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Perron's Method and Barrier Functions for the Viscosity Solutions of the Dirichlet Problem for some Non-Linear Partial Differential Equations

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Abstract. The Dirichlet problem for some non-linear partial differential equations via Perron's method is studied in the viscosity set up, by considering two families of functions, instead of one, as considered by others before. The notion of barrier at a boundary point is introduced to study the regularity of boundary points. Barriers for some non-linear operators are also constructed.

Keywords: *Viscosity super- and subsolutions, Perron families, resolutive functions, barriers*
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1. Introduction

In this paper, we try to adopt the classical Perron method of solutions of the Dirichlet problem for the Laplacian to the case of some non-linear partial differential equations and discuss the regularity of the boundary points using barrier functions.

Perron's approach in the viscosity set up was studied earlier by Ishii [2] and several others (see the references in [5]). In these works, only one family of functions was used to get a solution. We follow here fully Perron's idea of introducing two families of functions to study the existence problem and the barrier functions for the boundary behaviour of the solutions.

2. Basic definitions

Let Ω be a bounded open set in $\mathbb{R}^n$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times M_n \longrightarrow \mathbb{R}$ a map, where M_n is the space of all $n \times n$ real symmetric matrices. We study the solutions of the non-linear partial differential equation

$$F\left(x, u(x), Du(x), D^2u(x)\right) = 0 \tag{2.1}$$

in the viscosity sense.

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Definition 1: An extended real-valued function u on Ω is said to be a *viscosity supersolution* (respectively, a *viscosity subsolution*) of equation $F = 0$ if, for all $\varphi \in C^2(\Omega), u - \varphi$ has a local minimum (maximum) at a point $x_0 \in \Omega$ implies

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \leq 0$$

respectively

$$F\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \geq 0.$$

If the function u is both a super- and subsolution of equation $F = 0$, then it is said to be a *viscosity solution*. All the solutions, super- and subsolutions considered here are only in the viscosity sense.

Let f be a real-valued function defined on $\partial\Omega$. We define the Perron families of functions Φ_f and Ψ_f as

$$\Phi_f = \left\{ v \;\middle|\; \begin{array}{c} v \text{ bounded below lower semi-continuous supersolu-} \\ \text{tion such that } \liminf\limits_{y \to X, y \in \Omega} v(y) \geq f(X) \ \forall\, X \in \partial\Omega \end{array} \right\}$$

and

$$\Psi_f = \left\{ u \;\middle|\; \begin{array}{c} u \text{ bounded above upper semi-continuous subsolu-} \\ \text{tion such that } \limsup\limits_{y \to X, y \in \Omega} u(y) \leq f(X) \ \forall\, X \in \partial\Omega \end{array} \right\}.$$

Define

$$\overline{H}_f = \inf_{v \in \Phi_f} v \qquad \text{and} \qquad \underline{H}_f = \sup_{u \in \Psi_f} u$$

with the convention that the infimum and supremum over a void family is $+\infty$ and $-\infty$, respectively.

3. Existence of solutions

Let F be degenerate elliptic:

$$F(x, r, p, X + Y) \leq F(x, r, p, X) \tag{3.1}$$

for all $(x, r, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times M_n$ and $Y \geq 0$, symmetric. Let us assume further that

$$F(x, 0, 0, 0) = 0 \tag{3.2}$$

for all $x \in \Omega$ and F is monotone non-decreasing in the sense that

$$F(x, t, p, A) \geq F(x, s, p, A) \tag{3.3}$$

for all $(x, p, A) \in \Omega \times \mathbb{R}^n \times M_n$ and $t, s \in \mathbb{R}$ such that $t \geq s$.

Remark 1: Conditions (3.2) and (3.3) together imply that any positive real number is a positive supersolution and any negative real number is a negative subsolution of equation $F = 0$. Also, it can be easily proved using condition (3.3) that if v is a supersolution and u a subsolution of equation $F = 0$, then so is $v + \varepsilon$ and $u - \varepsilon$, respectively, for all $\varepsilon > 0$.

