Spectral Estimates for Compact Hyperbolic Space Forms and the Selberg Zeta Function for p-Spectra II

R. Schuster

Abstract. We prove an asymptotic estimation for the spectrum of the Laplace operator for compact hyperbolic space forms. Thereby we use estimations of the Selberg zeta function by **methods of analytic number theory.**

Keywords: *Eigenvolue spectrum, Laplace operator, hyperbolic space form, length spectrum, Selberg zeta function, analytic number theory techniques*

AMS subject classification: 11F72, 11M06, 35Q05, 58F19, 58F25

Introduction

First we recall the main definitions, motivations and results of Part I. Let G be a properly discontinuous group of orientation preserving isometries of the n-dimensional hyperbolic space H_n of constant curvature -1 without fixed points (with the exception of the identity map *id*) with compact fundamental domain. We consider the related Killing-Hopf space form $V = H_n/G$. Let Ω be the set of non-trivial free homotopy classes of V. In every class $\omega \in \Omega$ there lies exactly one closed geodesic line. We denote by $l(\omega)$ and $\nu(\omega)$ its length and muliplicity, respectively. The parallel displacement along a closed geodesic line induces an isometry of the tangent space in every point of that geodesic line with the eigenvalues $\beta_1(\omega),\ldots,\beta_{n-1}(\omega),1$ with $|\beta_i(\omega)| = 1$ $(i = 1,\ldots,n-1)$. Let $e_p(\omega)$ be the p^{th} elementary symmetric function of the $\beta_i(\omega)$ $(i = 1, ..., n - 1)$, and put $e_0(\omega) = 1$. Further on, we introduce the weight tions, motivation

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$$
\sigma(\omega) = \frac{e^{N l(\omega)}}{\nu(\omega)} \prod_{j=1}^{n-1} \frac{1}{(e^{l(\omega)} - \beta_j(\omega))}
$$

with $N = (n-1)/2$. Let S_p denote the p-spectrum of the Laplace operator $\Delta = d\delta + \delta d$.

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Thereby we have used the differential operatord and the codifferential operator δ = $(-1)^{pn+n-1}$ * d * for differential p-forms, where * denotes the Hodge dualization. Let $d_p^d(\mu)$ and $d_p^{\delta}(\mu)$ denote the dimension of the eigenspaces of closed $(d\alpha = 0)$ and coclosed $(\delta \alpha = 0)$ eigenforms α of Δ with eigenvalues μ , respectively. The dimension of the space of harmonic *p*-eigenforms is the p^{th} Betty number B_p of the space form *V*.

Our results are based on the Selberg trace formula as a duality statement between the *p*-eigenvalue spectrum and the geometric spectrum of *V* (expressed by $l(\omega)$, $\nu(\omega)$, $\sigma(\omega)$ and $e_p(\omega)$).

Theorem 1 (Selberg trace formula): Let $h = h(r)$ be an analytic function in the *strip* $|Im r| < N + \delta$ with $N = (n - 1)/2, 0 < \delta < 1/2$, which is even, $h(r) =$ $h(-r)$, and satisfies $|h(r)| \leq A(1+|r|)^{-n-\delta}$. By the help of the Fourier transform $g(u) =$ $\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru} dr$ of h we can state the trace formula **n** 1 (Selberg trace formula): Let $h =$
 $\langle N + \delta$ with $N = (n - 1)/2, 0 <$
 ttisfies $|h(r)| \le A(1 + |r|)^{-n-\delta}$. By the h
 i^{*u*} dr of *h* we can state the trace formul
 $\sum_{\mu \in S_p} d_p^*(\mu) h(r_p(\mu)) = vol V(S_p^n, g) + \sum_{\omega \in \Omega}$
 $n - 1$ wi

$$
\sum_{\mu \in S_p} d_p^*(\mu) h(r_p(\mu)) = vol \ V\langle S_p^n, g \rangle + \sum_{\omega \in \Omega} l(\omega) \sigma(\omega) e_p(\omega) g(l(\omega))
$$

for $p=0,\ldots,n-1$ with $r_p(\mu)=\sqrt{\mu-(p-N)^2}$, where

and
$$
e_p(\omega)
$$
).

\nTheorem 1 (Selberg trace formula): Let $h = h(r)$ be an analytic function string $|Im \ r| < N + \delta$ with $N = (n-1)/2, 0 < \delta < 1/2$, which is even, $h(h(-r))$, and satisfies $|h(r)| \leq A(1+|r|)^{-n-\delta}$. By the help of the Fourier transform g .

\n $\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)e^{-iru} dr$ of h we can state the trace formula

\n
$$
\sum_{\mu \in S_p} d_p^*(\mu)h(r_p(\mu)) = vol \ V(S_p^n, g) + \sum_{\omega \in \Omega} l(\omega)\sigma(\omega)e_p(\omega)g(l(\omega))
$$

\nfor $p = 0, \ldots, n-1$ with $r_p(\mu) = \sqrt{\mu - (p-N)^2}$, where

\n
$$
\int_{\omega \in \Omega} \int_{\omega = 1/2}^{\infty} \left(\prod_{\omega = 1/2}^{\infty} (r^2 + u^2) h(r) dr \right) dr
$$
 for n odd

\n
$$
\int_{\omega = 1/2}^{\infty} \left(\prod_{\omega = 1/2}^{\infty} (r^2 + u^2) h(r) \right) h(r) \text{tr} \tan(\pi r) dr
$$
 for n even.

