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Asymptotic Formulas for Small Sessile Drops

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Abstract. We will prove asymptotic formulas for the wetted disk of a drop with small volume resting on a horizontal plane which is in a vertical gravity field. These formulas are generalizations of results of Finn. There is a non-uniformity in the asymptotic behaviour depending on whether the boundary contact angle is near π or not. If the contact angle is different from π we get a complete asymptotic expansion of the wetted disk in powers of the volume. These results are consequences of the strong non-linearity of the problem.

Keywords: *Capillarity, drops, asymptotic expansions*

AMS subject classification: Primary 76B45, secondary 41A60, 35J65, 35J70

1. Introduction

We consider a connected drop of liquid of volume V resting on a horizontal plane Π in a vertical gravity field g directed downward Π. We suppose the plane to be of homogeneous material so that the contact angle γ will be a constant, $0 < \gamma \leq \pi$.

Let S be the surface which defines the drop, N_Π the unit normal on S directed toward and N_S the unit normal on S directed into the fluid (see Fig.1.1). The surface S satisfies the following boundary value problem (see Finn [1] concerning the derivation of the equations and for historical remarks):

$$2H = \kappa u + \lambda \qquad \text{on} \quad S \tag{1.1}$$

$$N_S \cdot N_\Pi = \cos\gamma \qquad \text{on} \quad \bar{S} \cap \Pi. \tag{1.2}$$

Here H denotes the mean curvature of S, $\kappa > 0$ the capillary constant, λ some other constant (Lagrange multiplier) and u the height of S above Π.

The symmetry of S was proved by Serrin [8], provided S is a graph over Π, and by Wente [9] without this restriction. Because of Wente's result, we may restrict attention

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to axially symmetric drops (see Fig.1.1).

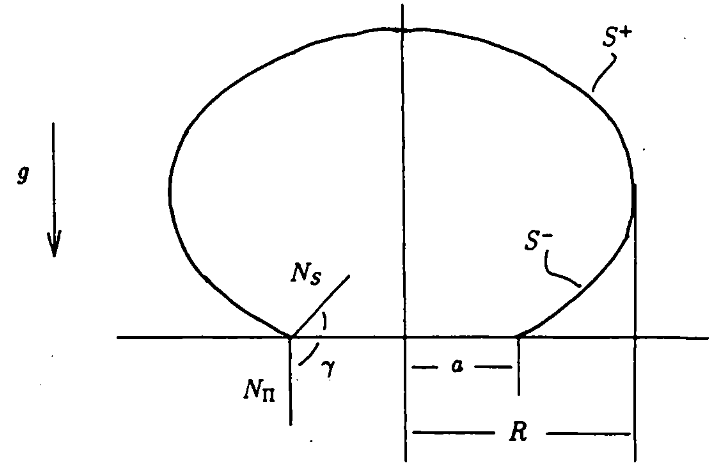

Fig. 1.1 Sessile drop

P.S. Laplace [5] gave a formal formula for the height of a large drop. No contribution to the analytical theory seems to have appeared prior to the paper [3] of Finn in 1980, in which upper and lower bounds for the height and for other quantities are given for small as well as for large drops.

The further discussion is based on the following two results due to Finn [3, 4].

Theorem 1.1. *There is a unique symmetric sessile drop with prescribed volume V, which meets Π in a prescribed constant contact angle γ, $0 < \gamma \leq \pi$. The solution curve (r, u) can be parametrizied by the inclination angle ψ, where $r(\psi)$ increases strictly in ψ, $0 < \psi \leq \pi/2$, and decreases strictly in ψ when $\psi > \pi/2$ and $u(\psi)$ increases strictly in ψ. Moreover, $r(\pi) = 0, u(\pi) = u_0$ and $a_\gamma = r(\pi - \gamma)$ is strictly positive (see Fig. 1.2).*

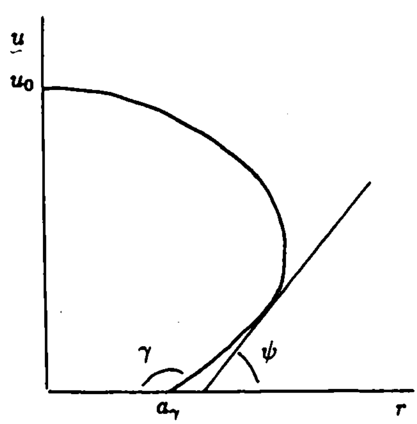

Fig. 1.2 Inclination angle

Theorem 1.2. *Let $R = r(\pi/2)$ for $\gamma \geq \pi/2$ and $R = r(\pi - \gamma)$ for $\gamma < \pi/2$. Then $R = O(V^{1/3})$ as the volume $V \to 0$, uniformly in γ, $0 < \gamma_0 \leq \gamma \leq \pi$.*

The first Theorem 1.1 shows that we can decompose S into two parts S^+, S^- which are graphs over the plane Π . Our proofs of the asymptotic formulas in the following sections are based on this decomposition. In the next sections we prove asymptotic formulas for (small) volumes V as $R \to 0$. At this point we need the above Theorem 1.2 of Finn which asserts that in fact R will be small for small V. This expansion implies expansions for R and for the radius $a = r(\pi - \gamma)$ of the wetted disk.

In particular, we are interested in asymptotic formulas of the type

$$a = F(\gamma, V) + O(V^\beta)$$

as $V \to 0$, where $F(\gamma, V)$ is an *explicitly* known expression and the remainder $O(V^\beta)$, $\beta > 0$, is *independent* of the boundary contact angle γ, $0 < \gamma_0 \leq \gamma \leq \pi$. This property is caused by the strong non-linearity of the problem. Thus, we can use these formulas for measurements of the unknown boundary contact angle γ.

We have shown such asymptotic formulas for the height rise of a fluid in a narrow capillary tube (see [6]). The method and results in this note depend strongly on the paper [6]. An inspection of the proofs in [7] shows that the remainder in the asymptotic expansion of the rise height in a narrow wedge is independent of the boundary contact angle, too.

In a later note we will make some asymptotic formulas more precise by calculation of the constants in the estimates for the remainder.

In this paper we obtain for the radius a of the wetted disk the existence of a complete asymptotic expansion uniform in γ when γ is in an interval (γ_0, γ_1) where $0 < \gamma_0 < \gamma_1 < \pi$:

$$a = \sum_{l=0}^{n} \beta_l(\gamma)\kappa^l V^{\frac{1}{3}(2l+1)} + R_{n+1} \tag{1.3}$$

if $V \leq V_0$, where $|R_{n+1}| \leq C V^{\frac{1}{3}(2n+3)}$, $C = C(\kappa, \gamma_0, \gamma_1, V_0)$. The constant $\beta_0(\gamma)$ is given by

$$\beta_0 = \left(\frac{\pi}{\sin^3 \gamma} \int_0^\gamma \sin^3 \theta \, d\theta \right)^{-1/3}.$$

Near $\gamma = \pi$ we will prove that

$$a = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3} \left(\frac{t}{2} + \sqrt{\frac{t^2}{4} + \kappa \frac{2}{3} \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}} \right) + R \tag{1.4}$$

if $t \equiv \sin\gamma \leq t_0$ and $V \leq V_0$, where $|R| \leq C(V + V^{2/3}t + V^{1/3}t^2)$, $C = C(\kappa, t_0, V_0)$. These formulas are generalizations of results of Finn [3, 4]. He proved that, for fixed γ with $0 < \gamma < \pi$,

$$a = \beta_0(\gamma) V^{1/3} + o(V^{1/3})$$

and, if $\gamma = \pi$, then

$$a = \left(\frac{2}{3}\kappa\right)^{1/2} \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} + o(V^{2/3})$$

as $V \to 0$.