We shall assume further that the map F has properties such that the following comparison theorem holds:

If u is an upper semi-continuous subsolution bounded from above and v a lower semi-continuous supersolution bounded from below of equation (2.1) *such that*

$$\limsup_n \{u(x_n) - v(y_n)\} \leq M$$

where $x_n, y_n \in \Omega, |x_n - y_n| \to 0$ and $\operatorname{dist}(x_n, \partial\Omega) \to 0$, then

$$u(x) - v(x) \leq M \qquad \text{for all} \quad x \in \Omega.$$

Several sets of conditions on the map F under which the above comparison property holds, appear in [4].

Let us assume that f is a bounded function defined on $\partial\Omega$. Then, by using similar arguments as in [2], it can be proved that both $\overline{H}_f$ and $\underline{H}_f$ are solutions of the equation $F = 0$. But, they need not be continuous. By the comparison theorem, it follows that every $u \in \Psi_f$ is less than or equal to every $v \in \Phi_f$. Hence $\underline{H}_f \leq \overline{H}_f$ on Ω.

Definition 2 (see [6]): The function f is said to be *semi-resolutive* if $\underline{H}_f = \overline{H}_f$. If f is semi-resolutive, then the above common value is denoted by H_f.

Definition 3: The function f is said to be *resolutive* if f is semi-resolutive and H_f is continuous.

Theorem 1: *If $\{f_n\}$ is a sequence of semi-resolutive (resp. resolutive) functions converging to f uniformly on $\partial\Omega$, then this limit function f is also semi-resolutive (resp. resolutive) and $H_{f_n} \to H_f$ uniformly on Ω.*

For the proof of this theorem, we need the following lemma which can be easily proved.

Lemma 1: *For all $\varepsilon > 0$, the inequalities $\overline{H}_{f+\varepsilon} \leq \overline{H}_f + \varepsilon$ and $\underline{H}_{f-\varepsilon} \geq \underline{H}_f - \varepsilon$ are true.*

Proof of Theorem 1: Since f_n converges to f uniformly on $\partial\Omega$, we have, for any $\varepsilon > 0$, the existence of an $N(\varepsilon) \in \mathbb{N}$ such that

$$f_n(X) - \varepsilon \leq f(X) \leq f_n(X) + \varepsilon \qquad \text{for } X \in \partial\Omega \text{ and } n \geq N(\varepsilon).$$

Thus $\underline{H}_{f_n - \varepsilon} \leq \underline{H}_f \leq \overline{H}_f \leq \overline{H}_{f_n + \varepsilon}$. Using Lemma 1, it follows that

$$\underline{H}_{f_n} - \epsilon \leq \underline{H}_f \leq \overline{H}_f \leq \overline{H}_{f_n} + \varepsilon. \tag{3.4}$$

Since f_n are semi-resolutive for all n, it follows that $|\overline{H}_f - \underline{H}_f| \leq 2\varepsilon$ for all $\varepsilon > 0$. This implies that $\overline{H}_f = \underline{H}_f$, proving that f is semi-resolutive ∎

From inequalities (3.4) we also get the estimate $|H_{f_n} - H_f| \le \varepsilon$ on Ω for all $n \ge N(\varepsilon)$. This implies that the sequence $\{H_{f_n}\}$ converges to H_f uniformly on Ω. Thus, if the functions f_n are resolutive for all n, so is also the function f.

Theorem 2: *Let $f \in C(\partial\Omega)$. Suppose f admits a continuous extension $f_1 : \Omega \to \mathbb{R}$ being a supersolution (resp. subsolution) of equation $F = 0$. Then f is semi-resolutive and H_f is upper semi-continuous (resp. lower semi-continuous).*

Proof: We shall prove the theorem assuming that the map f_1 is a supersolution. The proof when f_1 is a subsolution is analogous.

