Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 13 (1994), No. 1, 155—170

Modular Convergence Theorems in Fractional Musielak-Orlicz Spaces

C. Bardaro and G. Vinti

Abstract: Here we study modular convergence in fractional Musielak-Orlicz spaces for sequences of moment type operators and convolution operators. To obtain the requested convergence properties we give some estimates for the involved operators, using a growth condition on the convex function φ generating the space $L^{\varphi,\alpha}$. Then the convergence theorems are obtained using a density theorem of Musielak type. For the convolution operators we also consider the line group setting.

Key words: *Fractional Musielak-Orlicz spaces, Riemann-Liouville and Weil fractional integrals, moment type operators, convolution operators, h-boundedness*

AMS subject classification: 41O16, 47O57, 47O93

1. Introduction

The aim of our investigations in this paper is to present some convergence theorems of certain linear operators defined on fractional Musielak-Orlicz spaces, in which the modular is defined by means of Riemann-Liouville or Weil fractional integrals (see [3, 11 - 13]). These fractional spaces were recently introduced in [3] where some modular estimates for "homogeneous" integral operators are given. Note that a different definition of fractional Orlicz spaces was given by H. Musielak and J. Musielak in [9], where the modular is given by means of a fractional derivative.

The main theorems of this paper are concerned with modular convergence for moment type operators (see [1 - 3, 5, 6, 13]), and for convolution operators of classical type (see [4]).

In Section 2 we give some notations and definitions and we prove (Lemma 1) a density property. Section 3 concernes the study of moment type operators, for functions defined on a bounded interval, while in Section 4 we take into consideration convolution integral operators in the periodic case. In the last Section 5, we study the general case of convolution integral operator in the line group setting.

For measurable space with infinite measure, in order to deal with fractional modulars, the assumption of local integrability of $\varphi(t,c)$, c a real constant (here $\varphi = \varphi(t,u)$ is the

C. Bardaro: Univ. di Perugia, Dip. di Mat., Via Vanvitelli 1, 06100 Perugia, Italia
G. Vinti: Dip. di Mat., Fac. di Ing., Viale delle Scienze, 90100 Palermo, Italia

ISSN 0232-2064 / $ 2.20 © Heldermann Verlag Berlin

generating function of the Musielak-Orlicz spaces) is no longer suitable. Actually we must assume the integrability of $t \mapsto \varphi(t,c)$ on $\mathbb{R}$. We remark that this stronger assumption is not meaningful for functions φ without parameter; so the results we have given in Section 5 may be considered in the framework of weighted Orlicz spaces. However, an alternative condition is discussed at the end of Section 5. For usual Musielak-Orlicz spaces, in the bounded case these results were given by J. Musielak for convolution operators (see [10]) and by ourself for moment type operators (see [2]). As final remark we note that, for moment type operators, the line group setting seems to us to be attached by means of a different approach.

2. Notations and definitions

Let $I = (0,1) \subset \mathbb{R}$ and let m (or dt) be the Lebesgue measure on the Lebesgue σ-algebra $\mathcal{L}$ in I. For $A \in \mathcal{L}$, let χ_A denote the characteristic function of A. Let Φ be the class of all functions $\varphi : I \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the following properties are fulfilled:

- **(i)** $\varphi(t,\cdot)$ is non-decreasing convex and $\varphi(t,0) := lim_{u\to 0^+}\varphi(t,u) = 0$.
- **(ii)** $\varphi(t,u) > 0$ for $u > 0$ and $t \in I$.
- **(iii)** $\varphi(\cdot,u)$ is measurable.
- **(iv)** $\varphi(\cdot,a)$ is integrable for every constant $a > 0$.

Let X be the class of locally integrable functions $x : I \to \mathbb{R}$ (that is $x \in L^1(0,t)$ for every $t < 1$) and let $\alpha \in (0,1)$ be fixed. Following [3, 11 - 13], we define for $x \in X$

$$(\mathcal{P}_\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x(u)}{(t-u)^\alpha}\, du \qquad (t \in I). \tag{1}$$

This operator is known as the *generalized fractional primitive of order* $(1-\alpha)$ of x. We note that, by the Titchmarsh theorem (see [8: pp. 22 - 23]), we have $x(t) = 0$ for a.e. $t \in I$, whenever $(\mathcal{P}_\alpha x)(t) = 0$ for each $t \in I$. For $\varphi \in \Phi, x \in X, \alpha \in I$ we define

$$\rho^\alpha(x) = \int_0^1 \varphi(t, (\mathcal{P}_\alpha|x|)(t))\, dt.$$

It is very easy to show that $\rho^\alpha : X \to [0,+\infty]$ is a convex modular on X and the subspace

$$L^{\varphi,\alpha} = \Big\{x \in X : \rho^\alpha(\lambda x) < +\infty \text{ for some } \lambda > 0\Big\}$$

is called the *generalized fractional Orlicz space* (see [3]). Moreover, $E^{\varphi,\alpha}$ denotes the space of all $x \in X$ such that $\rho^\alpha(\lambda x) < +\infty$ for every $\lambda > 0$. This subspace of $L^{\varphi,\alpha}$ is usually called the *space of the finite elements* of X with respect to ρ^α.

Definition. A sequence $\{x_k\} \subset L^{\varphi,\alpha}$ is said to be *modular convergent* to $x \in L^{\varphi,\alpha}$, if for some $\lambda > 0$ the relation $\rho^\alpha(\lambda(x_k - x)) \to 0$ as $k \to +\infty$ is true. We denote this by $x_k \xrightarrow{\rho^\alpha} x$.

This notion of convergence induces a topology in $L^{\varphi,\alpha}$. Precisely, we say that a subset $V \subset L^{\varphi,\alpha}$ is *ρ^α-closed*, if $x_k \in V$ and $x_k \xrightarrow{\rho^\alpha} x$ imply $x \in V$. We denote by $\overline{V}^{\rho^\alpha}$ the ρ^α-closure of V. Finally, V is *ρ^α-dense* in $L^{\varphi,\alpha}$, if $\overline{V}^{\rho^\alpha} = L^{\varphi,\alpha}$; for further details see [10: p. 19].

From now on we denote by S the class of all simple functions in I.

Lemma 1. *For $\varphi \in \Phi$, we have*

(j) $S \subset E^{\varphi,\alpha}$

(jj) $\overline{S}^{\rho^\alpha} = L^{\varphi,\alpha}$.

Proof. Step (j). Let $\lambda > 0$ be fixed and let $s(t) = \sum_{i=1}^{N} a_i \chi_{F_i}(t)$ be the standard representation of s, where $F_i = \{t \in I : s(t) = a_i\}$. Then, by properties (i) and (iv) we have

$$\begin{aligned}
\rho^\alpha(\lambda s) &= \int_0^1 \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)} \sum_{i=1}^{N} \int_{F_i \cap (0,t)} \frac{|a_i|}{(t-v)^\alpha}\, dv\right) dt \\
&\le \int_0^1 \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)} \sum_{i=1}^{N} |a_i| \int_0^t \frac{1}{(t-v)^\alpha}\, dv\right) dt \\
&\le \int_0^1 \varphi(t, \eta)\, dt < +\infty,
\end{aligned}$$

where $\eta = \lambda \Gamma(1-\alpha)^{-1}(1-\alpha)^{-1} \sum_{i=1}^{N} |a_i|$.