2. Drops without an overhang

In this section we assume that the contact angle satisfies the inequalities

$$0 < \gamma \leq \pi/2 . \tag{2.1}$$

According to Theorem 1.1 there exists an R for each given V and because of (2.1) the surface $S: \; z = u(x)$ is a graph over the supporting plane Π (see Figure 2.1).

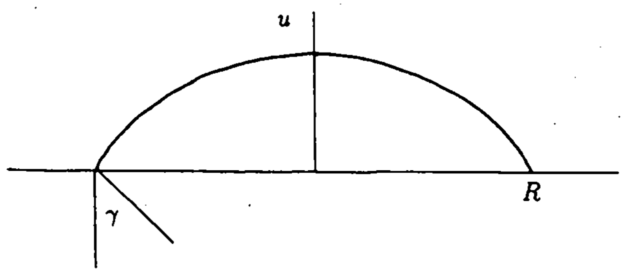

Fig. 2.1 Drop without overhang

Thus we can write the boundary value problem (1.1) - (1.2) as

$$\begin{aligned} divTu &= \kappa u + \lambda \quad &\text{in} \quad B_R \\ \nu \cdot Tu &= -\sin\gamma \quad &\text{on} \quad \partial B_R \end{aligned}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ,$$

$B_R = \{x \in \mathbf{R}^2 : x_1^2 + x_2^2 < R\}$ and ν is the outer normal at ∂B_R . After the mappings

$$u = v - \frac{\lambda}{\kappa} \quad \text{and} \quad v = Rw\left(\frac{x}{R}\right)$$

the above problems change to

$$divTw = Bw \quad \text{in} \quad B_1 \tag{2.2}$$

$$\nu \cdot Tw = -\sin\gamma \quad \text{on} \quad \partial B_1 . \tag{2.3}$$

Here $B = \kappa R^2$ is the *Bond number* and u is given by

$$u = Rw\left(\frac{x}{R}\right) - \frac{\lambda}{\kappa} \tag{2.4}$$

provided w is a solution of the boundary value problem (2.2) - (2.3).

There exists a complete asymptotic expansion of w in powers of B. More precisely, we have shown in [6] the following

Lemma 2.1. *For each non-negative integer n there exist n radially symmetric functions $\psi_l(x,\gamma)$ analytic in B_1 and bounded on the closed disk, such that*

$$w(x,\gamma,B) = -\frac{2\sin\gamma}{B} + \sum_{l=0}^{n} \psi_l(x,\gamma)B^l + O(B^{n+1})$$

holds uniformly in B_1 and γ, $0 < \gamma \le \pi/2$, as $B \to 0$.

The crucial point here is that the remainder is uniformly bounded with respect to γ in $0 < \gamma \le \pi/2$ despite the fact that $|\nabla w|$ becomes unbounded if $\gamma \to \pi/2$. The reason for this behaviour is the strong non-linearity of the problem. The proof of Lemma 2.1 is based on a maximum principle of Concus and Finn [1]. The functions ψ_l can be calculated by using the formulas in [6]. In particular, one obtaines for ψ_0 the upper hemisphere

$$\psi_0(x,\gamma) = -\frac{2(1-\cos^3\gamma)}{3\sin^3\gamma} + \frac{1}{\sin\gamma}\sqrt{1-r^2\sin^2\gamma}\ ,$$

where $r = \sqrt{x_1^2 + x_2^2}$.

From Theorem 1.2 we see that the asymptotic expansion in Lemma 2.1 makes sense for sufficiently small volume of the drop since R is then small and hence also $B = \kappa R^2$. Formula (2.4) and Lemma 2.1 imply

$$u(x,\gamma,R) = R\left(-\frac{2\sin\gamma}{B} + \sum_{l=0}^{n} \psi_l(x,\gamma)B^l + O(B^{n+1})\right) - \frac{\lambda}{\kappa}\ .$$

Since $u(R,\gamma,R) = 0$, we have

$$u(x,\gamma,R) = \sum_{l=0}^{n} \left(\psi_l\left(\frac{x}{R},\gamma\right) - \psi_l(1,\gamma)\right)\kappa^l R^{2l+1} + O(R^{2n+3})\ .$$

From this expansion we obtain for the volume

$$V = \int_{\Omega_R} u(x,\gamma,R)\,dx$$

the asymptotic expansion

$$V = \sum_{l=0}^{n} c_l(\gamma)\kappa^l R^{2l+3} + O(R^{2n+5}) \tag{2.5}$$

uniformly in γ, $0 < \gamma \le \pi/2$, as $R \to 0$. The remainder depends on the capillary constant κ. In particular, one has

$$c_0(\gamma) = \frac{\pi}{\sin^3\gamma}\int_0^{\gamma} \sin^3\theta\,d\theta$$

and $c_0(\pi/2) = \frac{2}{3}\pi$; $c_1(\pi/2) = -\frac{1}{6}\pi$.

From (2.5) and an inverse function lemma for asymptotic expansions one obtaines the asymptotic expansion of R in powers of V :

$$R = \sum_{l=0}^{n} \alpha_l(\gamma)\kappa^l V^{\frac{1}{3}(2l+1)} + O\left(V^{\frac{1}{3}(2n+3)}\right) \tag{2.6}$$

uniformly in γ, $0 < \gamma_0 \le \gamma \le \pi/2$, as $V \to 0$, where, for example,

$$\alpha_0(\gamma) = c_0(\gamma)^{-1/3} \quad \text{and} \quad \alpha_1(\gamma) = -\frac{1}{3}c_0(\gamma)^{-2}c_1(\gamma) .$$

3. Drops with an overhang

In this section we are interested in asymptotic expansions analogous to those of the previous section, when the boundary contact angle γ satisfies

$$\frac{\pi}{2} < \gamma < \pi . \tag{3.1}$$

The question as to what happens if $\gamma \to \pi/2$ will be discussed in Subsection 3.3 (the other borderline case $\gamma \to \pi$ is considered in Section 4).

According to Theorem 1.1 the drop has an overhang if the boundary contact angle satisfies (3.1); the free surface S can then be decomposed into two graphs S^+ and S^- over the supporting plane Π. Let u^+ and u^- be the graphs of S^+ and S^-, respectively. Then

$$\begin{aligned} \mathrm{div}Tu^+ &= \kappa u^+ + \lambda \quad &\text{in} \quad |x| < R \\ \nu \cdot Tu^+ &= -1 \quad &\text{on} \quad |x| = R \end{aligned}$$

and

$$\begin{aligned} -\mathrm{div}Tu^- &= \kappa u^- + \lambda \quad &\text{in} \quad a < |x| < R \\ \nu \cdot Tu^- &= 1 \quad &\text{on} \quad |x| = R \\ \nu \cdot Tu^- &= \sin\gamma \quad &\text{on} \quad |x| = a. \end{aligned}$$

After the mapping

$$u^{\pm} = Rw^{\pm}\left(\frac{x}{R}\right) - \frac{\lambda}{\kappa} \tag{3.2}$$

we are led to the following boundary value problems where we again denote x/R by x:

$$\mathrm{div}Tw^+ = Bw^+ \quad \text{in} \quad |x| < 1 \tag{3.3}$$

$$\nu \cdot Tw^+ = -1 \quad \text{on} \quad |x| = 1 \tag{3.4}$$

and

$$\mathrm{div}Tw^- = -Bw^- \quad \text{in} \quad q < |x| < 1 \tag{3.5}$$

$$\nu \cdot Tw^- = \sin\gamma \quad \text{on} \quad |x| = q \tag{3.6}$$

$$\nu \cdot Tw^- = 1 \quad \text{on} \quad |x| = 1. \tag{3.7}$$

Here $B = \kappa R^2$ is the Bond number and q is defined by $q = a/R$ (see Fig.3.1).