\nTherefore, we have used $N = \frac{n-1}{2}$ and

\n
$$
d_p^*(\mu) = \begin{cases} d_p^{\delta}(\mu) & \text{for } \mu > 0 \\ (-1)^p (B_0 - B_1 + \ldots + (-1)^p B_p) + K_p & \text{for } \mu = 0 \end{cases}
$$

\nwith

\n
$$
K_p = \begin{cases} (-1)^{p+1-n/2} \pi^{-(n+1)/2} \Gamma(\frac{n+1}{2}) vol V & \text{for } p \geq n/2 \ (n \text{ even}) \\ 0 & \text{for the other cases.} \end{cases}
$$

\n

and

have used
$$
N = \frac{n-1}{2}
$$
 and
\n
$$
d_p^*(\mu) = \begin{cases} d_p^{\delta}(\mu) & \text{for } \mu > 0 \\ (-1)^p (B_0 - B_1 + ... + (-1)^p B_p) + K_p & \text{for } \mu = 0 \end{cases}
$$

with

$$
K_p = \begin{cases} (-1)^{p+1-n/2} \pi^{-(n+1)/2} \Gamma(\frac{n+1}{2}) \, vol \, V & \text{for } p \ge n/2 \, (n \, even) \\ 0 & \text{for the other cases.} \end{cases}
$$

Further on, vol V denotes the volume of the space form V.

For $a > 0$ we define $E(t, a) = \int_1^t e^{at} s \, ds$. The main result of Part I was the following theorem.

Theorem A: *We can estimate the sum*

for the old
es the volume of the space form V.

$$
E(t, a) = \int_1^t e^{as}/s \, ds.
$$
 The main result of
can estimate the sum

$$
P_p(T) = \sum_{\substack{e \in \Omega \\ \cosh(l, \omega) \le T}} e_p(\omega) \qquad \text{for} \quad T \to \infty
$$

by

$$
Spectral Estimates for Compact Hyperbolic Space Forms 26
$$
\n
$$
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$$
\n
$$
P_p(T) = \begin{cases}\nO(T^{N + \frac{(n-1)^2}{2n}} / \ln T) & \text{for } 1 \leq p \leq n-2 \\
E(t, n-1) + \sum_{\substack{\mu \in S_p \\ 0 < \mu < N^2(2n-1)/n^2}} d_p^{\delta}(\mu) E(t, N + \sqrt{N^2 - \mu}) & \text{for } p = 0, n-1 \\
+ O(T^{N + \frac{(n-1)^2}{2n}} / \ln T) & \text{for } p = 0, n-1\n\end{cases}
$$
\nwith $N = \frac{n-1}{2}$, cosh $t = T$, $T > 1$. We get the same estimation if we replace $P_p(T)$ by

with $N = \frac{n-1}{2}$, cosh $t = T$, $T > 1$. We get the same estimation if we replace $P_p(T)$ by

$$
P_p^{\#}(T) = \sum_{\substack{\omega \in \Omega \\ \cosh{(\omega)} \leq T, \nu(\omega) = 1}} e_p(\omega)
$$

In Part I we have proved Theorem A using a Landau difference method and a solution of an Euler-Poisson-Darboux equation (cf. Section 3 in Part I) as a special function which we can use in the Selberg trace formula. In some cases these functions are better adapted to the geometric situation than functions which are usually taken in trace formulas when a Selberg zeta function is considered.

In Section 6 we will introduce a Selberg zeta function in a natural way with respect to our version of the Selberg trace formula. This zeta function is well known for the case $n = 2$, $p = 0$. Gangolli [9] treats zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one from the viewpoint of representation theory.' To see differences to our treatment one should compare the zeros and poles of the analytic continuation of the zeta function to the whole complex plane. The Selberg trace formula bears a striking resemblance to the explicite formulas of prime number theory. The Selberg zeta function is analogous in many ways to the classical Riemann zeta function. This enables us to study the asymptotic behaviour of the p-spectrum using techniques of analytic number theory. As a consequence of the well-known Weyl type asymptotic formula (cf. [2, 28] and Section 4) we have *i* introduce a Selberg zeta function in a natura

elberg trace formula. This zeta function is v

golli [9] treats zeta functions of Selberg's type

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Section 4) we have
 $= \sum_{\mu \in S_p} d_p^{\delta}(\mu)$
 $n_p = \frac{\binom{n-1}{p} \text{ vol } V}{(4\pi)^{n/2} \Gamma(\frac{n+2}{2})}$

fo

$$
\mathcal{N}_{p}(T) = \sum_{\substack{\mu \in S_{p} \\ \mu < T^{2} + (p - N)^{2}}} d_{p}^{6}(\mu) \sim n_{p} T^{n} \quad \text{for} \quad T \to \infty
$$