Obviously, $f_1 \in \Phi_f$ and $\overline{H}_f \le f_1$. As f_1 is continuous, $(\overline{H}_f)^* \le f_1$. Thus, for $X \in \partial\Omega$,

$$\limsup_{x \to X, x \in \Omega} (\overline{H}_f)^*(x) \le \limsup_{x \to X, x \in \Omega} f_1(x) = f(X). \tag{3.5}$$

This implies that $(\overline{H}_f)^* \in \Psi_f$. Hence, $\underline{H}_f \ge (\overline{H}_f)^*$. Thus, $(\overline{H}_f)^* \ge \overline{H}_f \ge \underline{H}_f \ge (\overline{H}_f)^*$, implying that $\overline{H}_f = \underline{H}_f = (\overline{H}_f)^*$. Hence the function f is semi-resolutive and H_f is upper semi-continuous ∎

Corollary (see [3: Theorem 3.2]): *Let the function $f \in C(\partial\Omega)$ admit two continuous extensions f_1 and f_2 such that f_1 is a subsolution and f_2 is a supersolution of equation $F = 0$. Then f is resolutive and $H_f(x) \to f(X)$ for all $X \in \partial\Omega$ as $x \to X, x \in \Omega$.*

Proof: It is immediate from Theorem 2 and the inequalities $\limsup_{y \to X, y \in \Omega} H_f(y) \le f(X) \le \liminf_{y \to X, y \in \Omega} H_f(y)$ ∎

Remark 2: When the function f admits a continuous extension $f_1 : \overline{\Omega} \to \mathbb{R}$ being a supersolution (resp. subsolution) of equation $F = 0$, then we get an upper semi-continuous (resp. lower semi-continuous) solution in $\overline{\Omega}$ of the Dirichlet problem

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega \tag{3.6}$$

by defining $H_f(X) = f(X)$ on $\partial\Omega$. The upper semi-continuity of this solution at the boundary points follows from (3.5) and (3.6).

Application: Consider a degenerate elliptic operator $F : \Omega \times M_n \to \mathbb{R}$ satisfying conditions (3.1) and (3.2). Then, if A is positive-definite,

$$F(x, A) \le 0 \quad \text{for all } x \in \Omega$$

as $F(x, 0) = 0$. Thus, if Ψ is a C^2 convex function on $\overline{\Omega}$, where Ω is convex, then $F(x, D^2\Psi(x)) \le 0$, implying that Ψ is a subsolution of equation $F = 0$. If $f = \Psi$ on $\partial\Omega$, then by Theorem 2 f is semi-resolutive and H_f is lower semi-continuous.

As every convex function Ψ is a uniform limit of C^2 convex functions, it follows from Theorem 1 that if $f = \Psi$ on $\partial\Omega$, then f is semi-resolutive and H_f is lower semi-continuous.

4. Regular points and barrier functions

Let F be as in (2.1) and the function f be semi-resolutive. Now, we seek conditions on Ω so that if f is continuous on $\partial\Omega$, then $H_f(y) \to f(X)$ as $y \in \Omega \to X$ for all $X \in \partial\Omega$. To this end, we define the notion of a barrier at a boundary point.

Definition 4: Let $X_0 \in \partial\Omega$. A pair of functions (v, u) is called a *barrier* at X_0 if the following conditions are satisfied:

(i) $v \geq 0$ and λv is a lower semi-continuous supersolution of $F = 0$ for all $\lambda \geq 0$.
(ii) $u \leq 0$ and λu is an upper semi-continuous subsolution of $F = 0$ for all $\lambda \geq 0$.
(iii) $\lim_{y\in\Omega, y\to X_0} v(y) = 0 = \lim_{y\in\Omega, y\to X_0} u(y)$
(iv) $\liminf_{y\in\Omega, y\to X} v(y) > 0 > \limsup_{y\in\Omega, y\to X} u(y)$ for all $X \in \partial\Omega, X \neq X_0$.

Definition 5: A point $X_0 \in \partial\Omega$ is said to be *regular* if there is a barrier at X_0.

From now onwards, we shall assume that F is a mapping from $\Omega \times \mathbb{R}^n \times M_n$ to $\mathbb{R}$. This will ensure that if v is a super- or subsolution of equation $F = 0$, then so is $v + c$ for every real number c.