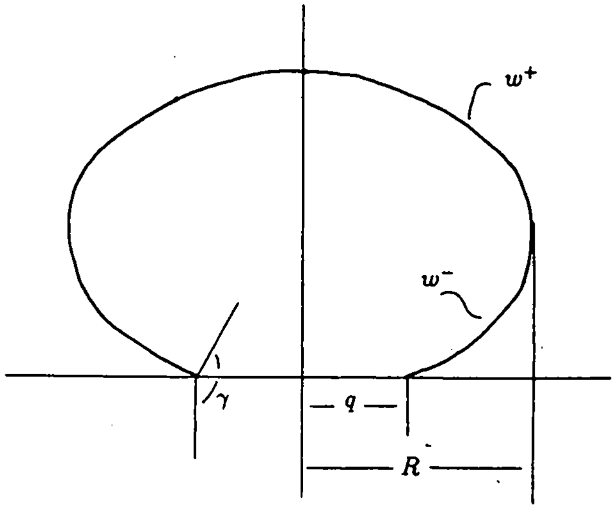

Fig. 3.1 Transformed drop

We split the proof of the existence of the asymptotic expansion into seven steps:

Step 1. There is an asymptotic expansion $w^+ = w_n^+ + O(B^{n+1})$ of w^+ as $B \to 0$.

Step 2. There is an *approximate solution* w_n^- in the sense that

$$\begin{aligned} \mathrm{div} T w_n^- &= -B w_n^- + O(B^{n+1}) && \text{in} \quad p < |x| < 1 \\ \nu \cdot T w_n^- &= 1 && \text{on} \quad |x| = 1 \\ \nu \cdot T w_n^- &= \sin\gamma && \text{on} \quad |x| = p \end{aligned}$$

for a given p, $0 < p < 1$.

Step 3. Determine a $p = q_n$ such that $w_n^+ = w_n^-$ on $|x| = 1$.

Step 4. Show that $w^- = w_n^- + O(B^{n+1})$ on $q_n - \epsilon < |x| < 1$ for an $\epsilon > 0$.

Step 5. Prove that $\nu \cdot Tw^- = \nu \cdot Tw_n^- + O(B^{n+1})$ on $q_n - \epsilon < |x| < 1$.

Step 6. Asymptotic expansion $q = q_n + O(B^{n+1})$ of the wetted disk.

Step 7. Asymptotic expansion of V in powers of B and, finally, expansions for a and R in powers of $V^{1/3}$.

3.1 Asymptotic expansion of w^+. The following lemma was shown in the paper [6] on the asymptotic expansion of the hight rise in a circular capillary tube.

Lemma 3.1. *The solution w^+ of problem (3.3) - (3.4) satisfies*

$$w^+(x, B) = w_n^+(x, B) + O(B^{n+1})$$

in $|x| < 1$ as $B \to 0$, where

$$w_n^+(x, B) = -\frac{2}{B} + \sum_{l=0}^{n} \psi_l(r) B^l ,$$

$r = \sqrt{x_1^2 + x_2^2}$. *The functions $\psi_l(r)$ are analytic in $|x| < 1$ and bounded on the closed region.*

In particular, one has

$$\psi_0(r) = -\frac{2}{3} + \sqrt{1 - r^2} \qquad \text{and} \qquad \psi_1(r) = \frac{1}{6} - \frac{1}{3} ln\left(1 + \sqrt{1 - r^2}\right)$$

and, for the boundary behaviour of ψ_l $(l \geq 1)$,

$$\sup |\psi_l(r)| < \infty$$
$$\sup \left((1 - r)^{1/2} |\psi_l'(r)|\right) < \infty$$
$$\sup \left((1 - r)^{3/2} |\psi_l''(r)|\right) < \infty$$

where the supremum is taken over $0 < r < 1$.

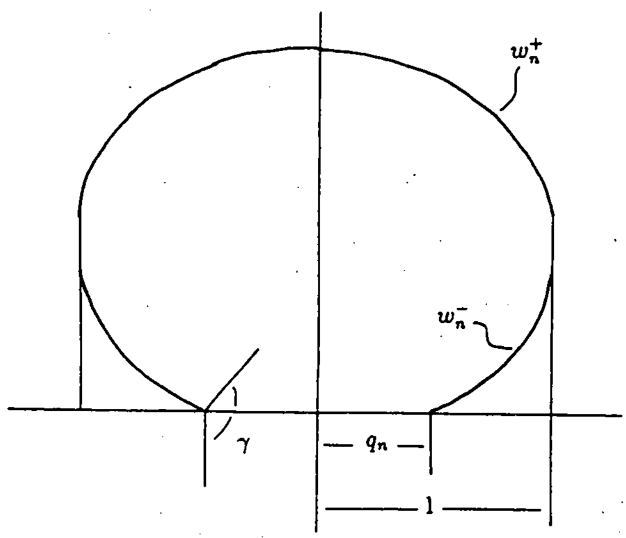

Fig. 3.2 Approximate drop

3.2 Approximate solution for w^-. In this subsection we will show that an approximate solution to the boundary value problem (3.5) - (3.6) exists in the sense of the following definition.

Definition. A function w_n^- is said to be an *approximate solution* to w^- if $w_n^- \in C^\infty$ on $p < |x| < 1$ for a given p in $0 < p < 1$, and

$$\text{div} T w_n^- = -B w_n^- + O(B^{n+1})$$

holds uniformly in x on $p < |x| < 1$ as $B \to 0$, and

$$\lim_{|x| \to 1} \nu \cdot T w_n^- = 1$$

$$\nu \cdot T w_n^- = \sin \gamma \qquad if \ |x| = p .$$

By the same method as in [6] one obtains the existence of such an approximate solution. The crucial point in this result is that the remainder $O(B^{n+1})$ remains bounded if $r \to 1$. The reason for this behaviour is the strong non-linearity of the boundary value problem (3.5) - (3.6). In contrast to Section 2 we have here an additional dependence on p. Therefore we have to pay attention to the limits $\gamma \to \pi/2$ and $\gamma \to \pi$ (see Subsection 3.3 and Section 4).

Lemma 3.2. *For each t, $0 < t < 1$, and for each non-negetive integer n there exist a positive constant η and $n+1$ functions $\varphi_l(r,p,t)$, $t \equiv \sin\gamma$, $l = 0, 1, \dots, n$, analytic in $t - \eta \le r < 1$ and bounded on the closed interval, such that*

$$w_n^-(r,p,t,B) = -\frac{C_{-1}(p,t)}{B} + \sum_{l=0}^{n}(-1)^l \varphi_l(r,p,t)B^l$$

defines an approximate solution to w^-, where

$$C_{-1}(p,t) = \frac{2(1-pt)}{1-p^2} \; .$$

Proof. Let

$$v = \frac{C}{B} + \sum_{l=0}^{n} \phi_l(r)B^l$$

be an approximate solution. Then the functions ϕ_l satisfy a recurrent system of boundary value problems. This follows from the expansion

$$\begin{aligned} r\,\mathrm{div}Tv &= \left(\frac{rv'}{\sqrt{1+v'^2}}\right)' \\ &= \frac{d}{dr}\left(rv'(1+\phi_0'^2)^{-1/2}\left(1 + \frac{2\phi_0'\sum_{l=1}^{n}\phi_l'B^l + \left(\sum_{l=1}^{n}\phi_l'B^l\right)^2}{1+\phi_0'^2}\right)^{-1/2}\right) \\ &= \left(\frac{r\phi_0'}{\sqrt{1+\phi_0'^2}}\right)' + \sum_{k=1}^{n}\left(\frac{r\phi_k'}{(1+\phi_0'^2)^{3/2}}\right)' B^k + \\ &\quad + \sum_{k=2}^{n}\left(rf_k(\phi_0',\dots,\phi_{k-1}')\right)' B^k + \left(r\tilde{f}_{n+1}(B,\phi_0',\dots,\phi_n')\right)' B^{n+1}. \end{aligned} \tag{3.8}$$