۰.

$$
\mathcal{N}_p(T) = \sum_{\substack{\mu \in S_p \\ \mu < T^2 + (p - N)^2}} d_p^{\delta}(\mu) \sim n_p T^n \quad \text{for} \quad T \to
$$
\nwith

\n
$$
n_p = \frac{\binom{n-1}{p} \text{ vol } V}{(4\pi)^{n/2} \Gamma\left(\frac{n+2}{2}\right)} \quad \text{and} \quad N = \frac{n-1}{2}.
$$

We will prove the following

Theorem B: *The error term* $\mathcal{R}_p(T)$ *defined by* $\mathcal{N}_p(T) = n_p T^n + \mathcal{R}_p(T)$ ($T > 1$) formula (cf. [2, 28] and Section 4) we have
 $\mathcal{N}_p(T) = \sum_{\mu \le T^2 + (p-N)^2} d_p^6(\mu) \sim$

with
 $n_p = \frac{\binom{n-1}{p} \text{ vol } V}{(4\pi)^{n/2} \Gamma(\frac{n+2}{2})}$

We will prove the following

Theorem B: The error term $\mathcal{R}_p(T)$ de

satisfies $|\math$

Hejhal [14] has given this estimation in the case $n = 2, p = 0$. Weaker results for more general spaces were proved by Gangolli [8] and Ivrii [17] for $n \geq 2, p = 0$. Hejhal $[14]$ remarked (for $n = 2$) that it seems hard to improve the estimation of Theorem *B*. The analogy between the Selberg and the Riemann zeta function is strongly apparent in our proofs. If one were able to improve the $T^{n-1}/\ln T$ - term in Theorem B, there would presumably be a corresponding improvement in the estimation arg $\zeta(1/2 + iT)$ = O(ln *T/* in in *T)* for the Riemann zeta function, assuming the Riemann hypothesis is valid. But no such improvement is known.

Section 7 deals with the analytic continuation of the Selberg zeta funktion and its logarithmic derivative. We state funtional equations for these functions. In Section 8 we give an estimation for the Selberg zeta function based on the Weyl estimation for the eigenvalue spectrum. We use the method of "good" and "admissible" numbers (cf. Hejhal [14] and Ingham [16]), but a straightforward generalization to the n-dimensional case would not be strong enought. In Section 9 we prove spectral estimations based on estimations for the zeta functions. We use these results in Section 10 in order to derive estimations for the logarithmic derivative of the Selberg zeta function. We essentially use these estimations to prove Theorem B in Section 11.

6. Definition of the Selberg zeta function and first properties

Since we have already noted that the Selberg trace formula (Theorem 1 in Part I) bears a striking resemblance to the so-called explicit formulas of prime number theory, it is quite natural to search for a function which is analogous to the classical Riemann zeta function. Nowadays this function *Z* is commonly known as the Selberg zeta function, the original reference is *[24]* (Part I). We generalize a method given by Hejhal [14] for the classical case $n = 2, p = 0$. In this special case the Selberg zeta funtion is defined by

$$
Z(s) = \prod_{\substack{\omega \in \Omega \\ \omega(s) = 1}} \prod_{m=0}^{\infty} \left(1 - e^{-l(\omega)(s+m)}\right)
$$

for $\text{Re } s > 1$. Elementary calculations show that $\frac{d}{ds} \ln Z(s)$ is the right-hand side of the Selberg trace formula in the version of Theorem 1 (Part I) with $g(u) = e^{-u(s-1/2)}$. In the general case we start with the definition of a function Ψ_p which later on (cf. Lemma 13) will be identified to be the logarithmic derivative of a Selberg zeta function Z_p . We define *Z* is commonly know We generalize a r
special case the S
 $Z(s) = \prod_{\substack{\omega \in \Omega \\ \omega(\omega)=1}} \prod_{m=-\infty}^{\infty} m$
y calculations show
the version of The
h the definition of
e logarithmic deriv
 $\Psi_p(s) = \kappa_p \sum_{\omega \in \Omega} l(\omega)$
 $(n-1)/2$ (as in

$$
\Psi_{p}(s) = \kappa_{p} \sum_{\omega \in \Omega} l(\omega) \sigma(\omega) e_{p}(\omega) e^{-i(\omega)(s-N)} \tag{62}
$$

for $\text{Re } s > 2N$ with $N = (n-1)/2$ (as in Part I), $\kappa_p = 1$ for n odd and $\kappa_p = \Gamma(p+1)\Gamma(n-p)$ for n even.

Lemma 12: *The series (62) converges absolutely and uniformly for s in the half space Re s* $\geq 2N + \epsilon_0, \epsilon_0 > 0$.

