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Estimating Remainder Functionals by the Moduli of Smoothness

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Abstract: For remainder functionals (e.g., approximation or quadrature errors), estimates by the moduli of smoothness are obtained. As a by-product, the constants in the estimate of the $\mathcal{K}$-functional by the moduli of smoothness are improved.

Keywords: *Error estimates, modulus of continuity, modulus of smoothness, $\mathcal{K}$-functional*

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1. Introduction

Let $R : C[a,b] \to \mathbb{R}$ be a bounded linear functional, and let $R[\mathcal{P}_{r-1}] = 0$, where $\mathcal{P}_{r-1}$ are the polynomials of degree less r (the setting in the following sections will be somewhat more general). The standard estimates for $R[f]$ are

$$|R[f]| \le \|R\|_j \|f^{(j)}\|_\infty \qquad \text{for } f \in C^j[a,b] \text{ and } 1 \le j \le r, \tag{1.1}$$

where

$$\|R\|_j = \sup\left\{\frac{|R[f]|}{\|f^{(j)}\|} : f \in C^j[a,b], \|f^{(j)}\|_\infty \ne 0\right\}.$$

But if additional information on f is available, e.g. $f^{(j)} \in Lip\,\alpha$, this information cannot be used by estimates of the type (1.1). Therefore, it is of interest to have estimates by the moduli of continuity, i.e., estimates of the form

$$|R[f]| \le c(t)\omega_r(f,t). \tag{1.2}$$

It is well known that such estimates exist, but, in general, no estimates for the constants $c(t)$ are available (see, e.g., Esser [3] and Ivanov [4]). The aim of this paper is to obtain

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$c(t)$ are available such estimates in the non-periodic case, and to improve them in the periodic case. Section 2 deals with functionals on $C^s_{2\pi}$ and $L^s_{p,2\pi}$, Section 3 with functionals on $C^s[a,b]$, and Section 4 with the $\mathcal{K}$-functional.

2. Functionals on $\mathbf{C}^s_{2\pi}$ and $\mathbf{L}^s_{p,2\pi}$

Let $C^s_{2\pi}$ be the class of s-times continuously differentiable, real-valued functions with period 2π, and $L^s_{p,2\pi}$ the class of 2π-periodic, real-valued functions with absolutely continuous $(s-1)$-th and p-integrable s-th derivative, with

$$\|f\|_p^p = \int_0^{2\pi} |f(x)|^p \, dx \quad (1 \le p < \infty) \qquad \text{and} \qquad \|f\|_\infty = \sup_{0 \le x \le 2\pi} |f(x)|.$$

For convenience, we denote these classes by X^s_p, i.e., $X^s_\infty = C^s_{2\pi}$ and $X^s_p = L^s_{p,2\pi}$ for $1 \le p < \infty$. We consider functionals

$$R : X^s_p \to \mathbb{R}$$

satisfying

$$|R[f+g]| \le |R[f]| + |R[g]| \tag{2.1}$$

and

$$\|R\|_{i,p} = \sup \left\{ \frac{|R[f]|}{\|f^{(i)}\|_p} : f \in X^i_p, \|f^{(i)}\|_p \ne 0 \right\} < \infty \tag{2.2}$$

for $i = s, \dots, r$, for some $r > s \ge 0$. We therefore have

$$|R[f]| \le \|R\|_{i,p} \|f^{(i)}\|_p \qquad \text{for } f \in X^i_p, \quad i = s, \dots, r. \tag{2.3}$$

Further let

$$\omega_j(f,t)_p = \sup_{|h| \le t} \|\Delta^j_h f\|_p, \quad \text{where} \quad \Delta^j_h f(x) = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(x+ih),$$

be the j-th modulus of smoothness (for the properties of ω_j, see Schumaker [12: Chap. 2.8]). To simplify notation, we introduce the following abbreviations. Let

$$\lceil x \rceil = \min\{i \in \mathbb{Z} : i \ge x\}, \qquad \beta_r = \binom{r}{\lceil r/2 \rceil}$$

and

$$\rho_r = \int_0^{r/2} \lceil x \rceil^r N_r\left(x + \frac{r}{2}\right) dx = \sum_{i=1}^{\lceil \frac{r}{2} \rceil} i^r N_{r+1}\left(i + \frac{r}{2}\right) \le \frac{1}{2} \left\lceil \frac{r}{2} \right\rceil^r \tag{2.4}$$

