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A Schur Type Analysis of the Minimal Unitary Hilbert Space Extensions of a Kreĭn Space Isometry whose Defect Subspaces are Hilbert Spaces

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Abstract. We consider a Kreĭn space isometry whose defect subspaces are Hilbert spaces and we show that its minimal unitary Hilbert space extensions are related to one-step isometric Hilbert space extensions and Schur parameters. These unitary extensions give rise to moments and scattering matrices defined on a scale subspace. By means of these notions we solve the labeling problem for the contractive intertwining liftings in the commutant lifting theorem for Kreĭn space contractions.

Keywords: *Isometries, unitary extensions, Kreĭn spaces, Schur parameters, commutant liftings*

AMS subject classification: 47A57, 47A20, 47B50

Introduction

Let V be an isometry in a Kreĭn space $\mathfrak{H}$ with domain $\mathfrak{D}$ and range $\mathfrak{R}$ such that the defect subspaces $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$ and $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{R}$ are Hilbert subspaces of $\mathfrak{H}$. In this paper we describe the class $\mathbf{U}(V)$ of all minimal unitary Hilbert space extensions of V. A pair $(U, \widetilde{\mathfrak{H}})$ consisting of a Kreĭn space $\widetilde{\mathfrak{H}}$ and a unitary operator $U \in \mathbf{L}(\widetilde{\mathfrak{H}})$ is called a *unitary Hilbert space extension* of V if $\mathfrak{H}$ is a regular subspace of $\widetilde{\mathfrak{H}}$ such that $\widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ is a Hilbert space, and U is an extension of V, and it is called *minimal* if $\widetilde{\mathfrak{H}}$ coincides with the closed linear span of all subspaces $U^n\mathfrak{H}$, $n = \cdots, -1, 0, 1, \cdots$.

The study of the set $\mathbf{U}(V)$ is carried out by means of a Schur type analysis. The approach leads to the set $\mathbf{V}(V)$ of all *one-step isometric Hilbert space extensions* $(V_1, \mathfrak{K}_1)$ of V. In Section 1 we show that there is a one-to-one correspondence between $\mathbf{V}(V)$ and the set of contractions in $\mathbf{L}(\mathfrak{N}, \mathfrak{M})$. Each $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ yields a sequence of continuous isometries $\{V_m\}$ acting in nested Kreĭn spaces $\mathfrak{K}_m$ such that, for all $m \geq 1$, $(V_m, \mathfrak{K}_m) \in \mathbf{V}(V_{m-1})$, $V_0 = V$, and a sequence of contractions $\{\gamma_m\}$, with $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$, the so called *Schur parameters*. The "degrees of freedom" in the construction of each $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$

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are given by its Schur parameters, in the sense that the construction can be carried out step by step starting from a contraction $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ and choosing in the m-th step a contraction γ_m. In Section 2 we describe this stepwise construction and we show that the pair $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ is, up to isomorphisms, uniquely determined by its Schur parameters. A subspace $\mathfrak{G} \subseteq \mathfrak{H}$ is called a *scale subspace* for V if it is regular and $\mathfrak{D} + \mathfrak{G}$ and $\mathfrak{R} + \mathfrak{G}$ are dense subspaces of $\mathfrak{H}$. In Section 3 we associate to each $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ the sequence of moments $\{P_{\mathfrak{G}}^{\widetilde{\mathfrak{H}}} U^n|_{\mathfrak{G}}\}$ relative to the scale subspace $\mathfrak{G}$, where $P_{\mathfrak{G}}^{\widetilde{\mathfrak{H}}}$ is the projection from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{G}$. We describe a Schur type algorithm which determines if a sequence of operators on a scale subspace $\mathfrak{G}$ is the sequence of moments relative to $\mathfrak{G}$ of a pair $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$. In Section 4 the notion of a scattering matrix with scale subspace is extended to the Kreĭn space situation. A model due to D.Z. Arov and I.Z. Grossman ([5] , [6]) for the Hilbert space setting is generalized to the Kreĭn space case. Finally, in Section 5, the methods discussed in the paper are applied to the study of the contractive intertwining liftings in the commutant lifting theorem for Kreĭn space contractions. The data in the commutant lifting theorem determine a Kreĭn space isometry V with the adequate properties such that there is a one-to-one correspondence between $\mathbf{U}(V)$ and the set of all contractive intertwining liftings.

In the Hilbert space case these methods were developed by R. Arocena ([2], [3]) in connection with a lifting problem for a class of translation invariant forms (see also [4]). For a detailed account on commutant lifting theorems in Hilbert spaces we refer to [14].

Familiarity with operator theory on Kreĭn spaces is assumed. We refer the reader to [1],[7],[8],[13] and [15] for more details about indefinite inner product spaces and to [13] for the Kreĭn space extensions of the Hilbert space notions of defect operators, minimal isometric dilations and minimal unitary dilations. We refer to [12] for various preliminary results.

In the sequel the standard Hilbert space notation is reserved for Kreĭn spaces and operators on them. So all notions are to be taken in their Kreĭn space versions, unless otherwise mentioned. We use the following notations: $\mathbf{Z}$ and $\mathbf{C}$ stand for the set of all integers and for the set of all complex numbers, respectively, while $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$. If $\mathfrak{H}_\lambda$ are linear subspaces of a Kreĭn space $\mathfrak{H}$, $\lambda \in \Lambda$, their linear span is indicated by l.s. $\{\mathfrak{H}_\lambda \mid \lambda \in \Lambda\}$. The symbol c.l.s. denotes closed linear span.

We recall that a *weak isomorphism* from a Kreĭn space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ is a densely defined linear mapping from $\mathfrak{H}$ onto a dense subspace of $\mathfrak{K}$ that preserves the scalar products. A weak isomorphism does not necessarily have a continuous extension to all of $\mathfrak{H}$. We will however frequently use the fact that an isometric linear mapping, densely defined in a closed subspace $\mathfrak{D}$ of a Kreĭn space $\mathfrak{H}$ with values in a Kreĭn space $\mathfrak{K}$, can be extended by continuity to all of $\mathfrak{D}$ if either the orthogonal complement of its domain or the orthogonal complement of its range is a Hilbert space.

A Kreĭn space *colligation* is a quadruple $\Delta = \{\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, H\}$, where $\mathfrak{H}$, $\mathfrak{F}$, $\mathfrak{G}$ are Kreĭn spaces and the operator matrix

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{F} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{G} \end{pmatrix},$$

called the *connecting* operator, belongs to $\mathbf{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{H} \oplus \mathfrak{G})$. A colligation is said to be *unitary* if the connecting operator is unitary. The *characteristic function* of Δ is the $\mathbf{L}(\mathfrak{F},\mathfrak{G})$-valued holomorphic function $\Theta_\Delta(z) = H + zG(I - zT)^{-1}F$ defined for those $z \in \mathbf{C}$ for which $(I - zT)^{-1} \in \mathbf{L}(\mathfrak{H})$. Two colligations $\Delta = \{\mathfrak{H},\mathfrak{F},\mathfrak{G};T,F,G,H\}$ and $\Delta' = \{\mathfrak{H}',\mathfrak{F},\mathfrak{G};T',F',G',H'\}$ are said to be *weakly isomorphic* if $H = H'$ and if there exists a weak isomorphism φ_0 from $\mathfrak{H}$ to $\mathfrak{H}'$ such that

$$\begin{pmatrix} \varphi_0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T & F \\ G & H \end{pmatrix} = \begin{pmatrix} T' & F' \\ G' & H' \end{pmatrix}\begin{pmatrix} \varphi_0 & 0 \\ 0 & I \end{pmatrix}$$

on $\operatorname{dom}\varphi_0 \oplus \mathfrak{F}$. If φ_0 can be extended by continuity to a unitary operator from $\mathfrak{H}$ onto $\mathfrak{H}'$, then Δ and Δ' are called *isomorphic*.

1 One-step isometric Hilbert space extensions

We assume that V is a continuous isometric operator acting in a Kreĭn space $\mathfrak{H}$ whose domain $\mathfrak{D}$ and range $\mathfrak{R}$ are regular subspaces of $\mathfrak{H}$ and that the defect subspaces $\mathfrak{N} = \mathfrak{H}\ominus\mathfrak{D}$ and $\mathfrak{M} = \mathfrak{H}\ominus\mathfrak{R}$ of V are Hilbert subspaces of $\mathfrak{H}$. In this section we label all one-step isometric Hilbert space extensions of V in terms of contractions from the Hilbert space $\mathfrak{N}$ into the Hilbert space $\mathfrak{M}$.

A pair $(V_1, \mathfrak{K}_1)$ is called a *one-step isometric Hilbert space extension* of the isometry V if

(a) $\mathfrak{K}_1$ is a Kreĭn space which contains $\mathfrak{H}$ as regular subspace such that $\mathfrak{K}_1 \ominus \mathfrak{H}$ is a Hilbert space,

(b) V_1 is a continuous isometric operator in $\mathfrak{K}_1$ with $\operatorname{dom} V_1 = \mathfrak{H}$ such that $V \subset V_1$,

(c) $\mathfrak{K}_1 = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1\}$.

We denote by $\mathfrak{N}_1 = \mathfrak{K}_1 \ominus \operatorname{dom} V_1$ and $\mathfrak{M}_1 = \mathfrak{K}_1 \ominus \operatorname{ran} V_1$ the defect spaces of the isometry V_1. Then $\mathfrak{N}_1 = \mathfrak{K}_1 \ominus \mathfrak{H}$, $\mathfrak{N}_1$ and $\mathfrak{M}_1$ are Hilbert spaces, and $V_1\mathfrak{N} \perp \mathfrak{R}$, so that $V_1\mathfrak{N} \subseteq (\mathfrak{K}_1 \ominus \mathfrak{H}) \oplus \mathfrak{M}$. Hence

(1.1) $P^{\mathfrak{K}_1}_{\mathfrak{M}} V_1 |_{\mathfrak{N}} = P^{\mathfrak{K}_1}_{\mathfrak{H}} V_1 |_{\mathfrak{N}}$.

The condition (c) is a minimality condition. Obviously, we have

$$\text{l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1\} = \text{l.s.}\,\{\mathfrak{H}, V\mathfrak{D}, V_1\mathfrak{N}\} = \text{l.s.}\,\{\mathfrak{H}, V_1\mathfrak{N}\}$$

and

$$\text{l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1\} = \text{l.s.}\,\{\mathfrak{M}, \mathfrak{R}, \operatorname{ran} V_1\} = \text{l.s.}\,\{\mathfrak{M}, \operatorname{ran} V_1\}.$$

Therefore

(1.2) $\mathfrak{K}_1 = \text{c.l.s.}\,\{\mathfrak{H}, V_1\mathfrak{N}\} = \text{c.l.s.}\,\{\mathfrak{M}, \operatorname{ran} V_1\}$.

The class of all one-step isometric Hilbert space extensions of V is denoted by $\mathbf{V}(V)$. With each $(V_1, \mathfrak{K}_1) \in \mathbf{V}(V)$ we associate an operator $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ defined by

$$\gamma_1 y = P_{\mathfrak{M}}^{\mathfrak{K}_1} V_1 y = P_{\mathfrak{H}}^{\mathfrak{K}_1} V_1 y, \quad y \in \mathfrak{N},$$

where the equality is due to (1.1). For all $y \in \mathfrak{N}$,

$$\begin{aligned} \langle \gamma_1 y, \gamma_1 y \rangle_{\mathfrak{K}_1} &= \langle P_{\mathfrak{H}}^{\mathfrak{K}_1} V_1 y, P_{\mathfrak{H}}^{\mathfrak{K}_1} V_1 y \rangle_{\mathfrak{K}_1} \\ &= \langle V_1 y, V_1 y \rangle_{\mathfrak{K}_1} - \langle P_{\mathfrak{K}_1 \ominus \mathfrak{H}}^{\mathfrak{K}_1} V_1 y, P_{\mathfrak{K}_1 \ominus \mathfrak{H}}^{\mathfrak{K}_1} V_1 y \rangle_{\mathfrak{K}_1} \leq \langle y, y \rangle_{\mathfrak{K}_1}, \end{aligned}$$

since V_1 is an isometry and $\mathfrak{K}_1 \ominus \mathfrak{H}$ is a Hilbert space. This shows that $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ is a contraction, and we have

$$P_{\mathfrak{H}}^{\mathfrak{K}_1} V_1 = P_{\mathfrak{H}}^{\mathfrak{K}_1} V_1 (P_{\mathfrak{D}}^{\mathfrak{H}} + P_{\mathfrak{N}}^{\mathfrak{H}}) = V P_{\mathfrak{D}}^{\mathfrak{H}} + \gamma_1 P_{\mathfrak{N}}^{\mathfrak{H}}.$$

There is a connection between the defect spaces $\mathfrak{N}_1$ and $\mathfrak{M}_1$ of V_1 and the defect spaces of γ_1. We recall (see [17] and also [13]) that by a *defect operator* for the contraction γ_1 we mean an operator $\widetilde{D}_{\gamma_1} \in \mathbf{L}(\widetilde{\mathfrak{D}}_{\gamma_1}, \mathfrak{N})$, where $\widetilde{\mathfrak{D}}_{\gamma_1}$ is a Hilbert space, called a *defect space*, such that $\widetilde{D}_{\gamma_1}$ has zero kernel, or equivalently, ran $\widetilde{D}_{\gamma_1}^*$ is dense in $\widetilde{\mathfrak{D}}_{\gamma_1}$, and $I - \gamma_1^* \gamma_1 = \widetilde{D}_{\gamma_1} \widetilde{D}_{\gamma_1}^*$. We denote by $D_{\gamma_1} \in \mathbf{L}(\mathfrak{D}_{\gamma_1}, \mathfrak{M})$ a defect operator for the contraction $\gamma_1^* \in \mathbf{L}(\mathfrak{M}, \mathfrak{N})$. These defect operators are unique up to isomorphisms.

Proposition 1.1 *Let $(V_1, \mathfrak{K}_1)$ belong to $\mathbf{V}_1(V)$ and let $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ be the corresponding contraction. The defect spaces of the isometry V_1 are given by*

(a) $\mathfrak{N}_1 = \mathfrak{K}_1 \ominus \mathfrak{H} = \overline{(V_1 - \gamma_1)\mathfrak{N}}$,
(b) $\mathfrak{M}_1 = \overline{(I - V_1 \gamma_1^*)\mathfrak{M}}$.

Let $\widetilde{D}_{\gamma_1} \in \mathbf{L}(\widetilde{\mathfrak{D}}_{\gamma_1}, \mathfrak{N})$ and $D_{\gamma_1} \in \mathbf{L}(\mathfrak{D}_{\gamma_1}, \mathfrak{M})$ be defect operators for γ_1 and γ_1^*, respectively. Then the mappings

(c) $(V_1 - \gamma_1) y \mapsto \widetilde{D}_{\gamma_1}^* y, \quad y \in \mathfrak{N}$,
(d) $(I - V_1 \gamma_1^*) x \mapsto D_{\gamma_1}^* x, \quad x \in \mathfrak{M}$,

can be extended by continuity to unitary operators from $\mathfrak{N}_1$ onto $\widetilde{\mathfrak{D}}_{\gamma_1}$ and from $\mathfrak{M}_1$ onto $\mathfrak{D}_{\gamma_1}$, respectively. In particular, $\mathfrak{N}_1 = \{0\}$ if and only if $\gamma_1^* \gamma_1 = I$, and $\mathfrak{M}_1 = \{0\}$ if and only if $\gamma_1 \gamma_1{}^* = I$.