Proof: As a consequence of Proposition 5 we get

\n The equation is represented by the equation
$$
\text{Var}(T) = \left| \sum_{\substack{\omega \in \Omega \\ \text{and } \mathfrak{l}(\omega) \leq T}} e_p(\omega) \, l(\omega) \, \sigma(\omega) \right| = O(T^N).
$$
\n

Using Stieltjes integration, we immediately get the assertion \blacksquare

We denote the *j*th power of the eigenvalues $\beta_i(\omega)$ given by the parallel displacement along the closed geodesic line (cf. Introduction to Part I) belonging to a non-trivial free along the closed geodesic line (cf. introduction to Part 1) belonging to a non-trivial free
homotopy class $\omega \in \Omega$ by $\beta_i^j(\omega) = (\beta_i(\omega))^j$. We can write the elementary symmetric
function e_p as a sum
 $e_p(\omega) = \sum_{k=1}^q \epsilon_{p,k$ function e_p as a sum er of the eigenvalues $\beta_i(\omega)$ given by the parallel displane (cf. Introduction to Part I) belonging to a non-triv $\beta_i^j(\omega) = (\beta_i(\omega))^j$. We can write the elementary syn

) with $q = \binom{n-1}{p}$ and $\epsilon_{p,k}(\omega) = \beta_{i_1}(\omega) \dots \beta_{i_p$

$$
e_p(\omega) = \sum_{k=1}^q \epsilon_{p,k}(\omega) \quad \text{with} \quad q = \binom{n-1}{p} \text{ and } \epsilon_{p,k}(\omega) = \beta_{i_1}(\omega) \ldots \beta_{i_p}(\omega)
$$

using $i_1 < i_2 < ... < i_p$ and $i_1, i_2, ..., i_p \in \{1, 2, ..., n-1\}$. Let M denote the set of multiindices $m = (m_1, \ldots, m_{n-1})$ with non-negative integers m_i . Further on we define in the usual way $\beta^{m}(\omega) = \beta_1^{m_1}(\omega) \dots \beta_{n-1}^{m_{n-1}}(\omega)$ and $|m| = m_1 + \dots + m_{n-1}$. $q = \binom{n-1}{p}$ and $\epsilon_{p,k}(\omega) = \beta_{i_1}(\omega) \dots \beta_{i_p}(\omega)$
 $\beta_{i_1}, \dots, \beta_{i_p} \in \{1, 2, \dots, n-1\}$. Let M denote the set of
 $\beta_{n-1}^{m}(\omega)$ and $|m| = m_1 + \dots + m_{n-1}$.
 $\prod_{m \in M} \left(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)} \right)^{k_p}$ (63)
 $\Psi_p(s) = \$

Lemma 13: *The zeta function*

$$
Z_p(s) = \prod_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \prod_{k=1}^{\nu} \prod_{m \in \mathbf{M}} \left(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)} \right)^{\kappa_p} \tag{63}
$$

satisfies

$$
\Psi_p(s) = \frac{d}{ds} \ln Z_p(s). \tag{64}
$$

The product on the right-hand side of (63) is absolutely convergent for Re s > 2N. We have $Z_p(\bar{s}) = \overline{Z_p(s)}$, $\Psi_p(\bar{s}) = \overline{\Psi_p(s)}$, where \bar{s} denotes the complex conjugate number of s .

Proof: The product on the right-hand side of (63) is absolutely convergent, if the series

$$
\sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \sum_{k=1}^{q} \sum_{m \in M} |\epsilon_{p,k}(\omega)| |\beta^{m}(\omega)| e^{-l(\omega)(Re s + |m|)}
$$
\n
$$
= \sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \sum_{k=1}^{q} \sum_{m \in M} e^{-l(\omega)(Re s + |m|)}
$$
\n
$$
= q \sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \left(\prod_{j=1}^{n-1} \frac{1}{1 - e^{-l(\omega)}} \right) e^{-l(\omega) Re s}
$$
\n
$$
\text{by we have used } |\epsilon_{p,k}(\omega)| = 1, |\beta^{m}(\omega)| = 1. \text{ and}
$$
\n
$$
\sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} e^{-l(\omega) Re s} \text{ for } Re s > 2N, \text{ which is a cons}
$$
\n
$$
\text{at primitive homotopy class, we can write}
$$
\n
$$
= \kappa_{p} \sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \sum_{j=1}^{\infty} l(\omega) e_{p}(\omega^{j}) \left(\prod_{i=1}^{n-1} \frac{1}{1 - \beta_{i}^{j}(\omega) e^{-l(\omega)j}} \right)
$$
\n
$$
\text{s, we get}
$$

is convergent. Thereby we have used $|\epsilon_{p,k}(\omega)| = 1$, $|\beta^m(\omega)| = 1$. It is sufficient to prove is convergent. Thereby we
the convergence of $\sum_{\substack{\omega \in \Omega \\ \omega_{\omega}=1}}$
5 (for $p = 0$) using Stieltjes
class is the power of a prin $e^{-(\omega)Re s}$ for $Re s > 2N$, which is a consequence of Proposition 5 (for $p = 0$) using Stieltjes integration. Using (3) and the fact that every free homotopy class is the power of a primitive homotopy class, we can write $\sum_{v(\omega)=1}^{\infty}$

