points

V. Bolotnikov

Abstract. In this paper we consider a generalization of the classical interpolation problem in a special class of analytic matrix-valued functions. The method used to solve this problem is that of the fundamental matrix inequality, suitably adapted to the present situation.

Keywords: *Carathéodory-Fejer interpolation, fundamental matrix inequalities*

AMS subject classification: Primary 47A57, secondary 30E05

1. Introduction

In this paper we consider the tangential Carathéodory-Fejer interpolation problem in the Stieltjes class.

Definition 1.1. A matrix-valued function w, holomorphic in the complex plane $\mathbb{C}$ with a cut along the semi-axis $\mathbb{R}_+ = [0, +\infty)$ is called a *Stieltjes function* if

$$1)\ \frac{w(z) - w(z)^*}{z - \overline{z}} \geq 0 \quad \text{for } z \neq \overline{z} \qquad \text{and} \qquad 2)\ w(x) \geq 0 \quad \text{for } x < 0.$$

The class of all $\mathbb{C}^{m\times m}$-valued Stieltjes functions is denoted by $\mathbf{S}_m$.

We formulate two theorems which are proved in [18].

Theorem 1.2. *The function w belongs to $\mathbf{S}_m$ if and only if*

$$\frac{w(z) - w(z)^*}{z - \overline{z}} \geq 0 \qquad \textit{and} \qquad \frac{zw(z) - \overline{z}w(z)^*}{z - \overline{z}} \geq 0 \qquad \textit{for } z \neq \overline{z}.$$

Theorem 1.3. *The function w belongs to $\mathbf{S}_m$ if and only if it admits the integral representation*

$$w(z) = A + \int_0^\infty \frac{d\sigma(t)}{t - z} \tag{1.1}$$

where $A \geq 0$ and $d\sigma \geq 0$ is a $\mathbb{C}^{m\times m}$-valued measure such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$.

In the class $\mathbf{S}_m$ we consider the following interpolation problem (IS):

V. Bolotnikov: Ben Gurion Univ. Negev, P.O. Box 653, 84105 Beer Sheva, Israel

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(IS) *Given a set of matrices $a_i, c_i \in \mathbb{C}^{s\times m}, \gamma_i \in \mathbb{C}^{s\times s}$ $(i = 0, \ldots, n)$ and a point $x < 0$ find necessary and sufficient conditions which ensure the existence of a function*

$$w(z) = \sum_{k=0}^{\infty}(z-x)^k w_k \tag{1.2}$$

in $\mathbf{S}_m$ *such that*

$$\sum_{i=0}^{k} a_{k-i}w_i = c_k \quad \text{and} \quad \sum_{i=0}^{k}\sum_{j=0}^{n} a_{k-i}w_{i+j+1}a^*_{n-j} = \gamma_k \quad (k = 0, \ldots, n) \tag{1.3}$$

and describe the set of all functions when these conditions preva il.

Since $w_k = w^{(k)}(x)/k!$, (1.3) are conditions on derivatives of the function w at the point x. Note that representation (1.1) implies $w_k \geq 0$ $(k = 0, 1, \ldots)$ because in case of real interpolation points the two-sided interpolation problem coincides with a tangential one. The tangential Carathéodory-Fejer problem can be set in other classes of analytic functions [3, 5 - 7, 10, 14]. In the paper [13] there was considered the non-degenerate case of the non-tangential problem in the Stieltjes class and in [9] its two-sided generalization for simple interpolating points.

Somewhat another interpolation problem in the same class is given in [1]. We also refer to the monographes [8, 11] where a number of approaches to solve some versions of the Nevanlinna-Pick problem have been developed. In the present paper a central role will be played by the method of the fundamental matrix inequality [12, 15 - 17].

The outline of the paper is as follows:

In Section 2, following ideas of I. Kovalishina and V. Potapov, we establish the system of matrix inequalities for the problem (IS). A necessary condition of the solvability of problem (IS) is the non-negativity of special matrices K and K_p defined in (2.12), (2.13). The description of the set of all solutions to the problem (IS) depends on whether K and K_p are degenerate or not. In Section 3 we consider the case where K and K_p both are strictly positive. The strict positivity of K and K_p is a sufficient condition for the problem (IS) to be solvable. The set of all solutions is parametrized by a linear fractional transformation of the form (3.16) with the resolvent matrix Θ of the class $\mathbf{W}_s$ (see Definition 3.1). Section 4 deals with the degenerate case; we still have a description of all the solutions as a linear fractional transformation which can be obtained by a suitable version of the Schur algorithm.

2. The fundamental matrix inequalities

In this section we characterize the solutions to the problem (IS) in terms of a system of matrix inequalities. Descriptions of the set of solutions in terms of the linear fractional transformation will be given in Sections 3 and 4. We begin with a preliminary lemma.

Lemma 2.1. *Let the function w be of the class $\mathbf{S}_m$ and let*

$$w_k \doteq \frac{w^{(k)}(x)}{k!} \qquad (k = 0, \ldots, 2n+1;\ x < 0). \tag{2.1}$$

Then for all $z \in \mathbb{C} \setminus \mathbb{R}_+$ the inequalities

$$\begin{pmatrix} T & \left(zI-\tilde{\Gamma}\right)^{-1}\left\{\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} w(z) - \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}\right\} \\ * & \dfrac{w(z)-w(z)^*}{z-\bar{z}} \end{pmatrix} \geq 0$$

$$\begin{pmatrix} T_p & \left(zI-\tilde{\Gamma}\right)^{-1}\left\{\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} zw(z) - \tilde{\Gamma}\begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}\right\} \\ * & \dfrac{zw(z)-\bar{z}w(z)^*}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{2.2}$$

are valid where $T, T_p \in \mathbb{C}^{m(n+1)\times m(n+1)}$ are Hankel block matrices defined by

$$(T)_{ij} = w_{i+j+1} \qquad \text{and} \qquad (T_p)_{ij} = xw_{i+j+1} + w_{i+j} \tag{2.3}$$

and $\tilde{\Gamma}$ is the $m(n+1)\times m(n+1)$-matrix given by

$$\tilde{\Gamma} = \begin{pmatrix} xI_m & & & 0 \\ I_m & \ddots & & \\ & \ddots & & \\ 0 & & I_m & xI_m \end{pmatrix}. \tag{2.4}$$

Proof: We first establish inequalities (2.2) for the simpliest functions $w \in \mathbf{S}_m$ such that a measure $\sigma(t)$ from its integral representation (1.1) has a single point of growth $\mu \geq 0$, i.e. for the function of the form

$$w(z) = (\mu - z)^{-1} D \tag{2.5}$$

with some non-negative matrix $D \in \mathbb{C}^{m\times m}$. It follows from (2.1), (2.5) that

$$w_k = (\mu - x)^{-k-1} D. \tag{2.6}$$

Substituting (2.3)–(2.6) into (2.2) we obtain the inequalities

$$WDW^* \geq 0 \qquad \text{and} \qquad \mu WDW^* \geq 0,$$

where

$$W^* = \left((\mu - x)^{-1} I_m, \ldots, (\mu - x)^{-n-1} I_m, (\mu - z)^{-1} I_m\right)$$

which are evidently valid for $\mu \geq 0$ and $D \geq 0$. Hence the inequalities (2.2) hold for the simpliest functions of the class $\mathbf{S}_m$. After summations and taking a limit we obtain that (2.2) hold for every function of the class $\mathbf{S}_m$ ∎

The following theorem characterizes all the solutions to the problem (IS) in terms of a system of matrix inequalities.

Theorem 2.2. Let w be a $\mathbb{C}^{m\times m}$-valued function analytic in $\mathbb{C}\setminus\mathbb{R}_+$. Then w is a solution to the problem (IS) if and only if it satisfies the system of inequalities

$$\begin{pmatrix} K & \Gamma(z)(A,-C)\begin{pmatrix} w(z) \\ I_m \end{pmatrix} \\ * & \dfrac{w(z)-w(z)^*}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{2.7}$$

$$\begin{pmatrix} K_p & \Gamma(z)(A,-\Gamma C)\begin{pmatrix} zw(z) \\ I_m \end{pmatrix} \\ * & \dfrac{zw(z)-\bar{z}w(z)^*}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{2.8}$$

for $z\neq\bar{z}$, where

$$A=\begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}, \qquad C=\begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} \tag{2.9}$$

$$\Gamma=\begin{pmatrix} xI_s & & & 0 \\ I_s & \ddots & & \\ & & \ddots & \\ 0 & & I_s & xI_s \end{pmatrix} \in \mathbb{C}^{s(n+1)\times s(n+1)} \tag{2.10}$$

$$\Gamma(z)=(zI-\Gamma)^{-1} \tag{2.11}$$

and K, K_p are $s(n+1)\times s(n+1)$ matrices defined by

$$K=\hat{A}\begin{pmatrix} c_1^* & \cdots & c_n^* & 0 \\ \vdots & & & \\ c_n^* & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} - \hat{C}\begin{pmatrix} a_1^* & \cdots & a_n^* & 0 \\ \vdots & & & \\ a_n^* & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \gamma_0 \\ \vdots & & & \gamma_1 \\ 0 & & & \vdots \\ \gamma_0 & \gamma_1 & \cdots & \gamma_n \end{pmatrix} \tag{2.12}$$

$$K_p=\Gamma K + AC^* \tag{2.13}$$

where $\hat{A}$ and $\hat{C}$ are $\mathbb{C}^{s(n+1)\times m(n+1)}$ Toeplitz block matrices defined by

$$\hat{A}=\begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & \ddots & & \vdots \\ \vdots & \ddots & & 0 \\ a_n & \cdots & a_1 & a_0 \end{pmatrix} \quad \text{and} \quad \hat{C}=\begin{pmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & \ddots & & \vdots \\ \vdots & \ddots & & 0 \\ c_n & \cdots & c_1 & c_0 \end{pmatrix}. \tag{2.14}$$

Proof. Let w be a solution to the problem (IS). Since w is of the class $\mathbf{S}_m$, it satisfies in view of Lemma 2.1 the inequalities (2.2). Multiplying (2.2) by the matrix

$\begin{pmatrix} \hat{A} & 0 \\ 0 & I_m \end{pmatrix}$ on the left, by its adjoint on the right and using relations

$$\hat{A}\begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A, \qquad \hat{A}\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = C, \qquad \hat{A}\tilde{\Gamma} = \Gamma\hat{A} \tag{2.15}$$

(which follow immediately from (1.3), (2.4), (2.9), (2.10) and (2.14)) we obtain (2.7), (2.8) with blocks $\hat{A}T\hat{A}^*$ and $\hat{A}T_p\hat{A}^*$ instead of K and K_p, respectively. But the equality

$$\hat{A}T\hat{A}^* = K \tag{2.16}$$

follows from (2.3), (2.12) and interpolation conditions (1.3), and after a multiplication of the evident identity

$$T_p = \tilde{\Gamma}T + \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}(w_0^*, \ldots, w_n^*)$$

by $\hat{A}$ on the left and by $\hat{A}^*$ on the right we obtain in view of (2.13), (2.15) and (2.16) that $\hat{A}T_p\hat{A}^* = \Gamma\hat{A}T\hat{A}^* + AC^* = K_p$ which ends the necessary part of the theorem.