Theorem 3: *Let a barrier exist at $X_0 \in \partial\Omega$ and let f be a bounded function on $\partial\Omega$ which is continuous at X_0. If f is semi-resolutive, then $H_f(y) \longrightarrow f(X_0)$ as $y \to X_0, y \in \Omega$.*

Proof: The idea is to construct, for all $\varepsilon > 0$, a supersolution $\omega^\varepsilon \in \Phi_f$ and a subsolution $\omega_\varepsilon \in \Psi_f$ such that

$$\limsup_{y\to X_0, y\in\Omega} \omega^\varepsilon(y) \leq f(X_0) + \varepsilon \qquad \text{and} \qquad \liminf_{y\to X_0, y\in\Omega} \omega_\varepsilon(y) \geq f(X_0) - \varepsilon.$$

Since f is continuous at X_0, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(X) - f(X_0)| < \varepsilon$ for all X with $|X - X_0| \leq \delta$. Let (v, u) be a barrier at X_0. The function $g : \partial\Omega \to \mathbb{R}$ defined as

$$g(X) = \liminf_{y\to X, y\in\Omega} v(y)$$

is lower semi-continuous and therefore attains its minimum m_1 on the compact set

$$K = \partial\Omega \cap \{X : |X - X_0| \geq \delta\}.$$

As $g(X) > 0$ for $X \in K, m_1 > 0$. Let $M > \sup_{X\in\partial\Omega} |f(X)|$. Define

$$\omega^\varepsilon = f(X_0) + \varepsilon + \frac{v}{m_1}(M - f(X_0)).$$

Then ω^ε is a supersolution.

Let us prove that $\omega^\varepsilon \in \Phi_f$. Let $X \in \partial\Omega$ be such that $|X - X_0| \leq \delta$. Then

$$\liminf_{y\to X, y\in\Omega} \omega^\varepsilon(y) \geq f(X_0) + \varepsilon \geq f(X).$$

Let $X \in \partial\Omega$ be such that $|X - X_0| \geq \delta$. Then

$$\liminf_{y\to X, y\in\Omega} \omega^\varepsilon(y) \geq f(X_0) + \varepsilon + (M - f(X_0)).$$

Thus, if $|X - X_0| \geq \delta$, then

$$\liminf_{y \to X, y \in \Omega} \omega^\varepsilon(y) \geq M + \varepsilon > f(X).$$

Hence $\omega^\varepsilon \in \Phi_f$. Therefore, $H_f \leq \omega^\varepsilon$ and

$$\limsup_{y \to X_0, y \in \Omega} H_f(y) \leq \limsup_{y \to X_0, y \in \Omega} \omega^\varepsilon(y) = f(X_0) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,

$$\limsup_{y \to X_0, y \in \Omega} H_f(y) \leq f(X_0).$$

Let $m < \inf_{x \in \partial\Omega} f(X)$ and $m_2 < 0$ be such that

$$\limsup_{y \to X, y \in \Omega} u(y) \leq m_2$$

for all X with $|X - X_0| \geq \delta$. Define

$$\omega_\varepsilon = f(X_0) - \varepsilon - \frac{u}{m_2}(f(X_0) - m).$$

Then ω_ε is a subsolution and let us prove that $\omega_\varepsilon \in \Psi_f$.

Let $X \in \partial\Omega$ be such that $|X - X_0| < \delta$. By the choice of m and m_2,

$$\omega_\varepsilon \leq f(X_0) - \varepsilon \qquad \text{and} \qquad \limsup_{y \to X, y \in \Omega} \omega_\varepsilon(y) \leq f(X_0) - \varepsilon \leq f(X).$$

Let $X \in \partial\Omega$ be such that $|X - X_0| \geq \delta$. Let $\lambda = -1/m_2$. Then $\lambda > 0$ and

$$\begin{aligned}
\limsup_{y \to X, y \in \Omega} \omega_\varepsilon(y) &= f(X_0) - \varepsilon + \lambda(f(X_0) - m) \limsup_{y \to X, y \in \Omega} u(y) \\
&\leq f(X_0) - \varepsilon + \lambda(f(X_0) - m)m_2 \\
&\leq f(X_0) - \varepsilon + (-1/m_2)(f(X_0) - m)m_2 \\
&= f(X_0) - \varepsilon - f(X_0) + m \\
&= m - \varepsilon \\
&< f(X).
\end{aligned}$$