Step (jj). Let $x \in L^{\varphi,\alpha}$ and let $\overline{\lambda} > 0$ be such that $\rho^\alpha(\overline{\lambda} x) < +\infty$. Firstly, we assume $x \ge 0$. Let $\{x_n\}$ be a sequence from S such that $0 \le x_n \le x_{n+1}$ for every $n \in I\!N$ and $x_n \to x$ a.e. on I. From property (i) we have $\varphi(t, \bar{\lambda}(\mathcal{P}_\alpha[x - x_n])(t)) \le \varphi(t, \bar{\lambda}(\mathcal{P}_\alpha x)(t))$. Moreover, $t \mapsto \varphi(t, \bar{\lambda}(\mathcal{P}_\alpha x)(t))$ is integrable because $x \in L^{\varphi,\alpha}$ and, by $\mathcal{P}_\alpha x_n \to \mathcal{P}_\alpha x$, we deduce $\lim_{n\to+\infty} \varphi(t, \bar{\lambda}\mathcal{P}_\alpha[x - x_n](t)) = 0$ for a.e. $t \in I$. From the Lebesgue dominated convergence theorem, we have $\rho^\alpha(\overline{\lambda}(x_n - x)) \to 0$ as $n \to +\infty$. Now, if we drop the assumption $x \ge 0$, we can split x into positive and negative parts, which belong again to $L^{\varphi,\alpha}$. Thus, let $\{x_n'\}$ and $\{x_n''\}$ be two sequences from S such that $x_n' \uparrow x^+$ and $x_n'' \uparrow x^-$ on I. Then, by choosing $\lambda > 0$ with $\lambda < \overline{\lambda}/2$, we have

$$\varphi(t, \lambda(\mathcal{P}_\alpha|x_n - x|)(t)) \le \frac{1}{2}\varphi\left(t, \bar{\lambda}\mathcal{P}_\alpha(x^+ - x_n')(t)\right) + \frac{1}{2}\varphi\left(t, \bar{\lambda}\mathcal{P}_\alpha(x^- - x_n'')(t)\right)$$

and so the assertion follows in the general case ∎

3. Approximation properties for the moment type operators

We denote by $\mathcal{F}$ the class of all the measurable functions $f : I \times I \to I\!R_0^+$ such that, putting $h(v) = \int_0^v f(t,v)\, dt$ for $v \in I$, we have $H := \sup_{v \in I} h(v) < +\infty$ and $h(v) \to 0$ as $v \to 1^-$ (see [10: p. 38, 2]).

Definition. A function $\varphi \in \Phi$ is called *h-bounded* if there exist $K_0 \in \mathbb{R}_0^+$ and $f \in \mathcal{F}$ such that

$$\varphi(tv^{-1}, u) \leq \varphi(t, K_0 u) + f(t, v) \tag{2}$$

for every $v \in I$, $t \in (0, v)$ and $u \in \mathbb{R}_0^+$.

The concept of h-boundedness is a modification of Musielak's τ-boundedness, and it is introduced in [2]. Clearly, every function $\varphi \in \Phi$ of the form $\varphi(t, v) \equiv \varphi(v)$ is h-bounded. Other non-trivial examples are given in [3].

Definition. A sequence of linear operators $T_n : L^{\varphi,\alpha} \to L^{\varphi,\alpha}$ is said to be $\mathbb{N}^\alpha$*-bounded*, if there are positive constants K_1, K_2 and a sequence $\{a_n\} \subset \mathbb{R}^+$ with $a_n \to 0$ such that

$$\rho^\alpha(T_n x) \leq K_1 \rho^\alpha(K_2 x) + a_n \qquad (x \in L^{\varphi,\alpha}). \tag{3}$$

For each $n \in \mathbb{N}$, we define a function $w_n : I \to \mathbb{R}_0^+$ with the following properties, in which H_1 and H_2 are two positive constants.

(W.1) $w_n(t)t^{\alpha-2} \in L^1(I)$ and $H_1 \leq \int_0^1 w_n(t)dt \leq H_2$ $(n \in \mathbb{N})$.

(W.2) $\lim_{n\to+\infty} \int_0^{1-\delta} w_n(t)t^{\alpha-2}dt = 0$ for every $\delta \in I$.

For each $n \in \mathbb{N}$, we consider the linear operators

$$(T_n x)(s) = \int_0^1 w_n(t)x(ts)\, dt \qquad (s \in I, x \in L^{\varphi,\alpha}(I)).$$

In the following we will prove that the operator T_n is well-defined and that the sequence $\{T_n\}$ is $\mathbb{N}^\alpha$-bounded for $\varphi \in \Phi$ h-bounded. A typical example is given by the "moment kernel" defined by the equation $w_n(t) = (n+1)t^n$ $(n \in \mathbb{N}, t \in I)$; see [1, 2, 5, 13] .The following lemma is a simple consequence of properties (W.1) and (W.2).

Lemma 2. *If the sequence $\{w_n\}$ verifies properties (W.1) and (W.2), then there is a constant $W > 0$ such that*

(W.3) $\int_0^1 w_n(t)t^{\alpha-2}\, dt \leq W \quad (n \in \mathbb{N})$.

Theorem 1. *Let $\varphi \in \Phi$ be an h-bounded function. Then $T_n x \in L^{\varphi,\alpha}$ for every $x \in L^{\varphi,\alpha}$ and $\{T_n\}$ is an $\mathbb{N}^\alpha$-bounded sequence.*

Proof. We limit ourself to prove the $\mathbb{N}^\alpha$-boundedness, because the remainder is a consequence. Let $x \in L^{\varphi,\alpha}$ be fixed and K_0 be the constant in (2). If $\rho^\alpha(WK_o x) = +\infty$, the theorem is obvious with $K_2 = WK_0$. Thus, we can assume $\rho^\alpha(WK_0 x) < +\infty$. We have, by the Fubini-Tonelli theorem,

$$\begin{aligned}
\rho^\alpha(T_n x) &\leq \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)} \int_0^1 w_n(s) \left\{\int_0^t \frac{|x(vs)|}{(t-v)^\alpha} dv\right\} ds\right) dt \\
&= \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{w_n(s)}{s^{1-\alpha}} \left\{\int_0^{ts} \frac{|x(u)|}{(ts-u)^\alpha} du\right\} ds\right) dt.
\end{aligned}$$

Put now $A_n = \int_0^1 w_n(s)s^{\alpha-1}ds$. By property (W.1) and Lemma 2, we have $H_1 \le A_n \le W$, and so by Jensen inequality and Fubini-Tonelli theorem

$$\begin{aligned}\rho^\alpha(T_n x) &\le \int_0^1 \left\{ \int_0^1 \frac{1}{A_n}\varphi\left(t, \frac{A_n}{\Gamma(1-\alpha)}\int_0^{ts} \frac{|x(u)|}{(ts-u)^\alpha}du\right)\frac{w_n(s)}{s^{1-\alpha}}ds\right\}dt \\ &\le \int_0^1 \left\{ \int_0^1 \frac{1}{H_1}\varphi\left(t, \frac{W}{\Gamma(1-\alpha)}\int_0^{ts} \frac{|x(u)|}{(ts-u)^\alpha}du\right)\frac{w_n(s)}{s^{1-\alpha}}dt\right\}ds.\end{aligned}$$