Conversely, let assume that a $\mathbb{C}^{m\times m}$-valued function w analytic in $\mathbb{C}\setminus\mathbb{R}_+$ satisfies the system of matrix inequalities (2.7) and (2.8). Then, in particular,

$$\frac{w(z) - w(z)^*}{z - \bar{z}} \geq 0 \qquad \text{and} \qquad \frac{zw(z) - \bar{z}w(z)^*}{z - \bar{z}} \geq 0$$

for $z \neq \bar{z}$. By Theorem 1.2, these inequalities imply that w belongs to $\mathbf{S}_m$.

The positivity of the matrix-valued function (2.7) implies the boundedness of its non-diagonal block, i.e. of the functions

$$(z-x)^{-k-1}\left\{\left(\sum_{i=0}^{k} a_i(z-x)^i\right)w(z) - \sum_{i=0}^{k} c_i(z-x)^i\right\} \qquad (k = 0, \ldots, n) \tag{2.17}$$

in compact neighbourhoods of the point x. Substituting into (2.17) the Taylor expansion (1.2) and setting $z \to x$ we obtain consequently

$$\sum_{j=0}^{k} a_{k-j}w_j = c_k \qquad (k = 0, \ldots, n). \tag{2.18}$$

Multiplying (2.7), (2.8) by the matrix-valued function $Q(z) = \begin{pmatrix} I_{s(n+1)} & 0 \\ -\Gamma(z) & \Gamma(z)A \end{pmatrix}$ on the left and by $Q(z)^*$ on the right we come to inequalities

$$\begin{pmatrix} K & W(z) \\ W(z)^* & R(z) \end{pmatrix} \geq 0 \qquad \text{and} \qquad \begin{pmatrix} K_p & W_p(z) \\ W_p(z)^* & R_p(z) \end{pmatrix} \geq 0 \tag{2.19}$$

where

$$W(z) = K\Gamma(\bar{z})^* + \Gamma(z)\{Aw(z) - C\}A^*\Gamma(\bar{z})^* \tag{2.20}$$

$$W_p(z) = -K_p\Gamma(\bar{z})^* + \Gamma(z)\{zAw(z) - \Gamma C\}A^*\Gamma(\bar{z})^* \tag{2.21}$$

$$\begin{aligned} R(z) = {} & \Gamma(\bar{z})K\Gamma(\bar{z})^* - \Gamma(\bar{z})\Gamma(z)\{Aw(z) - C\}A^*\Gamma(\bar{z})^* \\ & - \Gamma(\bar{z})A\{w(z)^*A^* - C^*\}\Gamma(z)^*\Gamma(\bar{z})^* \\ & + \frac{1}{z-\bar{z}}\Gamma(\bar{z})A\{w(z) - w(z)^*\}A^*\Gamma(\bar{z})^* \end{aligned} \tag{2.22}$$

$$\begin{aligned} R_p(z) = {} & \Gamma(\bar{z})K_p\Gamma(\bar{z})^* - \Gamma(\bar{z})\Gamma(z)\{zAw(z) - \Gamma C\}A^*\Gamma(\bar{z})^* \\ & - \Gamma(\bar{z})A\{\bar{z}w(z)^*A^* - C^*\Gamma^*\}\Gamma(z)^*\Gamma(\bar{z})^* \\ & + \frac{1}{z-\bar{z}}\Gamma(\bar{z})A\{zw(z) - \bar{z}w(z)^*\}A^*\Gamma(\bar{z})^*. \end{aligned} \tag{2.23}$$

Lemma 2.3. *Let W, W_p, R, R_p be matrix-valued functions defined by (2.20)–(2.23). Then*

$$W_p(z) = zW(z) + K \tag{2.24}$$

$$R(z) = \frac{W(z) - W(z)^*}{z - \bar{z}} \tag{2.25}$$

$$R_p(z) = \frac{zW(z) - \bar{z}W(z)^*}{z - \bar{z}}. \tag{2.26}$$

Proof. Using (2.13), (2.20), (2.21) and the identity

$$I + \Gamma\Gamma(z) = z\Gamma(z) \tag{2.27}$$

we obtain

$$\begin{aligned} W_p(z) &= -(K\Gamma^* + CA^*)\Gamma(\bar{z})^* + z\Gamma(z)Aw(z)A^*\Gamma(\bar{z})^* + (I - z\Gamma(z))\,CA^*\Gamma(\bar{z})^* \\ &= K(I - z\Gamma(\bar{z})^*) + z\Gamma(z)(Aw(z) - C)A^*\Gamma(\bar{z})^* \\ &= zW(z) + K. \end{aligned}$$

To prove (2.25) we use the identity

$$\Gamma K - K\Gamma^* = CA^* - AC^* \tag{2.28}$$

which follows immediately from (2.13). Multiplying (2.28) by $\Gamma(\bar{z})$ on the left, by $\Gamma(\bar{z})^*$ on the right and using (2.27) we obtain

$$(z - \bar{z})\Gamma(\bar{z})K\Gamma(\bar{z})^* = \Gamma(\bar{z})K - K\Gamma(\bar{z})^* + \Gamma(\bar{z})\{AC^* - CA^*\}\Gamma(\bar{z})^*. \tag{2.29}$$

Substituting (2.29) into relation (2.23) and taking into account the resolvent identity

$\Gamma(z) - \Gamma(\bar z) = (\bar z - z)\Gamma(\bar z)\Gamma(z)$ we obtain

$$\begin{aligned} R(z) &= (z-\bar z)^{-1}\Big\{\Gamma(\bar z)K - K\Gamma(\bar z)^* + (\Gamma(z)-\Gamma(\bar z))(Aw(z)-C)A^*\Gamma(\bar z^*) \\ &\quad + \Gamma(\bar z)A(w(z)^*A^* - C^*)(\Gamma(\bar z)^* - \Gamma(z)^*) \\ &\quad + \Gamma(\bar z)A(w(z)-w(z)^*)A^*\Gamma(\bar z)^*\Big\} \\ &= (z-\bar z)^{-1}\Big\{\Gamma(\bar z)K - K\Gamma(\bar z)^* + \Gamma(z)(Aw(z)-C)A^*\Gamma(\bar z)^* \\ &\quad - \Gamma(\bar z)A(w(z)^*A^* - C^*)\Gamma(z)^*\Big\} \\ &= \frac{W(z)-W(z)^*}{z-\bar z}. \end{aligned}$$

By the same way, with help of the identity $\Gamma K_p - K_p\Gamma^* = \Gamma CA^* - AC^*\Gamma^*$ which follows from (2.16), we obtain

$$R_p(z) = \frac{W_p(z)-W_p(z)^*}{z-\bar z} = \frac{zW(z)-\bar zW(z)^*}{z-\bar z},$$

which ends the proof of the lemma ∎

In view of (2.24)–(2.26) the inequalities (2.19) can be rewritten as

$$\begin{pmatrix} K & W(z) \\ W(z)^* & \dfrac{W(z)-W(z)^*}{z-\bar z} \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} K_p & zW(z)+K \\ \bar zW(z)^*+K & \dfrac{zW(z)-\bar zW(z)}{z-\bar z} \end{pmatrix} \geq 0$$

and imply

$$\frac{W(z)-W(z)^*}{z-\bar z} \geq 0 \quad \text{and} \quad \frac{zW(z)-\bar zW(z)^*}{z-\bar z} \geq 0$$

for $z \neq \bar z$. By Theorem 1.2, W is of the class $\mathbf{S}_{(n+1)}$, and therefore is analytic in some neighborhood of x. Substituting into (2.20) the Taylor expansion (1.2) of w and using (2.15) and (2.18) we obtain

$$\begin{aligned} W(z) &= K\Gamma(\bar z)^* + \hat A\Gamma(z)\begin{pmatrix} w(z)-w_0 \\ -w_1 \\ \vdots \\ -w_n \end{pmatrix} A^*\Gamma(\bar z)^* \\ &= -K\Gamma(\bar z)^* + \hat A\begin{pmatrix} w_1 + w_2(z-x) + \dots \\ w_2 + w_3(z-x) + \dots \\ \vdots \\ w_{n+1} + w_{n+2}(z-x) + \dots \end{pmatrix} A^*\Gamma(\bar z)^* \qquad (2.30) \\ &= -K\Gamma(\bar z)^* + \hat A\begin{pmatrix} w_1 + w_2(z-x) + \dots \\ \vdots \\ w_{n+1} + w_{n+2}(z-x) + \dots \end{pmatrix}\left(\frac{I_m}{(z-x)}, \dots, \frac{I_m}{(z-x)^{n+1}}\right)\hat A^* \\ &= -K\Gamma(\bar z)^* + \hat A\begin{pmatrix} w_1 & \dots & w_{n+1} \\ \vdots & & \vdots \\ w_{n+1} & \dots & w_{2n+1} \end{pmatrix}\hat A^*\Gamma(\bar z)^* + O(|z-x|). \end{aligned}$$

Setting $z \to x$ in (2.30) we obtain in view of analyticity of W

$$K = \hat{A} \begin{pmatrix} w_1 & \dots & w_{n+1} \\ \vdots & & \vdots \\ w_{n+1} & \dots & w_{2n+1} \end{pmatrix} \hat{A}^*. \tag{2.31}$$

Substituting (2.12) and (2.14) into (2.31) and comparing there s right rows we obtain the equality

$$\begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_n \end{pmatrix} = \hat{A} \begin{pmatrix} w_1 & \dots & w_{n+1} \\ \vdots & & \vdots \\ w_{n+1} & \dots & w_{2n+1} \end{pmatrix} \begin{pmatrix} a_n^* \\ \vdots \\ a_0^* \end{pmatrix}$$

which implies

$$\gamma_k = \sum_{i=0}^{k} \sum_{j=0}^{n} a_{k-i} w_{i+j+1} a_{n-j}^* \qquad (k = 0, \dots, n). \tag{2.32}$$

Equalities (2.19) and (2.33) mean that w is a solution to the problem (IS) , which ends the proof of theorem ∎

Setting in Theorem 2.2 $s = m$, $a_0 \doteq I_m$, $a_i = 0$ for $i > 0$ and $c_i = w_i$ ($i = 0, \dots, 2n+1$) we obtain the following inversion of Lemma 2.1.

Corollary 2.4. *Let the $\mathbb{C}^{m \times m}$-valued function w analytic in $\mathbb{C} \setminus \mathbb{R}$ satisfy the inequalities (2.2). Then w belongs to $\mathbf{S}_m$ and $w^{(k)}(x)/k! = w_k$ ($k = 0, \dots, 2n+1$).*

Remark 2.5. It follows from (2.7), (2.8) that the non-negativity of matrices K and K_p is a necessary condition to ensure that the problem (IS) has a solution.

As will be proved in Section 3 , the condition $\left(\begin{smallmatrix} K & 0 \\ 0 & K_p \end{smallmatrix}\right) > 0$ is a sufficient one. The matrices K and K_p will be called *the information matrices of the problem* (IS).