Proof. By (1.2), (a) follows from

$$(V_1 - \gamma_1)\mathfrak{N} = (I - P_{\mathfrak{H}}^{\mathfrak{K}_1}) V_1 \mathfrak{N} = P_{\mathfrak{K}_1 \ominus \mathfrak{H}}^{\mathfrak{K}_1} V_1 \mathfrak{N} = P_{\mathfrak{K}_1 \ominus \mathfrak{H}}^{\mathfrak{K}_1} (\mathfrak{H} + V_1 \mathfrak{N}).$$

To prove (b) we observe that

$$\langle (I - V_1 \gamma_1^*)\mathfrak{M}, V_1 \mathfrak{D} \rangle_{\mathfrak{K}_1} = \langle \mathfrak{M}, \mathfrak{R} \rangle - \langle V_1 \gamma_1^* \mathfrak{M}, V_1 \mathfrak{D} \rangle_{\mathfrak{K}_1} = \{0\},$$

since $\mathfrak{M}$ and $\mathfrak{N}$ are orthogonal to $\mathfrak{R}$ and $\mathfrak{D}$, respectively. Also, on account of (a),

$$\langle (I - V_1 \gamma_1^*)\mathfrak{M}, V_1 \mathfrak{N} \rangle_{\mathfrak{K}_1} = \langle \mathfrak{M}, (V_1 - \gamma_1)\mathfrak{N} \rangle_{\mathfrak{K}_1} = \{0\}.$$

These equalities show that $(I - V_1\gamma_1^*)\mathfrak{M}$ is contained in $(\operatorname{ran} V_1)^\perp = \mathfrak{M}_1$. To see that it is dense in $\mathfrak{M}_1$ we let $z \in \mathfrak{M}_1$ be orthogonal to $(I - V_1\gamma_1^*)\mathfrak{M}$. Then, for all $x \in \mathfrak{M}$,

$$\langle x, z\rangle = \langle (I - V_1\gamma_1^*)x, z\rangle_{\mathfrak{K}_1} = 0,$$

that is, z is orthogonal to $\mathfrak{M}$, which by (1.2) implies that $z = 0$. Hence (b) holds. It is easily verified that the mappings in (c) and (d) are isometric. By (a) and (b), these mappings are weak isomorphisms, and, since all spaces involved are Hilbert spaces, they can actually be extended by continuity to unitary operators. This completes the proof of the proposition. □

The following result shows that each contraction in $\mathbf{L}(\mathfrak{N}, \mathfrak{M})$ can be obtained in the way described above. We say that $(V_1, \mathfrak{K}_1), (V_1', \mathfrak{K}_1') \in \mathbf{V}(V)$ are *isomorphic* if there exists a unitary operator $\varphi : \mathfrak{K}_1 \to \mathfrak{K}_1'$ such that $\varphi|_{\mathfrak{H}} = I$ and $\varphi V_1 = V_1'\varphi$.

Theorem 1.2 *The formula*

$$(V_1, \mathfrak{K}_1) \in \mathbf{V}(V) \mapsto \gamma_1 = P_{\mathfrak{M}}^{\mathfrak{K}_1} V_1|_{\mathfrak{N}} \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$$

establishes, up to isomorphisms, a one-to-one correspondence between $\mathbf{V}(V)$ *and the set of all contractions in* $\mathbf{L}(\mathfrak{N}, \mathfrak{M})$. *The inverse is given by* $\gamma_1 \mapsto (V_1, \mathfrak{K}_1)$ *with*

$$\mathfrak{K}_1 = \mathfrak{H} \oplus \widetilde{\mathfrak{D}}_{\gamma_1}, \quad V_1 h = \begin{pmatrix} \widetilde{D}_{\gamma_1}^* P_{\mathfrak{N}}^{\mathfrak{H}} h \\ V P_{\mathfrak{D}}^{\mathfrak{H}} h + \gamma_1 P_{\mathfrak{N}}^{\mathfrak{H}} h \end{pmatrix}, \quad h \in \mathfrak{H},$$

where $\widetilde{D}_{\gamma_1} \in \mathbf{L}(\widetilde{\mathfrak{D}}_{\gamma_1}, \mathfrak{N})$ *is the defect operator for the contraction* $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$. *As to the defect subspaces of* V_1 *we have* $\mathfrak{N}_1 = \widetilde{\mathfrak{D}}_{\gamma_1}$ *and* $\mathfrak{M}_1$ *is isomorphic to the defect space* $\mathfrak{D}_{\gamma_1}$ *of* γ_1^*. *In fact,*

$$\mathfrak{M}_1 = \{\{x, u\} \in \mathfrak{M} \oplus \widetilde{\mathfrak{D}}_{\gamma_1} \mid \gamma_1^* x + \widetilde{D}_{\gamma_1} u = 0\},$$

and the mapping

$$\{x, u\} \in \mathfrak{M}_1 \mapsto D_{\gamma_1}^* x - \gamma_1 u \in \mathfrak{D}_{\gamma_1},$$

where $D_{\gamma_1} \in \mathbf{L}(\mathfrak{D}_{\gamma_1}, \mathfrak{M})$ *is the defect operator for* γ_1^*, *is unitary.*

Proof. If $(V_1, \mathfrak{K}_1)$ is a one-step isometric Hilbert space extension of V, we have shown before that the operator γ_1 is a contraction. Suppose that $(V_1, \mathfrak{K}_1), (V_1', \mathfrak{K}_1') \in \mathbf{V}(V)$ are such that $\gamma_1 = P_{\mathfrak{M}}^{\mathfrak{K}_1} V_1|_{\mathfrak{N}}$ and $\gamma_1' = P_{\mathfrak{M}}^{\mathfrak{K}_1'} V_1'|_{\mathfrak{N}}$ are equal. Since V_1 and V_1' are isometric operators and $\gamma_1 = \gamma_1'$,

$$\langle h + V_1 y, h + V_1 y\rangle_{\mathfrak{K}_1} = \langle h + V_1' y, h + V_1' y\rangle_{\mathfrak{K}_1'}, \quad h \in \mathfrak{H}, \quad y \in \mathfrak{N}.$$

It follows from (1.2) and the corresponding version for $(V_1', \mathfrak{K}_1')$ that the mapping

$$\varphi_0(h + V_1 y) = h + V_1' y, \quad h \in \mathfrak{H}, \quad y \in \mathfrak{N},$$

is a well defined weak isomorphism from $\mathfrak{K}_1$ into $\mathfrak{K}_1'$. Since $\mathfrak{H} \subseteq \operatorname{dom}\varphi_0$ and $\mathfrak{K}_1 \ominus \mathfrak{H}$ is a Hilbert space, $\varphi = \overline{\varphi_0}$ is a unitary operator, and $(V_1, \mathfrak{K}_1), (V_1', \mathfrak{K}_1')$ are isomorphic under φ.

If $\gamma_1 \in \mathbf{L}(\mathfrak{N},\mathfrak{M})$ is a contraction then the pair $(V_1,\mathfrak{K}_1)$ defined in the statement is an isometric extension of V and it is minimal since the defect operator has zero kernel. Moreover, $\gamma_1 = P_{\mathfrak{M}}^{\mathfrak{K}_1} V_1|_{\mathfrak{N}}$. Hence the second assertion holds. Clearly, $\mathfrak{N}_1 = \widetilde{\mathfrak{D}}_{\gamma_1}$, and the defining formula for V_1 implies that $\mathfrak{M}_1$ coincides with the kernel of the operator

$$\begin{pmatrix} \gamma_1^* & \tilde{D}_{\gamma_1} \end{pmatrix} : \begin{pmatrix} \mathfrak{M} \\ \widetilde{\mathfrak{D}}_{\gamma_1} \end{pmatrix} \to \mathfrak{N}.$$

That $\mathfrak{M}_1$ and $\mathfrak{D}_{\gamma_1}$ are isomorphic follows from the fact that the operator

$$\begin{pmatrix} \gamma_1 & D_{\gamma_1} \\ \tilde{D}_{\gamma_1} & -\gamma_1^* \end{pmatrix} : \begin{pmatrix} \mathfrak{N} \\ \mathfrak{D}_{\gamma_1} \end{pmatrix} \to \begin{pmatrix} \mathfrak{M} \\ \widetilde{\mathfrak{D}}_{\gamma_1} \end{pmatrix}$$

is unitary (see [17]). This completes the proof. □

2 Schur parameters

As in Section 1 we assume that V is a continuous isometric operator acting in a Kreĭn space $\mathfrak{H}$ such that $\mathfrak{D} = \operatorname{dom} V$ and $\mathfrak{R} = \operatorname{ran} V$ are regular subspaces of $\mathfrak{H}$ and the defect subspaces $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$ and $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{R}$ are Hilbert subspaces of $\mathfrak{H}$. We recall that a pair $(U, \widetilde{\mathfrak{H}})$ is said to be a *unitary Hilbert space extension* of V if $\widetilde{\mathfrak{H}}$ is a Kreĭn space which contains $\mathfrak{H}$ as regular subspace such that $\widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ is a Hilbert space, and $U \in \mathbf{L}(\widetilde{\mathfrak{H}})$ is a unitary operator such that $V \subset U$. The symbol $\mathbf{U}(V)$ denotes the family of all *minimal* unitary Hilbert space extensions of V, that is, the unitary Hilbert space extensions $(U, \widetilde{\mathfrak{H}})$ of V for which $\widetilde{\mathfrak{H}} = \text{c.l.s.}\{U^n\mathfrak{H} \mid n \in \mathbf{Z}\}$ (minimality property). We say that $(U, \widetilde{\mathfrak{H}}), (U', \widetilde{\mathfrak{H}}') \in \mathbf{U}(V)$ are *isomorphic* if there exists a unitary operator $\varphi : \widetilde{\mathfrak{H}} \to \widetilde{\mathfrak{H}}'$ such that $\varphi|_{\mathfrak{H}} = I$ and $\varphi U = U'\varphi$.

Let $(U, \widetilde{\mathfrak{H}})$ belong to $\mathbf{U}(V)$ and set, for $m \in \mathbf{N}$,

$$\mathfrak{K}_m = \text{c.l.s.}\{U^n\mathfrak{H} \mid 0 \le n \le m\}.$$

Then $\{\mathfrak{K}_m\}$ is a sequence of nested Kreĭn spaces between $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$:

$$\mathfrak{H} = \mathfrak{K}_0 \subseteq \mathfrak{K}_1 \subseteq \cdots \subseteq \mathfrak{K}_m \subseteq \mathfrak{K}_{m+1} \subseteq \cdots \subseteq \widetilde{\mathfrak{H}}.$$

Since $\widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ is a Hilbert space, it follows that, for all $m \ge 1$, $\mathfrak{K}_m \ominus \mathfrak{K}_{m-1}$ is a Hilbert space. Set

$$\mathfrak{K}_{-1} = \mathfrak{D}, \quad V_0 = V.$$

For $m \ge 1$, an operator V_m acting in $\mathfrak{K}_m$ is defined by setting

$$\operatorname{dom} V_m = \mathfrak{K}_{m-1}, \quad V_m = U|_{\mathfrak{K}_{m-1}}.$$

It turns out that, for all $m \ge 1$, V_m is an isometric extension of V_{m-1} to all of $\mathfrak{K}_{m-1}$, such that

$$\mathfrak{K}_m = \text{c.l.s.}\{\mathfrak{K}_{m-1}, \operatorname{ran} V_m\} = \text{c.l.s.}\{\mathfrak{H}, \operatorname{ran} V_1, \cdots, \operatorname{ran} V_m\}.$$

The above discussion shows that a sequence $\{(V_m, \mathfrak{K}_m)\}$ can be associated to each $(U, \tilde{\mathfrak{H}}) \in \mathcal{U}(V)$ in such a way that, for all $m \geq 1$, $(V_m, \mathfrak{K}_m)$ belongs to $\mathbf{V}(V_{m-1})$.

For $m \in \mathbb{N}$, let $\mathfrak{N}_m$ and $\mathfrak{M}_m$ denote the defect spaces of V_m. That is, $\mathfrak{N}_0 = \mathfrak{N}$, $\mathfrak{M}_0 = \mathfrak{M}$, and, for $m \geq 1, \mathfrak{N}_m = \mathfrak{K}_m \ominus \mathfrak{K}_{m-1}$, $\mathfrak{M}_m = \mathfrak{K}_m \ominus U\mathfrak{K}_{m-1}$. Clearly, $\mathfrak{N}_n$ and $\mathfrak{M}_n$ are Hilbert spaces, and $V_m \mathfrak{N}_{m-1} \perp V_{m-1}\mathfrak{K}_{m-2}$, so that

$$V_m \mathfrak{N}_{m-1} \subseteq (\mathfrak{K}_m \ominus \mathfrak{K}_{m-1}) \oplus \mathfrak{M}_{m-1}.$$

Define $\gamma_m \in \mathbf{L}(\mathfrak{N}_{m-1}, \mathfrak{M}_{m-1})$ by

$$\gamma_m = P^{\mathfrak{K}_m}_{\mathfrak{M}_{m-1}} V_m \mid \mathfrak{N}_{m-1} = P^{\mathfrak{K}_m}_{\mathfrak{K}_{m-1}} V_m \mid \mathfrak{N}_{m-1}, \quad m \geq 1,$$

where the equality is due to the above inclusion. Then, for all $m \geq 1$ and $y \in \mathfrak{N}_{m-1}$,

$$\begin{aligned} \langle \gamma_m y, \gamma_m y \rangle_{\mathfrak{K}_m} &= \langle P^{\mathfrak{K}_m}_{\mathfrak{K}_{m-1}} V_m y, P^{\mathfrak{K}_m}_{\mathfrak{K}_{m-1}} V_m y \rangle_{\mathfrak{K}_m} \\ &= \langle V_m y, V_m y \rangle_{\mathfrak{K}_m} - \langle P^{\mathfrak{K}_m}_{\mathfrak{K}_m \ominus \mathfrak{K}_{m-1}} V_m y, P^{\mathfrak{K}_m}_{\mathfrak{K}_m \ominus \mathfrak{K}_{m-1}} V_m y \rangle_{\mathfrak{K}_m}. \end{aligned}$$

Since V_m is an isometry and $\mathfrak{K}_m \ominus \mathfrak{K}_{m-1}$ is a Hilbert space, γ_m is a contraction. Furthermore,

$$P^{\mathfrak{K}_m}_{\mathfrak{K}_{m-1}} V_m = V_{m-1} P^{\mathfrak{K}_{m-1}}_{\mathfrak{K}_{m-2}} + \gamma_m P^{\mathfrak{K}_{m-1}}_{\mathfrak{N}_{m-1}}.$$

The defect spaces $\mathfrak{N}_m$ and $\mathfrak{M}_m$ of V_m, $m \geq 1$, satisfy

$$(2.1)\quad \mathfrak{N}_m = \mathfrak{K}_m \ominus \mathfrak{K}_{m-1} = \overline{(V_m - \gamma_m)\mathfrak{N}_{m-1}}, \quad \mathfrak{M}_m = \overline{(I - V_m \gamma_m^*)\mathfrak{M}_{m-1}}.$$

We adopt the same terminology as in [2] and we say that $\{\gamma_m\}$ is the sequence of *Schur parameters* of $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$.

Now we consider the converse process. Let $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ be a contraction. According to Theorem 1.2, a one-step isometric Hilbert space extension $(V_1, \mathfrak{K}_1)$ of V is uniquely determined by γ_1, up to isomorphisms. Denote by $\mathfrak{N}_1$ and $\mathfrak{M}_1$ the defect subspaces of V_1 and choose a contraction $\gamma_2 \in \mathbf{L}(\mathfrak{N}_1, \mathfrak{M}_1)$. Another application of Theorem 1.2 leads to a one-step isometric Hilbert space extension $(V_2, \mathfrak{K}_2)$ of V_1. We proceed inductively: assume that, for some $m \geq 1$, $(V_m, \mathfrak{K}_m) \in \mathbf{V}(V_{m-1})$ is defined. Let $\mathfrak{N}_m$ and $\mathfrak{M}_m$ be the defect subspaces of V_m. Choose a contraction $\gamma_{m+1} \in \mathbf{L}(\mathfrak{N}_m, \mathfrak{M}_m)$. Then there exists a one-step isometric Hilbert space extension $(V_{m+1}, \mathfrak{K}_{m+1})$ of V_m. In this way a repeated application of Theorem 1.2 leads to a sequence of contractions $\{\gamma_m\}$. The main result of this section is contained in the following theorem.