e used $|\epsilon_{p,k}(\omega)| =$
 $\sum_{r=1}^{\infty}$
 $\sum_{r=1}^{\infty}$ for $Re s > 2h$

gration. Using (3
 $\sum_{r=1}^{\infty} l(\omega) e_p(\omega^j) \left(\prod_{r=1}^{n-1} \right)$ *l* $\sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \left(\prod_{j=1}^{n-1} \frac{1}{1 - e^{-l(\omega)}} \right) e^{-l(\omega) Re s}$

sed $| \epsilon_{p,k}(\omega) | = 1, |\beta^m(\omega)| = 1.$
 f for *Re s* > 2*N*, which is a cons

ation. Using (3) and the fact the

protopy class, we can write
 $l(\omega) e_p(\omega^j) \left($ have used $|\epsilon_p$
have used $|\epsilon_p$
 $e^{-l(\omega)Re\phi}$ for *F*
integration.
itive homotop
 $\sum_{p=1}^{\infty}\sum_{j=1}^{\infty}l(\omega)e_p$
 ϵ_n
it

$$
\Psi_p(s) = \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \sum_{j=1}^{\infty} l(\omega) e_p(\omega^j) \left(\prod_{i=1}^{n-1} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} \right) e^{-l(\omega)js}.
$$

Using the definitions, we get

 \mathbf{I}

$$
= \kappa_p \sum_{\substack{\omega \in \Omega \\ \omega(\omega) = 1}} \sum_{j=1}^{p} \ell(\omega) e_p(\omega^r) \Big(\prod_{i=1}^{n} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} \Big)
$$
\n, we get

\n
$$
e_p(\omega^j) = \sum_{k=1}^{q} \epsilon_{p,k}^j(\omega) \quad \text{with } \epsilon_{p,k}^j(\omega) = (\epsilon_{p,k}(\omega))^j
$$

and thereby it follows

$$
\begin{aligned}\n\text{Schuster} \\
\Psi_p(s) &= \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega) = 1}} \sum_{j=1}^{\infty} \sum_{k=1}^{q} l(\omega) \epsilon_{p,k}^j(\omega) \Big(\prod_{i=1}^{n-1} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} \Big) e^{-l(\omega)js}.\n\end{aligned}
$$
\n
$$
\text{eometric series we get}
$$
\n
$$
\prod_{i=1}^{n-1} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} = \sum_{m_1=0}^{\infty} \dots \sum_{m_{n-1}=0}^{\infty} e^{-j(m_1 + \dots + m_{n-1})l(\omega)} \prod_{i=1}^{n-1} \beta_i^{jm_i}(\omega)
$$

Applying geometric series we get

Schuster
\nby it follows
\n
$$
\Psi_p(s) = \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \sum_{j=1}^{\infty} \sum_{k=1}^{q} l(\omega) \epsilon_{p,k}^j(\omega) \left(\prod_{i=1}^{n-1} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} \right) e^{-l(\omega)js}.
$$
\ngeometric series we get
\n
$$
\prod_{i=1}^{n-1} \frac{1}{1 - \beta_i^j(\omega) e^{-l(\omega)j}} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_{n-1}=0}^{\infty} e^{-j(m_1 + \cdots + m_{n-1})l(\omega)} \prod_{i=1}^{n-1} \beta_i^{jm_i}(\omega).
$$

It follows

$$
\Psi_p(s) = \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega) = 1}} \sum_{j=1}^{\infty} \sum_{k=1}^{q} \sum_{m \in M} l(\omega) \epsilon_{p,k}^j(\omega) \beta^{jm}(\omega) e^{-l(\omega)j(s+|m|)}
$$

\n
$$
= \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega) = 1}} \sum_{k=1}^{q} \sum_{m \in M} l(\omega) \frac{\epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)}}{1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)}}
$$

\nproves (64)
\n**Lemma 14:** Using $s = \sigma^* + iT$, it holds
\n(i) $|\ln Z_p(s)| \le c^* \sum_{\omega \in \Omega, \nu(\omega) = 1} e^{-l(\omega)\sigma^*} \text{ for } \sigma^* > n - 1 \text{ with } c^* = \kappa_p {n-1 \choose p} (1)$
\nand $l_0 = \inf_{\omega \in \Omega} {l(\omega) \choose \omega}$ uniformly for $\sigma^* > n - 1 + \epsilon, 0 < \epsilon < 1, |s| \to \infty$

This proves (64) .

Lemma 14: *Using* $s = \sigma^* + iT$ *, it holds* **for the set of** (64) **I**
 emma 14: Using s

(i) $|\ln Z_p(s)| \leq c^* \sum_{\omega}$

and $l_0 = \inf_{\omega \in \Omega} \{l\}$ *for* $\sigma^* > n - 1$ *with* $c^* = \kappa_p {n-1 \choose p} (1$ *and* $l_0 = \inf_{\omega \in \Omega} \{l(\omega)\}\$ (iii) $Z_p(s) = 1 + O(e^{-\log^2 t})$ uniformly for $\sigma^* > n - 1 + \epsilon, 0 < \epsilon < 1, |s| \to \infty$.