($\rho_1 = 1/2$, $\rho_2 = 1/2$, $\rho_3 = 31/48$, $\rho_4 = 9/8$). Here, N_r is the B-spline of degree $r-1$ for the knots $0, 1, \dots, r$, satisfying

$$\int_0^r N_r(x) \, dx = 1. \tag{2.5}$$

Further, there holds

$$\int_{t-1}^{t} N_r(x)\,dx = N_{r+1}(t) \tag{2.6}$$

(Schoenberg [11: p. 12]; this has been used to obtain the second formula for ρ_r given in (2.4)), and

$$\int_0^r F^{(r)}(x+\alpha v)N_r(v)\,dv = \alpha^{-r}\Delta_\alpha^r F(x) \qquad \text{for } F \in X_1^r \tag{2.7}$$

and $\alpha > 0$ (Schumaker [12: p. 54]). The following lemma is well known (see, e.g., DeVore [2]), but we have modified the proof such that better constants are obtained.

Lemma 2.1. *Let $r \geq 1$ and $1 \leq p \leq \infty$. Then for all $f \in X_p^0$ and all $t > 0$, there exists a function $g \in X_p^r$ such that*

a) $\|f-g\|_p \leq 2\beta_r^{-1}\rho_r\omega_r(f,t)_p$ and $\|g^{(r)}\|_p \leq \beta_r^{-1}(2^r-\beta_r)t^{-r}\omega_r(f,t)_p$,

or

b) $\|f-g\|_p \leq \beta_r^{-1}\omega_r(f,t)_p$ and $\|g^{(r)}\|_p \leq \lceil\frac{r}{2}\rceil^r\delta_r t^{-r}\omega_r(f,t)_p$, where $\delta_1 = \delta_2 = 1$ and $\delta_r = 2$ for $r \geq 3$.

Proof. Let $\alpha_r = \lceil r/2\rceil$, and let g be defined by

$$\begin{aligned} g(x) &= \beta_r^{-1}\int_{-r/2}^{r/2}\Big(\beta_r f(x) + (-1)^{r-\alpha_r-1}\Delta_{vt}^r f(x-\alpha_r vt)\Big)N_r\Big(v+\frac{r}{2}\Big)\,dv \\ &= \beta_r^{-1}\sum_{\substack{i=0\\ i\neq\alpha_r}}^{r}\binom{r}{i}(-1)^{i-\alpha_r-1}((i-\alpha_r)t)^{-r}\Delta_{(i-\alpha_r)t}^r F\Big(x+\frac{r}{2}(\alpha_r-i)t\Big), \end{aligned}$$

by (2.7), where $F^{(r)} = f$.

a) Minkowski's inequality and the periodicity of f yield

$$\begin{aligned} \|f-g\|_p &\leq \beta_r^{-1}\int_{-r/2}^{+r/2}\|\Delta_{vt}^r f\|_p N_r\Big(v+\frac{r}{2}\Big)\,dv \\ &\leq \beta_r^{-1}\int_{-r/2}^{+r/2}\omega_r(f,|vt|)_p N_r\Big(v+\frac{r}{2}\Big)\,dv \\ &\leq \beta_r^{-1}\int_{-r/2}^{+r/2}\lceil|v|\rceil^r\omega_r(f,t)_p N_r\Big(v+\frac{r}{2}\Big)\,dv \\ &= 2\beta_r^{-1}\rho_r\omega_r(f,t)_p, \end{aligned} \tag{2.8}$$

while, by the triangle inequality,

$$\|g^{(r)}\|_p \leq \beta_r^{-1}\sum_{\substack{i=0\\ i\neq\alpha_r}}^{r}\binom{r}{i}|i-\alpha_r|^{-r}t^{-r}\|\Delta_{(i-\alpha_r)t}^r F^{(r)}\|_p$$