Now, making use of the substitution $z = ts$, and by h-boundedness of the function φ, we have

$$\begin{aligned}\rho^\alpha(T_n x) &\le \frac{1}{H_1}\int_0^1 \left\{ \int_0^s \varphi\left(zs^{-1}, \frac{W}{\Gamma(1-\alpha)}\int_0^{z} \frac{|x(u)|}{(z-u)^\alpha}du\right)\frac{w_n(s)}{s^{2-\alpha}}dz\right\}ds \\ &\le \frac{1}{H_1}\int_0^1 \frac{w_n(s)}{s^{2-\alpha}}\left\{ \int_0^s \varphi\left(z, \frac{WK_0}{\Gamma(1-\alpha)}\int_0^{z} \frac{|x(u)|}{(z-u)^\alpha}du\right)dz\right\}ds \\ &\quad + \frac{1}{H_1}\int_0^1 \frac{w_n(s)}{s^{2-\alpha}}\left\{\int_0^s f(z,s)\,dz\right\}ds \\ &=: I_1(n) + I_2(n).\end{aligned}$$

Now, $I_1(n) \le H_1^{-1}W\rho^\alpha(WK_0x)$. Next, for a fixed $\varepsilon > 0$, we can choose $\delta \in (0,1)$ such that $s^{\alpha-2}h(s) < H_1H_2^{-1}\varepsilon$, $s \in (1-\delta, 1)$. Then, we write

$$I_2(n) = \frac{1}{H_1}\left\{\int_0^{1-\delta} + \int_{1-\delta}^1\right\} w_n(s)s^{\alpha-2}h(s)\,ds =: I_2^1(n) + I_2^2(n).$$

Now, $I_2^1(n) \le \frac{H}{H_1}\int_0^{1-\delta} w_n(s)s^{2-\alpha}ds$, and so, by property (W.2), $I_2^1(n) \to 0$ for $n \to +\infty$. Finally, $I_2^2(n) \le \frac{\varepsilon}{H_2}\int_{1-\delta}^1 w_n(s)\,ds \le \varepsilon$. Thus we deduce $\overline{\lim}_{n\to+\infty} I_2^2(n) \le \varepsilon$ and hence $\lim_{n\to+\infty} I_2(n) = 0$ and the assertion follows ∎

We will make use of the following general theorem given in [10: p. 24]. We report this result in a suitable form, for $L^{\varphi,\alpha}$ spaces.

Theorem 2 (J. Musielak [10: p. 24]). *Let $\{G_n\}$ be an $\mathbb{N}^\alpha$-bounded sequence of linear operators $G_n : L^{\varphi,\alpha} \to L^{\varphi,\alpha}$. Let $X_0 \subset L^{\varphi,\alpha}$ be a fixed set and $S(X_0)$ the set of all the finite linear combinations of elements of X_0. Suppose that $S(X_0)$ is ρ^α-dense in $L^{\varphi,\alpha}$. Then $G_n x \xrightarrow{\rho^\alpha} x$ for every $x \in L^{\varphi,\alpha}$ implies $G_n x \xrightarrow{\rho^\alpha} x$ for every $x \in L^{\varphi,\alpha}$.*

Now, we are ready to prove the main theorem of this section.

Theorem 3. *Let $\varphi \in \Phi$ be an h-bounded function. Suppose that $H_1 = H_2 = 1$ in property (W.1). Then, for every $x \in L^{\varphi,\alpha}$, there is a constant $a > 0$ such that $\rho^\alpha(a[T_n x - x]) \to 0$ as $n \to +\infty$.*

Proof. According with Theorems 1 and 2 and Lemma 1, we limit ourself to prove the theorem for $x = \chi_A$, where $A \in \mathcal{L}$. We will prove that $T_n\chi_A$ strongly converges to χ_A, that is $\rho^\alpha(\lambda(T_n\chi_A - \chi_A)) \to 0$ for every $\lambda > 0$. Let $\lambda > 0$ be fixed. We have

$$\rho^\alpha(\lambda(T_n\chi_A - \chi_A)) = \rho(\lambda\mathcal{P}_\alpha|T_n\chi_A - \chi_A|).$$

We firstly evaluate $\mathcal{P}_\alpha|T_n\chi_A - \chi_A|$. By Fubini-Tonelli theorem,

$$\begin{aligned}
(\mathcal{P}_\alpha|T_n\chi_A - \chi_A|)(t) &\leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 w_n(s)\left\{\int_0^t \frac{|\chi_A(vs) - \chi_A(v)|}{(t-v)^\alpha}\,dv\right\}ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^1 w_n(s)\left\{\int_{(0,t)\cap(A_{s^{-1}}\Delta A)} \frac{1}{(t-v)^\alpha}\,dv\right\}ds \\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 w_n(s)\left\{\int_0^t \frac{1}{(t-v)^\alpha}\,dv\right\}ds,
\end{aligned}$$

where $A_{s^{-1}} = s^{-1}A$. So,

$$(\mathcal{P}_\alpha|T_n\chi_A - \chi_A|)(t) \leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 w_n(s)\frac{t^{1-\alpha}}{1-\alpha}\,ds \leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}.$$

Hence, by the properties of the function φ, we have

$$\varphi(t, \lambda(\mathcal{P}_\alpha|T_n\chi_A - \chi_A|)(t) \leq \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)(1-\alpha)}\right) =: \beta(t), \tag{4}$$

and the function β is integrable on I.

Next, we will prove that $\mathcal{P}_\alpha|T_n\chi_A - \chi_A| \to 0$ a.e. on I. We have

$$\begin{aligned}
I(n) &:= (\mathcal{P}_\alpha|T_n\chi_A - \chi_A|)(t) \\
&\leq \int_0^1 w_n(s)\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{|\chi_A(vs) - \chi_A(v)|}{(t-v)^\alpha}\,dv\right\}ds \\
&= \left\{\int_0^{1-\delta_t} + \int_{1-\delta_t}^1\right\}\left(w_n(s)\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{|\chi_A(vs) - \chi_A(v)|}{(t-v)^\alpha}\,dv\right\}ds\right) \\
&=: I_1(n) + I_2(n).
\end{aligned}$$

Here, $\delta_t \in I$ is a constant depending on t, which will be fixed later. First, consider $I_1(n)$. We easily deduce

$$I_1(n) \leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\int_0^{1-\delta_t} w_n(s)\,ds \leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\int_0^{1-\delta_t} w_n(s)s^{\alpha-2}ds$$

and, by property (W.2), $I_1(n) \to 0$ as $n \to +\infty$. Concerning $I_2(n)$, we have

$$I_2(n) \leq \int_{1-\delta_t}^1 w_n(s)\left\{\frac{1}{\Gamma(1-\alpha)}\int_{(0,t)\cap(A_{s^{-1}}\Delta A)} \frac{1}{(t-v)^\alpha}\,dv\right\}ds.$$