3. Solution to the problem (IS) : the non-degenerate case

In this section we suppose that the information matrices K and K_p are strictly positive and describe the set of all solutions to the problem (IS) under this hypothesis To begin with we recall the necessary definitions .

Definition 3.1. A $\mathbb{C}^{2m \times 2m}$-valued meromorphic function Θ is of the *class* $\mathbf{W}_s$ if

$$\Theta(z) J \Theta(z)^* = J \quad (z \in \mathbb{R}) \qquad \text{and} \qquad \Theta(z) J \Theta(z)^* \geq J \quad (z \in \mathbb{C}_+) \tag{3.1}$$

and

$$\Theta(x) J_\pi \Theta(x)^* \geq J_\pi \qquad (x < 0)$$

where

$$J = \begin{pmatrix} 0 & iI_m \\ -iI_m & 0 \end{pmatrix} \qquad \text{and} \qquad J_\pi = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \tag{3.2}$$

and is of the *class* W if it satisfies only conditions (3.1).

Lemma 3.2 (see [12]). *The classes W and $\mathbf{W}_s$ are closed under multiplication:*
(i) Let $\Theta_1, \Theta_2 \in \mathbf{W}_s$ and $\Theta = \Theta_1\Theta_2$. Then, $\Theta \in \mathbf{W}_s$ and, for $i = 1, 2$,

$$\begin{aligned} \Theta(z)J\Theta(z)^* &\geq \Theta_i(z)J\Theta_i(z)^* \qquad (z \in \mathbb{C}_+) \\ \Theta(x)J_\pi\Theta(x) &\geq \Theta_i(x)J\Theta_i(x)^* \qquad (x < 0). \end{aligned} \tag{3.3}$$

(ii) If $\Theta_1, \Theta_2 \in \mathbf{W}_s$, then $\Theta = \Theta_1\Theta_2 \in \mathbf{W}$ and the first inequalities of (3.3) hold.

The following theorem establishes the connection between classes $\mathbf{W}$ and $\mathbf{W}_s$.

Theorem 3.3 (see [13]). *The $\mathbb{C}^{2m\times 2m}$-valued function Θ belongs to $\mathbf{W}_s$ if and only if*

$$\Theta \in \mathbf{W} \qquad \text{and} \qquad \theta_p(z) = P(z)\Theta(z)P(z)^{-1} \in \mathbf{W} \tag{3.4}$$

where

$$P(z) = \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}. \tag{3.5}$$

The following two lemmas which in fact are contained in [12] describe a number of functions of the classes $\mathbf{W}$ and $\mathbf{W}_s$.

Lemma 3.4. *Let H be a strictly positive $m \times m$ matrix which is a solution of the Lyapunov equation $\Gamma H - H\Gamma^* = -iGJG^*$, where J is a matrix defined in (3.2), $G \in \mathbb{C}^{r\times m}$ and $\Gamma \in \mathbb{C}^{m\times m}$. Then the $\mathbb{C}^{2m\times 2m}$-valued function $\hat{\Theta}(z) = I + iG^*(zI - \Gamma^*)^{-1}H^{-1}GJ$ is of the class $\mathbf{W}$ and*

$$\hat{\Theta}(z)J\hat{\Theta}(w)^* - J = i(\bar{w} - z)G^*(zI - \Gamma^*)^{-1}H^{-1}(\bar{w}I - \Gamma)^{-1}G.$$

Lemma 3.5. *Let H_1, H_2 be strictly positive $m \times m$ matrices such that*

$$H_2 - \Gamma H_1 = G_2G_1^* \tag{3.6}$$

for some matrices $G_1, G_2 \in \mathbb{C}^{r\times m}$ and $\Gamma \in \mathbb{C}^{m\times m}$. Then the $\mathbb{C}^{2m\times 2m}$-valued function

$$\Theta(z) = I + i\begin{pmatrix} G_1^* & G_1^*\Gamma^* \\ G_2^* & zG_2^* \end{pmatrix}\begin{pmatrix} \Gamma(\bar{z})^*H_1^{-1}G_1 & 0 \\ 0 & \Gamma(\bar{z})^*H_2^{-1}G_2 \end{pmatrix}J$$

(where $\Gamma(z)$ is defined by (2.11)) is of the class $\mathbf{W}_s$ and

$$\begin{aligned} \Theta(z)J\Theta(w)^* - J &= i(\bar{w} - z)\begin{pmatrix} G_1^* \\ G_2^* \end{pmatrix}\Gamma(\bar{z})^*H_1^{-1}\Gamma(\bar{w})(G_1, G_2) \\ \Theta_p(z)J\Theta_p(w)^* - J &= i(\bar{w} - z)\begin{pmatrix} G_1^*\Gamma^* \\ G_2^* \end{pmatrix}\Gamma(\bar{z})^*H_2^{-1}\Gamma(\bar{w})(\Gamma G_1, G_2). \end{aligned}$$

Note that under assumption (3.6) the function Θ admits the following representation which can be checked by a direct computation:

$$\Theta(z) = \left\{I + i\begin{pmatrix} G_1^* \\ G_2^* \end{pmatrix}\Gamma(\bar{z})^*H_1^{-1}(G_1, G_2)J\right\}\begin{pmatrix} I & 0 \\ G_2^*H_2^{-1}G_2 & I \end{pmatrix} \tag{3.7}$$

where the first factor is the function of the class $\mathbf{W}$ and the second one is a J-unitary matrix. In view of (2.16) matrices

$$H_1 = K, \qquad H_2 = K_p, \qquad G_1 = C, \qquad G_2 = A$$

satisfy the conditions of Lemma 3.5 and hence, the $\mathbb{C}^{2m\times 2m}$-valued function

$$\Theta(z) = I + i\begin{pmatrix} C^* & 0 \\ 0\&A^* & \end{pmatrix}\begin{pmatrix} \Gamma(\bar z)^* & \Gamma^*\Gamma(\bar z)^* \\ \Gamma(\bar z)^* & z\Gamma(\bar z)^* \end{pmatrix}\begin{pmatrix} K^{-1} & 0 \\ 0 & K_p^{-1} \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & A \end{pmatrix} J \tag{3.8}$$

is of the class $\mathbf{W}_s$ and

$$\begin{aligned} \Theta(z)J\Theta(w)^* - J &= i(\bar w - z)\begin{pmatrix} C^* \\ A^* \end{pmatrix}\Gamma(\bar z)^* K^{-1}\Gamma(\bar w)(C, A) \\ \Theta_p(z)J\Theta_p(w)^* - J &= i(\bar w - z)\begin{pmatrix} C^*\Gamma^* \\ A^* \end{pmatrix}\Gamma(\bar z)^* K_p^{-1}\Gamma(\bar w)(\Gamma C, A). \end{aligned} \tag{3.9}$$

In view of (3.7) Θ admits a representation

$$\Theta(z) = \hat\Theta(z)\begin{pmatrix} I & 0 \\ A^*K_p^{-1}A & I \end{pmatrix} \tag{3.10}$$

where the function

$$\hat\Theta(z) = I + i\begin{pmatrix} C^* \\ A^* \end{pmatrix}\Gamma(\bar z)^* K^{-1}(C, A)J$$

belongs, in view of (2.28) and Lemma 3.4, to the class $\mathbf{W}$. Since Θ and Θ_p are both J-unitary on the real axis, the symmetry relations

$$\Theta^{-1}(z) = J\Theta(\bar z)^* J \qquad \text{and} \qquad \Theta_p^{-1}(z) = J\Theta_p(\bar z)^* J$$

hold and together with (3.9) imply

$$\begin{aligned} J - \Theta(z)^{-*}J\Theta^{-1}(z) &= J(J - \Theta(\bar z)J\Theta(\bar z)^*)J \\ &= i(\bar z - z)\begin{pmatrix} A^* \\ -C^* \end{pmatrix}\Gamma(z)^* K^{-1}\Gamma(z)(A, -C) \end{aligned} \tag{3.11}$$

$$J - \Theta_p(z)^{-*}J\Theta_p^{-1}(z) = i(\bar z - z)\begin{pmatrix} A^* \\ -C^*\Gamma^* \end{pmatrix}\Gamma(z)^* K_p^{-1}\Gamma(z)(A, -\Gamma C). \tag{3.12}$$

Since $K, K_p > 0$, inequalities (2.7), (2.8) are equivalent to the following ones:

$$(w(z)^*, I)\left\{\frac{J}{i(\bar z - z)} - \begin{pmatrix} A^* \\ -C^* \end{pmatrix}\Gamma(z)^* K^{-1}\Gamma(z)(A, -C)\right\}\begin{pmatrix} w(z) \\ I \end{pmatrix} \geq 0$$

$$(\bar z w(z)^*, I)\left\{\frac{J}{i(\bar z - z)} - \begin{pmatrix} A^* \\ -C^*\Gamma^* \end{pmatrix}\Gamma(z)^* K_p^{-1}\Gamma(z)(A, -\Gamma C)\right\}\begin{pmatrix} zw(z) \\ I \end{pmatrix} \geq 0$$

which in view of (3.11), (3.12) can be rewritten as

$$(w(z)^*, I)\frac{\Theta(z)^{-*}J\Theta^{-1}(z)}{i(\bar z - z)}\begin{pmatrix} w(z) \\ I \end{pmatrix} \geq 0 \tag{3.13}$$

$$(\bar z w(z)^*, I)\frac{\Theta_p(z)^{-*}J\Theta_p^{-1}(z)}{i(\bar z - z)}\begin{pmatrix} zw(z) \\ I \end{pmatrix} \geq 0.$$

Using (3.4) and (3.5) one can express the last inequality in the form

$$(w(z)^*, I)\frac{\Theta(z)^{-*}P(z)^*JP(z)\Theta^{-1}(z)}{i(\bar z - z)}\begin{pmatrix} w(z) \\ I \end{pmatrix} \geq 0. \tag{3.14}$$

The description of all solutions of the system of inequalities (3.13),(3.14) for the non-tangential problem (IS) was obtained in [12]. A generalization of this result to the two-sided problem with simple points of interpolation is given in [9].

The presence of multiple points has no influence on the character of the description, and the same arguments lead to a similary description for the present problem, that's why the proof of Theorem 3.7 will be omitted. To formulate this theorem we need some definitions.

Definition 3.6. Let $\{p, q\}$ be a pair of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$.

(i) $\{p, q\}$ is called a *Stieltjes pair* if

(α) $\det(p(z)^*p(z) + q(z)^*q(z)) \not\equiv 0$

(β) $(q(z)^*p(z) - p(z)^*q(z))/(z - \bar z) \geq 0$ for $\mathrm{Im} z \neq 0$

(γ) $(zq(z)^*p(z) - \bar z p(z)^*q(z))/(z - \bar z) \geq 0$ for $\mathrm{Im} z \neq 0$.

(ii) $\{p, q\}$ is said to be *equivalent* to the pair $\{p_1, q_1\}$ if there exists a $\mathbb{C}^{m\times m}$-valued function Ω $(\det\Omega(z) \not\equiv 0)$ meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$ such that $p_1 = p\,\Omega$ and $q_1 = q\,\Omega$.