$$\begin{aligned} &\le \beta_r^{-1} t^{-r} \sum_{\substack{i=0 \\ i\ne\alpha_r}}^{r} \binom{r}{i} |i-\alpha_r|^{-r} \omega_r(f, |i-\alpha_r| t)_p \qquad (2.9) \\ &\le \beta_r^{-1} t^{-r} \sum_{\substack{i=0 \\ i\ne\alpha_r}}^{r} \binom{r}{i} \omega_r(f,t)_p \\ &= \beta_r^{-1} t^{-r} (2^r - \beta_r) \omega_r(f,t)_p. \end{aligned}$$

b) From (2.9), we also get

$$\|g^{(r)}\|_p \le \beta_r^{-1} t^{-r} \sum_{\substack{i=0 \\ i\ne\alpha_r}}^{r} \binom{r}{i} |i-\alpha_r|^{-r} \omega_r(f, \alpha_r t)_p \le \delta_r t^{-r} \omega_r(f, \alpha_r t)_p,$$

since $\beta_r^{-1} \sum_{\substack{i=0 \\ i\ne\alpha_r}}^{r} \binom{r}{i} |i-\alpha_r|^{-r} \le \delta_r$, by some elementary calculations, while (2.8) together with (2.5) gives

$$\|f-g\|_p \le \beta_r^{-1} \omega_r\left(f, \frac{r}{2}t\right)_p \le \beta_r^{-1} \omega_r(f, \alpha_r t)_p.$$

Replacing $u = \alpha_r t$ proves part b) ∎

From Lemma 2.1, we get the following estimates for $|R[f]|$ in terms of the moduli of smoothness.

Theorem 2.1. *Let $R : X_p^s \to \mathbb{R}$ be a functional satisfying (2.1) and (2.3), and let $t > 0$ and $1 \le j \le r-s$. Then*

$$|R[f]| \le \beta_j^{-1} \left(2\rho_j \|R\|_{s,p} + (2^j - \beta_j) t^{-j} \|R\|_{j+s,p}\right) \omega_j(f^{(s)}, t)_p \qquad \text{for } f \in X_p^s.$$

Proof. Let $F = f^{(s)} \in X_p^0$. By Lemma 2.1, there exists $G \in X_p^j$ such that

$$\|F-G\|_p \le c_1 \omega_j(F,t)_p \qquad \text{and} \qquad \|G^{(j)}\|_p \le c_2 \omega_j(F,t)_p. \qquad (2.10)$$

Now choose $g \in X_p^{j+s}$ with $g^{(s)} = G$. Then

$$\begin{aligned} |R[f]| &\le |R[f-g]| + |R[g]| \\ &\le \|R\|_{s,p} \|f^{(s)} - g^{(s)}\|_p + \|R\|_{j+s,p} \|g^{(j+s)}\|_p \\ &= \|R\|_{s,p} \|F-G\|_p + \|R\|_{j+s,p} \|G^{(j)}\|_p. \end{aligned}$$

Inserting the estimates from (2.10), together with the constants of Lemma 2.1a) (with r replaced by j) completes the proof ∎

Example 2.1. Let $E_n[f] = \inf_{g\in T_n} \|f-g\|_\infty$ be the error in the approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree lesser or equal n. By the Theorem of Favard-Achieser-Krein,

$$\|E_n\|_{r,\infty} = K_r (n+1)^{-r} \qquad \text{where } K_r = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \left(\frac{(-1)^\nu}{2\nu+1}\right)^{r+1} \le \frac{\pi}{2}$$

is Favard's constant. Choosing $t = 2/(n+1)$ in Theorem 2.1, we obtain

$$E_n[f] \le \frac{\pi c_j}{(n+1)^s}\omega_j\left(f^{(s)}, \frac{2}{n+1}\right)_\infty \qquad \text{for } f \in C_{2\pi}^s,$$

$s \ge 0$ and $j \ge 1$, where

$$c_j = \frac{1}{\beta_j}\left(\rho_j + \frac{1}{2}\right) - \frac{1}{2^{j+1}} \tag{2.11}$$

($c_1 = 3/4$, $c_2 = 3/8$, $c_3 = 23/72$ and $c_4 = 23/96$, but $c_j \to \infty$ for $j \to \infty$). The asymptotic behaviour of c_r is probably be given by

$$c_r \sim \frac{\sqrt{\pi r}}{2}\left(\frac{r}{4\sqrt{3e}}\right)^r \qquad \text{for } r \to \infty,$$

but we have not been able to prove this.