Proof: Definition (63) guarantees $Re Z_p(s) > 0$ for $Re s > n - 1$. We use the branch of $\ln Z_p(s)$ with $-\pi \leq Im \ln Z_p(s) < \pi$ for $Re s > n - 1$. The proof is a straightforward generalization of Hejhal [14, Proposition 4.13] using Proposition A (Part I), Lemma 12 and Lemma 13. For more details cf. Schuster [35] \blacksquare

As a direct consequence from the definition (62) and Theorem A we get

Lemma 15: Let $s = \sigma^* + iT$. Then it holds $|\Psi_p(s)| = O(1)$ uniformly for $\sigma^* \geq$ $n-1+\epsilon, \epsilon > 0, |s| \to \infty.$

An other version of a Selberg zeta function is given by Christian [29, 30] (cf. also [31 -33]).

7. Analytic continuation of the Selberg zeta function

Theorem 1 will come in handy if we try to find an analytic continuation of the Selberg zeta function. Till now we have only used the right-hand side of the Selberg trace formula for the definition of the logarithmic derivative of that zeta funtion: We will apply standard

analytic number theory techniques which already have been used by Hejhal [14] in the classical case $n = 2, p = 0$. It is possible to carry this idea through, but we will have much more technical problems. vitic number theory techniques which already have been used

ical case $n = 2, p = 0$. It is possible to carry this idea through,

itechnical problems.

Using $\alpha_i \neq \alpha_j$ for $i \neq j$ $(i, j = 1, ..., m)$, $m > n/2$ we define
 $\mathcal{P}(i$ imates for Compact Hyperbolic Space Form

th already have been used by Hejhal

to carry this idea through, but we will

., *m*), $m > n/2$ we define
 s^2 , $\mathcal{P}^+(i, s) = \prod_{\substack{j=2 \ j \neq i}}^m (\alpha_j^2 + s^2)$,
 s^2), $\mathcal{Q}^+(i, s$ heory techniques which already have been used by Hejhal [14, *p* = 0. It is possible to carry this idea through, but we will have beens.

for $i \neq j$ $(i, j = 1, ..., m)$, $m > n/2$ we define
 $\mathcal{P}(i, s) = \prod_{\substack{j=2 \ j \neq i}}^{m} (\alpha_j^2 - s$

$$
\mathcal{P}(i,s) = \prod_{\substack{j=2 \\ j \neq i}}^{m} (\alpha_j^2 - s^2), \quad \mathcal{P}^+(i,s) = \prod_{\substack{j=2 \\ j \neq i}}^{m} (\alpha_j^2 + s^2), \tag{65}
$$

$$
Q(i, s) = \prod_{\substack{i=1 \\ i \neq i}}^{m} (\alpha_j^2 - s^2), \quad Q^+(i, s) = \prod_{\substack{i=1 \\ i \neq i}}^{m} (\alpha_j^2 + s^2), \quad (66)
$$

Special Estimates for Compact Hyperbolic Space Forms 267
\nsory techniques which already have been used by Hejhal [14] in the
\n
$$
p = 0
$$
. It is possible to carry this idea through, but we will have much
\nlems.
\n $(i, s) = \prod_{\substack{j=2\\j \neq i}}^{m} (\alpha_j^2 - s^2), \quad \mathcal{P}^+(i, s) = \prod_{\substack{j=2\\j \neq i}}^{m} (\alpha_j^2 + s^2),$ \n (65)
\n $(i, s) = \prod_{\substack{j=2\\j \neq i}}^{m} (\alpha_j^2 - s^2), \quad \mathcal{Q}^+(i, s) = \prod_{\substack{j=1\\j \neq i}}^{m} (\alpha_j^2 + s^2),$ \n
$$
\mathcal{R}(s) = \prod_{\substack{j=1\\j \neq i}}^{m} (\alpha_j^2 + s^2).
$$
\n66)
\nshall have the value 1. From now on we use the following
\n16. We use $m = \frac{n}{2} + 1$ for n, even, and $m = \frac{n+1}{2}$ for n, odd. Put

An empty product shall have the value 1. From now on we use the following

Assumption 16: *We use m* = $\frac{n}{2}$ + 1 *for n even and m* = $\frac{n+1}{2}$ *for n odd. Put* $\alpha_1 = s - N = \sigma + iT$, $N = (n - 1)/2$. We suppose $\alpha_i \in \mathbb{R}$ for $i = 2, ..., m$, and $1+N<\alpha_2<\alpha_3<\ldots<\alpha_m$. Futher on we suppose $\alpha_i\neq N+k$ for $i=2,\ldots,m$, $k\in\mathbb{Z}$ **Assumption 16:** We
 $\alpha_1 = s - N = \sigma + iT, N$
 $1 + N < \alpha_2 < \alpha_3 < ... < \alpha_m$

and $\alpha_1 \neq \alpha_i$ for $i = 2,...,m$. now on we use the state of $\alpha_i \neq N$
 $\alpha_i \neq N$