By integrability of $v \mapsto (t-v)^{-\alpha}$ and taking into account of the relation $m(A_{s^{-1}} \Delta A) \to 0$ as $s \to 1^-$ (see [2: Lemma 1]), for a fixed $\varepsilon > 0$, we can choose $\delta_t \in I$ such that the relation $\int_{(0,t)\cap(A_{s^{-1}}\Delta A)} (t-v)^{-\alpha} dv \le \varepsilon\, \Gamma(1-\alpha)$ holds for $1 - \delta_t < s < 1$. Then $\overline{\lim}_{n\to+\infty} I_2(n) \le \varepsilon$, and so $\lim_{n\to+\infty} I(n) = 0$. The assertion follows from (4), the continuity of $\varphi(t,\cdot)$ and the Lebesgue dominated convergence theorem ∎

4. Approximation by convolution integral operators: bounded case

In this section we will refer to the notations and definitions of previous sections. Let $\varphi \in \Phi$ be a given function; we will extend the function $t \mapsto \varphi(t,u)$ to the whole real axis by 1-periodic way, that is $\varphi(t+1,u) = \varphi(t,u)$ for every $t \in \mathbb{R}$ and $u \in \mathbb{R}_0^+$. We denote by $\mathcal{F}^*$ the class of all the measurable functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ such that, putting $h(v) = \int_0^1 f(t,v)\,dt$ for $v \in \mathbb{R}$, we have $H := \sup_{v\in\mathbb{R}} h(v) < +\infty$ and $h(v) \to 0$ for $v \to 0^+$ and $v \to 1^-$.

Definition. A function φ is said to be *τ-bounded* (see [10: pp. 37 - 38]) if there are a constant $K_0 \in \mathbb{R}_0^+$ and a function $f \in \mathcal{F}^*$ such that

$$\varphi(t-v,u) \le \varphi(t, K_0 u) + f(t,v) \tag{5}$$

for $v,t \in \mathbb{R}$ and $u \in \mathbb{R}_0^+$.

Let now $x \in X$ be fixed. We denote by x^* the 1-periodic extension of x to the whole real axis. We need of a reasonable periodic extension of $\mathcal{P}_\alpha|x|$. So, we introduce a function $g_\alpha : I \times \mathbb{R}^+ \to \mathbb{R}_0^+$ by

$$g_\alpha(t,v) = \begin{cases} (t-v)^{-\alpha} & \text{if } t \in I,\ v \in (0,t) \\ 0 & \text{if } t \in I,\ v \ge t \end{cases} \qquad (0 < \alpha < 1)$$

and finally, we extend by 1-periodic way the function $t \mapsto g_\alpha(t,v)$ on putting

$$g_\alpha(t \pm 1, v) = g_\alpha(t,v) \quad \text{for every } v \in \mathbb{R}^+, t \in I$$

so that

$$g_\alpha(t \pm 1, v) = \begin{cases} (t-v)^{-\alpha} & \text{if } t \in I,\ v \in (0,t) \\ 0 & \text{if } t \in I,\ v \ge t. \end{cases} \qquad (0 < \alpha < 1)$$

Finally, we define the *extended fractional primitive of order* $1-\alpha$ of x by means of

$$(\mathcal{P}_\alpha^* x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t g_\alpha(t,v) x^*(v)\,dv$$

for every $x \in X^* = \{x \in X : |x^*| \in \mathrm{Dom}\mathcal{P}_\alpha^*\}$. We will denote by ρ_*^α the restriction of ρ^α to $X^*$and the corresponding modular space by $L_*^{\varphi,\alpha}$.

We will use the following

Lemma 3. *Suppose that $\varphi \in \Phi$ is τ-bounded. Then the translation operator $(\tau_\xi x)(s) = x^*(\xi + s)$ $(\xi \in I, s \in I)$ verifies the boundedness property*

$$\rho^\alpha_*(\tau_\xi x) \le \rho^\alpha_*(K_0 x) + h(\xi). \tag{6}$$

Proof. We have, for $\xi \in I$,

$$\begin{aligned}
\rho^\alpha_*(\tau_\xi x) &= \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)} \int_0^t |x^*(v+\xi)| g_\alpha(t,v)\, dv\right) dt \\
&= \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)} \int_\xi^{\xi+t} |x^*(u)| g_\alpha(t, u-\xi)\, du\right) dt \\
&= \int_\xi^{\xi+1} \varphi\left(s-\xi, \frac{1}{\Gamma(1-\alpha)} \int_\xi^s |x^*(u)| g_\alpha(s-\xi, u-\xi)\, du\right) ds.
\end{aligned}$$

Now, the function $\beta_\xi(s) = \Gamma(1-\alpha)^{-1} \int_\xi^s |x^*(u)| g_\alpha(s-\xi, u-\xi)\, du$ is 1-periodic for every fixed $\xi \in I$. Indeed,

$$\begin{aligned}
\beta_\xi(s+1) &= \frac{1}{\Gamma(1-\alpha)} \int_\xi^{s+1} |x^*(u)| g_\alpha(s+1-\xi, u-\xi)\, du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_\xi^{s+1} |x^*(u)| g_\alpha(s-\xi, u-\xi)\, du.
\end{aligned}$$

But, for $u \ge s$, we have $g_\alpha(s-\xi, u-\xi) = 0$ and so

$$\beta_\xi(s+1) = \frac{1}{\Gamma(1-\alpha)} \int_\xi^s |x^*(u)| g_\alpha(s-\xi, u-\xi)\, du = \beta_\xi(s).$$

Then, by 1-periodicity of $t \mapsto \varphi(t,u)$, we deduce

$$\begin{aligned}
\rho^\alpha_*(\tau_\xi x) &= \int_0^1 \varphi\left(s-\xi, \frac{1}{\Gamma(1-\alpha)} \int_\xi^s |x^*(u)| g_\alpha(s-\xi, u-\xi)\, du\right) ds \\
&= \int_0^1 \varphi\left(s-\xi, \frac{1}{\Gamma(1-\alpha)} \int_\xi^s \frac{|x^*(u)|}{(s-u)^\alpha}\, du\right) ds \\
&\le \int_0^1 \varphi\left(s-\xi, \frac{1}{\Gamma(1-\alpha)} \int_0^s \frac{|x^*(u)|}{(s-u)^\alpha}\, du\right) ds.
\end{aligned}$$

Now, by τ-boundedness of the function φ, we have

$$\rho^\alpha_*(\tau_\xi x) \le \int_0^1 \varphi\left(s, \frac{K_0}{\Gamma(1-\alpha)} \int_0^s \frac{|x^*(u)|}{(s-u)^\alpha}\, du\right) ds + \int_0^1 f(s,\xi)\, ds = \rho^\alpha_*(K_0 x) + h(\xi) \quad \blacksquare$$

Next, we introduce the *convolution operators* as follows: for each $n \in \mathbb{N}$, we define a function $R_n : I \to \mathbb{R}^+$ with the following properties:

(R.1) $R_n \in L^1(I), H_1 \le \int_0^1 R_n(s)\,ds \le H_2$ for two constants $H_1, H_2 > 0$.