The set of all Stieltjes pairs will be denoted by $\bar{\mathrm{S}}_m$.

Using the matrix A defined in (2.9) we introduce the following subset of $\bar{\mathrm{S}}_m$:

$$\bar{\mathrm{S}}_m^0 = \Big\{\{p, q\} \in \bar{\mathrm{S}}_m \ : \det\Big(p(z)^*A^*Ap(z) + q(z)^*q(z)\Big) \not\equiv 0\Big\}. \tag{3.15}$$

Theorem 3.7. *Under the hypothesis $K, K_p > 0$, let Θ be the function defined by (3.10) and let $\Theta = (\theta_{ij})$ be the block decomposition of Θ into four $\mathbb{C}^{m\times m}$-valued functions. Then the linear fractional transformation*

$$w(z) = \Big(\theta_{11}(z)p(z) + \theta_{12}(z)q(z)\Big)\Big(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)\Big)^{-1} \tag{3.16}$$

gives a parametrization of all solutions to the problem (IS) (or, equivalently, of the system (3.13), (3.14)) when the parameter $\{p, q\}$ varies in $\bar{\mathrm{S}}_m^0$.

More precisely:

(i) Every solution w to the problem (IS) is of the form (3.16) for some pair $\{p, q\} \in \bar{\mathrm{S}}_m^0$.

(ii) For every pair $\{p,q\} \in \bar{S}_m^0$ the transformation (3.16) is well defined (the relation $\det(\theta_{21}(z)p(z)+\theta_{22}(z)q(z)) \not\equiv 0$ is true) and the $\mathbb{C}^{m\times m}$-valued function w defined by (3.16) is a solution to the problem (IS).

(iii) Pairs $\{p,q\},\{p_1,q_1\} \in \bar{S}_m^0$ are equivalent if and only if they lead under the transformation (3.16) to the same w.

Corollary 3.8. *Since the class $\bar{S}_m^0$ defined in (3.15) is non-empty, the conditions $K, K_p > 0$ are sufficient to ensure the problem (IS) to be solvable.*

Definition 3.9. The matrix of the linear fractional transformation describing all the solutions of the interpolation problem is called the *resolvent matrix* of this problem.

To conclude this section we note that the problem (IS) can be set in the class $\bar{S}_m$ of Stieltjes pairs ; we consider the following problem (ISP):

(ISP) *Given a set of matrices $a_i, c_i \in \mathbb{C}^{s\times m}, \gamma_i \in \mathbb{C}^{s\times s}$ $(i=0,\ldots,n)$ and a point $x<0$, describe all Stieltjes pairs $\{p,q\} \in \bar{S}_m$ with Taylor expansions*

$$p(z)=\sum_{k=0}^{\infty}(z-x)^k p_k \qquad \text{and} \qquad q(z)=\sum_{k=0}^{\infty}(z-x)^k q_k$$

such that

$$\sum_{i=0}^{k} a_{k-i}p_i=\sum_{i=0}^{k} c_{k-i}q_i$$

and

$$\sum_{i=0}^{k}\sum_{j=0}^{n}(a_{k-i}p_{i+j+1}-c_{k-i}q_{i+j+1})g_{n-j}^*=\gamma_k \qquad (k=0,\ldots,n)$$

where $(g_0,\ldots,g_n)\in\mathbb{C}^{s\times m(n+1)}$ is the u nique solution of the system

$$\sum_{i=0}^{k} p_i g_{k-i}^*=c_k^*, \qquad \sum_{i=0}^{k} q_i g_{k-i}^*=a_k^* \qquad (k=0,\ldots,n).$$

Similary interpolation problem on the set of Nevanlinna pairs was considered in [1, 10] and in much more general classes in [3]). The following theorem shows that problem (ISP) is also equivalent to a system of matrix inequalities.

Theorem 3.10. *A pair $\{p,q\}$ of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C}\setminus\mathbb{R}_+$ is a solution to the problem (ISP) if and only if it satisfies the inequalities*

$$\begin{pmatrix} K & \Gamma(z)(A,-C)\begin{pmatrix}p(z)\\ q(z)\end{pmatrix} \\ * & \dfrac{q(z)^*p(z)-p(z)^*q(z)}{z-\bar{z}}\end{pmatrix}\geq 0, \quad \begin{pmatrix} K_p & \Gamma(z)(A,-\Gamma C)\begin{pmatrix}p(z)\\ q(z)\end{pmatrix} \\ * & \dfrac{zq(z)^*p(z)-\bar{z}p(z)q(z)}{z-\bar{z}}\end{pmatrix}\geq 0 \quad (3.17)$$

for $z \neq \bar{z}$ *, where the matrices* K, K_p, A, C, Γ *are as in (2.9), (2.10), (2.12) and (2.13).*

A representation of inequalities (3.17) as

$$(p(z)^*, q(z)^*)\frac{\Theta(z)^{-*}J\Theta^{-1}(z)}{i(z-\bar{z})}\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq 0$$

$$(p(z)^*, q(z)^*)\frac{\Theta(z)^{-*}P(z)^*JP(z)\Theta^{-1}(z)}{i(z-\bar{z})}\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq 0$$

(analogue of (3.13), (3.14)) with P, Θ defined in (3.5), (3.8) leads to the following "projective" analogue of Theorem 3.7.

Theorem 3.11. *Under the hypothesis* $K, K_p > 0$ *the following statements are true:*

(i) All the solutions $\{p, q\}$ *of the system (3.17) are parametrized by the linear transformation*

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Theta(z)\begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} \tag{3.18}$$

with the resolvent matrix $\Theta \in \mathbf{W}_s$ *defined by (3.10) and parameters* $\{\tilde{p}, \tilde{q}\}$ *varying in* $\bar{\mathbf{S}}_m$.

(ii) Pairs $\{p, q\}$ *and* $\{u, v\}$ *are equivalent if and only if the corresponding parameters* $\{\tilde{p}, \tilde{q}\}$ *and* $\{\tilde{u}, \tilde{v}\}$ *of the linear transformation (3.18) are equivalent.*

Note that the subclass $\bar{\mathbf{S}}_m^0$ does not appear in Theorem 3.11, since the non-singularity of q in "projective" case have not to be insured.

4. The Schur algorithm to the degenerate case

The problem (IS) as a number of classical interpolation problems can be solved by a recursive algorithm. This originates with the work of Schur [19]. We also refer to papers [4, 16] where some generalizations of this recursion were considered.

In this section we construct a suitable version of the Schur algorithm to the degenerate case of the problem (IS) where information blocks K, K_p being possibly singular: $K, K_p \geq 0$. As usual, the Schur algorithm will be concluded in a gradual (step by step) decrease of a number of interpolation conditions both with a simaltaneous recounting of the interpolation data.

Let a pair $\{p_l, q_l\} \in \bar{\mathbf{S}}_m$ satisfy the following system of matrix inequalities:

$$R^{(l)} = \begin{pmatrix} K^{(l)} & \Gamma_l(z)(A^{(l)}, -C^{(l)})\begin{pmatrix} p_l(z) \\ q_l(z) \end{pmatrix} \\ * & \dfrac{q_l(z)^*p_l(z) - p_l(z)^*q_l(z)}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{4.1}$$

$$R_p^{(l)} = \begin{pmatrix} K_p^{(l)} & \Gamma_l(z)(A^{(l)}, -\Gamma_l C^{(l)})\begin{pmatrix} p_l(z) \\ q_l(z) \end{pmatrix} \\ * & \dfrac{zq_l(z)^*p_l(z) - \bar{z}p_l(z)q_l(z)}{z-\bar{z}} \end{pmatrix} \geq 0 \tag{4.2}$$

where $\Gamma_l \in \mathbb{C}^{(n-l+1)s\times(n-l+1)s}$ is the matrix defined as

$$\Gamma_l = \begin{pmatrix} xI_s & & & 0 \\ I_s & \ddots & & \\ & & \ddots & \\ 0 & & I_s & xI_s \end{pmatrix} \tag{4.3}$$

with the resolvent

$$\Gamma_l(z) = \left(zI_{(n-l+1)s} - \Gamma_l\right)^{-1} \tag{4.4}$$

and matrices $A^{(l)}, C^{(l)} \in \mathbb{C}^{(n-l+1)s\times m}$ and $K^{(l)}, K_p^{(l)} \in \mathbb{C}^{(n-l+1)s\times(n-l+1)s}$ are defined after data $a_i^{(l)}, c_i^{(l)} \in \mathbb{C}^{s\times m}, \gamma_i^{(l)} \in \mathbb{C}^{s\times s}$ $(i = 0,\dots,n-l)$ similarly to (2.9), (2.12) and (2.13), i.e.

$$A^{(l)} = \begin{pmatrix} a_0^{(l)} \\ \vdots \\ a_{n-l}^{(l)} \end{pmatrix}, \qquad C^{(l)} = \begin{pmatrix} c_0^{(l)} \\ \vdots \\ c_{n-l}^{(l)} \end{pmatrix} \tag{4.5}$$

$$K^{(l)} = \left(\sum_{i=0}^{\min(r,n-j-1)} \left(a_{r-j}^{(l)} c_{j+i+1}^{(l)*} - c_{r-j}^{(l)} a_{j+i+1}^{(l)*}\right)\right)_{r,j=0}^{n-l} + \begin{pmatrix} 0 & \dots & 0 & \gamma_0^{(l)} \\ \vdots & & & \vdots \\ \gamma_0^{(l)} & \gamma_1^{(l)} & \dots & \gamma_{n-l}^{(l)} \end{pmatrix} \tag{4.6}$$

$$K_p^{(l)} = \Gamma_l K^{(l)} + A^{(l)} C^{(l)*}. \tag{4.7}$$

In particular, $\{p_l, q_l\}$ satisfy the inequalities

$$\begin{pmatrix} k_l & (z-x)^{-1}(a_0^{(l)}, -c_0^{(l)}) \begin{pmatrix} p_l(z) \\ q_l(z) \end{pmatrix} \\ * & \dfrac{q_l(z)^* p_l(z) - p_l(z)^* q_l(z)}{z - \bar{z}} \end{pmatrix} \geq 0 \tag{4.8}$$

$$\begin{pmatrix} k_{pl} & (z-x)^{-1}(za_0^{(l)}, -xc_0^{(l)}) \begin{pmatrix} p_l(z) \\ q_l(z) \end{pmatrix} \\ * & \dfrac{zq_l(z)^* p_l(z) - \bar{z} p_l(z)^* q_l(z)}{z - \bar{z}} \end{pmatrix} \geq 0 \tag{4.9}$$

where

$$k_l = a_0^{(l)} c_1^{(l)*} - c_0^{(l)} a_1^{(l)*} \qquad \text{and} \qquad k_{pl} = xk_l - a_0^{(l)} c_0^{(l)*}. \tag{4.10}$$