We apply the Selberg trace formula with the function

$$
h^{\#}(r) = \frac{1}{\mathcal{R}(r)}\tag{68}
$$

Then the assumptions of Theorem 1 are fulfilled for $Re s \ge 2N + \epsilon_0$, $\epsilon_0 > 0$. Gangolli [9] takes another special function (and another version of the trace formula) using a cut-off function. The resulting formulas are not explicite enough to use those methods of analytic number theory which we will apply. By partial fraction expansion we get ed for *Re*
ersion of
ite enoug
al fraction
 $\neq j$ (*i*, *j* :
 $\frac{1}{n}$, $\frac{1}{n^2}$ + *h*[#](*r*) = $\frac{1}{\mathcal{R}(r)}$
 n 2 + *n* 2 = *n* 2 + *n* 2 = *n* 2 = *n* 2 + *n* 2 = 2*N* + *c*₀, *c*₀ > 0. Gangoll (and another version of the trace formula) using a cut sa are not explicite enough to use those met

Lemma 17: We suppose $\alpha_i \neq \alpha_j$ for $i \neq j$ $(i,j = 1,...,m)$. Then we get.

$$
\frac{1}{\mathcal{R}(r)} = \sum_{j=1}^{m} \frac{1}{\mathcal{Q}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
 (69)

We remark, that the summands in (69) allone do not fulfill the growth assumption of Theorem 1. We easily obtain the Fourier transform.

(and another version of the trace formulae) using a det of a
as are not explicit enough to use those methods of analytic
apply. By partial fraction expansion we get

$$
\alpha_i \neq \alpha_j \text{ for } i \neq j \ (i,j = 1,...,m). \text{ Then we get}
$$

$$
\frac{1}{\mathcal{R}(r)} = \sum_{j=1}^{m} \frac{1}{\mathcal{Q}(j,\alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$

$$
\text{mands in (69) alone do not fulfill the growth assumption}
$$

$$
n \text{ the Fourier transform.}
$$

$$
g^*(u) = \sum_{j=1}^{m} \frac{1}{\mathcal{Q}(j,\alpha_j)} \frac{e^{-\alpha_j|u|}}{2\alpha_j}.
$$

$$
= 2, \ m = 2). \text{ We define (cf. Part I) } r_p(\mu) = \sqrt{\mu - (N - p)^2}.
$$

of $h^{\#}(r) = \frac{1}{\mathcal{R}(r)}$
for $\mu \geq (N - p)$ *y* (*cf.* [14] for n=2, m=2). We define (*cf.* Part I) $r_p(\mu) = \sqrt{\mu - (N - p)^2}$ for $\mu \ge (N - p)^2$ and $r_p(\mu) = \sqrt{(N - p)^2 - \mu}$ *i* for $\mu < (N - p)^2$ with the imaginary unit i.

Proposition **18:** *The logarithmic derivative*

The logarithmic derivative
\n
$$
\Psi_p(s) = \kappa_p \sum_{\omega \in \Omega} l(\omega) \sigma(\omega) e_p(\omega) e^{-l(\omega)(s-N)}
$$

of the Selberg zeta function

$$
Z_p(s) = \prod_{\substack{\omega \in \Omega \\ \nu(\omega)=1}} \prod_{k=1}^q \prod_{m \in \mathbf{M}} \left(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)} \right)^{\kappa_p}
$$

satisfies the equation

position 18: The logarithmic derivative
\n
$$
\Psi_p(s) = \kappa_p \sum_{\omega \in \Omega} l(\omega) \sigma(\omega) e_p(\omega) e^{-l(\omega)(s-N)}
$$
\n
$$
lberg \text{ zeta function}
$$
\n
$$
Z_p(s) = \prod_{\substack{\omega \in \Omega \\ \nu(\omega) = 1}} \prod_{k=1}^q \prod_{m \in M} \left(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)} \right)^{\kappa_p}
$$
\n
$$
the \text{ equation}
$$
\n
$$
\frac{\Psi_p(s)}{2(s-N)\kappa_p} = \sum_{i=2}^m \frac{\mathcal{P}(i, s-N)}{\mathcal{P}(i, \alpha_i)} \frac{\Psi_p(\alpha_i+N)}{2\alpha_i \kappa_p}
$$
\n
$$
+ \sum_{\mu \in S_p} d_p^*(\mu) \frac{\mathcal{Q}(1, s-N)}{\mathcal{R}(r_p(\mu))} - \langle S_p^n, g^* \rangle \mathcal{Q}(1, s-N) \text{ vol } V
$$
\n(71)

for $\sigma > N$ *supposing Assumption 16.*

Proof: The functions $h^*(r)$ and $g^*(u)$ satisfy the assumptions of Theorem 1 if we suppose Assumption 16 using $\frac{Q(1,s-N)}{Q(j,\alpha_j)} = -\frac{p(j,s-N)}{p(j,\alpha_j)}$. The application of the Selberg trace formula (5) and Lemma 17 proves the proposition