Theorem 2.1 *For any choice of contractions $\{\gamma_m\}$ there exists, up to isomorphisms, a unique pair $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ whose Schur parameters are $\{\gamma_m\}$.*

Before proving Theorem 2.1, we consider some special cases of sequences $\{\gamma_m\}$. It is straightforward to show that $VP^{\mathfrak{H}}_{\mathfrak{D}}$ is a bicontractive isometric operator on $\mathfrak{H}$. Therefore its minimal unitary dilation $(U_V, \tilde{\mathfrak{H}}_V)$ is unique, up to isomorphisms, and hence can be identified with the canonical one (see [12]). Clearly, $(U_V, \tilde{\mathfrak{H}}_V)$ belongs to $\mathbf{U}(V)$.

Proposition 2.2 *The minimal unitary dilation $(U_V, \tilde{\mathfrak{H}}_V)$ of $VP^{\mathfrak{H}}_{\mathfrak{D}}$ is, up to isomorphisms, the unique element in $\mathbf{U}(V)$ whose Schur parameters are equal to zero.*

Proof. The Kreĭn space $\tilde{\mathfrak{H}}_V$ is given by

$$\tilde{\mathfrak{H}}_V = \left(\bigoplus_1^\infty \mathfrak{M}\right) \oplus \mathfrak{H} \oplus \left(\bigoplus_1^\infty \mathfrak{N}\right),$$

and the operator U_V is of the form

$$U_V = \begin{pmatrix} & & & & \cdots & & & & \\ \cdots & I & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \iota^{\mathfrak{M}}_{\mathfrak{H}} & \boxed{VP^{\mathfrak{H}}_{\mathfrak{D}}} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & P^{\mathfrak{H}}_{\mathfrak{N}} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I & 0 & \cdots \\ & & & & \cdots & & & & \end{pmatrix},$$

where $\iota^{\mathfrak{M}}_{\mathfrak{H}}$ is the inclusion mapping from $\mathfrak{M}$ into $\mathfrak{H}$. Note that, for all $m \in \mathbf{N}$,

$$\mathfrak{K}_m(U_V) = \text{c.l.s.}\,\{U_V^n \mathfrak{H} \mid 0 \le n \le m\} = \mathfrak{H} \oplus \left(\bigoplus_1^m \mathfrak{N}\right),$$

$$\mathfrak{N}_{m+1}(U_V) = \mathfrak{K}_{m+1}(U_V) \ominus \mathfrak{K}_m(U_V) = U_V \mathfrak{N}_m(U_V)$$

and

$$\mathfrak{M}_{m+1}(U_V) = \mathfrak{K}_{m+1}(U_V) \ominus U_V \mathfrak{K}_m(U_V) = \mathfrak{M}_m(U_V).$$

Hence we have that, for all $m \ge 1$,

$$P^{\mathfrak{K}_m(U_V)}_{\mathfrak{M}_{m-1}(U_V)} U_V|_{\mathfrak{N}_{m-1}(U_V)} = P^{\mathfrak{K}_m(U_V)}_{\mathfrak{M}} U_V^m|_{\mathfrak{N}} = 0.$$

This completes the proof. □

The pair $(U_V, \tilde{\mathfrak{H}}_V)$ is, up to isomorphisms, the unique element in $\mathbf{U}(V)$ if either $\mathfrak{N} = \{0\}$ or $\mathfrak{M} = \{0\}$. If, for some $m \ge 1$ either $\mathfrak{N}_{m-1} = \{0\}$ or $\mathfrak{M}_{m-1} = \{0\}$, then, by Proposition 1.1, either $\gamma_m^* \gamma_m = I$ or $\gamma_m \gamma_m^* = I$, respectively. In both cases the element $(U, \mathfrak{H})$ in $\mathbf{U}(V)$ that is obtained by means of the step by step construction is isomorphic to the canonical minimal unitary dilation of $V_{m-1} P^{\mathfrak{K}_{m-1}}_{\mathfrak{K}_{m-2}}$. Its first m Schur parameters are $\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$, and, for all $n \ge m$, $\gamma_n = 0$. By Theorem 1.2, any other pair in $\mathbf{U}(V)$ with the same Schur parameters is isomorphic to $(U, \tilde{\mathfrak{H}})$ and, in this sense, $(U, \tilde{\mathfrak{H}})$ is the unique element in $\mathbf{U}(V)$ with this (finite) sequence of Schur parameters. For $m = 1$ the above discussion is summarized in the following result.

Proposition 2.3 *Let $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$ be a contraction such that either $\gamma_1^* \gamma_1 = I$ or $\gamma_1 \gamma_1^* = I$. Then there exists, up to isomorphisms, a unique element $(U, \mathfrak{H})$ in $\mathbf{U}(V)$ with first Schur parameter equal to γ_1 and such that, for all $n \ge 2$, $\gamma_n = 0$.*

The *proof* of Theorem 2.1 will be given in the following series of lemmas. Assume that a sequence of pairs $\{(V_m, \mathfrak{K}_m)\}$ is defined, starting from the contraction $\gamma_1 \in \mathbf{L}(\mathfrak{N}, \mathfrak{M})$. Each V_m is a one-step isometric Hilbert space extension of V_{m-1} to all of

$$\mathfrak{K}_{m-1} = \mathfrak{D} \oplus \mathfrak{N} \oplus \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_{m-1},$$

which acts in $\mathfrak{K}_m = \mathfrak{K}_{m-1} \oplus \mathfrak{N}_m$ and satisfies

$$\mathfrak{K}_m = \text{c.l.s.}\,\{\mathfrak{K}_{m-1}, \operatorname{ran} V_m\} = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1, \cdots, \operatorname{ran} V_m\}.$$

Moreover, each V_m was obtained from V_{m-1} via a contraction $\gamma_m \in \mathbf{L}(\mathfrak{N}_{m-1}, \mathfrak{M}_{m-1})$ in such a way that $\gamma_m = P^{\mathfrak{K}_m}_{\mathfrak{M}_{m-1}} V_m|_{\mathfrak{N}_{m-1}}$, and $\mathfrak{N}_m$, $\mathfrak{M}_m$ are isomorphic to the defect spaces $\widetilde{\mathfrak{D}}_{\gamma_m}$, $\mathfrak{D}_{\gamma_m}$, respectively. Consider the Kreĭn space

$$\mathfrak{K} = \mathfrak{H} \oplus \left(\bigoplus_{j=1}^{\infty} \mathfrak{N}_j \right).$$

The following result is an immediate consequence of the construction.

Lemma 2.4 *We have* $\mathfrak{K} = \text{c.l.s.}\,\{\mathfrak{K}_m \mid m \in \mathbb{N}\}$.

From Lemma 2.4 it is easily shown that a densely defined isometry W_0 in the Kreĭn space $\mathfrak{K}$ is obtained by setting

(2.2) $W_0|_{\mathfrak{K}_m} = V_{m+1}, \quad m \in \mathbb{N}.$

Since $\mathfrak{H} \subseteq \operatorname{dom} W_0$ and $\mathfrak{K} \ominus \mathfrak{H}$ is a Hilbert space, it follows that W_0 is continuous. Therefore $W = \overline{W_0}$ is a continuous isometry defined on all of $\mathfrak{K}$.

Lemma 2.5 $W \in \mathbf{L}(\mathfrak{K})$ *is an isometric extension of* V*, it is a bicontraction and*

$$\mathfrak{K} = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} W\}.$$

Proof. Since $\mathfrak{R} \subseteq \operatorname{ran} V_1 \subseteq \operatorname{ran} W$ and $\mathfrak{K} \ominus \mathfrak{R}$ is a Hilbert space, we have that $\mathfrak{K} \ominus \operatorname{ran} W$ is a Hilbert space. For all $k \in \mathfrak{K}$,

$$\begin{aligned} \langle W^*k, W^*k \rangle_{\mathfrak{K}} &= \langle WW^*k, k \rangle_{\mathfrak{K}} = \langle P^{\mathfrak{K}}_{\operatorname{ran} W} k, k \rangle_{\mathfrak{K}} \\ &= \langle k, k \rangle_{\mathfrak{K}} - \langle P^{\mathfrak{K}}_{\mathfrak{K} \ominus \operatorname{ran} W} k, P^{\mathfrak{K}}_{\mathfrak{K} \ominus \operatorname{ran} W} k \rangle_{\mathfrak{K}} \leq \langle k, k \rangle_{\mathfrak{K}}. \end{aligned}$$

This shows that W^* is a contraction. Since

$$\mathfrak{K}_m = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_n \mid 0 \leq n \leq m\}, \quad m \in \mathbb{N},$$

Lemma 2.4 implies that

$$\mathfrak{K} = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_n \mid n \in \mathbb{N}\} = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} W\}.$$

This completes the proof. □

Since $\operatorname{dom} W = \mathfrak{K}$, the minimal unitary Hilbert space extension of W is unique up to isomorphisms and hence can be identified with the canonical minimal unitary dilation $(U, \tilde{\mathfrak{H}})$ of W given by

$$(2.3)\ \tilde{\mathfrak{H}} = \mathfrak{K} \oplus \left(\bigoplus_{1}^{\infty} (\mathfrak{K} \ominus \operatorname{ran} W) \right),$$

and

$$(2.4)\ U = \begin{pmatrix} W & \iota_{\mathfrak{K}}^{\mathfrak{K}\ominus\operatorname{ran} W} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & I & 0 & 0 & \cdots \\ & & & \cdots & & & \end{pmatrix},$$

where $\iota_{\mathfrak{K}}^{\mathfrak{K}\ominus\operatorname{ran} W}$ denotes the inclusion operator from $\mathfrak{K} \ominus \operatorname{ran} W$ into $\mathfrak{K}$.

Lemma 2.6 *For all $m \in \mathbb{N}$,*

$$\operatorname{ran} V_{m+1} \subseteq \text{c.l.s.}\,\{\operatorname{ran} V_m, U^{m+1}\mathfrak{H}, U^m\mathfrak{H}, \cdots, U\mathfrak{H}\}.$$

Proof. The lemma is easily checked for $m = 0$. For $m \geq 1$ we notice that

$$\begin{aligned} \operatorname{ran} V_{m+1} &= V_{m+1}\mathfrak{K}_m = V_{m+1}(V_m\mathfrak{K}_{m-1} \oplus \mathfrak{M}_m) \\ &= V_{m+1}V_m\mathfrak{K}_{m-1} \oplus V_{m+1}\mathfrak{M}_m \\ &= V_{m+1}V_m \cdots V_1\mathfrak{H} \oplus \left(\bigoplus_{n=1}^{m} V_{m+1}V_m \cdots V_{n+1}\mathfrak{M}_n \right) \\ &= U^{m+1}\mathfrak{H} \oplus \left(\bigoplus_{n=1}^{m} V_{m+1}V_m \cdots V_{n+1}\mathfrak{M}_n \right). \end{aligned}$$

From (2.1) it follows that, for each n, $1 \leq n \leq m$,

$$V_{m+1}V_m \cdots V_{n+1}\mathfrak{M}_n \subseteq \overline{V_{m+1}V_m \cdots V_{n+1}(I - V_n\gamma_n^*)\mathfrak{M}_{n-1}}.$$

Since $V_1 \subset V_2 \subset \cdots \subset V_m \subset V_{m+1}$ and $\mathfrak{M}_{k-1} \subseteq \mathfrak{K}_{k-1} = \operatorname{dom} V_k$, $k = 1, 2, \cdots$, we get

$$V_{m+1}V_m \cdots V_{n+1}\mathfrak{M}_{n-1} = V_mV_{m-1} \cdots V_n\mathfrak{M}_{n-1} \subseteq \operatorname{ran} V_m.$$

Also, since $\gamma_n^*\mathfrak{M}_{n-1} \subseteq \mathfrak{N}_{n-1} \subseteq \mathfrak{K}_{n-1}$, the inclusion

$$V_{m+1}V_m \cdots V_{n+1}V_n\gamma_n^*\mathfrak{M}_{n-1} \subseteq V_{m+1}V_m \cdots V_{n+1}V_n\mathfrak{K}_{n-1}$$

holds true. As $\mathfrak{K}_{n-1} = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1, \cdots, \operatorname{ran} V_{n-1}\}$, this inclusion implies that

$$V_{m+1}V_m \cdots V_{n+1}V_n\gamma_n^*\mathfrak{M}_{n-1} \subseteq \text{c.l.s.}\,\{U^{m+1}\mathfrak{H}, U^m\mathfrak{H}, \cdots, U^{m-n+2}\mathfrak{H}\}.$$

Consequently we have that, for $1 \leq n \leq m$,

$$V_{m+1}V_m \cdots V_{n+1}\mathfrak{M}_n \subseteq \text{c.l.s.}\,\{\operatorname{ran} V_m, U^{m+1}\mathfrak{H}, U^m\mathfrak{H}, \cdots, U^{m-n+1}\mathfrak{H}\},$$

and the lemma follows. □

Lemma 2.7 *For all $n \in \mathbf{N}$, $\mathfrak{K}_n = \text{c.l.s.}\,\{U^m\mathfrak{H} \mid 0 \le m \le n\}$.*

Proof. Since $\mathfrak{K}_1 = \text{c.l.s.}\,\{\mathfrak{H}, \operatorname{ran} V_1\}$ and $\operatorname{ran} V_1 = U\mathfrak{H}$, the result holds for $n = 1$. Now assume that

$$\mathfrak{K}_m = \text{c.l.s.}\,\{U^j\mathfrak{H} \mid 0 \le j \le m\}, \quad 0 \le m \le n.$$

We show that the result also holds for $m = n+1$. Let $k \in \mathfrak{K}_{n+1}$ be such that $\langle k, U^j\mathfrak{H}\rangle_{\mathfrak{K}} = \{0\}$, $0 \le j \le n+1$. Then the induction hypothesis implies that $\langle k, \mathfrak{K}_j\rangle_{\mathfrak{K}} = \{0\}$, $0 \le j \le n$, where $\mathfrak{K}_0 = \mathfrak{H}$. Consequently, $\langle k, \mathfrak{H}\rangle_{\mathfrak{K}} = \{0\}$ and $\langle k, \operatorname{ran} V_j\rangle_{\mathfrak{K}} = \{0\}$, $1 \le j \le n$. Hence $\langle k, \mathfrak{K}_n\rangle_{\mathfrak{K}} = \{0\}$ and $\langle k, \operatorname{ran} V_n\rangle_{\mathfrak{K}} = \{0\}$. From these equalities, the equality $\mathfrak{K}_{n+1} = \text{c.l.s.}\,\{\mathfrak{K}_n, \operatorname{ran} V_{n+1}\}$ and Lemma 2.6, we conclude that $k = 0$. □

Lemma 2.8 *We have $\widetilde{\mathfrak{H}} = \text{c.l.s.}\,\{U^n\mathfrak{H} \mid n \in \mathbf{Z}\}$.*

Proof. Suppose that $\widetilde{h} \in \widetilde{\mathfrak{H}}$ is such that, for all $n \in \mathbf{Z}$, $\langle \widetilde{h}, U^n\mathfrak{H}\rangle_{\widetilde{\mathfrak{H}}} = \{0\}$. By Lemma 2.7, we get that, for all $n \in \mathbf{N}$, $\langle \widetilde{h}, \mathfrak{K}_n\rangle_{\widetilde{\mathfrak{H}}} = \{0\}$. Hence, according to Lemma 2.4, $\langle \widetilde{h}, \mathfrak{K}\rangle_{\widetilde{\mathfrak{H}}} = \{0\}$. From this relation and the definition of $(U, \widetilde{\mathfrak{H}})$ given in (2.3), (2.4), we conclude that $\widetilde{h} \in \bigoplus_1^\infty(\mathfrak{K} \ominus \operatorname{ran} W)$. That is, $\widetilde{h} = \{0, z_1, z_2, \cdots\}$, where $\{z_n\} \subseteq \mathfrak{K} \ominus \operatorname{ran} W$. We claim that, for all $n \ge 1$, $z_n = 0$. According to Lemma 2.5, $\mathfrak{K} \ominus \operatorname{ran} W = \overline{P^{\mathfrak{K}}_{\mathfrak{K}\ominus\operatorname{ran} W}\mathfrak{H}}$. So, to prove the claim, it is enough to show that each z_n is orthogonal to $P^{\mathfrak{K}}_{\mathfrak{K}\ominus\operatorname{ran} W}\mathfrak{H}$. Since

$$U^{-1} = \begin{pmatrix} W^* & 0 & 0 & 0 & \cdots \\ P^{\mathfrak{K}}_{\mathfrak{K}\ominus\operatorname{ran} W} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ & & \cdots & & \end{pmatrix},$$

we have that, for all $h \in \mathfrak{H}$, $\langle z_1, P^{\mathfrak{K}}_{\mathfrak{K}\ominus\operatorname{ran} W}h\rangle_{\widetilde{\mathfrak{H}}} = \langle \widetilde{h}, U^{-1}h\rangle_{\widetilde{\mathfrak{H}}} = 0$. In a similar way it can be seen that, for all $n > 1$ and $h \in \mathfrak{H}$, $\langle z_n, P^{\mathfrak{K}}_{\mathfrak{K}\ominus\operatorname{ran} W}h\rangle_{\widetilde{\mathfrak{H}}} = 0$. Hence $\widetilde{h} = 0$. □

The following lemma provides the proof of Theorem 2.1.