From this calculation we see that f_k and $\tilde{f}_{n+1}$ are bounded on $p - \eta < r < 1$, provided for $l = 0, 1, \dots, n$ the inequalities

$$\sup \frac{|\phi_l'|}{(1+\phi_0'^2)^{1/2}} < \infty \tag{3.9}$$

are satisfied, where the supremum is taken over r in $p - \eta < r < 1$. Moreover, if we assume

$$\sup \frac{|\phi_l''|}{(1+\phi_0'^2)^{3/2}} < \infty \; , \tag{3.10}$$

then f'_k and f'_{n+1} are also bounded. This follows from the formulas

$$
\begin{aligned}
r\,\mathrm{div}Tv &= \left(\frac{rv'}{\sqrt{1+v'^2}}\right)' \\
&= v'(1+v'^2)^{-1/2} + rv''(1+v'^2)^{-3/2} \\
&= \frac{v'}{\sqrt{1+\phi_0'^2}}\left(1+\frac{2\phi_0'\sum_{l=1}^n \phi_l' B^l + \left(\sum_{l=1}^n \phi_l' B^l\right)^2}{1+\phi_0'^2}\right)^{-1/2} \\
&\quad + \frac{rv''}{(1+\phi_0'^2)^{3/2}}\left(1+\frac{2\phi_0'\sum_{l=1}^n \phi_l' B^l + \left(\sum_{l=1}^n \phi_l' B^l\right)^2}{1+\phi_0'^2}\right)^{-3/2} .
\end{aligned}
\tag{3.11}
$$

The expansion (3.8) and a similar expansion for $\nu\cdot Tv$ show that v defines an approximate solution to problem (3.5) - (3.7) if the functions ϕ_l ($l = 0, 1, \ldots, n$) satisfy (3.9) - (3.10),

$$\sup|\phi_l| < \infty \tag{3.12}$$

and if

$$\varphi_l = (-1)^l \phi_l$$

are solutions of the following recurrent system (3.13) - (3.18) of boundary value problems:

$$\left(\frac{r\varphi_0'}{\sqrt{1+\varphi_0'^2}}\right)' = -Cr \tag{3.13}$$

on $p < r < 1$ with the boundary conditions

$$\frac{\varphi_0'(p)}{\sqrt{1+\varphi_0'(p)^2}} = \sin\gamma \qquad \text{and} \qquad \lim_{r\to 1}\frac{\varphi_0'(r)}{\sqrt{1+\varphi_0'(r)^2}} = 1\,, \tag{3.14}$$

$$\left(\frac{r\varphi_1'}{(1+\varphi_0'^2)^{3/2}}\right)' = r\varphi_0 \tag{3.15}$$

on $p < r < 1$ with

$$\varphi_1'(p) = 0 \qquad \text{and} \qquad \lim_{r\to 1}\frac{\varphi_1'}{(1+\varphi_0'^2)^{3/2}} = 0\,, \tag{3.16}$$

and, for $2 \le k \le n$,

$$\left(\frac{r\varphi_k'}{(1+\varphi_0'^2)^{3/2}}\right)' + (rf_k)' = r\varphi_{k-1} \tag{3.17}$$

on $p < r < 1$ with the boundary conditions

$$\varphi_k'(p) = \qquad \text{and} \qquad \lim_{r\to 1}\frac{\varphi_k'}{(1+\varphi_0'^2)^{3/2}} = 0\,. \tag{3.18}$$

From the boundary value problem (3.13) - (3.14) we conclude that

$$C = C_{-1} = \frac{2(1-pt)}{1-p^2}\ , \quad t \equiv \sin\gamma \tag{3.19}$$

and

$$\varphi_0'(r,p,t) = \frac{1-p^2-(1-r^2)(1-pt)}{\left(r^2(1-p^2)^2-(1-p^2-(1-r^2)(1-pt))^2\right)^{1/2}}\ . \tag{3.20}$$

We define

$$\tilde{\varphi}_0(r,p,t) = -\int_r^1 \varphi_0'(s,p,t)\,ds\ . \tag{3.21}$$

where φ_0' is given by (3.20). Then

$$\varphi_0 = \tilde{\varphi}_0 + C_0 \tag{3.22}$$

with

$$C_0(p,t) = -\frac{2}{1-p^2}\int_p^1 s\tilde{\varphi}_0(s,p,t)\,ds\ . \tag{3.23}$$

This follows from the boundary value problem (3.15) - (3.16). We continue this procedure and obtain from (3.15) - (3.16)

$$\varphi_1' = -r^{-1}(1+\varphi_0'^2)^{3/2}\int_r^1 s\varphi_0(s,p,t)\,ds\ . \tag{3.24}$$

Set

$$\tilde{\varphi}_1 = -\int_r^1 \varphi_1'(s,p,t)\,ds. \tag{3.25}$$

Then

$$\varphi_1 = \tilde{\varphi}_1 + C_1 \tag{3.26}$$

with

$$C_1(p,t) = -\frac{2}{1-p^2}\int_p^1 s\tilde{\varphi}_1(s,p,t)\,ds\ . \tag{3.27}$$

For $k \geq 2$ we have the formulas

$$\varphi_k' = -r^{-1}(1+\varphi_0'^2)^{3/2}\left(\int_r^1 (rf_k)'\,ds + \int_r^1 s\varphi_{k-1}(s,p,t)\,ds\right) \tag{3.28}$$

$$\tilde{\varphi}_k = -\int_r^1 \varphi_k'(s,p,t)\,ds \tag{3.29}$$

$$\varphi_k = \tilde{\varphi}_k + C_k \tag{3.30}$$

$$C_k(p,t) = -\frac{2}{1-p^2}\int_p^1 s\tilde{\varphi}_k(s,p,t)ds. \tag{3.31}$$

The functions f_{lk} are analytic in their arguments, and

$$f_{lk}(\zeta_0, \zeta_1, \cdots, \zeta_l) = 0 \tag{3.32}$$

if $\zeta_m = 0$ for at least one $m \geq 1$ (see the expansion (3.8)). Let t_0, t_1 be fixed positive constants such that $0 < t_0 < t_1 < 1$. We consider here all t with $t_0 \leq t \leq t_1$. From the formula (3.20) one concludes that there exists a constant $\eta = \eta(t_0, t_1) > 0$ such that, for p with $|t - p| < \eta$ and r from the interval $t - \eta < r < 1$,

$$c_1(1-r)^{-1/2} \leq \varphi_0'(r, p, t) \leq c_2(1-r)^{-1/2} \ , \tag{3.33}$$

$$|\varphi_0''| \leq c_3(1-r)^{-3/2} \tag{3.34}$$

with positive constants c_l depending only on t_1 and t_2. Because of (3.33) the function $\tilde{\varphi}_0$ is well defined by (3.21) and (see (3.21) - (3.23)) we have with a constant depending only on t_0 and t_1

$$|\varphi_0| < c\ . \tag{3.35}$$

These and the following inequalities in this subsection hold on the interval $t - \eta < r < 1$. From (3.34) and (3.33) we conclude

$$|\varphi_1'| \leq c(1-r)^{-1/2}\ . \tag{3.36}$$

For the second derivative we obtain from (3.24) - (3.26) the inequality

$$|\varphi_1''| \leq c(1-r)^{-3/2}\ . \tag{3.37}$$

Equation (3.28) implies

$$\frac{\varphi_2'}{(1+\varphi_0'^2)^{3/2}} = O(1-r)$$

as $r \to 1$ since f_k' remains bounded if $r \to 1$. This last property follows from the formula (3.11) and from the inequalities (3.33) - (3.34) and (3.36) - (3.37). Then we obtain by induction for $l = 1, \ldots, n$ the inequalities