Taking into account a possible singularity of k_l, k_{pl} we introduce orthogonal projections $P_{\operatorname{Ker} k_l}$ and $P_{\operatorname{Ker} k_{pl}}$ on their kernels and set

$$Q_l = I_s - P_{\operatorname{Ker} k_l} \qquad \text{and} \qquad Q_{pl} = I_s - P_{\operatorname{Ker} k_{pl}}. \tag{4.11}$$

Since the transformations $k_l : \operatorname{Ran} k_l \to \operatorname{Ran} k_l$ and $k_{pl} : \operatorname{Ran} k_{pl} \to \operatorname{Ran} k_{pl}$ are one-to-one, the pseudoinverse operators

$$\begin{aligned} k_l^{[-1]} f &= \begin{cases} (Q_l k_l|_{\operatorname{Ran} k_l})^{-1} f & \text{for } f \in \operatorname{Ran} k_l \\ 0 & \text{for } f \in \operatorname{Ker} k_l \end{cases} \\ k_{pl}^{[-1]} f &= \begin{cases} (Q_{pl} k_{pl}|_{\operatorname{Ran} k_{pl}})^{-1} f & \text{for } f \in \operatorname{Ran} k_{pl} \\ 0 & \text{for } f \in \operatorname{Ker} k_{pl} \end{cases} \end{aligned} \tag{4.12}$$

are well defined on $\mathbb{C}^s$ and

$$k_l k_l^{[-1]} = k_l^{[-1]} k_l = Q_l \qquad \text{and} \qquad k_{pl} k_{pl}^{[-1]} = k_{pl}^{[-1]} k_{pl} = Q_{pl}. \tag{4.13}$$

Lemma 4.1. *A non-degenerate pair $\{p_l, q_l\}$ of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$ satisfies the system of matrix inequalities (4.8), (4.9) if and only if it admits a representation*

$$\begin{pmatrix} p_l(z) \\ q_l(z) \end{pmatrix} = \Theta_l(z) \begin{pmatrix} p_{l+1}(z) \\ q_{l+1}(z) \end{pmatrix} \tag{4.14}$$

where

$$\Theta_l(z) = \left\{ I + (z-x)^{-1} \begin{pmatrix} c_0^{(l)*} \\ a_0^{(l)*} \end{pmatrix} k_l^{[-1]} (a_0^{(l)}, -c_0^{(l)}) \right\} \begin{pmatrix} I & 0 \\ a_0^{(l)*} k_{pl}^{[-1]} a_0^{(l)} & I \end{pmatrix} \tag{4.15}$$

and $\{p_{l+1}, q_{l+1}\}$ is some Stieltjes pair such that

$$P_{\operatorname{Ker} k_{pl}} a_0^{(l)} p_{l+1}(z) \equiv 0 \qquad \textit{and} \qquad P_{\operatorname{Ker} k_l} c_0^{(l)} q_{l+1}(z) \equiv 0. \tag{4.16}$$

Proof. In the proof the index l will be omitted. Let a pair $\{p_l, q_l\}$ satisfy the inequalities (4.8), (4.9). According to a lemma about a non-negative block matrix [13] the inequality (4.8) is equivalent to the system

$$P_{\operatorname{Ker} k} (a_0, -c_0) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0 \tag{4.17}$$

$$(p(z)^*, q(z)^*) \left\{ \frac{J}{z-\bar{z}} - |z-x|^{-2} \begin{pmatrix} a_0^* \\ -c_0^* \end{pmatrix} k^{[-1]} (a_0, -c_0) \right\} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq 0. \tag{4.18}$$

Using (4.15) and Lemma 3.4 we obtain

$$\Theta(z) J \Theta(z)^* - J = i(z-\bar{z}) |z-x|^{-2} \begin{pmatrix} c_0^* \\ a_0^* \end{pmatrix} k^{[-1]} (c_0, a_0). \tag{4.19}$$

Hence the function Θ is J-unitary on the real axis, $\Theta^{-1}(z) = J\Theta(\bar{z})^* J$, and, in view of (4.19),

$$\begin{aligned} \Theta(z)^{-*} J \Theta^{-1}(z) - J &= J(\Theta(\bar{z}) J \Theta(\bar{z})^* - J) J \\ &= i(z-\bar{z}) |z-x|^{-2} \begin{pmatrix} a_0^* \\ -c_0^* \end{pmatrix} k^{[-1]} (a_0, -c_0). \end{aligned} \tag{4.20}$$

Using (4.20) we can rewrite (4.18) as

$$(p(z)^*, q(z)^*)\frac{\Theta(z)^{-*}J\Theta(z)^{-1}}{z-\bar{z}}\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq 0.$$

It follows from the last inequality that $\{p, q\}$ admits a representation (4.14) with some non-degenerate (since $\det\Theta(z) \not\equiv 0$) pair $\{p_1, q_1\}$ such that

$$\frac{q_1(z)^* p_1(z) - p_1(z)^* q_1(z)}{z-\bar{z}} \geq 0. \tag{4.21}$$

It remains to show that the pair $\{p_1, q_1\}$ belongs to $\bar{\mathbf{S}}_m$ and satisfies conditions (4.16). According to the cited lemma about a non-negative block matrix the inequality (4.9) is equivalent to the system

$$P_{\operatorname{Ker} k_p}(za_0, -xc_0)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0 \tag{4.22}$$

$$(\bar{z}p(z)^*, q(z)^*)\left\{\frac{J}{z-\bar{z}} - |z-x|^{-2}\begin{pmatrix} a_0^* \\ -xc_0^* \end{pmatrix} k_p^{[-1]}(a_0, -xc_0)\right\}\begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} \geq 0. \tag{4.23}$$

Using (4.10)–(4.13) and (4.15) we obtain

$$\begin{aligned}(a_0, -c_0)\Theta(z) &= (a_0, -c_0)\begin{pmatrix} I & 0 \\ a_0^* k_p^{[-1]} a_0 & I \end{pmatrix} \\ &= \left(\left(I - (k_p - xk)k_p^{[-1]}\right)a_0, -c_0\right) \\ &= \left(\left(P_{\operatorname{Ker} k_p} + xkk_p^{[-1]}\right)a_0, -c_0\right)\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}(za_0, -xc_0)\Theta(z) &= (((z-x)a_0, 0) + x(a_0, -c_0))\Theta(z) \\ &= ((z-x)a_0, 0) + xP_{\operatorname{Ker} k}\left(P_{\operatorname{Ker} k_p}a_0, -c_0\right) \\ &\quad + k_p\left(\left(k^{[-1]}P_{\operatorname{Ker} k_p} + xQk_p^{[-1]}\right)a_0, -k^{[-1]}c_0\right).\end{aligned} \tag{4.25}$$

Substitution of (4.14), (4.24) into (4.17) gives

$$P_{\operatorname{Ker} k}\left(P_{\operatorname{Ker} k_p}a_0, -c_0\right)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \equiv 0. \tag{4.26}$$

Substituting (4.14) into (4.22) and taking into account (4.25), (4.26) we obtain the identity $P_{\operatorname{Ker} k_p}a_0 p_1(z) \equiv 0$, which both with (4.26) implies $P_{\operatorname{Ker} k}c_0 q_1(z) \equiv 0$. Using two last identities and substituting (4.14) into (4.23) we obtain the inequality

$$(\bar{z}p_1(z)^*, q_1(z)^*)\frac{J}{z-\bar{z}}\begin{pmatrix} zp_1(z) \\ q_1(z) \end{pmatrix} \geq 0$$

which both with (4.21) means that $\{p_1, q_1\} \in \bar{\mathbf{S}}_m$.

The necessary part of the lemma can be obtained by a direct substitution of (4.14) into (4.8), (4.9) and use of (4.16) ∎

The solution $\{p_l, q_l\}$ of the system (4.8), (4.9) (i.e. a Stieltjes pair of the form (4.14)) to be a solution of the "full" system (4.1), (4.2), its parameter $\{p_{l+1}, q_{l+1}\}$ in (4.14) has to satisfy some additional conditions which are given in the following lemma

Lemma 4.2. *Let $\{p_{l+1}, q_{l+1}\}$ be a non-degenerate pair of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C}\backslash\mathbb{R}_+$ which satisfy (4.16). Let Θ_l be a $\mathbb{C}^{2m\times 2m}$-valued function defined by (4.15) and let $\{p_l, q_l\}$ be a pair defined by the linear transformation (4.14). Then $\{p_l, q_l\}$ satisfies (4.1), (4.2) if and only if $\{p_{l+1}, q_{l+1}\}$ satisfies the system*

$$R^{(l+1)} \geq 0, \qquad R_p^{(l+1)} \geq 0 \tag{4.27}$$

where matrices $R^{(l+1)}$ ans $R_p^{(l+1)}$ are defined by (4.1) and (4.2), and ($j \geq 0$)

$$a_j^{(l+1)} = \begin{cases} a_{j+1}^{(l)} - (xc_{j+2}^{(l)}a_0^{(l)^*} - xa_{j+2}^{(l)}c_0^{(l)^*} + c_{j+1}^{(l)}a_0^*)k_{pl}^{[-1]}a_0^{(l)} & if\ j < n-l \\ a_{j+1}^{(l)} - (x\gamma_0^{(l)^*} + c_{j+1}^{(l)}a_0^{(l)^*})k_{pl}^{[-1]}a_0^{(l)} & if\ j = n-l \end{cases} \tag{4.28}$$

$$c_j^{(l+1)} = \begin{cases} c_{j+1}^{(l)} - (c_{j+2}^{(l)}a_0^{(l)^*} - a_{j+2}^{(l)}c_0^{(l)^*})k_l^{[-1]}c_0^{(l)} & if\ j < n-l \\ c_{j+1}^{(l)} - \gamma_0^{(l)^*}k_l^{[-1]}c_0^{(l)} & if\ j = n-l \end{cases} \tag{4.29}$$

$$\gamma_j^{(l+1)} = \begin{cases} \gamma_{j+1}^{(l)} - (c_{j+2}^{(l)}a_0^{(l)^*} - a_{j+2}^{(l)}c_0^{(l)^*})k_l^{[-1]}\gamma_0^{(l)} & if\ j < n-l \\ \gamma_{j+1}^{(l)} - \gamma_0^{(l)^*}k_l^{[-1]}\gamma_0^{(l)} & if\ j = n-l. \end{cases} \tag{4.30}$$

Proof. In the proof the index l will be omitted. Let

$$K = \begin{pmatrix} k & B \\ B^* & \tilde{K} \end{pmatrix} \quad \text{and} \quad K_p = \begin{pmatrix} k_p & B_p \\ B_p^* & \tilde{K}_p \end{pmatrix} \tag{4.31}$$

be block decompositions of information matrices K and K_p. The non-negativity of K ans K_p implies $P_{\mathrm{Ker}\,k}B = 0$ and $P_{\mathrm{Ker}\,k_p}B_p = 0$ which in view of (4.10) can be rewritten as $B = QB$ and $B_p = Q_pB_p$. Therefore, K and K_p admit the factorizations

$$K = \begin{pmatrix} I_s & 0 \\ B^*k^{[-1]} & I_{ns} \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & \tilde{K} - B^*k^{[-1]}B \end{pmatrix} \begin{pmatrix} I_s & k^{[-1]}B \\ 0 & I_{ns} \end{pmatrix} \tag{4.32}$$

$$K_p = \begin{pmatrix} I_s & 0 \\ B_p^*k_p^{[-1]} & I_{ns} \end{pmatrix} \begin{pmatrix} k_p & 0 \\ 0 & \tilde{K}_p - B_p^*k_p^{[-1]}B_p \end{pmatrix} \begin{pmatrix} I_s & k_p^{[-1]}B_p \\ 0 & I_{ns} \end{pmatrix}. \tag{4.33}$$