(R.2) $\lim_{n\to+\infty} \int_\delta^{1-\delta} R_n(s)\,ds = 0$ for every $\delta \in (0, 1/2)$.

If we extend with 1-periodicity the kernels R_n to the whole real axis, we can define

$$(U_n x)(t) = \int_0^1 R_n(s-t)x(s)\,ds \qquad (x \in L_*^{\varphi,\alpha}).$$

The following theorem shows that $U_n x$ is well-defined for every $x \in L_*^{\varphi,\alpha}$.

Theorem 4. *Let $\varphi \in \Phi$ be a τ-bounded function. Then $U_n x \in L_*^{\varphi,\alpha}$ for every $x \in L_*^{\varphi,\alpha}$ and $\{U_n\}$ is an $\mathbb{N}^\alpha$-bounded sequence.*

Proof. By using Lemma 3, we can apply similar reasonings of Theorem 1. It is sufficient to prove $\mathbb{N}^\alpha$-boundedness of the sequence $\{U_n\}$. Moreover, we can suppose that $\rho_*^\alpha(K_0 H_2 x) < +\infty$. We easily deduce that, by periodicity assumptions,

$$\begin{aligned}
\mathcal{P}_\alpha^*|U_n x| &= \frac{1}{\Gamma(1-\alpha)}\int_0^t g_\alpha(t,v)|U_n x|(v)\,dv \\
&\le \frac{1}{\Gamma(1-\alpha)}\int_0^1 R_n(\xi)\left[\int_0^t g_\alpha(t,v)|x^*(v+\xi)|\,dv\right]d\xi
\end{aligned}$$

so that

$$\begin{aligned}
\rho_*^\alpha(U_n x) &= \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)}\int_0^t |U_n x|(v) g_\alpha(t,v)\,dv\right)dt \\
&\le \int_0^1 \varphi\left(t, \frac{1}{\Gamma(1-\alpha)}\int_0^1 R_n(\xi)\left[\int_0^t g_\alpha(t,v)|x^*(v+\xi)|\,dv\right]d\xi\right)dt.
\end{aligned}$$

Now, by Jensen inequality and Fubini-Tonelli theorem, it is easy to show that

$$\rho_*^\alpha(U_n x) \le \frac{1}{H_1}\int_0^1 R_n(\xi)\rho_*^\alpha(H_2 \tau_\xi x)\,d\xi.$$

By Lemma 3, we obtain

$$\begin{aligned}
\rho_*^\alpha(U_n x) &\le \frac{1}{H_1}\int_0^1 R_n(\xi)\rho_*^\alpha(H_2 K_0 x)\,d\xi + \frac{1}{H_1}\int_0^1 R_n(\xi)h(\xi)\,d\xi \\
&\le \frac{H_2}{H_1}\rho_*^\alpha(H_2 K_0 x) + J_n,
\end{aligned}$$

where $J_n = H_1^{-1}\int_0^1 R_n(\xi)h(\xi)\,d\xi$. We will prove that $J_n \to 0$ as $n \to +\infty$. Let $\varepsilon > 0$ be fixed. There is $\delta \in (0,1/2)$ such that $h(\xi) < \varepsilon$ for $\xi \in (0,\delta)$ and $\xi \in (1-\delta, 1)$, so we can write

$$J_n = \frac{1}{H_1}\left\{\int_0^{\delta} + \int_{\delta}^{1-\delta} + \int_{1-\delta}^{1}\right\} R_n(\xi)h(\xi)\,d\xi \leq \varepsilon\frac{H_2}{H_1} + \varepsilon\frac{H_2}{H_1} + \frac{H}{H_1}\int_{\delta}^{1-\delta} R_n(\xi)\,d\xi,$$

that is $\overline{\lim}_{n\to+\infty} J_n \leq 2\frac{H_2}{H_1}\varepsilon$ and the assertion follows by the arbitrariety of $\varepsilon > 0$ ∎

Finally we are ready to prove the convergence theorem for convolution operators.

Theorem 5. *Let $\varphi \in \Phi$ be a τ-bounded function. Suppose that $H_1 = H_2 = 1$ in property (R.1). Then, for every $x \in L_*^{\varphi,\alpha}$ there exists a constant $\lambda > 0$ such that $\rho_*^{\alpha}(\lambda(U_n x - x)) \to 0$ as $n \to +\infty$.*

Proof. By $I\!N^{\alpha}$-boundedness of the sequence $\{U_n\}$ and Musielak density theorem, it is sufficient to prove the theorem for $x = \chi_A$, $A \in \mathcal{L}$. Since, by Fubini-Tonelli theorem,

$$\begin{aligned}
(\mathcal{P}_{\alpha}^{*}|U_n\chi_A - \chi_A|)(t) &\leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 R_n(s)\left\{\int_0^t \frac{|\chi_{A^*}(s+v) - \chi_A(v)|}{(t-v)^{\alpha}}\,dv\right\} ds \\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 R_n(s)\left\{\int_{(0,t)\cap((A-s)\Delta A)} (t-v)^{-\alpha}dv\right\} ds \\
&\leq \frac{1}{\Gamma(1-\alpha)\,(1-\alpha)},
\end{aligned}$$

we have, by properties of the function $\varphi = \varphi(t,u)$,

$$\varphi(t, \lambda\mathcal{P}_{\alpha}^{*}|U_n\chi_A - \chi_A|(t)) \leq \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)(1-\alpha)}\right) \tag{7}$$

for every $\lambda > 0$ and the function $\eta(t) = \varphi(t, \frac{\lambda}{\Gamma(1-\alpha)(1-\alpha)})$ is integrable. Moreover, we have

$$\begin{aligned}
&(\mathcal{P}_{\alpha}^{*}|U_n\chi_A - \chi_A|)(t) \\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^1 R_n(s)\left[\int_0^t \frac{|\chi_{A^*}(s+v) - \chi_A(v)|}{(t-v)^{\alpha}}\,dv\right] ds \\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\int_0^{\delta} + \int_{\delta}^{1-\delta} + \int_{1-\delta}^{1}\right\} R_n(s)\left[\int_0^t \frac{|\chi_{A^*}(s+v) - \chi_A(v)|}{(t-v)^{\alpha}}\,dv\right] ds \\
&=: I_1(n) + I_2(n) + I_3(n),
\end{aligned}$$

where $\delta \in (0,1/2)$ will be chosen later. We have

$$I_1(n) \leq \frac{1}{\Gamma(1-\alpha)}\int_0^{\delta} R_n(s)\left\{\int_{(0,t)\cap((A-s)\Delta A)} (t-v)^{-\alpha}\,dv\right\} ds$$

and by integrability of $(t-v)^{-\alpha}$ on the interval $(0,t)$, and by the convergence $m[(A-s)\Delta A] \to 0$ as $s \to 0$, it is possible to choose $\delta > 0$ such that the inner integral is less than $\varepsilon > 0$. Thus, $I_1(n) < \varepsilon/\Gamma(1-\alpha)$.