$$|\varphi_l| \leq c \tag{3.38}$$

$$|\varphi_l'| \leq c(1-r)^{-1/2} \tag{3.39}$$

$$|\varphi_l''| \leq c(1-r)^{-3/2}. \tag{3.40}$$

Consequently, according to our previous discussion, the function w_n^- of Lemma 3.2 defines an approximate solution if the boundary condition

$$\lim_{|x| \to 1} \nu \cdot Tv = 1 \tag{3.41}$$

holds. The proof follows from the expansion

$$\nu \cdot Tv = \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} + \sum_{k=1}^{n} \frac{\varphi_k'}{(1+\varphi_0'^2)^{3/2}} B^k + \sum_{k=2}^{n} f_k B^k + \tilde{f}_{n+1} B^{n+1} \tag{3.42}$$

where the functions f_k, $\tilde{f}_{n+1}$ are bounded on the interval $p - \eta < r < 1$ and $\tilde{f}_{n+1}$ are uniformly bounded with respect to $B \leq B_0$. Thus, there exists a subsequence $r_i \to 1$ such that $f_k(r_j) \to \hat{f}_k$. Set $k = 1$ in (3.42). Then

$$0 = \sum_{k=2}^{n} \hat{f}_k B^k + O(B^{n+1})$$

as $B \to 0$. Hence $\lim_{r \to 1} f_k(r) = 0$ for $k = 2, \ldots, n$. Then (3.42) implies that $\lim_{r \to 1} \tilde{f}_{n+1}(r, B) = 0$ holds. Induction over n completes the proof of equation (3.41) ∎

We will need the next lemma for an estimate of $w^- - w_n^-$ in Subsection 3.4. The first part is a direct consequence of the previous Lemma 3.2.

Lemma 3.3. *There exist positive constants r_0, c_0, c_1 such that on $r_0 < r < 1$*

$$c_0(1-r)^{-1/2} \le w_{n,r}^- \le c_1(1-r)^{-1/2} \tag{3.43}$$

$$c_0(1-r)^{-1/2} \le w_r^- \le c_1(1-r)^{-1/2}. \tag{3.44}$$

Proof. The inequalities (3.43) follow from Lemma 3.2 in conjunction with the estimates (3.33), (3.36) and (3.38) - (3.40). We derive the second inequality as follows. From

$$\left(\frac{rw_r^-}{\sqrt{1+(w_r^-)^2}}\right)' = -Brw^-(r)$$

it follows

$$\frac{w_r^-(r)}{\sqrt{1+(w_r^-(r))^2}} = \frac{1}{r}\left(1 + B\int_r^1 sw^-(s)\,ds\right) \equiv G(r)\ .$$

Thus

$$w_r^-(r) = \frac{G(r)}{\sqrt{1-G^2(r)}}\ .$$

We have

$$1 - G^2 = 2G'(1)(1-r) + o(1-r) \tag{3.45}$$

as $r \to 1$, and from the definition of $G(r)$ we will see that

$$G'(1) = 1 + O(B) \tag{3.46}$$

as $B \to 0$. Thus the inequalities hold. Concerning (3.46), we obtain from the definition of G that $G(r) + rG'(r) = -Brw^-(r)$. This implies $G'(1) = -1 - Bw^-(1)$. Since $w^-(1) = w^+(1)$ and $w^+(1) = -2/B + O(1)$ as $B \to 0$ (see Lemma 3.1) we obtain formula (3.46) ∎

3.3 Approximation of the wetted disk. Now we determine a p near $t \equiv \sin\gamma$ such that $w_n^+ = w_n^-$ on $|x| = 1$. According to Lemma 3.1 and Lemma 3.2 p has to be a solution of the equation

$$-\frac{2}{B} + \sum_{l=0}^{n} \psi_l(1)B^l = -\frac{C_{-1}(p,t)}{B} + \sum_{l=0}^{n}(-1)^l\varphi_l(1,p,t)B^l\ .$$

Because of the definition of $C_l(p,t)$ (compare Subsection 3.2) we have $C_l = \varphi_l(1,p,t)$, $l \ge 0$. We recall that $C_{-1} = 2(1-pt)(1-p^2)^{-1}$. Set $A_l = \psi_l(1)$ and

$$f(p,t,B) = 2p^2 - 2pt - \sum_{l=0}^{n}\Big(-(1-p^2)A_l + (-1)^l(1-p^2)C_l(p,t)\Big)B^{l+1}. \tag{3.47}$$

Since $f(t,t,0) = 0$, $f_p(t,t,0) = -2t$ and $C_k(p,t)$ is in C^∞ in a neighbourhood of $p = t$ we have

$$p = p(t,B) = t + \sum_{l=1}^{n+1} a_l(t)B^l + O(B^{n+2})$$

as a solution of (3.47). The properties of $a_l(t)$ and of the remainder $R(t,B)$ as $t \to 1$ listed in the following lemma follow after some calculations by using the formulas (3.20) - (3.31).

Lemma 3.4. *For any fixed constant t_0, $0 < t_0 < 1$, there exist constants $a_l(t)$ such that, for $p = q_n(t,B)$ defined by*

$$q_n = t + \sum_{l=1}^{n+1} a_l(t)B^l + R(t,B),$$

the equation $w_n^+ = w_n^-$ hold on $|x| = 1$ as $B \to 0$, where

$$a_1(t) = \frac{1}{3t}(1-t^2)\left(1 + (1-t^2)^{1/2}\right) \tag{3.48}$$

$a_l(t) = O(1-t)$, $1 \le l \le n$, and $R(t,B) = O\left((1-t)B^{n+2}\right)$. The functions a_l are independent of t_0. If

$$p = Q_{n+1} = t + \sum_{l=1}^{n+1} a_l(t)B^l,$$

then $w_n^+ = w_n^- + O(B^{n+1})$.

Concerning the behaviour if $t \to 1$ we obtain from the formulas (3.19) - (3.32) after some calculations the following

Lemma 3.5. *For any given t_0, $0 < t_0 < 1$, there exists a B_0 such that for $\eta = -2a_1(t)B$, $0 < B < B_0$, where a_1 is defined by (3.48), the inequalities (3.33) - (3.40) and therefore (3.9) - (3.10) and (3.12) hold on the interval $(t - \eta(t), 1)$ uniformly in t, $t_0 < t < 1$.*

One important consequence of this lemma is that for $p = Q_{n+1}$ the remainder $O(B^{n+1})$ in the definition of an approximate solution (see Subsection 3.2) is independent of t and therefore it remains bounded if $t \to 1$.

3.4 Estimate of $w^- - w_n^-$. On $|x| = 1$ we have according to Lemma 3.1

$$w_n^+ = w^+ + O(B^{n+1})$$

and (see Lemma 3.4)

$$w_n^- = w_n^+ + O(B^{n+1}),$$

when we choose $p = q_n$, where q_n is defied in Lemma 3.4. Combining these equations with $w^+ = w^-$ on $|x| = 1$, we find that

$$w_n^- = w^- + O(B^{n+1}) \tag{3.49}$$

holds on $|x| = 1$. Let $\bar{q} = \bar{q}(B)$, such that $w_r^-(r,t,B) = 0$ at $r = \bar{q}$ and $w_r^-(r,t,B) > 0$ if $\bar{q} < r < 1$. According to a result of Finn (see Theorem 1.1) there exists such a $\bar{q} > 0$, even if $t = 0$. From (3.49) we obtain the following estimate, where $m = max(\bar{q}, t - \eta)$.

Lemma 3.6. *There exists a constant $\eta > 0$ such that*

$$|w^-(r,t,B) - w_n^-(r,Q_{n+1};t,B)| \le cB^{n+1}$$

for $r \in [m,1]$ and $0 < B \le B_0$, B_0 small enough, where $Q_{n+1} = t + \sum_{l=1}^{n+1} a_l(t)B^l$ (see Lemma 3.4).