It can be shown that Theorem 2.1 remains true if $\|R\|_{i,p}$ is replaced by

$$|||R|||_{i,p} = \sup\left\{\frac{|R[f]|}{\|f^{(i)}\|_p} : f \in X_p^i, \|f^{(i)}\|_p \ne 0, \int_0^{2\pi} f^{(i)}(x)\,dx = 0\right\}$$

(obviously, $|||R|||_{i,p} \le \|R\|_{i,p}$, but with equality for $i \ge 1$ because of the periodicity of the functions considered). E.g., for quadrature errors

$$R[f] = \int_a^b f(x)\,dx - \sum_{i=1}^n a_i f(x_i)$$

there holds

$$\|R\|_{0,\infty} = b - a + \sum i = 1^n |a_i|, \quad \text{but} \quad |||R|||_{0,\infty} = \sum_{i=1}^n |a_i|$$

(for an example and some other details omitted here, see [6]).

3. Functionals on $C^s[a, b]$

Let $C^s[a,b]$ be the class of s-times continuously differentiable, real-valued functions on $[a,b]$, and $\|f\| = \sup|f(x)|$. We consider functionals $R : C^s[a,b] \to \mathbb{R}$ satisfying (2.1), and assume that, for $f \in C^i[a,b]$ and $i = s, \ldots, r$,

$$|R[f]| \le \|R\|_i \|f^{(i)}\| \tag{3.1}$$

holds, where $\|R\|_i$ is defined analogous to (2.2), and $\|R\|_i < \infty$. Further, let

$$\omega_j(f,t) = \sup\left\{|\Delta_h^j f(x)| : |h| \le t \text{ and } x, x+jh \in [a,b]\right\}$$

denote the j-th modulus of continuity of f. The proof of an analogue of Lemma 2.1 is more complicated, since f is not defined outside $[a,b]$. A standard method to overcome this difficulty is to extend f in a suitable way (see DeVore [2]), but then one has to know the constants related to this extension. For the case of the sup-norm considered in this section, this difficulties can be avoided by a modification of the step size of the differences involved in the proof (this does not work for the L_p-norms, $1 \le p < \infty$, which is the reason why we do not treat this case here). This modification is a useful tool to obtain explicit constants, when Steklov functions are applied to non-periodic functions. It was derived by the author some years ago for a first version of this paper, but afterwards, I discovered that it had already been used by Sendov in [9]; inserting $t = (b-a)/r^2$ in part b) of the following lemma, gives the estimate of Sendov. Let

$$\tau_r = \int_0^r \lceil x \rceil^r N_r(x)\,dx = \sum_{i=1}^r i^r N_{r+1}(i) \le r^r \tag{3.2}$$

($\tau_1 = 1$, $\tau_2 = 5/2$, $\tau_3 = 10$, $\tau_4 = 331/6$; the second formula for τ follows from (2.6)).

Lemma 3.1. *Let $f \in C[a,b]$, $r \ge 1$ and $t \in (0,(b-a)/r^2]$. Then there exists a function $g \in C^r[a,b]$ such that*

a) $\|f-g\| \le \tau_r \omega_r(f,t)$ and $\|g^{(r)}\| \le (2^r-1)t^{-r}\omega_r(f,t)$,

or

b) $\|f-g\| \le \omega_r(f,rt)$ and $\|g^{(r)}\| \le (r+1)t^{-r}\omega_r(f,rt)$.