Lemma 2.9 *The pair $(U, \widetilde{\mathfrak{H}})$ is, up to isomorphisms, the unique element in* $\mathbf{U}(V)$ *whose Schur parameters are given by the sequence $\{\gamma_m\}$.*

Proof. The Kreĭn space $\widetilde{\mathfrak{H}}$ contains $\mathfrak{H}$ as regular subspace and

$$\widetilde{\mathfrak{H}} \ominus \mathfrak{H} = (\widetilde{\mathfrak{H}} \ominus \mathfrak{K}) \oplus (\mathfrak{K} \ominus \mathfrak{H}),$$

where

$$\widetilde{\mathfrak{H}} \ominus \mathfrak{K} = \bigoplus_1^\infty(\mathfrak{K} \ominus \operatorname{ran} W), \quad \mathfrak{K} \ominus \mathfrak{H} = \bigoplus_{j=1}^\infty \mathfrak{N}_j,$$

are Hilbert spaces. The operator $U \in \mathbf{L}(\widetilde{\mathfrak{H}})$ is a unitary extension of V, since, for all $h \in \mathfrak{D}$, $Uh = Wh = Vh$. By Lemma 2.8, $(U, \widetilde{\mathfrak{H}})$ is minimal. Hence $(U, \widetilde{\mathfrak{H}})$ belongs to $\mathbf{U}(V)$. Clearly, $\{\gamma_m\}$ is the sequence of Schur parameters of $(U, \widetilde{\mathfrak{H}})$. The uniqueness easily follows from the construction. □

3 On a Schur type algorithm

We still assume that V is a continuous isometric operator acting in a Kreĭn space $\mathfrak{H}$ such that $\mathfrak{D} = \operatorname{dom} V$ and $\mathfrak{R} = \operatorname{ran} V$ are regular subspaces of $\mathfrak{H}$ whose orthogonal complements $\mathfrak{N}$ and $\mathfrak{M}$, respectively, are Hilbert subspaces of $\mathfrak{H}$. A subspace $\mathfrak{G}$ of $\mathfrak{H}$ is called a *scale subspace* of V if it is regular and if

(3.1) $\mathfrak{H} = \text{c.l.s.}\,\{\mathfrak{D}, \mathfrak{G}\} = \text{c.l.s.}\,\{\mathfrak{R}, \mathfrak{G}\}$.

In the sequel we fix a scale space and denote it by $\mathfrak{G}$. The minimality of $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ can be expressed in terms of $\mathfrak{G}$ in the following way.

Lemma 3.1 *We have* $\widetilde{\mathfrak{H}} = \text{c.l.s.}\,\{\mathfrak{D}, U^n\mathfrak{G} \mid n \in \mathbf{Z}\} = \text{c.l.s.}\,\{\mathfrak{R}, U^n\mathfrak{G} \mid n \in \mathbf{Z}\}$.

Proof. From (3.1), it follows that, for all $n \geq 1$,

$$U^n\mathfrak{H} \subseteq \text{c.l.s.}\,\{\mathfrak{H}, U\mathfrak{G}, \cdots, U^n\mathfrak{G}\}$$

and

$$U^{-n}\mathfrak{H} \subseteq \text{c.l.s.}\,\{\mathfrak{H}, U^{-1}\mathfrak{G}, \cdots, U^{-n}\mathfrak{G}\}.$$

Since $\widetilde{\mathfrak{H}} = \text{c.l.s.}\,\{U^n\mathfrak{H} \mid n \in \mathbf{Z}\}$, these inclusions imply the lemma. □

If $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$, then the sequence $\{G_m\} \subseteq \mathbf{L}(\mathfrak{G})$ defined by

(3.2) $G_m = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{G}} U^m|_{\mathfrak{G}}, \quad m \geq 1,$

is called the sequence of *moments* of U *relative to the scale space* $\mathfrak{G}$.

Proposition 3.2 *The mapping* $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V) \mapsto \{G_m\} \subseteq \mathbf{L}(\mathfrak{G})$ *is one-to-one, up to isomorphisms.*

Proof. Let $(U, \widetilde{\mathfrak{H}})$, $(U', \widetilde{\mathfrak{H}}') \in \mathbf{U}(V)$ be such that

$$G_m = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{G}} U^m|_{\mathfrak{G}} = P^{\widetilde{\mathfrak{H}}'}_{\mathfrak{G}} U'^m|_{\mathfrak{G}} = G'_m, \quad m \geq 1.$$

Then we have that, for all $g, g' \in \mathfrak{G}$ and $n, m \in \mathbf{Z}$, $n > m$,

$$\begin{aligned} \langle U'^n g, U'^m g' \rangle_{\widetilde{\mathfrak{H}}'} &= \langle U'^{n-m} g, g' \rangle_{\widetilde{\mathfrak{H}}'} = \langle G'_{n-m} g, g' \rangle_{\mathfrak{H}} \\ &= \langle G_{n-m} g, g' \rangle_{\mathfrak{H}} = \langle U^n g, U^m g' \rangle_{\widetilde{\mathfrak{H}}}. \end{aligned}$$

From this and (3.1) it can be seen that, for all $h \in \mathfrak{H}$, $g \in \mathfrak{G}$ and $n \in \mathbf{Z}$,

$$\langle U'^n g, h \rangle_{\widetilde{\mathfrak{H}}'} = \langle U^n g, h \rangle_{\widetilde{\mathfrak{H}}}.$$

Hence the linear mapping φ_0 defined by

$$\varphi_0(h + U^n g) = h + U'^n g, \quad h \in \mathfrak{H}, \ g \in \mathfrak{G}, \ n \in \mathbf{Z},$$

is well defined and isometric, and, by Lemma 3.1, φ_0 is a weak isomorphism from $\tilde{\mathfrak{H}}$ into $\tilde{\mathfrak{H}}'$. Since $\mathfrak{H} \subseteq \tilde{\mathfrak{H}}$ and $\tilde{\mathfrak{H}} \ominus \mathfrak{H}$ is a Hilbert space, it follows that φ_0 is continuous and hence $\varphi = \overline{\varphi_0}$ is a unitary operator from $\tilde{\mathfrak{H}}$ onto $\tilde{\mathfrak{H}}'$. By construction $(U, \tilde{\mathfrak{H}})$ and $(U', \tilde{\mathfrak{H}}')$ are isomorphic under φ. □

In general, the mapping in Proposition 3.2 will not be surjective: not every sequence $\{G_m\} \subseteq \mathbf{L}(\mathfrak{G})$ can be "realised" as the sequence of moments of some $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ with scale subspace $\mathfrak{G}$. This gives rise to the following problems..

The *moment problem with scale subspace* or the M- *problem*:
Given a sequence $\{G_m\} \subseteq \mathbf{L}(\mathfrak{G})$, determine if it is the sequence of moments of some $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ relative to $\mathfrak{G}$. If this is the case, find all solutions $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$.

The *truncated moment problem with scale subspace* or the M_m-*problem*:
Given $\{G_1, G_2, \cdots, G_m\} \subseteq \mathbf{L}(\mathfrak{G})$, determine if there exists a pair $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ such that

$$G_n = P_{\mathfrak{G}}^{\tilde{\mathfrak{H}}} U^n |_{\mathfrak{G}}, \quad 1 \leq n \leq m.$$

If this is the case, find all solutions $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$.

Proposition 3.2 implies that the M-problem either has no solution or, up to isomorphisms, a unique solution.

The above problems can be treated by means of a Schur type algorithm, which is the same as the one given in [2] for the corresponding problems in the Hilbert space setting. We reproduce it here to make our treatment selfcontained.

To relate the moments of $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ to the associated Schur parameters γ_m, we recall the following definitions:

$$\mathfrak{K}_{-1} = \mathfrak{D}, \quad \mathfrak{K}_0 = \mathfrak{H}, \quad V_0 = V, \quad \mathfrak{N}_0 = \mathfrak{N}, \quad \mathfrak{M}_0 = \mathfrak{M},$$

$$\mathfrak{K}_m = \text{c.l.s.}\{U^n \mathfrak{H} \mid 0 \leq n \leq m\}, \quad V_m = U|_{\mathfrak{K}_{m-1}}, \quad m \geq 1,$$

$$\mathfrak{N}_m = \mathfrak{K}_m \ominus \mathfrak{K}_{m-1}, \quad \mathfrak{M}_m = \mathfrak{K}_m \ominus U\mathfrak{K}_{m-1}, \quad m \geq 1,$$

$$(3.3)\quad \gamma_m = P_{\mathfrak{M}_{m-1}}^{\mathfrak{K}_m} V_m|_{\mathfrak{N}_{m-1}} = P_{\mathfrak{K}_{m-1}}^{\mathfrak{K}_m} V_m|_{\mathfrak{N}_{m-1}}, \quad m \geq 1.$$

We consider V_m as an isometry acting in the space $\mathfrak{K}_m$. The following properties are easily checked:

$$(V_m, \mathfrak{K}_m) \in \mathbf{V}(V_{m-1}),$$

$$\mathfrak{K}_m = \text{c.l.s.}\{\mathfrak{G}, \operatorname{ran} V_m\} = \text{c.l.s.}\{\mathfrak{K}_{m-1}, V_m^m \mathfrak{G}\},$$

$$(3.4)\quad \mathfrak{N}_m = \overline{P_{\mathfrak{N}_m}^{\mathfrak{K}_m} V_m^m \mathfrak{G}}, \quad \mathfrak{M}_m = \overline{P_{\mathfrak{M}_m}^{\mathfrak{K}_m} \mathfrak{G}}.$$

Lemma 3.3 *Let $(U,\tilde{\mathfrak{H}})$ belong to $\mathbf{U}(V)$. Then the associated isometries $V_1,\cdots,V_m$ and Schur parameters $\gamma_1,\cdots,\gamma_m$ are, up to isomorphisms, uniquely determined by its moments $G_1,\cdots,G_m$. Moreover, for $j=0,\cdots,m-1$,*

$$(a)_{j+1}\quad \begin{cases} \left|\langle G_{j+1}g - V_j P^{\mathfrak{K}_j}_{\mathfrak{K}_{j-1}} V_j^j g, g'\rangle\right|^2 \leq \langle P^{\mathfrak{K}_j}_{\mathfrak{N}_j} V_j^j g, P^{\mathfrak{K}_j}_{\mathfrak{N}_j} V_j^j g\rangle \times \\ \times\langle P^{\mathfrak{K}_j}_{\mathfrak{M}_j} g', P^{\mathfrak{K}_j}_{\mathfrak{M}_j} g'\rangle, \quad g,g'\in\mathfrak{G}, \end{cases}$$

and

$$(b)_{j+1}\quad G_{j+1} = P^{\mathfrak{K}_j}_{\mathfrak{G}}(V_j P^{\mathfrak{K}_j}_{\mathfrak{K}_{j-1}} + \gamma_{j+1} P^{\mathfrak{K}_j}_{\mathfrak{N}_j}) V_j^j|_{\mathfrak{G}}.$$

Proof. The uniqueness of the pairs $(V_j,\mathfrak{K}_j)$, up to an isomorphism, can be proved in a similar way as Proposition 3.2, and by (3.3) these pairs determine the γ_j's. The formula $(a)_{j+1}$ follows from $(b)_{j+1}$ and the Cauchy-Schwartz inequality. Finally, $(b)_{j+1}$ follows from the definitions of V_j and γ_j. □

Suppose the operators $G_1,\cdots,G_m,G_{m+1},\cdots$ in $\mathbf{L}(\mathfrak{G})$ are given. To describe the $(m+1)$-step of the algorithm we assume that the M_m-problem is solvable. By Lemma 3.3 the data give rise to the isometries $V_1,\cdots,V_m$ and Schur parameters $\gamma_1,\cdots,\gamma_m$. There are two possibilities:

(a) the defect-subspaces of V_m are nontrivial, or

(b) at least one of them is trivial.

In case (b) the algorithm terminates after the m-th step: the M_m-problem has, up to isomorphisms, a unique solution; this is also the only solution of the M_{m+j}-problems and the M-problem if and only if its moments match the data $G_{m+1},\cdots,G_{m+j},\cdots$.

In case (a) the M_{m+1}-problem is solvable if and only if $(a)_{m+1}$ holds. The necessity follows from Lemma 3.3. If $(a)_{m+1}$ holds then, by property (3.4) above, the formula

$$\{P^{\mathfrak{K}_m}_{\mathfrak{N}_m} V_m^m g, P^{\mathfrak{K}_m}_{\mathfrak{M}_m} g'\} \mapsto \langle G_{m+1}g - V_m P^{\mathfrak{K}_m}_{\mathfrak{K}_{m-1}} V_m^m g, g'\rangle_{\mathfrak{K}_m}$$

defines a bounded sesquilinear form $\varepsilon_{m+1}:\mathfrak{N}_m\oplus\mathfrak{M}_m\to\mathbf{C}$, and there exists a contraction $\gamma_{m+1}\in\mathbf{L}(\mathfrak{N}_m,\mathfrak{M}_m)$ such that

$$\varepsilon_{m+1}\{y_m,x_m\} = \langle\gamma_{m+1}y_m,x_m\rangle_{\mathfrak{K}_m},\quad y_m\in\mathfrak{N}_m,\quad x_m\in\mathfrak{M}_m,$$

and $(b)_{m+1}$ is valid. From $\gamma_{m+1}\in\mathbf{L}(\mathfrak{N}_m,\mathfrak{M}_m)$ and Theorem 1.2 we obtain a pair $(V_{m+1},\mathfrak{K}_{m+1})\in\mathbf{V}(V_m)$ such that (3.3) is valid with m replaced by $m+1$. It is unique up to isomorphisms. Again by $(b)_{m+1}$ and Theorem 1.2, we get

$$\langle V_{m+1}^{m+1}g,g'\rangle_{\mathfrak{K}_{m+1}} = \langle G_{m+1}g,g'\rangle_{\mathfrak{H}},\quad g,g'\in\mathfrak{G},$$

and the M_{m+1}-problem is solvable.