Next, we estimate $I_3(n)$ as follows:

$$\begin{aligned} I_3(n) &= \frac{1}{\Gamma(1-\alpha)}\int_0^\delta R_n(1-u)\left\{\int_0^t \frac{|\chi_{A^*}(1+v-u)-\chi_A(v)|}{(t-v)^\alpha}\,dv\right\}du \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^\delta R_n(1-u)\left\{\int_0^t \frac{|\chi_{A^*}(v-u)-\chi_A(v)|}{(t-v)^\alpha}dv\right\}du \end{aligned}$$

and then, we proceed as before for obtaining $I_3(n) < \varepsilon/\Gamma(1-\alpha)$. Finally, it is easy to show that

$$I_2(n) \le \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\int_\delta^{1-\delta} R_n(s)\,ds,$$

and hence we deduce $\lim_{n\to+\infty}\mathcal{P}_\alpha^*|U_n\chi_A - \chi_A|(t) = 0$ for a.e. $t \in I$. By continuity of the function $\varphi(t,\cdot)$ and (7) we obtain the assertion from the Lebesgue dominated convergence theorem ∎

5. Approximation by convolution integral operators: unbounded case

We will refer to the notations and definitions of previous sections.

Let now $\Phi_{\mathbb{R}}$ be the class of all the functions $\varphi : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that conditions i) - iv) are satisfied with $I = \mathbb{R}$ instead of $I = (0,1)$. Let X be the class of all the functions which are integrable on every interval $(-\infty, a)$. For a fixed $\alpha \in (0,1)$, we define the *Weil fractional primitive of order* $(1-\alpha)$ (see [6, 11]), the operator defined on X by the equation

$$(\tilde{\mathcal{P}}_\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^t \frac{x(u)}{(t-u)^\alpha}\,du \qquad (t \in \mathbb{R}).$$

It is not difficult to see that $x(u) = 0$ for a.e. $t \in \mathbb{R}$ whenever $(\tilde{\mathcal{P}}_\alpha x)(t) = 0$ for every $t \in \mathbb{R}$.

As in Section 2, for $\varphi \in \Phi_{\mathbb{R}}, x \in X, \alpha \in (0,1)$ we define

$$\tilde{\rho}^\alpha(x) = \int_{-\infty}^{+\infty} \varphi(t, (\tilde{\mathcal{P}}_\alpha|x|)(t))\,dt. \tag{8}$$

It is easy to see that $\tilde{\rho}^\alpha : X \to [0,+\infty]$ is a convex modular on X and the subspaces $\tilde{L}^{\varphi,\alpha}$ and $\tilde{E}^{\varphi,\alpha}$ are similarly defined, as well as the notion of modular convergence. Hence, the topological concepts related to this convergence are similarly introduced.

Moreover, we denote by S the class of all simple integrable functions $s : \mathbb{R} \to \mathbb{R}$. We remark that the set where the function s is not vanishing is bounded. In this case, the assumption iv) on $\varphi \in \Phi_{\mathbb{R}}$ is not satisfied when $\varphi(t,u) \equiv \varphi(u)$. Thus, this assumption is not meaningful in this case. Next, we discuss an alternative condition on φ. In this setting, the class $\tilde{L}^{\varphi,\alpha}$ may be regarded as a generalized version of fractional Orlicz spaces, with suitable weights; for example we may take $\varphi(t,u) = \psi(t)\varphi(u)$ with suitable ψ and φ.

We denote by $\tilde{\mathcal{F}}$ the class of all measurable functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ such that, putting $h(v) = \int_{-\infty}^{+\infty} f(t,v)\,dt$ for $v \in \mathbb{R}$, we have $H := \sup_{v\in\mathbb{R}} h(v) < +\infty$ and $h(v) \to 0$ as $v \to 0$. A function φ is said to be *τ-bounded* if it verifies the same definition of the bounded case, with $\tilde{\mathcal{F}}$ instead of $\mathcal{F}^*$.

Lemma 4. *For every interval $[a,b]$ and number $\alpha \in (0,1)$ we have*

(a) $J(t) := \int_a^b (t-v)^{-\alpha}\chi_{(-\infty,t)}(v)\,dv \leq \frac{1}{(1-\alpha)}[b-a]^{1-\alpha}$ *for every $t \in \mathbb{R}$.*

(b) $J(t,z) := \int_{(A-z)\Delta A} (t-v)^{-\alpha}\chi_{(-\infty,t)}(v)\,dv \leq \frac{1}{(1-\alpha)}(b+|z|-a)^{1-\alpha}$ *for every $t,z \in \mathbb{R}$ and for every measurable subset $A \subset [a,b]$.*

Proof. (a) It is sufficient to assume $t > a$. If $a < t < b$, the assertion is obvious. Suppose now $t > b$. Then we have

$$J(t) = \int_a^b (t-v)^{-\alpha}dv \leq \int_a^b (b-v)^{-\alpha}dv = \frac{1}{(1-\alpha)}(b-a)^{1-\alpha},$$

and hence the proof of statement (a) is complete.

(b) Let $z \geq 0$. If $t < b$, the assertion is trivial. So, we assume $t > b$. In this case

$$J(t,z) \leq \int_{a-z}^{b} (t-v)^{-\alpha}dv \leq \int_{a-z}^{b} (b-v)^{-\alpha}\,dv = \frac{1}{(1-\alpha)}[b+z-a]^{1-\alpha}.$$

Let now $z \leq 0$. By considering the cases $t < b-z$ and $t > b-z$, the assertion follows with similar arguments ∎

Lemma 5. *For $\varphi \in \Phi_{\mathbb{R}}$ and $\alpha \in (0,1)$, we have*

(j) $S \subset \tilde{E}^{\varphi,\alpha}$

(jj) $\overline{S}^{\tilde{\rho}^\alpha} = \tilde{L}^{\varphi,\alpha}$.

Proof. The proof of statement (jj) is the same as of the bounded case. So, we will prove statement (j). Let $\lambda > 0$ be fixed and let $s(t) = \sum_{i=1}^N a_i\chi_{F_i}(t)$ where $a_i \neq 0$ and the sets $F_i = \{t \in \mathbb{R} : s(t) = a_i\}$ are all contained in a bounded interval [a,b]. Then, by properties i), iv) and Lemma 4/a), we have

$$\begin{aligned}
\tilde{\rho}^\alpha_*(\lambda s) &\leq \int_{-\infty}^{+\infty} \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)}\sum_{i=1}^N |a_i| \int_{F_i} \frac{\chi_{(-\infty,t)}(v)}{(t-v)^\alpha}\,dv\right)dt \\
&\leq \int_{-\infty}^{+\infty} \varphi\left(t, \frac{\lambda}{\Gamma(1-\alpha)}\sum_{i=1}^N |a_i| \int_a^b \frac{\chi_{(-\infty,t)}(v)}{(t-v)^\alpha}\,dv\right)dt \\
&\leq \int_{-\infty}^{+\infty} \varphi(t,\eta)\,dt < +\infty,
\end{aligned}$$

where $\eta := \frac{\lambda(b-a)^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sum_{i=1}^N |a_i|$ ∎