Proof. Let

$$g(x) = \int_0^r \big(f(x) + (-1)^{r+1}\Delta_u^r f(x)\big) N_r(v)\,dv \qquad \text{with} \qquad u = vt - \frac{x-a}{b-a} rt.$$

Here, u and the restriction for t stated in the lemma have been chosen such that $x + iu \in [a,b]$ always (i.e., for $x \in [a,b]$ and $i = 0,\dots,r$).

a) We obtain

$$|f(x) - g(x)| \le \int_0^r \omega_r\Big(f, t\Big|v - \frac{x-a}{b-a} r\Big|\Big) N_r(v)\,dv. \tag{3.3}$$

For simplicity, let $a = 0$ and $b = 1$, and let

$$\phi(x) = \omega_r(f,tx) \qquad \text{and} \qquad \psi(x) = \int_0^r \phi(|v - rx|) N_r(v)\,dv.$$

From the monotonicity of $\omega_r(f,\cdot)$, it follows that ϕ is an increasing function, and the symmetry of N_r yields $\psi(1-x) = \psi(x)$. Using the symmetry properties of N_r and ψ, and the monotonicity of ϕ and of N_r on $[0,r/2]$, we obtain for $x \in [0,1/2]$

$$\begin{aligned}
\psi(0) - \psi(x) &= \int_0^{rx} (\phi(v) - \phi(rx - v)) N_r(v)\,dv + \int_{rx}^r (\phi(v) - \phi(v - rx)) N_r(v)\,dv \\
&\ge \int_0^{rx} (\phi(v) - \phi(rx - v)) N_r(v)\,dv \\
&= \int_0^{rx/2} (\phi(rx - v) - \phi(v))(N_r(rx - v) - N_r(v))\,dv \\
&\ge 0.
\end{aligned}$$

Because of the symmetry, the same holds for $x \in [1/2, 1]$, so that

$$\|f-g\| \le \psi(0) = \int_0^r \omega_r(f,tv)N_r(v)\,dv \le \int_0^r \lceil v\rceil^r N_r(v)\,dv\,\omega_r(f,t) = \tau_r\omega_r(f,t).$$

For the r-th derivative of g, we obtain, using (2.7),

$$\begin{aligned}
|g^{(r)}(x)| &= \left|\sum_{i=1}^{r}\binom{r}{i}(-1)^{1-i}(it)^{-r}\left(1-\frac{irt}{b-a}\right)^r \Delta_{it}^r f\left(x-\frac{x-a}{b-a}irt\right)\right| \\
&\le \sum_{i=1}^{r}\binom{r}{i}(it)^{-r}\omega_r(f,it) \qquad (3.4)\\
&\le \sum_{i=1}^{r}\binom{r}{i}t^{-r}\omega_r(f,t) \\
&= (2^r-1)t^{-r}\omega_r(f,t).
\end{aligned}$$

b) From (3.4), we also get

$$|g^{(r)}(x)| \le \sum_{i=1}^{r}\binom{r}{i}i^{-r}t^{-r}\omega_r(f,rt) \le (r+1)t^{-r}\omega_r(f,rt).$$

Further, $\|f-g\| \le \omega_r(f,rt)$, by (3.3) and (2.5) ∎

In the same way as Theorem 2.1, we obtain the following one.

Theorem 3.1. *Let $R : C^s[a,b] \to \mathbb{R}$ be a functional satisfying (2.1) and (3.1), and let $1 \le j \le r-s$ and $t \in (0,(b-a)/j^2]$. Then*

$$|R[f]| \le \left(\tau_j\|R\|_s + (2^j-1)t^{-j}\|R\|_{j+s}\right)\omega_j(f^{(s)},t) \qquad \textit{for } f \in C^s[a,b].$$

The estimate of Theorem 3.1 makes use of $\|R\|_s$ and $\|R\|_{j+s}$, which, however, may not be known. This difficulty can be overcome if $\|R\|_m$ and $\|R\|_r$ are known for some $m < s$ and some $r > j+s$, by using the estimate

$$\|R\|_i \le K_{i-m}\|R\|_m^{(r-i)/(r-m)}\left(\frac{\|R\|_r}{K_{r-m}}\right)^{(i-m)/(r-m)} \qquad (3.5)$$

which holds for $0 \le m \le i \le r$ (see Ligun [7] and Köhler [5]; the K_i are again Favard's constants).