Substituting (4.14), (4.15) into (4.1), (4.2) and multiplying matrices from (4.1),(4.2) on the left by matrices

$$L = \begin{pmatrix} I & 0 & 0 \\ -B^*k^{[-1]} & I & 0 \\ 0 & 0 & I_m \end{pmatrix} \quad \text{and} \quad L_p = \begin{pmatrix} I & 0 & 0 \\ -B_p^*k_p^{[-1]} & I & 0 \\ 0 & 0 & I_m \end{pmatrix},$$

and on the right by by L^* and L_p^*, respectively, we obtain in view of factorizations (4.32) and (4.33) the inequalities

$$\begin{pmatrix} k & 0 & \psi(z) \\ 0 & \tilde{K} - B^*k^{[-1]}B & \Psi(z) \\ \psi(z)^* & \Psi(z)^* & \dfrac{q_1(z)^*p_1(z) - p_1(z)^*q_1(z)}{z-\bar{z}} + \psi(z)^*k^{[-1]}\psi(z) \end{pmatrix} \geq 0 \tag{4.34}$$

and

$$\begin{pmatrix} k_p & 0 & \psi_p(z) \\ 0 & \tilde{K}_p - B_p^* k_p^{[-1]} B_p & \Psi_p(z) \\ \psi_p(z)^* & \Psi_p(z)^* & \dfrac{zq_1(z)^*p_1(z) - \bar{z}p_1(z)^*q_1(z)}{z-\bar{z}} + \psi_p(z)^* k_p^{[-1]} \psi_p(z) \end{pmatrix} \geq 0 \tag{4.35}$$

where

$$\psi(z) = (z-x)^{-1}(a_0, -c_0)\Theta(z)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \tag{4.36}$$

$$\psi_p(z) = (z-x)^{-1}(za_0, -xc_0)\Theta(z)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \tag{4.37}$$

$$\Psi(z) = -Bk^{[-1]}\psi(z) + (0, I_{ns})\Gamma(z)(A, -C)\Theta(z)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \tag{4.38}$$

$$\Psi_p(z) = -B_p k_p^{[-1]}\psi_p(z) + (0, I_{ns})\Gamma(z)(zA, -\Gamma C)\Theta(z)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}. \tag{4.39}$$

Substituting (4.24) and (4.25) into (4.36) and (4.37), respectively, and taking into account (4.16) we obtain

$$\psi(z) = (z-x)^{-1}\left(xkk_p^{[-1]}a_0p_1(z) - c_0q_1(z)\right) \tag{4.40}$$

$$\psi_p(z) = a_0p_1(z) + (z-x)^{-1}k_p\left(xQk_p^{[-1]}a_0p_1(z) - k^{[-1]}c_0q_1(z)\right). \tag{4.41}$$

To transform (4.38) and (4.39) we consider block decompositions

$$A = \begin{pmatrix} a_0 \\ \tilde{A} \end{pmatrix}, \; C = \begin{pmatrix} c_0 \\ \tilde{C} \end{pmatrix}, \; \Gamma = \begin{pmatrix} xI_s & 0 \\ G & \Gamma_1 \end{pmatrix}, \Gamma(z) = \begin{pmatrix} (z-x)^{-1}I_s & 0 \\ G(z) & \Gamma_1(z) \end{pmatrix} \tag{4.42}$$

where

$$\tilde{A} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \; \tilde{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \; G = \begin{pmatrix} I_s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; G(z) = \begin{pmatrix} (z-x)^{-2}I_s \\ (z-x)^{-3}I_s \\ \vdots \\ (z-x)^{-n-1}I_s \end{pmatrix} \tag{4.43}$$

and Γ_1 ($=\Gamma_{l+1}$) and $\Gamma_1(z)$ ($=\Gamma_{l+1}(z)$) are matrices defined according to (4.3) and (4.4), respectively. Substituting decompositions (4.31), (4.42) into (4.7) and comparing in the obtained identity non-diagonal blocks we have

$$B_p - xB = a_0\tilde{C}^* \tag{4.44}$$

$$B_p^* - \Gamma_1 B^* - Gk = \tilde{A}c_0^* \tag{4.45}$$

$$\tilde{K}_p - \Gamma_1\tilde{K} - GB = \tilde{A}\tilde{C}^*. \tag{4.46}$$

Substituting (4.15), (4.40), (4.41) into (4.38), (4.39) and using (4.44)–(4.46) we obtain

$$\Psi(z) = \Gamma_1(z)\left(\tilde{A} - B_p^* k_p^{[-1]} a_0, -\tilde{C} + B^* k^{[-1]} c_0\right)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}. \tag{4.47}$$

$$\Psi_p(z) = \Gamma_1(z)\left(\tilde{A} - B_p^* k_p^{[-1]} a_0, -\Gamma_1(\tilde{C} - B^* k^{[-1]} c_0)\right)\begin{pmatrix} zp_1(z) \\ q_1(z) \end{pmatrix}. \tag{4.48}$$

It follows from (4.6), (4.7), (4.31) that

$$B = (a_0 c_2^* - c_0 a_2^*, \ldots, a_0 c_n^* - c_0 a_n^*, \gamma_0) \tag{4.49}$$

$$B_p = xB + a_0(c_1^*, \ldots, c_n^*). \tag{4.50}$$

Comparing (4.49) and (4.50) with (4.28) and (4.29), respectively, we obtain

$$A^{(1)} = \tilde{A} - B_p^* k_p^{[-1]} a_0 \qquad \text{and} \qquad C^{(1)} = \tilde{C} - B^* k^{[-1]} c_0 \tag{4.51}$$

and therefore (4.47), (4.48) can be rewritten as

$$\Psi(z) = \Gamma_1(z)\left(A^{(1)}, -C^{(1)}\right)\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \tag{4.52}$$

$$\Psi_p(z) = \Gamma_1(z)\left(A^{(1)}, -\Gamma_1 C^{(1)}\right)\begin{pmatrix} zp_1(z) \\ q_1(z) \end{pmatrix}. \tag{4.53}$$

It follows from (4.6), (4.28)–(4.30), (4.49), (4.50) that

$$K^{(1)} = \tilde{K} - Bk^{[-1]}B^*. \tag{4.54}$$

Moreover, we show that

$$K_p^{(1)} = \tilde{K}_p - B_p k_p^{[-1]} B_p^*. \tag{4.55}$$

Really, substituting (4.51), (4.54) into (4.7) and using (4.44)–(4.46) we have

$$\begin{aligned} K_p^{(1)} &= \Gamma_1 \tilde{K} - \Gamma_1 B^* k^{[-1]} B + \tilde{A}\tilde{C}^* - \tilde{A} c_0^* k^{[-1]} B \\ &\quad - B_p^* k_p^{[-1]} a_0 \tilde{C}^* + B_p^* k_p^{[-1]} a_0 c_0^* k^{[-1]} B \\ &= \Gamma_1 \tilde{K} - \Gamma_1 B^* k^{[-1]} B + \tilde{A}\tilde{C}^* - \left(B_p^* - \Gamma_1 B^* - Gk\right) k^{[-1]} B \\ &\quad - B_p^* k_p^{[-1]} (B_p - xB) + B_p^* k^{[-1]} B - x B_p^* k_p^{[-1]} B \\ &= \Gamma_1 \tilde{K} + \tilde{A}\tilde{C}^* - GB - B_p^* k_p^{[-1]} B_p \\ &= \tilde{K}_p - B_p^* k^{[-1]} B_p. \end{aligned}$$

To conclude the proof we note that according to the lemma about a non-negative block matrix the system of inequalities (4.34), (4.35) is equivalent to the system

$$\begin{pmatrix} \tilde{K} - B^* k^{[-1]} B & \Psi(z) \\ \Psi(z)^* & \dfrac{q_1(z)^* p_1(z) - p_1(z)^* q_1(z)}{z - \bar{z}} \end{pmatrix} \geq 0$$

$$\begin{pmatrix} \tilde{K}_p - B_p^* k_p^{[-1]} B_p & \Psi_p(z) \\ \Psi_p(z)^* & \dfrac{zq_1(z)^* p_1(z) - \bar{z} p_1(z)^* q_1(z)}{z - \bar{z}} \end{pmatrix} \geq 0$$

which in view of (4.1), (4.2) and (4.52)–(4.55) coincides with (4.27) ∎

In view of Lemma 4.2 the conditions (4.27) on the parameter $\{p_{l+1}, q_{l+1}\}$ preserve the structure of inequalities (4.1), (4.2) and, thus, the process is continuable. Under the process we obtain a sequence of resolvent matrices Θ_l, matrices $a_j^{(l)}, c_j^{(l)} \in \mathbb{C}^{s\times m}$, matrices $\gamma_j^{(l)}, k_l, k_{pl} \in \mathbb{C}^{s\times s}$ and parameters $\{p_l, q_l\} \in \bar{S}_m$ for $l = 0, \ldots, n$ and $j = 0, \ldots, n-l$ (recursive formulas for $a_j^{(l)}, c_j^{(l)}, \gamma_j^{(l)}$ and $\{p_l, q_l\}$ are given correspondingly by (4.28) - (4.30) and (4.14); in formula (4.10) $k_n = \gamma_0^{(n)}, k_{pn} = x\gamma_0^{(n)} + a_0^{(n)}c_0^{(n)*}$ for $l = n$).

We notice that in conditions (4.16) and in the formula (4.15) for the resolvent matrix Θ_l only matrices $a_0^{(l)}, c_0^{(l)}$ $(i = 0, \ldots, n)$ present. The following lemmas establish some properties of the last ones.