Proposition 3.4 *Assume that the M_m-problem is solvable and that the isometry V_m has nontrivial defect subspaces. Then the M_{m+1}-problem is solvable if and only if $(a)_{m+1}$ holds. In this case the solutions are given by the pairs $(U,\tilde{\mathfrak{H}})\in\mathbf{U}(V)$ with first m Schur parameters equal to $\{\gamma_1,\gamma_2,\cdots,\gamma_m\}$ and $(m+1)$-Schur parameter γ_{m+1} determined by G_{m+1} by means of $(b)_{m+1}$.*

Proposition 3.5 *If $\gamma_m^*\gamma_m = I$ and $\gamma_m\gamma_m^* = I$ never hold, every* M_m*-problem has infinitely many solutions and the* M*-problem has exactly one solution, namely, the pair* $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ *with Schur parameters* $\{\gamma_m\}$ *determined by* $\{G_m\}$ *via* $(b)_m$.

4 Scattering matrices with scale subspace

In this section we introduce *scattering matrices* for isometries and their unitary extensions. We present a parametrization for the scattering matrices associated with unitary extensions. This parametrization is due to [5] (see also [6]) for the Hilbert space case. Here we treat the Kreĭn space case. We assume that the isometry V has the same properties as in the forgoing sections and that $\mathfrak{G}$ is a scale space for V.

Corresponding to the defect spaces $\mathfrak{N}$ and $\mathfrak{M}$ of V we introduce the Hilbert spaces $\widehat{\mathfrak{N}}$ and $\widehat{\mathfrak{M}}$ and the *output* and *input* isometries $\Gamma_o : \widehat{\mathfrak{N}} \to \mathfrak{H}$ and $\Gamma_i : \widehat{\mathfrak{M}} \to \mathfrak{H}$ with the properties $\operatorname{ran} \Gamma_o = \mathfrak{N}$, $\operatorname{ran} \Gamma_i = \mathfrak{M}$, so that $\ker \Gamma_o^* = \mathfrak{D}$, $\ker \Gamma_i^* = \mathfrak{R}$. Then $\{\mathfrak{H}, \widehat{\mathfrak{M}}, \widehat{\mathfrak{N}}; \widehat{W}\}$ with

$$\widehat{W} = \begin{pmatrix} VP_{\mathfrak{D}}^{\mathfrak{H}} & \Gamma_i \\ \Gamma_o^* & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \widehat{\mathfrak{M}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \widehat{\mathfrak{N}} \end{pmatrix}$$

is a unitary colligation, whose characteristic function is usually called *the characteristic function* of the isometric operator V.

Corresponding to the scale space $\mathfrak{G}$ we introduce the Kreĭn space $\mathfrak{G}_e$ and the isometry $\Gamma_e : \mathfrak{G}_e \to \mathfrak{H}$ with the property $\operatorname{ran} \Gamma_e = \mathfrak{G}$, so that $\ker \Gamma_e^* = \mathfrak{H} \ominus \mathfrak{G}$. Rewrite $\widehat{W}$ as a 3×3 block matrix with respect to the decompositions

$$\mathfrak{H} \oplus \widehat{\mathfrak{M}} = (\mathfrak{H} \ominus \mathfrak{G}) \oplus \mathfrak{G} \oplus \widehat{\mathfrak{M}}, \quad \mathfrak{H} \oplus \widehat{\mathfrak{N}} = (\mathfrak{H} \ominus \mathfrak{G}) \oplus \mathfrak{G} \oplus \widehat{\mathfrak{N}}.$$

With $\widehat{W}$ we associate the unitary operator defined by

$$W_{\mathfrak{G}} = \begin{pmatrix} I & 0 & 0 \\ 0 & \Gamma_e^* & 0 \\ 0 & 0 & I \end{pmatrix} \widehat{W} \begin{pmatrix} I & 0 & 0 \\ 0 & \Gamma_e & 0 \\ 0 & 0 & I \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \ominus \mathfrak{G} \\ \mathfrak{G}_e \\ \widehat{\mathfrak{M}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \ominus \mathfrak{G} \\ \mathfrak{G}_e \\ \widehat{\mathfrak{N}} \end{pmatrix}.$$

For $W_{\mathfrak{G}}$ we introduce the following block decomposition

$$W_{\mathfrak{G}} = \begin{pmatrix} T_{\mathfrak{G}} & F_{\mathfrak{G}} \\ G_{\mathfrak{G}} & H_{\mathfrak{G}} \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \ominus \mathfrak{G} \\ \mathfrak{G}_e \oplus \widehat{\mathfrak{M}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \ominus \mathfrak{G} \\ \mathfrak{G}_e \oplus \widehat{\mathfrak{N}} \end{pmatrix},$$

where

(a) $T_{\mathfrak{G}} = P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} V P_{\mathfrak{D}}^{\mathfrak{H}}|_{\mathfrak{H}\ominus\mathfrak{G}} : \mathfrak{H} \ominus \mathfrak{G} \to \mathfrak{H} \ominus \mathfrak{G}$,

(b) $F_{\mathfrak{G}} = (P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} V P_{\mathfrak{D}}^{\mathfrak{H}} \Gamma_e\ ,\ P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i) : \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{M}} \end{pmatrix} \to \mathfrak{H} \ominus \mathfrak{G}$,

(c) $G_{\mathfrak{G}} = \begin{pmatrix} \Gamma_e^* V P_{\mathfrak{D}}^{\mathfrak{H}}|_{\mathfrak{H}\ominus\mathfrak{G}} \\ \Gamma_o^*|_{\mathfrak{H}\ominus\mathfrak{G}} \end{pmatrix} : \mathfrak{H} \ominus \mathfrak{G} \to \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{N}} \end{pmatrix}$,

(d) $H_{\mathfrak{G}} = \begin{pmatrix} \Gamma_e^* V P_{\mathfrak{D}}^{\mathfrak{H}} \Gamma_e & \Gamma_e^* \Gamma_i \\ \Gamma_o^* \Gamma_e & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{M}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{N}} \end{pmatrix}.$

Observe that $\{\mathfrak{H} \ominus \mathfrak{G}, \mathfrak{G}_e \oplus \widehat{\mathfrak{M}}, \mathfrak{G}_e \oplus \widehat{\mathfrak{N}}; W_{\mathfrak{G}}\}$ is a unitary colligation of operators on Kreĭn spaces. Its characteristic function $\Theta_{V,\mathfrak{G}}$ is the *scattering matrix* of V *with respect to the scale subspace* $\mathfrak{G}$. It is an $\mathbf{L}(\mathfrak{G}_e \oplus \widehat{\mathfrak{M}}, \mathfrak{G}_e \oplus \widehat{\mathfrak{N}})$-valued function defined and holomorphic for those $z \in \mathbf{C}$ for which $(I - zT_{\mathfrak{G}})^{-1} \in \mathbf{L}(\mathfrak{H} \ominus \mathfrak{G})$. For each z in the domain of holomorphy, $\Theta_{V,\mathfrak{G}}(z)$ can be represented as a 2×2 block matrix

$$\begin{pmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{pmatrix} : \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{M}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{G}_e \\ \widehat{\mathfrak{N}} \end{pmatrix}.$$

Write $T = V P_{\mathfrak{D}}^{\mathfrak{H}}$. We obtain the following expressions for the Θ_{jk}'s:

$$(4.1) \quad \begin{cases} \Theta_{11}(z) : \mathfrak{G}_e \to \mathfrak{G}_e, \\ \Theta_{11}(z) = \Gamma_e^* T(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} \Gamma_e = \Gamma_e^*(I - zTP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}})^{-1} T \Gamma_e, \end{cases}$$

$$(4.2) \quad \begin{cases} \Theta_{12}(z) : \widehat{\mathfrak{M}} \to \mathfrak{G}_e, \\ \Theta_{12}(z) = \Gamma_e^*(I - zTP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}})^{-1} \Gamma_i, \end{cases}$$

$$(4.3) \quad \begin{cases} \Theta_{21}(z) : \mathfrak{G}_e \to \widehat{\mathfrak{N}}, \\ \Theta_{21}(z) = \Gamma_o^*(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} \Gamma_e, \end{cases}$$

$$(4.4) \quad \begin{cases} \Theta_{22}(z) : \widehat{\mathfrak{M}} \to \widehat{\mathfrak{N}}, \\ \Theta_{22}(z) = z\Gamma_o^*(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i = z\Gamma_o^* P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}}(I - zTP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}})^{-1} \Gamma_i. \end{cases}$$

Next we consider a pair $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$. Since, by definition, $\mathfrak{H} \subseteq \widetilde{\mathfrak{H}}$ is a regular subspace, we may interpret Γ_o, Γ_i and Γ_e as isometries $\Gamma_o : \widehat{\mathfrak{N}} \to \widetilde{\mathfrak{H}}$, $\Gamma_i : \widehat{\mathfrak{M}} \to \widetilde{\mathfrak{H}}$ and $\Gamma_e : \mathfrak{G}_e \to \widetilde{\mathfrak{H}}$, so that Γ_o^*, Γ_i^* and Γ_e^* are bounded linear mappings from $\widetilde{\mathfrak{H}}$ to $\widehat{\mathfrak{N}}$, $\widehat{\mathfrak{M}}$ and $\mathfrak{G}_e$, respectively. The operators $\Gamma_o\Gamma_o^*$, $\Gamma_i\Gamma_i^*$ and $\Gamma_e\Gamma_e^*$ are the projections from $\widetilde{\mathfrak{H}}$ onto $\mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{G}$, respectively. Also, with these interpretations, we have

$$(4.5) \quad U - TP_{\mathfrak{H}}^{\widetilde{\mathfrak{H}}} = \Gamma_i \Gamma_i^* U.$$

The unitary operator $U \in \mathbf{L}(\widetilde{\mathfrak{H}})$ can be regarded as an isometric operator with zero defect subspaces. Consequently, also for U a scattering matrix relative to $\mathfrak{G}$ is well defined. More precisely, set $\mathfrak{H}_U = \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ and $\mathfrak{H}_{U,\mathfrak{G}} = \widetilde{\mathfrak{H}} \ominus \mathfrak{G} = \mathfrak{H}_U \oplus (\mathfrak{H} \ominus \mathfrak{G})$. Then U admits a 2×2 block matrix representation with respect to the decomposition $\widetilde{\mathfrak{H}} = \mathfrak{H}_{U,\mathfrak{G}} \oplus \mathfrak{G}$. Let $X_{U,\mathfrak{G}}$ be the unitary operator in $\mathfrak{H}_{U,\mathfrak{G}} \oplus \mathfrak{G}_e$ obtained from U by

$$X_{U,\mathfrak{G}} = \begin{pmatrix} I & 0 \\ 0 & \Gamma_e^* \end{pmatrix} U \begin{pmatrix} I & 0 \\ 0 & \Gamma_e \end{pmatrix},$$

that is,

$$X_{U,\mathfrak{G}} = \begin{pmatrix} P_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U|_{\mathfrak{H}_{U,\mathfrak{G}}} & P_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U\Gamma_e \\ \Gamma_e^* U|_{\mathfrak{H}_{U,\mathfrak{G}}} & \Gamma_e^* U \Gamma_e \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_{U,\mathfrak{G}} \\ \mathfrak{G}_e \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_{U,\mathfrak{G}} \\ \mathfrak{G}_e \end{pmatrix}.$$

Then $\{\mathfrak{H}_{U,\mathfrak{G}}, \mathfrak{G}_e, \mathfrak{G}_e; X_{U,\mathfrak{G}}\}$ is a unitary colligation and its characteristic function $\Theta_{U,\mathfrak{G}}$ is the *scattering matrix* of $(U,\tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ *with respect to the scale subspace* $\mathfrak{G}$. A straightforward computation gives

$$(4.6)\quad \Theta_{U,\mathfrak{G}}(z) = \Gamma_e^* U(I - zP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)^{-1}\Gamma_e = \Gamma_e^*(I - zUP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}})^{-1}U\Gamma_e,$$

for those $z \in \mathbf{C}$ such that $(I - zP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U\,|\,\mathfrak{H}_{U,\mathfrak{G}})^{-1} \in \mathbf{L}(\mathfrak{H}_{U,\mathfrak{G}})$. This formula implies that the scattering matrix determines $(U,\tilde{\mathfrak{H}}) \in \mathbf{U}(V)$, up to isomorphisms.

We associate another operator-valued holomorphic function to the pair $(U,\tilde{\mathfrak{H}}) \in \mathbf{U}(V)$. The unitary operator $U \in \mathbf{L}(\tilde{\mathfrak{H}})$ admits the 3×3 block matrix representation

$$U = \begin{pmatrix} T_U & F_U & 0 \\ G_U & H_U & 0 \\ 0 & 0 & V \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_U \\ \mathfrak{N} \\ \mathfrak{D} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_U \\ \mathfrak{M} \\ \mathfrak{R} \end{pmatrix},$$

and $\{\mathfrak{H}_U, \mathfrak{N}, \mathfrak{M}; T_U, F_U, G_U, H_U\}$ is a unitary colligation of operators in Hilbert spaces. Its characteristic function $\Theta_U(z) = H_U + zG_U(I - zT_U)^{-1}F_U$ belongs to the Schur class $\mathbf{S}(\mathfrak{N},\mathfrak{M})$ of all $\mathbf{L}(\mathfrak{N},\mathfrak{M})$-valued functions ε which are defined and holomorphic on $\mathbf{D}$ and such that, for each $z \in \mathbf{D}$, $\varepsilon(z)$ is a contraction. The function Θ_U has the following representation:

$$\Theta_U(z) = P^{\tilde{\mathfrak{H}}}_{\mathfrak{M}}(I - zUP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_U})^{-1}U\,|\,\mathfrak{N} = P^{\tilde{\mathfrak{H}}}_{\mathfrak{M}}U(I - zP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_U}U)^{-1}\,|\,\mathfrak{N}, \quad z \in \mathbf{D}.$$

The operator block

$$\begin{pmatrix} \widehat{T}_U & \widehat{F}_U \\ \widehat{G}_U & \widehat{H}_U \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \Gamma_i^* \end{pmatrix}\begin{pmatrix} T_U & F_U \\ G_U & H_U \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \Gamma_o \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_U \\ \widehat{\mathfrak{N}} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_U \\ \widehat{\mathfrak{M}} \end{pmatrix}$$

defines a unitary colligation $\{\mathfrak{H}_U, \widehat{\mathfrak{N}}, \widehat{\mathfrak{M}}; \widehat{T}_U, \widehat{F}_U, \widehat{G}_U, \widehat{H}_U\}$ with characteristic function $\varepsilon_U(z) = \Gamma_i^*\Theta_U(z)\Gamma_o$, $z \in \mathbf{D}$. The main result of this section is the following theorem. In the Hilbert space case it can be found in [5], Theorem 2; for proofs see [6].

Theorem 4.1 *There exists a neighborhood* $\mathcal{V}_0 \subseteq \mathbf{D}$ *of* 0 *such that the formula*

$$(4.7)\quad \Theta_{U,\mathfrak{G}}(z) = \Theta_{11}(z) + \Theta_{12}(z)\varepsilon(z)(I - \Theta_{22}(z)\varepsilon(z))^{-1}\Theta_{21}(z), \quad z \in \mathcal{V}_0,$$

with $\varepsilon(z) = \varepsilon_U(z)$, $z \in \mathbf{D}$, *sets up a bijection between* $\mathbf{S}(\widehat{\mathfrak{N}},\widehat{\mathfrak{M}})$ *and the set of all scattering matrices* $\{\Theta_{U,\mathfrak{G}} \mid (U,\tilde{\mathfrak{H}}) \in \mathbf{U}(V)\}$.