Example 3.1. Let $R_n^G[f] = \int_{-1}^1 f(x)\,dx - Q_n^G[f]$ be the error of the n-point Gauss-Legendre quadrature formula. It is well known that

$$\|R_n^G\|_0 = 4 \qquad \text{and} \qquad \|R_n^G\|_{2n} = \frac{2^{2n+1}n!^4}{(2n+1)(2n)!^3}.$$

Using Stirling's and Wallis' formula, it can be shown that

$$t_{2n} := \left(\frac{\|R_n^G\|_{2n}}{\|R_n^G\|_0 K_{2n}}\right)^{1/(2n)} \le \frac{e}{4n}.$$

Using (3.5) to estimate $\|R_n^G\|_s$ and $\|R_n^G\|_{j+s}$ by $\|R_n^G\|_0$ and $\|R_n^G\|_{2n}$, and applying Theorem 3.1 with $r = 2n$ and $t = t_{2n}$, yields

$$|R_n^G[f]| \le 4\left(\frac{e}{4n}\right)^s \left(\tau_j K_s + (2^j - 1)K_{j+s}\right)\omega_j\left(f^{(s)}, \frac{e}{4n}\right) \qquad \text{for } f \in C^s[a,b],$$

$0 \le s < j + s \le 2n$ and $n \ge ej^2/8$.

Finally, let us shortly consider compound functionals, and state an estimate given by Sendov and Popov [10: p. 49] for the τ-moduli, in the framework of this paper. Let the function $R : C^s[0,1] \to \mathbb{R}$ satisfy (2.1) and (3.1), and let the N-compound functional $R_N : C^s[a,b] \to \mathbb{R}$ be defined by

$$R_N[f] = \frac{b-a}{N}\sum_{i=0}^{N-1} R\left[f\left(a + (i + \cdot\,)\frac{b-a}{N}\right)\right] \qquad \text{for } f \in C^s[a,b].$$

Theorem 3.2. *Let $1 \le j \le r - s$. Then*

$$|R_N[f]| \le 6\frac{(b-a)^{s+1}}{N^s}\|R\|_s\,\omega_j\left(f^{(s)}, \frac{b-a}{N(j+1)}\right) \qquad \text{for } f \in C^s[a,b].$$

Especially, this can be applied to compound quadrature rules. Better (and partly sharp) estimates by $\omega_r(f, \cdot\,)$ have been obtained by Büttgenbach, Lüttgens and Nessel [1] for the compound midpoint and the first four compound Newton-Cotes rules, using representations for the error by differences of order r.

4. Estimates for the $\mathcal{K}$-functional

a) Let us first consider the $\mathcal{K}$-functional of Peetre in the periodic case, i.e.,

$$\mathcal{K}_r(f,u)_p = \inf_{g \in X_p^r}\left(\|f-g\|_p + u\|g^{(r)}\|_p\right) \qquad \text{for } f \in X_p^0.$$

Specializing to g from Lemma 2.1a), and choosing $t = 2u$, yields the following theorem.

Theorem 4.1. *Let $1 \le p \le \infty$, $r \ge 1$ and $u > 0$. Then*

$$\mathcal{K}_r(f,u^r)_p \le 2c_r\omega_r(f,2u)_p \qquad \text{for } f \in X_p^0,$$

with c_r as in (2.11).

For $r = 1, 2$, this coincides with the estimate given by Maligranda [8]. The unmodified proof of Lemma 2.1, as given by DeVore [2], yields $\mathcal{K}_r(f, u^r)_p \leq (r^r + 1 - 2^{-r})\omega_r(f, 2u)_p$ (see also Maligranda [8]).

b) For the $\mathcal{K}$-functional in the non-periodic case,

$$\mathcal{K}_r(f, u) = \inf_{g \in C^r[a,b]} \left(\|f - g\| + u\|g^{(r)}\| \right) \qquad \text{for } f \in C[a, b],$$

choose, e.g., g from Lemma 3.1a) with $t = 2u$.

Theorem 4.2. *Let $r \geq 1$ and $0 < u \leq (b - a)/(2r^2)$. Then*

$$\mathcal{K}_r(f, u^r) \leq (\tau_r + 1 - 2^{-r})\omega_r(f, 2u) \qquad \textit{for } f \in C[a, b].$$

It is well known that $\mathcal{K}_r$ can be estimated by ω_r, but, as far as we know, no explicit constants have been given for $r > 2$.

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