Hence $\omega_\varepsilon \in \Psi_f$ and $\underline{H}_f \geq \omega_\varepsilon$. Therefore

$$\liminf_{y \to X_0, y \in \Omega} \underline{H}_f(y) \geq \liminf_{y \to X_0, y \in \Omega} \omega_\varepsilon(y) = f(X_0) - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $\liminf_{y \to X_0, y \in \Omega} \underline{H}_f(y) \geq f(X_0)$. Thus,

$$f(X_0) \leq \liminf_{y \to X_0, y \in \Omega} \underline{H}_f(y) \leq \limsup_{y \to X_0, y \in \Omega} \overline{H}_f(y) \leq f(X_0),$$

implying $\lim_{y \to X_0, y \in \Omega} H_f(y) = f(X_0)$ ∎

5. Barrier functions for some Ω and some operators

In this section, we shall construct barriers for some operators F and some domains Ω.

Example 1: Consider the quasilinear degenerate elliptic equations of the type

$$F(x, Du, D^2u) = -Tr(a(Du)D^2u) = 0 \qquad \text{in } \Omega \tag{5.1}$$

where $a \in C(\mathbb{R}^n, M_n)$. This type is considered in [4]. We shall construct a barrier at every boundary point of a ball $B(y;r) \subset \Omega$.

Let $\Psi(x) = |x-y|^2 - r^2$ and $d(x) = |x - X_0|^2$ where $X_0 \in \partial B(y;r)$. Let

$$v = d(x) - \Psi(x).$$

Then $D^2(\lambda v) = 0$ for all $\lambda \geq 0$. Hence λv is a solution in $B(y;r)$ of equation (5.1). Further, $v \geq 0$ and $v(X_0) = 0, v(X) > 0$ for points $X \in \partial B(y;r)$ with $X \neq X_0$.

Consider $u = \Psi(x) - d(x) = -v$. Then λu is a solution in $B(y;r)$ of equation (5.1). Further $u \leq 0, u(X_0) = 0$ and $u(X) < 0$ for points $X \in \partial B(y;r)$ with $X \neq X_0$. Hence (v, u) is a barrier at X_0. Thus, every point on $\partial B(y;r)$ is regular.

Example 2: Let Ω be an open set in $\mathbb{R}^n$ $(n \geq 2)$. The set Ω is said to *satisfy an exterior sphere condition at* $X_0 \in \partial\Omega$ if there exists an $R > 0$ such that $\overline{B(y;R)} \cap \overline{\Omega} = \{X_0\}$. We shall construct a barrier at such a point X_0 for equations of type (5.1) for some suitable function a.

Without loss of generality, let us assume that $y = 0$. We define on Ω, for all $\sigma > 0$, the function $v_\sigma(x) = 1/R^\sigma - 1/|x|^\sigma$. Then we have, for all $x \in \Omega$ and $\lambda \in \mathbb{R}$,

$$D(\lambda v_\sigma)(x) = \frac{\lambda\sigma}{|x|^{\sigma+2}}x \qquad \text{and} \qquad D^2(\lambda v_\sigma)(x) = \frac{\lambda\sigma}{|x|^{\sigma+2}}\, I_n - \frac{\lambda\sigma(\sigma+2)}{|x|^{\sigma+4}}\, B(x)$$

where I_n is the $n \times n$ identity matrix and $B(x)$ is the positive definite matrix $[b_{ij}]$ with $b_{ij} = x_i x_j$. Let us assume that, for some $l \in \mathbb{N}$,

$$a(k\xi) = k^{2l}a(\xi) \qquad \text{for all } k > 0. \tag{5.2}$$

Then it is seen easily that

$$F\left(x, D(\lambda v_\sigma), D^2(\lambda v_\sigma)\right) = \frac{(\lambda\sigma)^{2l+1}}{|x|^{2l(\sigma+2)+\sigma+4}}\left\{(\sigma+2)\langle(x)x, x\rangle - |x|^2 Tr(a(x))\right\}.$$

Let us assume further that

$$\inf_{x\in\Omega}\{\langle a(x)x, x\rangle\} = m > 0. \tag{5.3}$$

This is always verified if $a(x)$ is a positive definite matrix, as $d(0,\Omega) > 0$.