The *proof* of Theorem 4.1 is given after the following lemmas. In the remaining part of this section, we freely use the notations introduced above.

Lemma 4.2 *If* $(U,\tilde{\mathfrak{H}}) \in \mathbf{U}(V)$, *then*

$$(4.8)\quad I - zP^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U = (I - z\widehat{T}_U - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{G}_U)P^{\tilde{\mathfrak{H}}}_{\mathfrak{H}_U} + \\ +(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)P^{\mathfrak{H}}_{\mathfrak{D}}P^{\tilde{\mathfrak{H}}}_{\mathfrak{H}} + (\Gamma_o - z\widehat{F}_U - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{H}_U)\Gamma_o^*, \quad z \in \mathbf{C}.$$

Proof. Let $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$. The 3×3 block decomposition of U gives

$$(4.9)\quad \begin{aligned} U &= T_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + F_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{N}} + G_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + H_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{N}} + V P^{\mathfrak{H}}_{\mathfrak{D}} P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}} \\ &= \widehat{T}_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + \widehat{F}_U \Gamma_o^* + \Gamma_i \widehat{G}_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + \Gamma_i \widehat{H}_U \Gamma_o^* + T P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}. \end{aligned}$$

Hence

$$P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U = \widehat{T}_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + \widehat{F}_U \Gamma_o^* + P^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} \Gamma_i \widehat{G}_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U} + P^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} \Gamma_i \widehat{H}_U \Gamma_o^* + P^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} T P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}},$$

and this implies (4.8). □

Lemma 4.3 *There exists a neighborhood $\mathcal{V}_0 \subseteq \mathbf{D}$ of 0 such that, for all $z \in \mathcal{V}_0$ and $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$,*

$$(I - \Theta_{22}(z)\varepsilon_U(z))^{-1} \in \mathbf{L}(\widehat{\mathfrak{N}}).$$

Proof. By (4.4), we have $\Theta_{22}(0) = 0$. Therefore, given $0 < r < 1$, there exists a $\delta > 0$ such that, for z with $|z| < \delta$, we have $\|\Theta_{22}(z)\| < \sqrt{r}$. For all $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ and z with $|z| < \delta$, it follows that

$$\langle \Theta_{22}(z)\varepsilon_U(z)\widehat{y}, \Theta_{22}(z)\varepsilon_U(z)\widehat{y}\rangle_{\widehat{\mathfrak{N}}} < r\langle \widehat{y}, \widehat{y}\rangle_{\widehat{\mathfrak{N}}}, \quad \widehat{y} \in \widehat{\mathfrak{N}}.$$

This means that, for all $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ and z with $|z| < \delta$, $\Theta_{22}(z)\varepsilon_U(z)$ is a strict contraction in the Hilbert space $\widehat{\mathfrak{N}}$. Clearly, $\mathcal{V}_0 = \{z \in \mathbf{D} \mid |z| < \delta\}$ verifies the claim. □

We may suppose that $\Theta_{V,\mathfrak{G}}$ is holomorphic in $\mathcal{V}_0$ and that, for all $z \in \mathcal{V}_0$, $(I - zT_{\mathfrak{G}})^{-1} \in \mathbf{L}(\mathfrak{H} \ominus \mathfrak{G})$. We also have, for $z \in \mathcal{V}_0$, $(I - zT_U)^{-1} \in \mathbf{L}(\mathfrak{H}_U)$.

Lemma 4.4 *For all $z \in \mathcal{V}_0$ and $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$,*

$$(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U|_{\mathfrak{H}_{U,\mathfrak{G}}})^{-1} \in \mathbf{L}(\mathfrak{H}_{U,\mathfrak{G}}).$$

Therefore, for all $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$, $\Theta_{U,\mathfrak{G}}$ is defined and holomorphic in $\mathcal{V}_0$.

Proof. First we show that, for each $z \in \mathcal{V}_0$, $I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U$ is an injective operator on $\mathfrak{H}_{U,\mathfrak{G}}$. Assume that, for some $\widetilde{h} \in \mathfrak{H}_{U,\mathfrak{G}}$, $(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)\widetilde{h} = 0$. Since $\mathfrak{H}_{U,\mathfrak{G}} = \mathfrak{H}_U \oplus (\mathfrak{H} \ominus \mathfrak{G})$, we obtain from (4.8) and our assumption that

$$(4.10)\quad 0 = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U}(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)\widetilde{h} = (I - z\widehat{T}_U)P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U}\widetilde{h} - z\widehat{F}_U \Gamma_o^* \widetilde{h}$$

and

$$(4.11)\quad \begin{aligned} 0 = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}\ominus\mathfrak{G}}(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)\widetilde{h} &= -zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} \Gamma_i \widehat{G}_U P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U}\widetilde{h} + \\ &+ P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}\ominus\mathfrak{G}}(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} T)P^{\mathfrak{H}}_{\mathfrak{D}} P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}\widetilde{h} + P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}\ominus\mathfrak{G}}(\Gamma_o - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}} \Gamma_i \widehat{H}_U)\Gamma_o^* \widetilde{h}. \end{aligned}$$

From (4.10) we obtain

$$(4.12)\quad P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U}\widetilde{h} = z(I - z\widehat{T}_U)^{-1}\widehat{F}_U \Gamma_o^* \widetilde{h}.$$

In (4.11) we observe that, since $\widetilde{h} \in \mathfrak{H}_{U,\mathfrak{G}}$,

$$P_{\mathfrak{D}}^{\mathfrak{H}} P_{\mathfrak{H}}^{\widetilde{\mathfrak{H}}} \widetilde{h} + \Gamma_o \Gamma_o^* \widetilde{h} = P_{\mathfrak{H}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = P_{\mathfrak{H}\ominus\mathfrak{G}}^{\widetilde{\mathfrak{H}}} \widetilde{h}, \quad T P_{\mathfrak{D}}^{\mathfrak{H}} P_{\mathfrak{H}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = T P_{\mathfrak{H}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = T P_{\mathfrak{H}\ominus\mathfrak{G}}^{\widetilde{\mathfrak{H}}} \widetilde{h}.$$

Hence, by (4.12) and (4.11),

$$(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T) P_{\mathfrak{H}\ominus\mathfrak{G}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i [\widehat{H}_U + z\widehat{G}_U (I - z\widehat{T}_U)^{-1} \widehat{F}_U] \Gamma_o^* \widetilde{h}.$$

Thus we get

(4.13) $P_{\mathfrak{H}\ominus\mathfrak{G}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = z(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i \varepsilon_U(z) \Gamma_o^* \widetilde{h}.$

As $\mathfrak{H}_U \subseteq \ker \Gamma_o^*$, this equality and (4.4) imply $\Gamma_o^* \widetilde{h} = \Theta_{22}(z)\varepsilon_U(z)\Gamma_o^* \widetilde{h}$, that is,

$$(I - \Theta_{22}(z)\varepsilon_U(z))\Gamma_o^* \widetilde{h} = 0.$$

By Lemma 4.3, $\Gamma_o^* \widetilde{h} = 0$. From (4.12) and (4.13) it follows that $\widetilde{h} = P_{\mathfrak{H}_U}^{\widetilde{\mathfrak{H}}} \widetilde{h} + P_{\mathfrak{H}\ominus\mathfrak{G}}^{\widetilde{\mathfrak{H}}} \widetilde{h} = 0$. Therefore $I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U$ is injective on $\mathfrak{H}_{U,\mathfrak{G}}$.

Next we show that, for $z \in \mathcal{V}_0$, the operator $I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U$ maps onto $\mathfrak{H}_{U,\mathfrak{G}}$. Let $\widetilde{h} \in \mathfrak{H}_{U,\mathfrak{G}}$. Then $\widetilde{h} = h_U + h$ with $h_U \in \mathfrak{H}_U$, $h \in \mathfrak{H} \ominus \mathfrak{G}$. We treat the elements h_U and h separately. First we consider the element $h \in \mathfrak{H} \ominus \mathfrak{G}$. Since $\operatorname{ran} \Gamma_o^* \subseteq \widehat{\mathfrak{N}}$, by Lemma 4.3, there exists an element $\widehat{y} \in \widehat{\mathfrak{N}}$ such that

$$\Gamma_o^* (I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} h = (I - \Theta_{22}(z)\varepsilon_U(z))\widehat{y}.$$

It follows from (4.4) that $\widehat{y} = \Gamma_o^* f$, where

$$f = (I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1}(h + zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i \varepsilon_U(z)\widehat{y}) \in \mathfrak{H} \ominus \mathfrak{G}.$$

We also define

$$f_U = z(I - z\widehat{T}_U)^{-1} \widehat{F}_U \widehat{y} \in \mathfrak{H}_U, \quad \widetilde{f} = f + f_U \in \mathfrak{H}_{U,\mathfrak{G}}.$$

As $\mathfrak{H}_U \subseteq \ker \Gamma_o^*$, we have $\Gamma_o^* \widetilde{f} = \widehat{y}$. From (4.8) we obtain

$$(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U)\widetilde{f} = (I - z\widehat{T}_U) f_U - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i \widehat{G}_U f_U +$$
$$+ (I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T) f - z\widehat{F}_U \Gamma_o^* \widetilde{f} - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i \widehat{H}_U \Gamma_o^* \widetilde{f},$$

which after substitution of f, f_U and $\Gamma_o^* \widetilde{f}$ leads to the equality

(4.14) $(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\widetilde{\mathfrak{H}}} U)\widetilde{f} = h.$

We now consider the element $h_U \in \mathfrak{H}_U$. Again, since $\operatorname{ran} \Gamma_o^* \subseteq \widehat{\mathfrak{N}}$, by Lemma 4.3, there exists an element $\widehat{w} \in \widehat{\mathfrak{N}}$ such that

$$z\Gamma_o^* (I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i \widehat{G}_U (I - z\widehat{T}_U)^{-1} h_U = (I - \Theta_{22}(z)\varepsilon_U(z))\widehat{w}.$$

It follows from (4.4) that $\widehat{w} = \Gamma_o^* g$, where

$$g = z(I - zP_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} T)^{-1} P_{\mathfrak{H}\ominus\mathfrak{G}}^{\mathfrak{H}} \Gamma_i [\widehat{G}_U (I - z\widehat{T}_U)^{-1} h_U + \varepsilon_U(z)\widehat{w}] \in \mathfrak{H} \ominus \mathfrak{G}.$$

We also define

$$g_U = (I - z\widehat{T}_U)^{-1}(h_U + z\widehat{F}_U\widehat{w}) \in \mathfrak{H}_U, \quad \widetilde{g} = g + g_U \in \mathfrak{H}_{U,\mathfrak{G}}.$$

As $\mathfrak{H}_U \subseteq \ker \Gamma_o^*$, we have $\Gamma_o^*\widetilde{g} = \widehat{w}$. From (4.8) we obtain

$$(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)\widetilde{g} = (I - z\widehat{T}_U)g_U - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{G}_U g_U +$$
$$+(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)g - z\widehat{F}_U\Gamma_o^*\widetilde{g} - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{H}_U\Gamma_o^*\widetilde{g},$$

which after substitution of g, g_U and $\Gamma_o^*\widetilde{g}$ leads to the equality

(4.15) $(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)\widetilde{g} = h_U.$

Finally, if we define the element $\widetilde{k} \in \mathfrak{H}_{U,\mathfrak{G}}$ by $\widetilde{k} = \widetilde{f} + \widetilde{g}$, then (4.14) and (4.15) imply that $(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)\widetilde{k} = \widetilde{h}$. Hence the operator $I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U$ is surjective. This completes the proof. □

We recall the following result; see [9], Theorem 5 and [10], Theorem 1, and see also [11], Theorem 1.1.

Lemma 4.5 *The mapping $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V) \mapsto \Theta_U \in \mathbf{S}(\mathfrak{N}, \mathfrak{M})$ establishes, up to isomorphisms, a bijection between $\mathbf{U}(V)$ and $\mathbf{S}(\mathfrak{N}, \mathfrak{M})$.*

Proof of Theorem 4.1. First we show that the scattering matrix $\Theta_{U,\mathfrak{G}}$ of any $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ can be obtained via (4.7) by setting $\varepsilon(z) \equiv \varepsilon_U(z)$. So let $(U, \widetilde{\mathfrak{H}})$ belong to $\mathbf{U}(V)$. According to (4.6) and (4.1), we have, for all $z \in \mathcal{V}_0$,

$$\begin{aligned}\Theta_{U,\mathfrak{G}}(z) - \Theta_{11}(z) &= \Gamma_e^*(I - zUP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}})^{-1}U\Gamma_e - \Gamma_e^*T(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}\Gamma_e \\ &= \Gamma_e^*(I - zUP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}})^{-1}(U - T)(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}\Gamma_e.\end{aligned}$$

From $(U - T)\,|_{\mathfrak{H}} = U(I - P^{\widetilde{\mathfrak{H}}}_{\mathfrak{D}})\,|_{\mathfrak{H}} = UP^{\widetilde{\mathfrak{H}}}_{\mathfrak{N}}\,|_{\mathfrak{H}}$ and $P^{\widetilde{\mathfrak{H}}}_{\mathfrak{N}} = \Gamma_o\Gamma_o^*$, in view of (4.3), we get

$$\begin{aligned}(4.16)\ \Theta_{U,\mathfrak{G}}(z) - \Theta_{11}(z) &= \Gamma_e^*(I - zUP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}})^{-1}U\Gamma_o\Theta_{21}(z) \\ &= \Gamma_e^*U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o\Theta_{21}(z).\end{aligned}$$

Now consider the term $\Gamma_e^*U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o$. Since $TP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}\Gamma_o = 0$ and $TP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}U = TP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U$, we have

$$\begin{aligned}(4.17)\ &\Gamma_e^*U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o \\ &= \Gamma_e^*U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o - \Gamma_e^*(I - zTP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}})^{-1}TP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}\Gamma_o \\ &= \Gamma_e^*(I - zTP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}})^{-1}(U - TP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}})(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o.\end{aligned}$$

From (4.5), (4.2), (4.16) and (4.17) we conclude that

$$\Theta_{U,\mathfrak{G}}(z) = \Theta_{11}(z) + \Theta_{12}(z)\Gamma_i^*U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_o\Theta_{21}(z).$$

To show that $\Theta_{U,\mathfrak{G}}$ is given by the formula (4.7) with $\varepsilon(z) \equiv \varepsilon_U(z)$ it remains to establish that, for all $z \in \mathcal{V}_0$,

(4.18) $\Gamma_i^* U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)^{-1}\Gamma_o = \varepsilon_U(z)(I - \Theta_{22}(z)\varepsilon_U(z))^{-1}$.

For $z \in \mathcal{V}_0$ and $\widehat{y} \in \widehat{\mathfrak{N}}$, we define $\widetilde{h} = (I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)^{-1}\Gamma_o\widehat{y} \in \widetilde{\mathfrak{H}}$ and we set

$$h_U = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_U}\widetilde{h}, \quad h = P^{\mathfrak{H}}_{\mathfrak{D}} P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}}\widetilde{h}, \quad y = P^{\widetilde{\mathfrak{H}}}_{\mathfrak{N}}\widetilde{h}.$$

We claim that

(4.19) $\widehat{y} = (I - \Theta_{22}(z)\varepsilon_U(z))\Gamma_o^* y$.