Lemma 4.3. *Let $a_j^{(l)}, c_j^{(l)} \in \mathbb{C}^{s\times m}$ and $\gamma_j^{(l)}, k_l, k_{pl} \in \mathbb{C}^{s\times s}$ be matrices defined in (4.28) - (4.30) and (4.9), respectively, and let $\{p_l, q_l\}$ be Stieltjes pairs defined by (4.14). Then*

(i) $P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} c_j^{(r)*} = 0$ *and* $P_{\mathrm{Ker}\, k_l} c_0^{(l)} a_j^{(r)*} = 0$ $(r > l; j = 0, \ldots, n-r)$ (4.56)

(ii) $P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} p_r(z) \equiv 0$ *and* $P_{\mathrm{Ker}\, k_l} c_0^{(l)} q_r(z) \equiv 0$ $(r > l)$. (4.57)

Proof. At first we prove (4.56) for $r = l+1$. Using (4.43), (4.44), (4.51) we obtain

$$\begin{aligned} P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} \left(c_0^{(l+1)}, \ldots, c_{n-l-1}^{(l+1)}\right) &= P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} C^{(l+1)*} \\ &= P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} \left(\tilde{C}^* - c_0^{(l)} k_l^{[-1]} B\right) \\ &= P_{\mathrm{Ker}\, k_{pl}} \left(B_p - xB - (k_{pl} - xk_l) k_l^{[-1]} B\right) \\ &= x P_{\mathrm{Ker}\, k_{pl}} (-B + QB) \\ &= 0. \end{aligned}$$

By the same way we get $P_{\mathrm{ker}\, k_l} c_0^{(l)} a_j^{(l+1)*} = 0$ for $j = 0, \ldots, n-l-1$. To obtain (4.56) for all r it sufficies to note that, in view of recursive formulas (4.28), (4.29),

$$\mathrm{Ran}\, a_j^{(r)} \subset \mathrm{Lin}\left(\mathrm{Ran}\, a_i^{(l)}\right)_{i=1,\ldots,n-l} \tag{4.58}$$

$$\mathrm{Ran}\, c_j^{(r)} \subset \mathrm{Lin}\left(\mathrm{Ran}\, c_i^{(l)}\right)_{i=1,\ldots,n-l} \tag{4.59}$$

for all $r > l$ and $j = 0, \ldots, n-r$, where Lin stands for linear span and Ran F denotes the left image of the $s \times m$ matrix F: $\mathrm{Ran}\, F = \{f \in \mathbb{C}^m : f = gF \text{ for some } g \in \mathbb{C}^s\}$.

To prove (4.57) we use induction. The assertions of the lemma are valid for $r = l+1$ in view of (4.16). Let (4.56) hold for $r = t$. Then, (4.14) implies

$$P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} p_t(z) = P_{\mathrm{Ker}\, k_{pl}} (a_0^{(l)}, 0) \Theta_t(z) \begin{pmatrix} p_{t+1}(z) \\ q_{t+1}(z) \end{pmatrix}. \tag{4.60}$$

Substituting into (4.60) the expression (4.15) for Θ_t and taking into account (4.56) we obtain

$$P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} p_t(z) = P_{\mathrm{Ker}\, k_{pl}} a_0^{(l)} p_{t+1}(z) \equiv 0.$$

The second identity in (4.57) can be obtained by the same way ∎

Lemma 4.4. *Let $\mathcal{M}, \mathcal{E}$ be subspaces in $\mathbb{C}^m$ defined by*

$$\mathcal{M} = \operatorname{Lin}\left(\operatorname{Ran} P_{\operatorname{Ker} k_{pl}} a_0^{(l)}\right)_{l=0}^{n} \quad \textit{and} \quad \mathcal{E} = \operatorname{Lin}\left(\operatorname{Ran} P_{\operatorname{Ker} k_l} c_0^{(l)}\right)_{l=0}^{n}. \tag{4.61}$$

Then,

(i) $\mathcal{M} \perp \mathcal{E}$ (4.62)

(ii) $\dim \mathcal{M} = \operatorname{rank} P_{\operatorname{Ker} k_p} A$ *and* $\dim \mathcal{E} = \operatorname{rank} P_{\operatorname{Ker} k} C$ (4.63)

where A, C are matrices defined in (2.13).

Proof. Relation (4.62) follows from (4.56) and the equality

$$P_{\operatorname{Ker} k_{pl}} a_0^{(l)} c_0^{(l)*} P_{\operatorname{Ker} k_{pl}} = P_{\operatorname{Ker} k_{pl}} (k_{pl} - z k_l) P_{\operatorname{Ker} k_l} = 0.$$

To prove relation (4.63) we note that, for any $(n-l+1)s \times (n-l+1)$ non-degenerate matrix T,

$$\operatorname{rank} P_{\operatorname{Ker} K^{(l)}} C^{(l)} = \operatorname{rank} P_{\operatorname{Ker} TK^{(l)}T^*} TC^{(l)} \tag{4.64}$$

($K^{(l)}$ is the informative matrix defined by (4.3)). Turning to the block decomposotion (4.31) of $K^{(l)}$ we put $T = \left(\begin{smallmatrix} I & 0 \\ -B^* k_l^{[-1]} & 0 \end{smallmatrix}\right)$. It follows from (4.32), (4.51) and (4.54) that

$$S = TK^{(l)}T^* = \begin{pmatrix} k_l & 0 \\ 0 & K^{(l+1)} \end{pmatrix} \quad \text{and} \quad TC^{(l)} = \begin{pmatrix} c_0^{(l)} \\ C^{(l+1)} \end{pmatrix}. \tag{4.65}$$

Substituting these equalities into (4.64) we obtain

$$\begin{aligned} \operatorname{rank} P_{\operatorname{Ker} K^{(l)}} C^{(l)} &= \operatorname{rank} P_{\operatorname{Ker} S} \begin{pmatrix} c_0^{(l)} \\ C^{(l+1)} \end{pmatrix} \\ &= \operatorname{rank} P_{\operatorname{Ker} k_l} c_0^{(l)} + \operatorname{rank} P_{\operatorname{Ker} K^{(l+1)}} C^{(l+1)}. \end{aligned}$$

Applying induction we receive $\operatorname{rank} P_{\operatorname{Ker} K} C = \sum_{l=0}^{n} \operatorname{rank} P_{\operatorname{Ker} k_l} c_0^{(l)} = \dim \mathcal{E}$. The first equality in (ii) can be obtained by the same way ∎

Lemma 4.5 *(see [9]). Let $\{p, q\} \in \bar{\mathbf{S}}_m$.*

(i) If $[0, I_\nu]p(z) \equiv 0$, then there exists a pair $\{p_1(z), q_1(z)\} \in \bar{\mathbf{S}}_{m-\nu}$ such that the pairs

$$\{p(z), q(z)\} \quad \textit{and} \quad \left\{ \begin{pmatrix} p_1(z) & 0 \\ 0 & 0_\nu \end{pmatrix}, \begin{pmatrix} q_1(z) & 0 \\ 0 & I_\nu \end{pmatrix} \right\}$$

are equivalent.

(ii) If $[0, I_\nu]q(z) \equiv 0$, then there exists a pair $\{p_2, q_2\} \in \bar{\mathbf{S}}_{m-\mu}$ such that the pairs

$$\{p(z), q(z)\} \quad \textit{and} \quad \left\{ \begin{pmatrix} p_2(z) & 0 \\ 0 & I_\mu \end{pmatrix}, \begin{pmatrix} q_2(z) & 0 \\ 0 & 0_\mu \end{pmatrix} \right\}$$

are equivalent.

The following theorem is a degenerate analogue of Theorem 3.10 and gives a description of all the solutions to the problem (ISP).

Theorem 4.6. *Let $A, C, K \geq 0, K_p \geq 0$ be matrices defined by (2.9), (2.12), (2.13) and let Θ_l $(l = 0, \dots, n)$ be $\mathbb{C}^{2m\times 2m}$-valued functions defined by (4.15). Then the linear transformation (3.18) with the resolvent matrix*

$$\Theta(z) = \prod_{l=0}^{n} \Theta_l(z) \tag{4.66}$$

gives a parametrization of all solutions to the problem (ISP) (or, equivalently, all the solutions to the system (3.17)) when the parameter $\{\tilde{p}, \tilde{q}\}$ varies in $\bar{\mathbf{S}}_m$ and is of the form

$$\tilde{p}(z) = U \begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix} \quad \text{and} \quad \tilde{q}(z) = U \begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \tag{4.67}$$

with unitary matrix $U \in \mathbb{C}^{m\times m}$ depending only on the interpolation data and Stieltjes pair $\{\hat{p}(z), \hat{q}(z)\} \in \bar{\mathbf{S}}_{m-\mu-\nu}$, where

$$\mu = \operatorname{rank} P_{\operatorname{Ker} K_p} A \quad \text{and} \quad \nu = \operatorname{rank} P_{\operatorname{Ker} K} C. \tag{4.68}$$

Proof. Setting $a_j^{(0)} = a_j, c_j^{(0)} = c_j, \gamma_j^{(0)} = \gamma_j$ $(j = 0, \dots, n)$ we apply $n+1$ times Lemmas 4.1 and 4.2 to the system (3.17). As a result we obtain that every solution $\{p, q\}$ of the system (3.17) admits a representation (3.18) with the resolvent matrix Θ defined by (4.66) and parameter $\{\tilde{p}, \tilde{q}\}$ $(= \{p_{n+1}, q_{n+1}\})$ in $\bar{\mathbf{S}}_m$. In view of (4.57)

$$P_{\mathcal{M}}\, \tilde{p}(z) \equiv 0 \quad \text{and} \quad P_{\mathcal{E}}\, q(z) \equiv 0 \tag{4.69}$$

where $P_{\mathcal{M}}$ and $P_{\mathcal{E}}$ are orthogonal projections on subspaces $\mathcal{M}$ and $\mathcal{E}$, respectively, defined by (4.61). To represent (4.69) in the form (4.67) we note that in view of (4.62) there exists a unitary matrix $U \in \mathbb{C}^{m\times m}$ such that

$$U^* P_{\mathcal{M}} U = \begin{pmatrix} 0_{m-\mu-\nu} & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \quad \text{and} \quad U^* P_{\mathcal{E}} U = \begin{pmatrix} 0_{m-\mu-\nu} & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix} \tag{4.70}$$

where

$$\mu = \dim \mathcal{M} \quad \text{and} \quad \nu = \dim \mathcal{E}. \tag{4.71}$$

Substituting (4.70) into (4.69) and applying Lemma 4.5 to the Stieltjes pair $\{U^*\tilde{p}, U^*\tilde{q}\}$ we obtain equivalence of the pairs

$$\{\tilde{p}(z), \tilde{q}(z)\} \quad \text{and} \quad \left\{ U \begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix}, U \begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \right\}.$$

Equalities (4.68) follow from (4.63) and (4.71).

To check that any pair $\{\tilde{p}, \tilde{q}\}$ of the form (4.67) (or, equivalently, satisfying the conditions (4.69)) under linear transformation leads to some solution $\{p, q\}$ to (3.17) we use a multiplicative representation (4.66) of Θ and apply step by step Lemmas 4.1 and 4.2. Finally, since functions Θ_l $(l = 0, \dots, n)$ belong to $\mathbf{W}$ (see (4.19)), the matrix Θ of the form (4.66) belongs to $\mathbf{W}$ as well ∎

Corollary 4.7. *The problem (ISP) is solvable if and only if the matrices K and K_p are both non-negative.*

Proof. The necessity follows from Theorem 3.10. The sufficiency follows from Theorem 4.6 since the set of elements of the form (4.67) is not empty ∎

Corollary 4.8. *The problem (ISP) has a unique solution if and only if $\mu + \nu = m$, where μ, ν are numbers defined in (4.68).*

To obtain the analogous description for the problem (IS) we have to care of the non-degeneracy of the "denominator" in the linear fractional transformation (3.16).