Lemma 6. *Suppose that $\varphi \in \Phi_{\mathbb{R}}$ is τ-bounded. Then the translation operator $(\tau_\xi x)(s) = x(\xi + s)$ $(\xi, s \in \mathbb{R})$ verifies the property $\tilde{\rho}^\alpha(\tau_\xi) \leq \tilde{\rho}^\alpha(K_0 x) + h(\xi)$ (here K_0 and h are related with the definition of τ-boundedness of $\varphi \in \Phi_{\mathbb{R}}$).*

Proof. By using suitable substitutions, it is easy to show that

$$\tilde{\rho}^\alpha(\tau_\xi x) = \int_{-\infty}^{+\infty} \varphi\left(s - \xi, \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{s} \frac{|x(u)|}{(s-u)^\alpha} du\right) ds.$$

Now, by τ-boundedness of the function φ, the assertion immediately follows ∎

Next, we will introduce the convolution operators, whose kernels have properties which seems to be very useful to describe the unbounded case.

For each $n \in \mathbb{N}$, we define a function $R_n : \mathbb{R} \to \mathbb{R}^+$ with the following properties:

$(\widetilde{\mathbf{R.1}})$ $R_n \in L^1(\mathbb{R})$ and $H_1 \leq \int_{-\infty}^{+\infty} R_n(t)\, dt \leq H_2$ $(n \in \mathbb{N})$ for two constants $H_1, H_2 > 0$.

$(\widetilde{\mathbf{R.2}})$ $\lim_{n\to\infty} \int_{|t|\geq\delta} R_n(t)\, dt = 0$ for every $\delta > 0$.

Now, we define the operators

$$(\tilde{U}_n x)(t) = \int_{-\infty}^{+\infty} R_n(s-t) x(s)\, ds \qquad (x \in \tilde{L}^{\varphi,\alpha}).$$

The following theorem shows that $\tilde{U}_n x$ is well-defined for every $x \in \tilde{L}^{\varphi,\alpha}$ when the generating function φ is τ-bounded.

Theorem 6. *Let $\varphi \in \Phi_{\mathbb{R}}$ be a τ-bounded function. Then $\tilde{U}_n x \in \tilde{L}^{\varphi,\alpha}$ for every $x \in \tilde{L}^{\varphi,\alpha}$ and the sequence $\{\tilde{U}_n\}$ is $\mathbb{N}^\alpha$-bounded.*

Proof. As in the previous section, we will prove only $\mathbb{N}^\alpha$-boundedness of $\{\tilde{U}_n\}$. By similar arguments, one has

$$\tilde{\rho}^\alpha(\tilde{U}_n x) \leq \int_{-\infty}^{+\infty} \varphi\left(t, \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} R_n(\xi) \left[\int_{-\infty}^{t} \frac{|x(\xi+v)|}{(t-v)^\alpha} dv\right] d\xi\right) dt.$$

Now, by Jensen inequality, Fubini-Tonelli theorem and property $(\widetilde{R.1})$, we have

$$\tilde{\rho}^\alpha(\tilde{U}_n x) \leq \frac{1}{H_1} \int_{-\infty}^{+\infty} R_n(\xi) \tilde{\rho}^\alpha(H_2 \tau_\xi x)\, d\xi.$$

By Lemma 6 and property $(\widetilde{R.1})$, we deduce

$$\tilde{\rho}^\alpha(\tilde{U}_n x) \leq \frac{H_2}{H_1} \tilde{\rho}^\alpha(H_2 K_0 x) + \frac{1}{H_1} \int_{-\infty}^{+\infty} R_n(\xi) h(\xi)\, d\xi.$$

Put now $J_n = H_1^{-1}\int_{-\infty}^{+\infty} R_n(\xi)h(\xi)\,d\xi$. By the definition of τ-boundedness, we know that $h(\xi) \to 0$ as $\xi \to 0$. So, for a fixed $\varepsilon > 0$, we can choose $\delta_\varepsilon > 0$ such that $h(\xi) < \varepsilon$ whenever $|\xi| < \delta_\varepsilon$. So, we can write

$$J_n := \frac{1}{H_1}\left\{\int\limits_{|\xi|\geq\delta_\varepsilon} + \int\limits_{|\xi|<\delta_\varepsilon}\right\} R_n(\xi)h(\xi)\,d\xi \leq \frac{H}{H_1}\int\limits_{|\xi|\geq\delta_\varepsilon} R_n(\xi)\,d\xi + \varepsilon\frac{H_2}{H_1}.$$

From this, the assertion easily follows by property $(\widetilde{R.2})$ and the arbitrariety of $\varepsilon > 0$ ∎

Now, we are ready to prove the main theorem of this section.

Theorem 7. *Let $\varphi \in \Phi_{\mathbb{R}}$ be a τ-bounded function and $\alpha \in (0,1)$. Assume that $H_1 = H_2 = 1$ in property $(\widetilde{R.1})$ and, for every $\delta > 0$,*

$(\widetilde{\mathbf{R.3}})$ *$R_n(\cdot)|\cdot|^{1-\alpha} \in L^1(\mathbb{R})$ and $\lim_{n\to+\infty}\int_{|t|\geq\delta} R_n(t)|t|^{1-\alpha}\,dt = 0$.*

Then, for every $x \in \tilde{L}^{\varphi,\alpha}$ there is a constant $\lambda > 0$ such that $\tilde{\rho}^\alpha[\lambda(\tilde{U}_n x - x)] \to 0$ as $n \to +\infty$.

Proof. Firstly we note that properties $(\widetilde{R.3})$ and $(\widetilde{R.1})$ imply the existence of a constant $K > 0$ such that

$$\int_{-\infty}^{+\infty} R_n(t)|t|^{1-\alpha}\,dt \leq K \quad \text{for every } n \in \mathbb{N}. \tag{9}$$

Moreover, it is clear that property $(\widetilde{R.3})$ implies property $(\widetilde{R.2})$. By $\mathbb{N}^\alpha$-boundedness of the sequence $\{\tilde{U}_n\}$, Lemma 5 and the Musielak density theorem, it is sufficient to prove the theorem for $x = \chi_A$ where A is a bounded measurable set. Let $[a,b]$ be an interval such that $A \subset [a,b]$. By Fubini-Tonelli theorem, Lemma 4/b) and (10), we have

$$\begin{aligned}
&(\tilde{\mathcal{P}}_\alpha|\tilde{U}_n\chi_A - \chi_A|)(t)\\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{+\infty} R_n(z)\left\{\int\limits_{(A-z)\Delta A} (t-v)^{-\alpha}\chi_{(-\infty,t)}(v)\,dv\right\}dz\\
&\leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\int_{-\infty}^{+\infty} R_n(z)[b-a+|z|]^{1-\alpha}dz\\
&\leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\left\{(b-a)\int_{-\infty}^{+\infty} R_n(z)\,dz + (b-a)^\alpha\int_{-\infty}^{+\infty} R_n(z)|z|^{1-\alpha}dz\right\}\\
&\leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)}\{(b-a)H_2 + (b-a)^\alpha K\} =: M.
\end{aligned}$$