Indeed, on account of (4.8), we have

$$\Gamma_o\widehat{y} = (I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)\widetilde{h} = (I - z\widehat{T}_U - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{G}_U)h_U + \\ +(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)h + (\Gamma_o - z\widehat{F}_U - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{H}_U)\Gamma_o^* y.$$

Projecting onto $\mathfrak{H}_U$ and onto $\mathfrak{H}$ we obtain

$$0 = (I - z\widehat{T}_U)h_U - z\widehat{F}_U\Gamma_o^* y$$

and

$$\Gamma_o\widehat{y} = -zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{G}_U h_U + (I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)h + (\Gamma_o - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\widehat{H}_U)\Gamma_o^* y,$$

respectively. The first equation implies that

(4.20) $h_U = z(I - z\widehat{T}_U)^{-1}\widehat{F}_U\Gamma_o^* y$,

and substituting this in the second equation and using $\Gamma_o\Gamma_o^* y = y$, we get

$$(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)h = \Gamma_o\widehat{y} + zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\varepsilon_U(z)\Gamma_o^* y - y.$$

Therefore

$$h = (I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}\Gamma_o\widehat{y} + \\ +z(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}P^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\varepsilon_U(z)\Gamma_o^* y - (I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}y.$$

Since $T\Gamma_o = 0$ and, for $y \in \mathfrak{N}$, $Ty = 0$, we find that the first and third summands are equal to

$$(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}\Gamma_o\widehat{y} = \Gamma_o\widehat{y}, \quad (I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}y = y.$$

Thus we obtain

$$h = \Gamma_o\widehat{y} + z(I - zP^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}T)^{-1}P^{\mathfrak{H}}_{\mathfrak{H}\ominus\mathfrak{G}}\Gamma_i\varepsilon_U(z)\Gamma_o^* y - y.$$

Projecting onto $\mathfrak{N}$ and using (4.4), we get

$$0 = \Gamma_o\widehat{y} + \Gamma_o\Theta_{22}(z)\varepsilon_U(z)\Gamma_o^* y - y = \Gamma_o(\widehat{y} + \Theta_{22}(z)\varepsilon_U(z)\Gamma_o^* y - \Gamma_o^* y),$$

which proves our claim (4.19). From (4.9), (4.20) and (4.19) we obtain

$$
\begin{aligned}
(4.21)\ \Gamma_i^* U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}} U)^{-1}\Gamma_o \widehat{y} &= \Gamma_i^* U\widetilde{h} \\
&= \Gamma_i^*[Th + (\widehat{T}_U + \Gamma_i \widehat{G}_U)h_U + (\widehat{F}_U + \Gamma_i \widehat{H}_U)\Gamma_o^* y] \\
&= \Gamma_i^* \Gamma_i[\widehat{H}_U \Gamma_o^* y + \widehat{G}_U h_U] = \widehat{H}_U \Gamma_o^* y + \widehat{G}_U h_U \\
&= \widehat{H}_U \Gamma_o^* y + z\widehat{G}_U(I - z\widehat{T}_U)^{-1}\widehat{F}_U \Gamma_o^* y = \varepsilon_U(z)\Gamma_o^* y \\
&= \varepsilon_U(z)(I - \Theta_{22}(z)\varepsilon_U(z))^{-1}\widehat{y},
\end{aligned}
$$

which establishes (4.18).

Now let ε belong to $\mathbf{S}(\widehat{\mathfrak{N}}, \widehat{\mathfrak{M}})$. Set $\Theta(z) = \Gamma_i \varepsilon(z)\Gamma_o^*$. Since Θ belongs to $\mathbf{S}(\mathfrak{N}, \mathfrak{M})$, by applying Lemma 4.5, we obtain a pair $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$, uniquely determined by Θ, up to isomorphisms, such that $\Theta(z) = \Theta_U(z)$ for all $z \in \mathbf{D}$. Since $\varepsilon(z) = \Gamma_i^* \Theta(z)\Gamma_o$ for all $z \in \mathbf{D}$, we have $\varepsilon(z) = \varepsilon_U(z)$ for all $z \in \mathbf{D}$. Hence the formula (4.7) applied to $\varepsilon(z)$ yields the scattering matrix $\Theta_{U,\mathfrak{G}}$ of $(U, \widetilde{\mathfrak{H}})$ relative to $\mathfrak{G}$.

To complete the proof it remains to show that $\Theta_{U,\mathfrak{G}}$ is uniquely determined by the function ε. Assume that $\varepsilon, \varepsilon' \in \mathbf{S}(\widehat{\mathfrak{N}}, \widehat{\mathfrak{M}})$ are such that, for all $z \in \mathcal{V}_0$,

$$
\begin{aligned}
(4.22)\ &\Theta_{11}(z) + \Theta_{12}(z)\varepsilon(z)(I - \Theta_{22}(z)\varepsilon(z))^{-1}\Theta_{21}(z) \\
&= \Theta_{11}(z) + \Theta_{12}(z)\varepsilon'(z)(I - \Theta_{22}(z)\varepsilon'(z))^{-1}\Theta_{21}(z).
\end{aligned}
$$

In particular, since $\Theta_{22}(0) = 0$, we have

$$\Theta_{12}(0)\varepsilon(0)\Theta_{21}(0) = \Theta_{12}(0)\varepsilon'(0)\Theta_{21}(0).$$

Then, according to (4.2) and (4.3), we get

$$\Gamma_e^* \Gamma_i(\varepsilon(0) - \varepsilon'(0))\Gamma_o^* \Gamma_e = 0.$$

This means that ran $\Gamma_i(\varepsilon(0) - \varepsilon'(0))\Gamma_o^* \Gamma_e \subseteq \mathfrak{H} \ominus \mathfrak{G}$. Since $\Gamma_i(\varepsilon(0) - \varepsilon'(0))\Gamma_o^* \Gamma_e \subseteq \mathfrak{M}$ and, by definition, $\mathfrak{M} \cap (\mathfrak{H} \ominus \mathfrak{G}) = \{0\}$, it follows that $(\varepsilon(0) - \varepsilon'(0))\Gamma_o^* \Gamma_e = 0$. Note that $\mathfrak{H} = \text{c.l.s.}\{\mathfrak{D}, \mathfrak{G}\}$, so that $(\varepsilon(0) - \varepsilon'(0))\Gamma_o^* = 0$ on $\mathfrak{H}$. We conclude that $\varepsilon(0) = \varepsilon'(0)$. By taking derivatives in (4.22) we get

$$\Theta_{12}(0)\frac{d\varepsilon}{dz}(0)\Theta_{21}(0) = \Theta_{12}(0)\frac{d\varepsilon'}{dz}(0)\Theta_{21}(0).$$

The argument used before to prove that $\varepsilon(0) = \varepsilon'(0)$ now shows that

$$\frac{d^n\varepsilon}{dz^n}(0) = \frac{d^n\varepsilon}{dz^n}(0),$$

for $n = 1$, and, inductively, for $n \in \mathbf{N}$. Therefore $\varepsilon(z)$ and $\varepsilon'(z)$ coincide in all of $\mathbf{D}$. This completes the proof of the theorem. □

5 On a parametrization of LIF(A)

In this section we give a complete labeling of all contractive intertwining liftings in the commutant lifting theorem for Kreĭn space contractions. With the data of the commutant lifting theorem, we associate an isometric operator in a Kreĭn space and a scale space of the type we met in Section 3. Our description is then obtained via the theory of scattering matrices in Section 4. In the Hilbert space case this procedure is due to Arocena ([2],[3]).

In order to solve the above labeling problem, we first carry the analysis of Section 4 a little further. We assume that V is a continuous isometric operator in the Kreĭn space $\mathfrak{H}$, whose domain $\mathfrak{D}$ and range $\mathfrak{R}$ are regular subspaces of $\mathfrak{H}$, and whose defect subspaces $\mathfrak{N}$ and $\mathfrak{M}$ are Hilbert subspaces of $\mathfrak{H}$. We also assume that $\mathfrak{G} \subseteq \mathfrak{H}$ is a scale subspace of V. Corresponding to $\mathfrak{G}$ we introduce a Kreĭn space $\mathfrak{G}_e$ and an isometry $\Gamma_e : \mathfrak{G}_e \to \mathfrak{H}$ with the property that ran $\Gamma_e = \mathfrak{G}$, so that ker $\Gamma_e^* = \tilde{\mathfrak{H}} \ominus \mathfrak{G}$. Let $(U, \tilde{\mathfrak{H}})$ belong to $\mathbf{U}(V)$, and let $\{G_m\}$ be the sequence of moments of $(U, \tilde{\mathfrak{H}})$ relative to $\mathfrak{G}$. The operators G_m, $m \geq 1$, are the Fourier coefficients of the function

(5.1) $G_{U,\mathfrak{G}}(z) = P_{\mathfrak{G}}^{\tilde{\mathfrak{H}}}(I - zU)^{-1}U|_{\mathfrak{G}} \in \mathbf{L}(\mathfrak{G})$,

defined and holomorphic in a neighborhood $\mathcal{V}_U \subseteq \mathbf{D}$ of 0.

Lemma 5.1 *Let $\mathcal{V}_0$ be the neighborhood of* 0 *in* $\mathbf{D}$ *given by Lemma 4.3. Then we have that, for every $(U, \tilde{\mathfrak{H}}) \in \mathbf{U}(V)$ and $z \in \mathcal{V}_U \cap \mathcal{V}_0$,*

(5.2) $\Theta_{U,\mathfrak{G}}(z) = (I + z\Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e)^{-1}\Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e$

and

(5.3) $\Gamma_e(I - z\Theta_{U,\mathfrak{G}}(z))^{-1}\Theta_{U,\mathfrak{G}}(z)\Gamma_e^* = G_{U,\mathfrak{G}}(z)$.

Proof. We recall that Γ_e is an isometry, and that $\Gamma_e\Gamma_e^*$ is the orthogonal projection from $\tilde{\mathfrak{H}}$ onto $\mathfrak{G}$. Since $\tilde{\mathfrak{H}} = \mathfrak{H}_{U,\mathfrak{G}} \oplus \mathfrak{G}$, we have $\Gamma_e\Gamma_e^* = I - P_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}$. It follows from the definition (5.1) that, for $z \in \mathcal{V}_U$,

(5.4) $I + z\Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e = I + z\Gamma_e^*(I - zU)^{-1}U\Gamma_e = \Gamma_e^*(I - zU)^{-1}\Gamma_e$,

and that, for $z \in \mathcal{V}_U \cap \mathcal{V}_0$,

$$
\begin{aligned}
&(I + z\Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e)\Theta_{U,\mathfrak{G}}(z) - \Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e \\
&= \Gamma_e^*(I - zU)^{-1}\Gamma_e\Gamma_e^* U(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U)^{-1}\Gamma_e - \Gamma_e^*(I - zU)^{-1}U\Gamma_e \\
&= \Gamma_e^*(I - zU)^{-1}[P_{\mathfrak{G}}^{\tilde{\mathfrak{H}}}U(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U)^{-1} - U]\Gamma_e \\
&= \Gamma_e^*(I - zU)^{-1}[P_{\mathfrak{G}}^{\tilde{\mathfrak{H}}}U - U(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U)](I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U)^{-1}\Gamma_e \\
&= \Gamma_e^*(I - zU)^{-1}(zU - I)P_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U(I - zP_{\mathfrak{H}_{U,\mathfrak{G}}}^{\tilde{\mathfrak{H}}}U)^{-1}\Gamma_e = 0.
\end{aligned}
$$

Now (5.2) follows, since we claim that, for each $z \in \mathcal{V}_U \cap \mathcal{V}_0$, the operator on the right hand side of (5.4) is invertible. First we show that this operator is injective. Assume that, for some $g_e \in \mathfrak{G}_e$, $\Gamma_e^*(I - zU)^{-1}\Gamma_e g_e = 0$. Then $h = (I - zU)^{-1}\Gamma_e g_e$ belongs to $\mathfrak{H}_{U,\mathfrak{G}}$. Hence $\Gamma_e g_e = (I - zU)h$ and $0 = (I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)h$. Lemma 4.4 implies $h = 0$. Consequently, $g_e = 0$. This proves that $\Gamma_e^*(I - zU)^{-1}\Gamma_e$ is injective. Next we show that it is surjective. For $g_e \in \mathfrak{G}_e$, we define

$$g_e' = g_e - z\Gamma_e^* U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e.$$

Then we obtain

$$\begin{aligned}
&\Gamma_e^*(I - zU)^{-1}\Gamma_e g_e' \\
&\quad = \Gamma_e^*(I - zU)^{-1}\Gamma_e g_e - z\Gamma_e^*(I - zU)^{-1}\Gamma_e\Gamma_e^* U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\quad = \Gamma_e^*(I - zU)^{-1}\Gamma_e g_e - z\Gamma_e^*(I - zU)^{-1}U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\qquad\quad + z\Gamma_e^*(I - zU)^{-1}P^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\quad = \Gamma_e^*(I - zU)^{-1}\Gamma_e g_e - \Gamma_e^*(I - zU)^{-1}(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\qquad\quad - \Gamma_e^*(I - zU)^{-1}\Gamma_e g_e + \Gamma_e^*(I - zU)^{-1}(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\qquad\quad + \Gamma_e^*(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e \\
&\quad = \Gamma_e^*(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e.
\end{aligned}$$

If we set $(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)^{-1}\Gamma_e g_e = h$, then $h \in \mathfrak{H}_{U,\mathfrak{G}}$ and

$$g_e = \Gamma_e^*\Gamma_e g_e = \Gamma_e^*(I - zP^{\widetilde{\mathfrak{H}}}_{\mathfrak{H}_{U,\mathfrak{G}}}U)h = \Gamma_e^* h.$$

This leads to the equality

(5.5) $\Gamma_e^*(I - zU)^{-1}\Gamma_e g_e' = g_e.$

Thus the operator $\Gamma_e^*(I - zU)^{-1}\Gamma_e$ is surjective. This completes the proof of our claim.

Finally, it follows from (5.2) that

$$(I + z\Gamma_e^* G_{U,\mathfrak{G}}(z)\Gamma_e)^{-1} = I - z\Theta_{U,\mathfrak{G}}(z), \quad z \in \mathcal{V}_U \cap \mathcal{V}_0.$$

Therefore $(I - z\Theta_{U,\mathfrak{G}}(z))^{-1}$ belongs to $\mathbf{L}(\mathfrak{G}_e)$ for all $z \in \mathcal{V}_U \cap \mathcal{V}_0$, and (5.3) is clear. □

Now we briefly recall the main facts about the commutant lifting theorem, as presented in [12]. We consider two Kreĭn spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, two contractions $T_1 \in \mathbf{L}(\mathfrak{H}_1)$ and $T_2 \in \mathbf{L}(\mathfrak{H}_2)$ with minimal isometric dilations $(W_1, \mathfrak{G}_1)$ and $(W_2, \mathfrak{G}_2)$, respectively, and a contraction $A \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ such that $AT_1 = T_2A$. The set $\mathbf{LIF}(A)$ of all contractive intertwining liftings of A is defined by

$$\mathbf{LIF}(A) = \{\widetilde{A} \in \mathbf{L}(\mathfrak{G}_1, \mathfrak{G}_2) \mid \widetilde{A} \text{ is a contraction}, \widetilde{A}W_1 = W_2\widetilde{A}, P_2\widetilde{A} = AP_1\},$$

where P_1 and P_2 denote the orthogonal projections from $\mathfrak{G}_1$ onto $\mathfrak{H}_1$ and from $\mathfrak{G}_2$ onto $\mathfrak{H}_2$, respectively. The operator A can be factorized as follows; see [12], Lemmas 1.1 and 1.2.

Proposition 5.2 *There exist a Kreĭn space* $\mathfrak{G}$ *and two continuous isometries* $\tau_1 : \mathfrak{H}_1 \to \mathfrak{G}$ *and* $\tau_2 : \mathfrak{H}_2 \to \mathfrak{G}$ *such that* $\mathfrak{G} = \text{c.l.s.}\{\tau_1\mathfrak{H}_1, \tau_2\mathfrak{H}_2\}$ *and* $A = \tau_2^*\tau_1$.