Proof. We have $\operatorname{div} Tw^- = -Bw^-$ on $(\bar{q},1)$ and from Lemma 3.2 $\operatorname{div} Tw_n^- = -Bw_n^- + O(B^{n+1})$ on $(t-\eta,1)$. Thus

$$\operatorname{div} Tw^- - \operatorname{div} Tw_n^- = -B(w^- - w_n^-) + O(B^{n+1})$$

on $(m,1)$, or

$$\left(\frac{rw_r^-}{\sqrt{1+(w_r^-)^2}} - \frac{rw_{n,r}^-}{\sqrt{1+(w_{n,r}^-)^2}}\right)' = rF(r) \ , \tag{3.50}$$

where $F(r) \equiv -B(w^- - w_n^-) + O(B^{n+1})$. We write (3.50) as

$$(r\rho(r)z')' = rF(r) \tag{3.51}$$

on $(m,1)$, where $z = w^- - w_n^-$ and $\rho(r) = (1+\zeta^2)^{-3/2}$ with $\zeta = w_r^- + \epsilon(w_{n,r}^- - w_r^-)$, $0 < \epsilon < 1$. From Lemma 3.3 we obtain

$$\rho(r) = \rho_0(r)(1-r)^{3/2} \ , \tag{3.52}$$

where $c_0 \le \rho_0(r) \le c_1$ on $m \le r \le 1$ with positive constants c_0 and c_1. Integration of equation (3.51) and the boundary conditions on $r = 1$ yield

$$-r\rho(r)z'(r) = \int_r^1 sF(s)\,ds \tag{3.53}$$

on $(m,1)$. Set

$$g(r) = \int_r^1 sF(s)\,ds.$$

Then we conclude from the definition of F that

$$|g(r)| \le \int_r^1 \left(Bs|w^- - w_n^-| + O(B^{n+1})\right) ds \le \left(B \ max\ |z| + O(B^{n+1})\right)(1-r)$$

where the maximum is taken over $[m,1]$. From (3.53) we have

$$z'(r) = -\frac{g(r)}{r\rho(r)}, \tag{3.54}$$

and since $r > t - \eta > 0$ for a given sufficiently small η, we obtain from (3.52)

$$|z'| \le c(1-r)^{-1/2}\left(B \ \max|z| + O(B^{n+1})\right) \ .$$

This inequality implies

$$|z(r)| \le |z(1)| + cB \ \max|z| + O(B^{n+1}).$$

Since $z(1) = O(B^{n+1})$ the assertion of Lemma 3.6 follows ∎

3.5 Estimate of $\nu \cdot Tw^- - \nu \cdot Tw_n^-$. By using the maximum estimate in the previous lemma we have

Lemma 3.7. *The estimate*

$$\nu \cdot Tw^- - \nu \cdot Tw_n^- = O(B^{n+1})$$

holds uniformly on $(m, 1)$ as $B \to 0$.

Proof. From $\operatorname{div} Tw^- - \operatorname{div} Tw_n^- = -B(w^- - w_n) + O(B^{n+1})$ and Lemma 3.6 we see that $\nu \cdot Tw^- - \nu \cdot Tw_n^- = r^{-1}O(B^{n+1})$ on $(m, 1)$. Since r is bounded from below, the lemma follows immediately. At this point we need a more careful consideration if $t \equiv \sin\gamma \to 0$ (see Section 4) ∎

3.6 Asymptotic expansion of the wetted disk. We will conclude from the previous lemma that q_n is an approximation of the radius q of the wetted disk up to $O(B^{n+1})$.

Theorem 3.1. *The radius of the wetted disk is given by*

$$q = q_n(B) + O(B^{n+1})$$

where q_n is defined in Lemma 3.4.

Remark. According to this result, we have $q = \sin\gamma + O(B)$ for *fixed* γ, $\pi/2 < \gamma < \pi$. If $\gamma = \pi$, then (see Section 4) $q = \sqrt{\frac{2}{3}}B^{1/2} + O(B)$.

Proof of Theorem 3.1. We will split the proof into a series of lemmas.

Lemma 3.8. *Set*

$$A(r) = (\nu \cdot Tw_n^-)(r, p, t, B) .$$

There exist positive constants η, c_0 and c_1 depending only on t_0, t_1 and B_0 such that

$$c_0 \le A'(r) \le c_1$$

holds on $t - \eta < r < 1$, uniformly in p and B, when $|p - t| < \eta$ and $0 < B < B_0$.

Proof. The proof follows from the properties of the functions $\varphi_l(r, p, t)$. The crucial point here is the special non-linearity of the problem. Set

$$V(r) = \sum_{l=0}^{n}(-1)^l \varphi_l(r, p, t)B^l .$$

Then

$$A(r) = \frac{V'(r)}{\sqrt{1 + V'(r)^2}} \qquad \text{and} \qquad A'(r) = \frac{V''}{(1 + V'^2)^{3/2}} .$$

From the properties of φ_l and an expansion analogous to (3.11) we conclude that

$$A'(r) = \frac{\varphi_0''(r, p, t)}{(1 + \varphi_0'(r, p, t)^2)^{3/2}} + O(B)$$

holds. The assertion of the lemma is then a consequence of the equation $\frac{\varphi_0''}{(1+\varphi_0'^2)^{3/2}} = \frac{1-pt}{1-p^2} + \frac{p(p-t)}{r^2(1-p^2)}$ ∎

From a result of Finn (see Theorem 1.1), the function

$$\nu \cdot Tw^- \equiv \frac{w_r^-(r,t,B)}{\sqrt{1+w_r^-(r,t,B)^2}}$$

increases strictly in r, $\bar{q}(B) < r < 1$, where $(\nu \cdot Tw^-)(\bar{q}) = 0$ with a $\bar{q} > 0$. Thus, we have $\bar{q}(B) \le q(B)$, where $(\nu \cdot Tw^-)(q) = \sin\gamma$. More precisely, we have $\bar{q}(B) < q(B)$ since $\pi/2 < \gamma < \pi$ is assumed in this section.

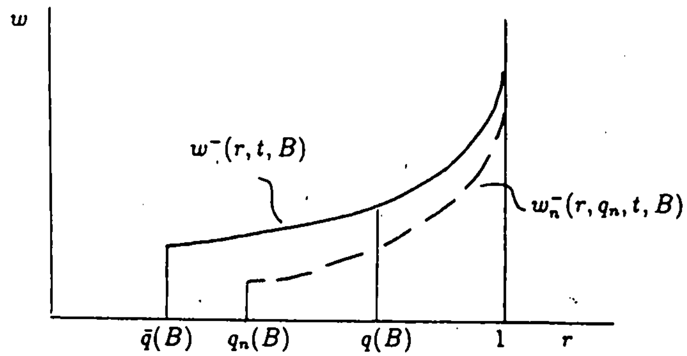

Fig. 3.3 Proof of Theorem 3.1

Lemma 3.9. *There exist positive constants* η *and* B_0 *such that*

$$\bar{q}(B) \le \sin\gamma - \eta \tag{3.55}$$

for all B, $0 < B \le B_0$.