Lemma 4.9. *Let $\Theta = (\theta_{ij})$ be the block decomposition of Θ defined by (4.66) into four $\mathbb{C}^{m\times m}$-valued functions and let $\{p,q\}$ be in $\bar{\mathbf{S}}_m$ and satisfy the conditions (4.69). Then*

$$\det(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \not\equiv 0$$

if and only if $\{p,q\}$ belongs to the subclass $\bar{\mathbf{S}}_m^0$ defined in (3.15).

Proof. In view of (4.28) $\operatorname{Ran} A = \operatorname{Lin}(\operatorname{Ran} a^{(l)})_{l=0,\ldots,n}$, and therefore, $\{p,q\}$ belongs to $\bar{\mathbf{S}}_m^0$ if and only if

$$\det\left(p(z)^*\left(\sum_{l=0}^{n} a_0^{(l)^*} a_0^{(l)}\right)p(z) + q(z)^* q(z)\right) \not\equiv 0. \tag{4.72}$$

Let $\{p,q\} \in \bar{\mathbf{S}}_m^0$. We introduce a pair

$$\begin{pmatrix} p_0(z) \\ q_0(z) \end{pmatrix} = \Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$$

and show that $\det q_0(z) \not\equiv 0$. Indeed, suppose that the point $z \in \mathbb{C}_+$ and the non-zero vector $h \in \mathbb{C}^m$ are such that

$$q_0(z)h = 0. \tag{4.73}$$

Since

$$h^*(p(z)^*, q(z)^*)\Theta(z)^* J\Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} h = (h^* p_0(z)^*, 0) J \begin{pmatrix} p_0(z)h \\ 0 \end{pmatrix} = 0$$

then, in view of (4.66) and Lemma 3.2,

$$\begin{aligned} 0 &\le h^*(p(z)^*, q(z)^*) J \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \\ &= h^*(p(z)^*, q(z)^*)(J - \Theta(z)^* J\Theta(z))\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} h \\ &\le h^*(p(z)^*, q(z)^*)(J - \Theta_l(z)^* J\Theta_l(z))\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} h \qquad (l = 0, \ldots, n). \end{aligned}$$

Substituting (4.15) into the last inequality and using (4.69) we obtain

$$h^*\left(xp(z)^* a_0^{(l)^*} k_{pl}^{[-1]} k_l - q(z)^* c_0^{(l)^*}\right) k_l^{[-1]} \left(xk_l k_{pl}^{[-1]} a_0^{(l)} p(z) - c_0^{(l)} q(z)\right) h = 0$$

which is equivalent to

$$k_l^{[-1]}\left(xk_lk_{pl}^{[-1]}a_0^{(l)}p(z)-c_0^{(l)}q(z)\right)h=0 \qquad (l=0,\dots,n). \tag{4.74}$$

Setting $\{p_n,q_n\}=\{p,q\}$ we apply the inverse Schur process using (4.14). In view of (4.69) and (4.74),

$$\begin{aligned}\begin{pmatrix}p_{n-1}(z)\\ q_{n-1}(z)\end{pmatrix}h &= \left(\begin{pmatrix}I & 0\\ a_0^{(n)*}k_{pn}^{[-1]}a_0^{(n)} & I\end{pmatrix}\right.\\ &\quad\left.+\frac{1}{z-x}\begin{pmatrix}c_0^{(n)*}\\ a_0^{(n)*}\end{pmatrix}k_n^{[-1]}\left(xk_nk_{pn}^{[-1]}a_0^{(n)},-c_0^{(n)}\right)\right)\begin{pmatrix}p(z)\\ q(z)\end{pmatrix}h\\ &=\begin{pmatrix}p(z)\\ a_0^{(n)*}k_{pn}^{[-1]}a_0^{(n)}p(z)+q(z)\end{pmatrix}h.\end{aligned}$$

By induction we obtain

$$\begin{pmatrix}p_0(z)\\ q_0(z)\end{pmatrix}h=\begin{pmatrix}p(z)\\ \sum_{l=0}^n a_0^{(l)*}k_{pl}^{[-1]}a_0^{(l)}p(z)+q(z)\end{pmatrix}h \tag{4.75}$$

and, hence, (4.73) contradicts to (4.72).

Let, conversely, (4.72) be broken, and for any $z\in\mathbb{C}^+$ there exists a vector $h\in\mathbb{C}^{m\times m}$ such that

$$q(z)h=0 \qquad \text{and} \qquad a_0^{(l)}p(z)h=0 \qquad (l=0,\dots,n). \tag{4.76}$$

Therefore equalities (4.74) hold for $l=0,\dots,n$ and after the inverse Schur process we obtain (4.75), which in view of (4.76) implies (4.73) ∎

Now we can state the main result of this section.

Theorem 4.10. *Let $A,C,K\geq 0$ and $K_p\geq 0$ be matrices defined by (2.9), (2.12), (2.13). Then the linear fractional transformation (3.16) with the resolvent matrix Θ defined by (4.66) gives a parametrization of all the solutions to the problem(IS) when the parameter $\{p,q\}$ varies in $\tilde{\mathrm{S}}_m^0$ and is of the form (4.67).*

Corollary 4.11. *Let* $\operatorname{rank}A=m$. *Then the problem (IS) is solvable if and only if $K\geq 0$ and $K_p\geq 0$.*

Proof. Since $\operatorname{rank}A=m$, then $\tilde{\mathrm{S}}_m^0=\tilde{\mathrm{S}}_m$ and we use arguments from the proof of Corollary 4.8 ∎

The following example shows that conditions $K,K_p\geq 0$ alone do not ensure for the problem (IS) to be solvable.

Example 4.12. Let $s=2, m=3, n=0, x<0$ and let

$$a_0=\begin{pmatrix}1&0&0\\1&1&0\end{pmatrix},\qquad c_0=\begin{pmatrix}-2x&2x&0\\0&0&1\end{pmatrix},\qquad \gamma_0=\begin{pmatrix}1&0\\0&0\end{pmatrix}. \tag{4.77}$$

Then, according to (2.12) and (2.13),

$$K=\begin{pmatrix}1&0\\0&0\end{pmatrix}\geq 0 \qquad \text{and} \qquad K_p=\begin{pmatrix}-x&0\\0&0\end{pmatrix}\geq 0 \tag{4.78}$$

and, therefore,

$$P_{\mathrm{Ker}\,K} = P_{\mathrm{Ker}\,K_p} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.79}$$

Let the problem (IS) with the interpolation data (4.77) has a solution. Then, in view of Theorem 4.10, there exists a Stieltjes pair $\{p,q\}$ such that

$$P_{\mathrm{Ker}\,K_p} a_0 p(z) \equiv 0 \quad \text{and} \quad P_{\mathrm{Ker}\,K}\, c_0 q(z) \equiv 0 \tag{4.80}$$

and

$$\det\Big(p(z)^* a_0^* a_0 p(z) + q(z)^* q(z)\Big) \not\equiv 0. \tag{4.81}$$

Substituting (4.77), (4.79) into (4.80) we obtain $(1,1,0)\,p(z) \equiv 0$ and $(0,0,1)\,q(z) \equiv 0$. Applying Lemma 4.5 to the last identities we obtain that, up to equivalence,

$$p(z) = \begin{pmatrix} \hat{p}(z) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad q(z) = \begin{pmatrix} \hat{q}(z) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.82}$$

Substituting (4.82) into (4.81) we obtain a contradiction. So, the problem (IS) has no solutions in spite of the non-negativity of K, K_p which have been established in (4.78).

References

[1] Alpay, D., Ball, J., Gohberg, I. and L. Rodman: *The two-sided residue interpolation in the Stieltjes class.* Lin. Alg. Appl. (submitted).

[2] Alpay, D. and V. Bolotnikov: *Two-sided Nevanlinna-Pick interpolation for a class of matrix-valued functions.* Z. Anal. Anw. 12 (1993), 211 - 238.

[3] Alpay, D., Bruinsma, P., Dijksma, A. and H. de Snoo: *Interpolation problems, extensions of symmetric operators and reproducing kernel spaces* I. Oper. Theory: Adv. Appl. 50 (1991), 35 - 82.

[4] Alpay, D. and H. Dym: *On applications of reproducing kernel spaces to the Schur algorithm and rational J-unitary factorization.* Oper. Theory: Adv. Appl. 18 (1986), 89 - 159.

[5] Alpay, D. and H. Dym: *On reproducing kernel spaces, the Schur algorithm and interpolation in a general class of domains.* Oper. Theory: Adv. Appl. (to appear).

[6] Ball, J.: *Interpolation problems of Pick-Nevanlinna and Leowner types for meromorphic matrix function.* Int. Equ. Oper. Theory 6 (1983), 804 - 840.

[7] Ball, J.: *Nevanlinna-Pick interpolation: Generalizations and applications* in: Surveys of some recent results in operator theory, volume 1 (J.B. Conway and B. B Morel Editors), Pitman Research Notes in Mathematics: Vol. 171, (1989), 51 -94. Longman Scientfic and Technical Churchill Livingstone Inc. 1560 Broadway, New York.

[8] Ball, J., Gohberg, I. and L. Rodman: *Interpolation of Rational Matrix Functions* (Oper. Theory: Adv. Appl.: Vol. 45). Basel et al: Birkhäuser Verlag 1990.

[9] Bolotnikov, V.: *The two-sided Nevanlinna-Pick problem in the Stieltjes class.* Oper. Theory: Adv. Appl. 62 (1993), 15 - 37.

[10] Bruinsma, P.: *Degenerate interpolation problems for Nevanlinna pairs.* Indag. Math. (to be published).

[11] Dym, H.: *J-contractive matrix functions, reproducing kernel Hilbert spaces and interpolation* (Regional Conf. Series in Math.: Vol. 71). Providence (R. I.): Amer. Math. Soc. 1989.

[12] Dyukarev, Y. and V. Katsnelson: *Multiplicative and additive classes of Stieltjes analytic matrix-valued functions and interpolation problems associated with them.* Amer. Math. Soc. Transl. 131 (1986), 55 - 70.

[13] Efimov, A. and V. Potapov: *J-expanding matrix functions and their role in the analytical theory of electrical circuits.* Russian Math. Surveys 28 (1973), 69 - 140.

[14] Fedchina, I. : *Tangential Nevalinna-Pick problem with multiple points.* Doklady Acad. Nauk Arm. SSR 61 (1975), 214 - 218.

[15] Katsnelson, V.: *Methods of J-theory in continuous interpolation problems of analysis.* Private transl. by T.Ando, Sapporo, 1985.

[16] Kovalishina, I.: *Analytic theory of a class of interpolation problems.* Math. USSR Izv. 22 (1984), 419 - 463.

[17] Kovalishina, I. and V. Potapov: *An indefinite metric in the Nevanlinna-Pick problem.* Amer. Math. Soc. Transl. 138 (1988), 15 - 19.

[18] Krein, M. and A. Nudelman: *Markov Moment Problem and External Problems* (Amer. Math. Soc. Transl. Math. Monographs: Vol. 50). Providence (R. I.): Amer. Math. Soc. 1977.

[19] Schur, I.: *Über Protenzreihen, die im Innern des Einheitskreises beschränkt sind.* J. Reine Ang. Math. 147 (1917), 205 - 232.

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