Let $(V_2, \mathfrak{F}_2)$ be the canonical minimal isometric dilation of T_2^*. Consider the Kreĭn space $\mathfrak{H} = \mathfrak{G} \oplus (\mathfrak{G}_1 \ominus \mathfrak{H}_1) \oplus (\mathfrak{F}_2 \ominus \mathfrak{H}_2)$. We define two continuous isometries $\sigma_1 : \mathfrak{G}_1 \to \mathfrak{H}$ and $\rho_2 : \mathfrak{F}_2 \to \mathfrak{H}$ by setting

$$\sigma_1|_{\mathfrak{H}_1} = \tau_1, \quad \sigma_1|_{\mathfrak{G}_1\ominus\mathfrak{H}_1} = I, \quad \rho_2|_{\mathfrak{H}_2} = \tau_2, \quad \rho_2|_{\mathfrak{F}_2\ominus\mathfrak{H}_2} = I.$$

Set $\mathfrak{D} = \text{c.l.s.}\{\sigma_1\mathfrak{G}_1, \rho_2V_2\mathfrak{F}_2\}$ and $\mathfrak{R} = \text{c.l.s.}\{\sigma_1W_1\mathfrak{G}_1, \rho_2\mathfrak{F}_2\}$. Let V_0 be the linear mapping determined by the relation

$$V_0(\sigma_1 g_1 + \rho_2V_2f_2) = \sigma_1W_1g_1 + \rho_2f_2, \quad g_1 \in \mathfrak{G}_1, \quad f_2 \in \mathfrak{F}_2.$$

Proposition 5.3 *The spaces* $\mathfrak{D}$ *and* $\mathfrak{R}$ *are regular subspaces of* $\mathfrak{H}$ *and* V_0 *can be extended to a continuous isometry* V *with* $\operatorname{dom} V = \mathfrak{D}$ *and* $\operatorname{ran} V = \mathfrak{R}$. *Furthermore,* $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$ *and* $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{R}$ *are Hilbert spaces, and* $\mathfrak{G}$ *is a scale subspace for* V.

Thus the data of the commutant lifting theorem give rise to an isometry V with a scale space $\mathfrak{G}$ of the type considered in the previous sections.

Let $(U, \widetilde{\mathfrak{H}})$ be a unitary Hilbert space extension of V. Put

$$\widetilde{\mathfrak{G}}_2 = \text{c.l.s.}\{U^n\tau_2\mathfrak{H}_2 \mid n \in \mathbf{N}\}.$$

Then $\widetilde{\mathfrak{G}}_2$ is a regular U-invariant subspace of $\widetilde{\mathfrak{H}}$ and $(U|_{\widetilde{\mathfrak{G}}_2}, \widetilde{\mathfrak{G}}_2)$ is a minimal isometric dilation of T_2. Consequently , there exists a unitary operator $\sigma_2 : \mathfrak{G}_2 \to \widetilde{\mathfrak{G}}_2$ such that $\sigma_2|_{\mathfrak{H}_2} = \tau_2$ and $\sigma_2W_2 = U\sigma_2$.

Theorem 5.4 *Let* $(U, \widetilde{\mathfrak{H}})$ *belong to* $\mathbf{U}(V)$. *Then the operator* $\widetilde{A} = \sigma_2^*\sigma_1$ *belongs to* $\mathbf{LIF}(\mathfrak{G}_1, \mathfrak{G}_2)$, *and the mapping* $(U, \widetilde{\mathfrak{H}}) \mapsto \widetilde{A}$ *is, up to isomorphisms, a bijection between* $\mathbf{U}(V)$ *and* $\mathbf{LIF}(A)$.

The fact that $\mathbf{LIF}(A)$ is nonempty was first proved by M. Dritschel in his doctoral dissertation (University of Virginia, May, 1989); a simplified version is given in [13]. The above results concerning the commutant lifting theorem can be found in [12], where references to the relevant literature are given. Another proof can be found in [16].

At this point we are ready to consider the problem of labeling the set $\mathbf{LIF}(A)$. To each $\widetilde{A} \in \mathbf{LIF}(A)$ we associate a formal power series $P_{\widetilde{A}}$ by

$$P_{\widetilde{A}}(z) = \sum_{m=1}^{\infty} z^m \widetilde{A}_m, \quad \widetilde{A}_m = P_2W_2^{*m}\widetilde{A}|_{\mathfrak{H}_1}, \quad m \geq 1.$$

For an operator-valued function φ, defined and holomorphic in a neighborhood $\mathcal{V}_\varphi \subseteq \mathbf{D}$ of 0, which is symmetric with respect to the real axis, it is convenient to introduce the notation $\varphi^\#(z) = \varphi(\bar{z})^*$, $z \in \mathcal{V}_\varphi$.

Proposition 5.5 *The mapping*

$$\widetilde{A} \in \mathbf{LIF}(A) \mapsto \{\widetilde{A}_m\} \subseteq \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$$

is one-to-one, up to isomorphisms. Moreover, if $\widetilde{A} \in \mathbf{LIF}(A)$ *and* $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ *are related via Theorem 5.4, then* $P_{\widetilde{A}}$ *is defined and holomorphic on* $\mathcal{V}_U$ *and*

$$(5.6)\quad P_{\widetilde{A}}(z) = z\tau_2^{-1} P^{\mathfrak{G}}_{\tau_2\mathfrak{H}_2} G^{\#}_{U,\mathfrak{G}}(z)\tau_1, \quad z \in \mathcal{V}_U.$$

Proof. From the definition of the set $\mathbf{LIF}(A)$ it follows that, for all $h_1 \in \mathfrak{H}_1$, $h_2 \in \mathfrak{H}_2$, $n, m \in \mathbf{N}$,

$$\langle \widetilde{A} W_1^n h_1, W_2^m h_2 \rangle_{\mathfrak{G}_2} = \langle W_2^n \widetilde{A} h_1, W_2^m h_2 \rangle_{\mathfrak{G}_2} = \begin{cases} \langle A T_1^{n-m} h_1, h_2 \rangle_{\mathfrak{H}_2}, & n \geq m, \\ \langle \widetilde{A}_{m-n} h_1, h_2 \rangle_{\mathfrak{H}_2}, & n < m. \end{cases}$$

Suppose that $\widetilde{A}, \widetilde{A}' \in \mathbf{LIF}(A)$ satisfy

$$\widetilde{A}_m = P_2 W_2^{*m} \widetilde{A}|_{\mathfrak{H}_1} = P_2 W_2^{*m} \widetilde{A}'|_{\mathfrak{H}_1} = \widetilde{A}'_m, \quad m \geq 1.$$

Then, clearly, for all $h_1 \in \mathfrak{H}_1$, $h_2 \in \mathfrak{H}_2$, $n, m \in \mathbf{N}$,

$$\langle \widetilde{A} W_1^n h_1, W_2^m h_2 \rangle_{\mathfrak{G}_2} = \langle \widetilde{A}' W_1^n h_1, W_2^m h_2 \rangle_{\mathfrak{G}_2}.$$

Since $\mathfrak{G}_1 = \text{c.l.s.}\,\{W_1^n \mathfrak{H}_1 \mid n \in \mathbf{N}\}$ and $\mathfrak{G}_2 = \text{c.l.s.}\,\{W_2^n \mathfrak{H}_2 \mid n \in \mathbf{N}\}$, it follows that $\widetilde{A} \equiv \widetilde{A}'$, so that the indicated mapping is injective.

To prove the last part of the proposition we observe that, for all $h_1 \in \mathfrak{H}_1$, $h_2 \in \mathfrak{H}_2$, $m \geq 1$,

$$\begin{aligned} \langle \widetilde{A}_m h_1, h_2 \rangle_{\mathfrak{H}_2} &= \langle \widetilde{A} h_1, W_2^m h_2 \rangle_{\mathfrak{G}_2} = \langle \tau_1 h_1, U^m \tau_2 h_2 \rangle_{\widetilde{\mathfrak{H}}} \\ &= \langle \tau_1 h_1, P^{\widetilde{\mathfrak{H}}}_{\mathfrak{G}} U^m \tau_2 h_2 \rangle_{\mathfrak{G}} = \langle \tau_1 h_1, G_m \tau_2 h_2 \rangle_{\mathfrak{G}}, \end{aligned}$$

so that

$$\widetilde{A}_m = \tau_2^{-1} P^{\mathfrak{G}}_{\tau_2\mathfrak{H}_2} G_m^* \tau_1, \quad m \geq 1.$$

This shows that the function $P_{\widetilde{A}}$ is defined and holomorphic in $\mathcal{V}_U$, and is related to the function $G_{U,\mathfrak{G}}$ in the indicated way. □

The connection between the sequences $\{\widetilde{A}_m\}$ and $\{G_m\}$ and the Schur algorithm described in Section 3 can be used to determine if the elements of a given sequence $\{\widetilde{A}_m\} \subseteq \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ are of the form $\widetilde{A}_m = P_2 W_2^{*m} \widetilde{A}\,|_{\mathfrak{H}_1}$, $m \geq 1$, for some $\widetilde{A} \in \mathbf{LIF}(A)$. We will not pursue this avenue, but concentrate on the labeling problem.

Theorem 5.6 *Let* $\widetilde{A} \in \mathbf{LIF}(A)$ *and* $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ *be related via Theorem 5.4. Then the function* $P_{\widetilde{A}}$ *is given by*

$$(5.7)\quad P_{\widetilde{A}}(z) = z\tau_2^{-1} P^{\mathfrak{G}}_{\tau_2\mathfrak{H}_2} \Gamma_e \varphi^{\#}(z)(I - z\varphi^{\#}(z))^{-1} \Gamma_e^* \tau_1, \quad z \in \mathcal{V}_U \cap \mathcal{V}_0,$$

where

$$(5.8)\quad \varphi(z) = \Theta_{11}(z) + \Theta_{12}(z)\varepsilon(z)(I - \Theta_{22}(z)\varepsilon(z))^{-1}\Theta_{21}(z), \quad z \in \mathcal{V}_0,$$

and $\varepsilon(z) = \varepsilon_U(z) = \Gamma_i^* \Theta_U(z) \Gamma_0$, $z \in \mathbf{D}$. *The mapping* $\varepsilon \mapsto P_{\widetilde{A}}$ *determined by the formulas (5.7) and (5.8) yields a bijection between* $\mathbf{S}(\widehat{\mathfrak{N}}, \widehat{\mathfrak{M}})$ *and the class* $\{P_{\widetilde{A}} \mid \widetilde{A} \in \mathbf{LIF}(A)\}$.

Proof. The first part of the theorem follows from (5.6), (5.3) and Theorem 4.1. Since every $\varepsilon \in \mathbf{S}(\widehat{\mathfrak{N}}, \widehat{\mathfrak{M}})$ is of the form $\varepsilon = \varepsilon_U$ for some pair $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ (see Lemma 4.5 and the observation halfway the proof of Theorem 4.1), the range of the map $\varepsilon \mapsto P_{\widetilde{A}}$ is contained in and coincides with the class $\{P_{\widetilde{A}} \mid \widetilde{A} \in \mathbf{LIF}(A)\}$. By Theorem 4.1, the fact that an element in $\mathbf{U}(V)$ is, up to isomorphisms, uniquely determined by its scattering matrix, Theorem 5.4 and Proposition 5.5, we have that the mappings

$$\varepsilon \mapsto \Theta_{U,\mathfrak{G}} \mapsto (U, \widetilde{\mathfrak{H}}) \mapsto \widetilde{A} \mapsto P_{\widetilde{A}}$$

are all one-to-one (up to isomorphisms as far as $(U, \widetilde{\mathfrak{H}}) \in \mathbf{U}(V)$ is concerned). This implies that the mapping $\varepsilon \mapsto P_{\widetilde{A}}$ is injective, This completes the proof. □

Finally, we note that (5.7) can be written as a Schur like formula:

$$P_{\widetilde{A}}(z) = a(z) + b(z)\varphi^{\#}(z)(I - c(z)\varphi^{\#}(z))^{-1}d(z), \quad z \in \mathcal{V} \cap \mathcal{V}_0,$$

where

$$a(z) = 0, \quad b(z) = z\tau_2^{-1}P^{\mathfrak{G}}_{\tau_2\mathfrak{H}_2}\Gamma_\varepsilon, \quad c(z) = z, \quad d(z) = \Gamma_\varepsilon^*\tau_1.$$

References

[1] Andô, T.: *Linear Operators on Krein Spaces.* Sapporo: Hokkaido University, Res. Inst. Appl. Electricity, Div. Appl. Math. 1979, pp. 59.

[2] Arocena, R.: *Schur analysis of a class of translation invariant forms.* Lect. Notes Pure Appl. Math. 122 (1990), 355 - 369.

[3] Arocena, R.: *Unitary extensions of isometries and contractive intertwining dilations.* Operator Theory: Adv. Appl. 41 (1989), 13 - 23.

[4] Arocena, R.: *On the extension problem for a class of translation invariant forms.* J. Oper. Theory 21 (1989), 323 - 347.

[5] Arov, D. Z. and I. Z. Grossman: *Scattering matrices in the theory of dilations of isometric operators.* Dokl. Akad. Nauk. SSSR 270 (1983) 17 - 20; Engl. transl.: Soviet Math. Dokl. 27 (1983), 518 - 522.

[6] Arov, D. Z. and I. Z. Grossman: *Scattering matrices in the theory of unitary extensions of isometric operators.* Math. Nachr. 157 (1992), 105 - 123.

[7] Azizov, T. and I. S. Iokhvidov: *Foundations of the Theory of Linear Operators in Spaces with Indefinite Metric.* Moscow: Nauka 1986; Engl. transl.: *Linear Operators in Spaces with Indefinite Metric.* New York: Wiley 1989.

[8] Bognár, J.: *Indefinite Inner Product Spaces.* Berlin: Springer - Verlag 1974.

[9] Brodskiĭ, V. M.: *On operator nodes and their characteristic functions.* Dokl. Akad. Nauk. SSSR 198 (1971), 16 - 19; Engl. transl.: Soviet Math. Dokl. 12 (1971), 696 - 700.

[10] Brodskiĭ V. M. and Ya. S. Shvartsman: *On invariant subspaces of contractions.* Dokl. Akad. Nauk. SSSR 201 (1971), 519 - 522; Engl. transl.: Soviet Math. Dokl. 12 (1971), 1659 - 1663.

[11] Dijksma, A., Langer, H. and H. S. V. de Snoo: *Generalized coresolvents of standard isometric operators and generalized resolvents of standard symmetric relations in Krein*

spaces. Operator Theory: Adv. Appl. 48 (1990), 261 - 274.

[12] Dijksma, A., Dritschel, M. A., Marcantognini, S. A. M. and H. S. V. de Snoo: *The commutant lifting theorem for contractions on Kreĭn spaces.* Operator Theory: Adv. Appl. 61 (1993), 65 - 83.

[13] Dritschel, M. A. and J. Rovnyak: *Extension theorems for contraction operators on Kreĭn spaces.* Operator Theory: Adv. Appl. 47 (1990), 221 - 305.

[14] Foiaş, C. and A. Frazho: *The Commutant Lifting Approach to Interpolation Problems* (Operator Theory: Adv. Appl.: Vol. 44). Basel: Birkhäuser Verlag 1990.

[15] Iokhvidov, I. S., Kreĭn, M. G. and H. Langer: *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric.* Berlin: Akademie - Verlag 1982.

[16] Marcantognini, S. A. M.: *The commutant lifting theorem in the Kreĭn space setting: a proof based on the coupling method.* Indiana Univ. Math. J. 41 (1992), 1303 - 1314.

[17] Sz.-Nagy, B. and C. Foiaş: *Harmonic Analysis of Operators on Hilbert Space.* North Holland, New York 1970.

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