Proof. We will prove this lemma by contradiction. Let

$$\bar{q}(B) > \sin\gamma - \eta \tag{3.56}$$

for each η, $0 < \eta \le \eta_0$, and for a sequence $B = B_j \to 0$. Because of (3.56) the number $\bar{q}(B)$ belongs to the interval $[m, 1]$ where $V(r)$ is defined (see Lemma 3.7). We recall that $m = \max(\bar{q}(B), t - \eta)$. Then Lemma 3.7 implies

$$(\nu \cdot Tw_n^-)(\bar{q}, q_n, t, B) = O(B^{n+1}) \tag{3.57}$$

and from the monotonicity property (see Lemma 3.8) we obtain

$$\nu \cdot Tw_n^-(\bar{q}, q_n, t, B) \ge \nu \cdot Tw_n^-(\sin\gamma - \eta, q_n, t, B). \tag{3.58}$$

We have

$$\nu \cdot Tw_n^-\left(q_n - \frac{1}{2}\eta, q_n, t, B\right) \le O(B^{n+1}), \tag{3.59}$$

which is a consequence of (3.57) - (3.58) and of the inequality $q_n - \eta/2 \le \sin\gamma - \eta$. From Lemma 3.8, the mean value theorem and $\nu \cdot Tw_n^-(q_n, q_n, t, B) = \sin\gamma$ there follows

$$\nu \cdot Tw_n^-(q_n + \epsilon, q_n, t, B) = \sin\gamma + O(\epsilon). \tag{3.60}$$

Combining (3.59) - (3.60), we obtain $\sin\gamma \le c(\eta + B^{n+1})$ which contradicts the assumption $\sin\gamma > 0$ if η and B are small enough ∎

Thus, we have shown that the interval $\sin\gamma - \eta \le r < 1$ is a subset of $[m, 1)$ for a (small) $\eta > 0$.

Lemma 3.10. *There exists a (small) positive constant η such that*

$$\sin\gamma - \eta \le q(B) \tag{3.61}$$

holds for any B , $0 < B < B_0$.

Proof. If not, then we have for each fixed η, $0 < \eta < \eta_0$, $\eta_0 > 0$ small enough, the inequality $q(B) < \sin\gamma - \eta$ for a sequence $B = B_j \to 0$. From Theorem 1.1, Lemma 3.7 and Lemma 3.8 we obtain with $t \equiv \sin\gamma$ the following sequence of inequalities:

$$\begin{aligned}\sin\gamma &= \nu \cdot Tw^-(q, t, B)\\ &\le \nu \cdot Tw^-(\sin\gamma - \eta, t, B)\\ &\le \nu \cdot Tw_n^-(\sin\gamma - \eta, q_n, t, B) + O(B^{n+1})\\ &\le \nu \cdot Tw_n^-(q_n - \eta/2, q_n, t, B) + O(B^{n+1})\\ &\le \nu \cdot Tw_n^-(q_n, q_n, t, B) - c_1\eta + O(B^{n+1})\end{aligned}$$

with a positive constant c_1. Since $\nu \cdot Tw_n^-(q_n, q_n, t, B) = \sin\gamma$, it follows $\eta = O(B^{n+1})$ which is a contradiction because $\eta > 0$ is fixed ∎

Expanding $(\nu \cdot Tw_n^-)(q_n + (q - q_n))$, we get from Lemma 3.8, Lemma 3.7 and since q belongs to the admissible interval (see Lemma 3.10) that

$$\begin{aligned}\sin\gamma + \rho(q - q_n) &= (\nu \cdot Tw^-)(q, q, t, B) + O(B^{n+1})\\ &= \sin\gamma + O(B^{n+1}) ,\end{aligned}$$

where $c_0 \le \rho \le c_1$ with positive constants c_0 and c_1. Thus, we have $q = q_n + O(B^{n+1})$ which proves Theorem 3.1.

3.7 Asymptotic expansion of the volume. From the asymptotic formulas for u^+, u^- and q of the previous sections we derive an asymptotic formula for the volume of the drop. Then an inverse function lemma implies asymptotic expansions for R and for the radius a of the wetted disk in powers of $V^{1/3}$ as $V \to 0$. For the volume V^+ of the upper part of the drop defined by

$$\Omega^+ = \left\{(x, z) \in \mathbf{R}^3 : x \in B_R,\ u^+(R) < z < u^+(x)\right\}$$

we get from Section 2

$$V^+ = \frac{2}{3}\pi R^3 - \frac{1}{6}\pi\kappa R^5 + \sum_{l=2}^{n} c_l^+ \left(\frac{\pi}{2}\right) \kappa^l R^{2l+3} + O(R^{2l+5}) \tag{3.62}$$

as $R \to 0$. For the volume of the lower part of the drop we have

$$V^- = 2\pi \int_a^R s\big(u^-(R, t, B) - u^-(s, t, B)\big)ds + a\pi\big(u^-(R, t, B) - u^-(a, t, B)\big)$$

or

$$V^- = 2\pi R^3 \int_q^1 r \left(w^-(1,t,B) - w^-(r,t,B)\right) dr$$
$$+ \pi q^3 R^3 \left(w^-(1,t,B) - w^-(q,t,B)\right) . \qquad (3.63)$$

According to Lemma 3.4 we have

$$w^-(1,t,B) - w^-(r,t,B) = w_n^-(1,q_n,t,B) - w_n^-(r,q_n,t,B) + O(B^{n+1}) \qquad (3.64)$$

on $t - \eta < r < 1$, $\eta > 0$ small enough, and from Theorem 3.1

$$q = q_n(B) + O(B^{n+1}) , \qquad (3.65)$$

where q_n is defined in Lemma 3.4. From the formulas of Subsection 3.2 for w_n^- we obtain

$$V^- = 2\pi R^3 \int_q^1 r \left(- \sum_{l=o}^{n} (-1)^l B^l \int_r^1 \varphi_l'(s,q_n,t)\, ds + O(B^{n+1}) \right) dr$$
$$+ q^2 \pi R^3 \int_q^1 \sum_{l=0}^{n} (-1)^l \varphi_l'(s,q_n,t) B^l\, ds + O(B^{n+1}) .$$

Integration by parts yields

$$V^- = \pi R^3 \int_q^1 r^2 \left(\sum_{l=0}^{n} (-1)^l \varphi_l'(r,q_n,t) B^l + O(B^{n+1}) \right) dr.$$

From the above formulas (3.63) - (3.65) we obtain for fixed γ in the interval $\pi/2 < \gamma < \pi$, that

$$V^- = \pi R^3 \left((1 - \sin^2\gamma)^{1/2} - \frac{1}{3}(1 - \sin^2\gamma)^{3/2} \right)$$
$$+ \sum_{l=1}^{n} c_l^-(\gamma) \kappa^l R^{2l+3} + O(R^{2n+5}) . \qquad (3.66)$$

Combining (3.66) with (3.62), we finally arrive at

$$V = \pi R^3 \int_0^\gamma \sin^3\theta\, d\theta + \sum_{l=1}^{n} c_l(\gamma) \kappa^l R^{2l+3} + O(R^{2n+5})$$

as $R \to 0$. Then we obtain an expansion of R in powers of $V^{1/3}$ from an inverse function lemma for asymptotic expansions. Since the radius a of the wetted disk is given by $a = Rq$ and $q = Q_{n+1} + O(B^{n+2})$ where $Q_{n+1} = t + \sum_{l=1}^{n+1} a_l(t) B^l$ and $B = \kappa R^2$, we obtain an asymptotic formula for a (see formula (1.3)).

4. Expansion near $\gamma = \pi$

In this section we will sketch the case that the contact angle γ is near π. There is a non-uniformity in asymptotic behaviour as volume $V \to 0$, depending on whether or not $\gamma = \pi$. The proofs of the following lemmas require some elementary calculations which we omit. The main part of the proofs consists in a careful inspection of the formulas which define $w_n^-(r, q_n, t, B)$, where q_n is given through (4.2). Moreover, we restrict our considerations to the leading term in the expansion and we let the question open whether or not there exists a complete expansion near $\gamma = \pi$.