So, by the properties of the function $\varphi = \varphi(t,u)$, for every $\lambda > 0$ we have

$$\varphi(t, \lambda\tilde{\mathcal{P}}_\alpha|\tilde{U}_n\chi_A - \chi_A|(t)) \leq \varphi(t, \lambda M) \tag{10}$$

and the function $\varphi(\cdot, \lambda M)$ is integrable. Moreover, we have

$$\begin{aligned}
&(\tilde{\mathcal{P}}_\alpha|\tilde{U}_n\chi_A - \chi_A|)(t) \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} R_n(z) \left\{ \int_{(A-z)\Delta A} \frac{\chi_{(-\infty,t)}(v)}{(t-v)^\alpha}\, dv \right\} dz \\
&= \frac{1}{\Gamma(1-\alpha)} \left\{ \int_{|z|\geq\delta} + \int_{|z|<\delta} \right\} R_n(z) \left[\int_{(A-z)\Delta A} \frac{\chi_{(-\infty,t)}(v)}{(t-v)^\alpha}\, dv \right] dz \\
&=: J_1^n + J_2^n,
\end{aligned}$$

where the constant will be chosen later. Then, by Lemma 4/b),

$$J_1^n \leq \frac{1}{\Gamma(1-\alpha)(1-\alpha)} \int_{|z|\geq\delta} R_n(z)(b-a+|z|)^{1-\alpha} dz$$

and so, by properties $(\widetilde{R.2})$ and $(\widetilde{R.3})$ it easily follows that $J_1^n \to 0$ as $n \to +\infty$.

Finally we estimate J_2^n. We have

$$J_2^n \leq \frac{1}{\Gamma(1-\alpha)} \int_{|z|<\delta} R_n(z) \left\{ \int_{(A-z)\Delta A} \frac{\chi_{(-\infty,t)}(v)}{(t-v)^\alpha}\, dv \right\} dz. \tag{11}$$

Now, by the integrability of the function $(t-v)^{-\alpha}$ on $[(A-z)\Delta A]\cap(-\infty,t)$ and by the convergence $m[(A-z)\Delta A] \to 0$ as $z \to 0$, it is possible to choose the constant δ in such a way that the inner integral in (11) is less then $\varepsilon > 0$. Thus, $J_2^n \leq \varepsilon/\Gamma(1-\alpha)$ and, from the arbitrariety of $\varepsilon > 0$, $J_2^n \to 0$ as $n \to +\infty$. We conclude that $\lim_{n\to+\infty}(\tilde{\mathcal{P}}_\alpha|\tilde{U}_n\chi_A - \chi_A|)(t) = 0$ for a.e. $t \in \mathbb{R}$. By continuity of the function $\varphi(t,\cdot)$ and (11), the assertion follows from the Lebesgue dominate convergence theorem ∎

Remarks: a). Suppose that the generating function $\varphi \in \Phi_{\mathbb{R}}$ verifies the condition $(iv)'$ below instead of (iv):

(iv)' Let $t \mapsto \varphi(t,a)$ be locally summable for every $a \in \mathbb{R}_0^+$ and, for every $g \in L^1_{loc}(\mathbb{R})$ such that $g(t) = O(t^{-\alpha})$ $(t \to +\infty, 0 < \alpha < 1)$ the function $t \mapsto \varphi(t, g(t))$ be integrable on $\mathbb{R}$.

Then it is possible to obtain all the results of Section 5. Indeed, if $\chi_{[a,b]}$ is the characteristic function of an interval $[a,b] \subset \mathbb{R}$, then $\mathcal{P}_\alpha(\chi_A) = O(t^{-\alpha})$. Thus, it is clear that the same is verified for χ_A, A a general bounded measurable subset of $\mathbb{R}$. So, from property $(iv)'$, $\chi_A \in \tilde{E}^{\varphi,\alpha}$, $A \in \mathcal{L}$. Then, we can show that, for every $\lambda > 0$, $\lim_{z\to 0} \tilde{\rho}^\alpha(\lambda(\tau_z\chi_A - \chi_A)) = 0$ and consequently, we can proceed as in Theorem 4 of [2].

b) We remark that for functions $x \in E^{\varphi,\alpha}$ we have a strong convergence for the sequences $\{T_n x\}, \{U_n x\}$ or $\{\widetilde{U}_n x\}$. Indeed, by using similar reasonings of Lemma 1 and Lemma 5, it results that S is dense in $E^{\varphi,\alpha}$ with respect to the norm.

In general, the modular convergence seems to us the more appropriate topological setting in order to study convergence problems for sequences of integral operators. On the other hand, this is usual in the classical case of Orlicz spaces L^φ (see [10: pp. 33 - 43]).

References

[1] Barbieri, F.: *Approssimazione mediante nuclei momento.* Atti Sem. Mat. Fis. Univ. Modena 32 (1983), 308 - 328.

[2] Bardaro, C. and G. Vinti: *Modular convergence in generalized Orlicz spaces for moment type operators.* Appl. Anal. 32 (1989), 265 - 276.

[3] Bardaro, C. and G. Vinti: *Some estimates of certain integral operators in generalized fractional Orlicz classes.* Numer. Funct. Anal. and Optimiz. 12 (1991), 443 - 453.

[4] Butzer, P.L. and R.J. Nessel: *Fourier Analysis and Approximation.* New York - London: Academic Press 1971.

[5] Degani Cattelani, F.: *Nuclei di tipo "distanza" che attutiscono i salti in una o piú variabili.* Atti Sem. Mat. Fis. Univ. Modena 30 (1981), 299 - 321.

[6] Fiocchi, C.: *Variazione di ordine α e dimensione di Hausdorff degli insiemi di Cantor.* Atti Sem. Mat. Fis. Univ. Modena 34 (1991), 649 - 667.

[7] Kozlowski, W.M.: *Modular Function Spaces* (Pure Appl. Math.). New York and Basel: Marcel Dekker 1988.

[8] Mikusinski, J.: *Operational Calculus.* Warszava: Pergamon Press 1959.

[9] Musielak, H. and J. Musielak: *An Application of a Bernstein-type inequality for fractional derivatives to some problems of modular spaces.* Constr. Funct. Theory '77, Sofia (1980), 427-432.

[10] Musielak, J.: *Orlicz spaces and modular spaces.* Lect. Notes Math. 1034 (1983), 1 - 222.

[11] Oldham, K.B. and J. Spanier: *The Fractional Calculus.* New York: Academic Press 1974.

[12] Zanelli, V.: *I polinomi di Stieltjes approssimanti in variazione di ordine non intero.* Atti Sem. Mat. Fis. Univ. Modena 30 (1981), 151 - 175.

[13] Zanelli, V.: *Funzioni momento convergenti dal basso in variazione di ordine non intero.* Atti Sem. Mat. Fis. Univ. Modena 30 (1981), 355 - 369.

Receivd 09.08.1993