Let $M = \sup_{x\in\Omega}|x|^2 Tr(a(x))$. Then, if $\lambda \geq 0$,

$$F\left(x, D(\lambda v_\sigma)(x), D^2(\lambda v_\sigma)(x)\right) \geq \frac{(\lambda\sigma)^{2l+1}}{|x|^{2l(\sigma+2)+\sigma+4}}\left\{(\sigma+2)m - M\right\}.$$

Hence, if σ is such that

$$(\sigma+2)m - M \geq 0, \tag{5.4}$$

then λv_σ is a supersolution of equation (5.1) for all $\lambda \geq 0$. Further $v_\sigma(X_0) = 0, v \geq 0, v > 0$ on $\partial\Omega\backslash\{X_0\}$. Similarly, $-\lambda v_\sigma$ is a subsolution of equation (5.1) for all $\lambda \geq 0, -v_\sigma(X_0) = 0, -v_\sigma \leq 0$ and $-v_\sigma < 0$ on $\partial\Omega\backslash\{X_0\}$. Thus, $(v_\sigma, -v_\sigma)$ is a barrier at X_0.

Now, we shall give a particular $a \in C(\mathbb{R}^n, M_n)$, satisfying (5.2) and (5.3). For example, let $a_{ij}(\xi) = \xi_i\xi_j$ as considered in [1]. This equation arises as the limiting equation for the p-Laplacian as $p \to \infty$ and is important in the study of plastic torsion. Then, relation (5.2) holds with $l = 1$. Further,

$$\langle a(x)x, x\rangle = \sum_i \sum_j x_i^2 x_j^2 = |x|^4 \geq R^4 \qquad \text{on } \overline{\Omega}.$$

Thus, $m = R^4$ in condition (5.3) and

$$F\left(x, D(\lambda v_\sigma)(x), D^2(\lambda v_\sigma)(x)\right) = \frac{(\lambda\sigma)^3}{|x|^{3\sigma+8}}\left\{(\sigma+2)|x|^4 - |x|^4\right\} = \frac{(\lambda\sigma)^3}{|x|^{3\sigma+4}}(\sigma+1).$$

Hence, $(v_\sigma - v_\sigma)$ is a barrier for any $\sigma > 0$.

Concluding Remark: Depending on the function F and the nature of comparison theorems available, one can modify the families Φ_f and Ψ_f. For example one can consider

$$(\Phi_f^{Lip}, \Psi_f) \qquad \text{or} \qquad (\Phi_f, \Psi_f^{Lip})$$

where

$$\Phi_f^{Lip} = \left\{ v \,\middle|\, \begin{array}{c} v \text{ a locally Lipschitz supersolution such} \\ \text{that } \liminf\limits_{y\to X, y\in\Omega} v(y) \geq f(X)\ \forall\, X \in \partial\Omega \end{array} \right\}$$

and

$$\Psi_f^{Lip} = \left\{ u \,\middle|\, \begin{array}{c} u \text{ a locally Lipchitz subsolution such} \\ \text{that } \limsup\limits_{y\to X, y\in\Omega} u(y) \leq f(X)\ \forall\, X \in \partial\Omega \end{array} \right\}.$$

Then, under some conditions listed in [4], the comparison result holds. Then again, we can prove similar results. For example, if we consider (Φ_f^{Lip}, Ψ_f), then one can prove that if f admits a continuous extension as a supersolution, then f is semi-resolutive, H_f is upper semi-continuous and

$$\limsup_{y\to X, y\in\Omega} H_f(y) \leq f(X)$$

for all $X \in \partial\Omega$.

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