After some calculation we obtain (for the definition of A_l and C_l ($l \geq 1$) see Subsections 3.2 and 3.3) that $A_0 = -2/3$, $C_0(0,0) = 2/3$, $C_0(p,t) = C_0(0,0) + O(|\zeta|)$ as $\zeta \to 0$, $|C_l(p,t)| \leq c$ if $|\zeta| < \epsilon_0$, where $\zeta = (p,t)$. Thus we have for the function f defined through formula (3.47) of Subsection 3.3 the expression

$$f(p,t,B) = 2p^2 - 2pt - \frac{4}{3}B + O(B^2) + O(|\zeta|^3) + O(|\zeta|)B \ .$$

Since $C_l(p,t)$ and the remainder in the above formula for C_0 are continuous in their arguments, we conclude that

$$p \equiv q_n = \frac{t}{2} + \sqrt{\frac{t^2}{4} + \frac{2}{3}B} + O(t^2 + B) \tag{4.1}$$

holds as t, $B \to 0$ for the solution of $f(p,t,B) = 0$ near $t = 0$, $B = 0$. For the next lemma we recall that w^- is defined on $[\bar{q}, 1]$ and it is easily seen that w_n^- is defined on $[\frac{1}{2}q_n, 1]$, where q_n is given through (3.67) (for the definition of $\bar{q}$ compare Subsection 3.4).

Lemma 4.1. *Set* $m = \max\left(\bar{q}, \frac{1}{2}q_n\right)$. *Then*

$$\max |w^-(r,t,B) - w_n^-(r, q_n, t, B)| \leq cB^{n+1},$$

where the maximum is taken over $r \in [m, 1]$.

Proof. We obtain this lemma analogously to Lemma 3.6. The difference here is that we have now $r \geq c_0(t + B^{1/2})$ as a lower bound for r in the formula (3.55) of Subsection 3.4. Thus, we obtain (see Subsection 3.4)

$$|z'| \leq \frac{c}{t + B^{1/2}} \left(B \max |z| + O(B^{n+1})\right)$$

which implies

$$\max |w^-(r,t,B) - w_n^-(r, q_n, t, B)| = O(B^{n+1/2}).$$

We replace n by $n + 1$. Then

$$\begin{aligned} w^- &= w_{n+1}^- + O(B^{n+3/2}) = w_n^- + (-1)^{n+1}\varphi_{n+1}B^{n+1} + O(B^{n+3/2}) \\ &= w_n^- + O(B^{n+1}) \end{aligned}$$

which is the assertion of the lemma ∎

Lemma 4.2. *We have the formula*

$$(\nu \cdot Tw^-)(r,t,B) - (\nu \cdot Tw_n^-)(r,q_n,t,B) = O(B^{n+1})$$

as $B \to 0$.

Proof. We argue analogously to the associated Lemma 3.7 where we here use the same lower bound for r as in the proof of the previous lemma. Thus, we obtain

$$\nu \cdot Tw^- - \nu \cdot Tw_n^- = O(B^{n+1/2})$$

on $(m,1)$. Replacing n by $n+1$, we have

$$\nu \cdot Tw^- - \nu \cdot Tw_{n+1}^- = O(B^{n+3/2}) .$$

Then the lemma is a concequence of the formula

$$\nu \cdot Tw_{n+1}^- = \nu \cdot Tw_n^- + O(B^{n+1})$$

on $(m,1)$ which follows from the properties of $w_{n,r}^-$ and $\varphi_{n+1,r}$ near $r=1$ and from the special non-linear structure of the boundary condition :

$$\begin{aligned}
\nu \cdot Tw_{n+1}^- &= \left(w_{n,r}^- + (-1)^{n+1}B^{n+1}\varphi_{n+1,r}\right) \\
&\quad \cdot \left(1 + \left(w_{n,r}^- + (-1)^{n+1}B^{n+1}\varphi_{n+1,r}\right)^2\right)^{-1/2} \\
&= \left(w_{n,r}^- + (-1)^{n+1}B^{n+1}\varphi_{n+1,r}\right)\left(1 + (w_{n,r}^-)^2\right)^{-1/2} \\
&\quad \times \left(1 + \frac{2w_{n,r}^-(-1)^{n+1}\varphi_{n+1,r}B^{n+1} + (\varphi_{n+1,r})^2 B^{2n+2}}{1 + (w_{n,r}^-)^2}\right)^{-1/2} .
\end{aligned}$$

Thus the lemma is proved ∎

From these two previous lemmas we obtain the following theorem by considerations analogous to those of Subsection 3.6.

Theorem 4.1. *The radius* q *of the wetted disk is given by*

$$q = q(t,B) = q_n + O(B^{n+1}). \tag{4.2}$$

In particular, we have

$$q = \frac{t}{2} + \sqrt{\frac{t^2}{4} + \frac{2}{3}B} + O(t^2) + O(B) .$$

Proof. For

$$A(r) \equiv (\nu \cdot Tw_n^-)(r,q_n,t,B) ,$$

where $0 < t \le t_0$, $0 < B \le B_0$ and $q_n/2 < r < 1$, we obtain after some calculation the same result as in Lemma 3.8 that there exist positive constants c_0 and c_1 such that

$$c_0 \le A'(r) \le c_1. \tag{4.3}$$

Moreover we need the inequality

$$\frac{1}{2}q_n(t,B) \le \bar{q}(t,B) \tag{4.4}$$

if $t \le t_0$ and $B \le B_0$. For the proof let $(t,B) \equiv (t_j, B_j) \to (0,0)$ for a sequence such that $\bar{q}(t,B) < \frac{1}{2}q_n(t,B)$. Then, because of the strict monotonicity of $\nu \cdot Tw^-(q,t,B)$ with respect to q, we have

$$0 = \nu \cdot Tw^-(\bar{q},t,B) \le \nu \cdot Tw^-\left(\frac{q_n}{2},t,B\right) = \nu \cdot Tw_n^-\left(q_n - \frac{q_n}{2},t,B\right) + O(B^{n+1}).$$

From Lemma 4.2 and Lemma 4.4 we obtain $0 = t - \frac{1}{2}\rho q_n + O(B^{n+1})$ and finally the equation $q_n = O(t) + O(B^{n+1})$ which is a contradiction to the definition of q_n. This means, we have $m = \bar{q}$ and w_n^-, w^- are defined on the interval $[\bar{q},1]$. Then Lemma 4.2 implies

$$(\nu \cdot Tw^-)(q,t,B) - (\nu \cdot Tw_n^-)(q,q_n,t,B) = O(B^{n+1}) .$$

We recall that $q \in [m,1]$ since $\bar{q} \le q$ holds. It follows

$$t - (\nu \cdot Tw_n^-)(q_n + (q-q_n), q_n, t, B) = O(B^{n+1})$$

$$t - (\nu \cdot Tw_n^-)(q_n, q_n, t, B) - \rho(q-q_n) = O(B^{n+1})$$

where $c_0 < \rho < c_1$, with positive constants (see (4.3)). We finally arrive at $q - q_n = O(R^{n+1})$ since $(\nu \cdot Tu_n^-)(q_n, q_n, t, B) = t$ and the proof of Theorem 4.1 is completed ∎

As in Subsection 3.7 we find for the volume of the lower part of the drop

$$V^- = \pi R^3 \int_{q_n}^{1} r^2 \varphi_0'(r,q_n,t)\,dr + O(R^5)$$

as $R \to 0$, which implies

$$V^- = \pi R^3 \left(\frac{2}{3} + O(R) + O(t)\right) + O(R^5) .$$

Then the volume of the drop is given by

$$V = \frac{4}{3}\pi R^3 + O(R^4) + O(tR^3).$$

This implies that

$$R = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}\left(1 + O(t) + O(V^{1/3})\right) \tag{4.5}$$

as $V \to 0$. Combining (4.5) and (4.2), we obtain for the radius q of the wetted disk of the transformed drop

$$q = \frac{t}{2} + \sqrt{\frac{t^2}{4} + \kappa\frac{2}{3}\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}} + O(t^2) + O(V^{2/3})$$

as t, $V \to 0$. We recall that $t \equiv \sin\gamma$. Since the radius of the wetted disk of the drop is $a = qR$, we finally arrive at the asymptotic formula (1.4) for a.

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