We remark that we can not split up the sum $\sum_{\mu \in S_p}$ in (71) using (69) as we have done it for Ψ_p using (70) because of the fact that we have to guarantee the convergence of the sum $\sum_{\mu \in S_p}$. Next we are interested in the meromorphic continuation of that sum. For $Re s > n - 1$ we define *A*_{$\mu \in S_p$
 *A*_{$\mu \in S_p$ *A*_{μ} *A*_{σ} *A*_{σ} *A*_{*A*_{*A*}_{*A*}^{*A*}*AA*_{*A*}^{*A*}*AAA*^{*A*}*AAA*^{*A*}*AAAAAAAAA}}}* $\frac{f(x)}{f(x)} = -\frac{f(x)}{f(x)}$

Es, the propos

Split up the fact

of the fact
 $\sum_{f \in S_p} d_p^*(\mu) \frac{2(x)}{f(x)}$

(b) give

$$
\mathcal{A}(s) = \sum_{\mu \in S_p} d_p^*(\mu) \frac{2(s-N) \mathcal{Q}(1, s-N)}{\mathcal{R}(r_p(\mu))}.
$$
 (72)

Standard considerations (çf.[9, 14]) give

Lemma 19: *The function A(s) has a meromorphic continuation into the whole complex plane. The poles are* $s^+(\mu) = N + i r_p(\mu)$ and $s^-(\mu) = N - i r_p(\mu)$ for $\mu \in S_p$ and $d_p^*(\mu) \neq 0$. These are simple poles with residues $d_p^*(\mu)$ if $\mu \neq (N - p)^2$ and with residue $2 d_p^*(\mu)$ *if* $\mu = (N - p)^2$. $M(s) = \sum_{\mu \in S_p} d_p^*(\mu) \frac{2(s - N)Q(1, s - N)}{\mathcal{R}(r_p(\mu))}$. (72)
 Cf.[9, 14]) give
 Punction $\mathcal{A}(s)$ has a meromorphic continuation into the whole
 are $s^+(\mu) = N + i r_p(\mu)$ and $s^-(\mu) = N - i r_p(\mu)$ for $\mu \in S_p$ and
 ple poles nction $A(s)$ has a meromorphic continuation into the whole
 $e^{st}(\mu) = N + i r_p(\mu)$ and $s^-(\mu) = N - i r_p(\mu)$ for $\mu \in S_p$ and
 e poles with residues $d_p^*(\mu)$ if $\mu \neq (N - p)^2$ and with residue

omorphic continuation of $\Psi_p(s)$

In order to get the meromorphic continuation of $\Psi_p(s)$ we define

$$
\mathcal{W}(s) = -\mathcal{Q}(1, s - N) \left\langle S_n^n, g^* \right\rangle \text{ vol } V \tag{73}
$$

for *Res* > 2N. We start with the discussion of the case of *n* even. Using (6) and (22) we also can write

$$
W(s) = c_W \int_{-\infty}^{+\infty} r \,\bar{h}(r) \, \tanh(\pi r) \, dr \tag{74}
$$

with

Special Estimates for Compact Hyperbolic Space Forms

\n
$$
\bar{h}(r) = P(r^2) \frac{\mathcal{Q}(1, s - N)}{\mathcal{R}(r)}, \quad P(s) = \prod_{\substack{u=1/2 \\ u \neq |N-p|}}^{N} (s + u^2), \quad c_W = \frac{(-1)^{\frac{n-2}{2}} \chi(V)}{2 \Gamma(p+1) \Gamma(n-p)}.
$$
\n(75)

\npolynomial P is of degree $n/2 - 1 = m - 2$.

The polynomial P is of degree $n/2 - 1 = m - 2$.

Lemma 20: For *n* even the function $W(s) = -Q(1, s - N)\langle S_n^n, g^* \rangle$ vol V has a

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\nwith
\n
$$
\bar{h}(r) = P(r^2) \frac{Q(1, s - N)}{\mathcal{R}(r)}, P(s) = \prod_{\substack{s=1/2 \\ s=b'=P}}^{N} (s + u^2), c_W = \frac{(-1)^{\frac{n-2}{2}} \chi(V)}{2\Gamma(p+1)\Gamma(n-p)}.
$$
\n(75)
\nThe polynomial P is of degree $n/2 - 1 = m - 2$.
\nLemma 20: For n even the function $W(s) = -Q(1, s - N) (S_p^m, g^{\#})$ vol V has a
\nmeromorphic continuation into the whole complex plane which is given by
\n
$$
\frac{W(s)}{c_W} = P(-(s - N)^2) \frac{-2|N-p|}{-(N-p)^2 + (s - N)^2}
$$
\n
$$
- \sum_{j=2}^{m} P(-\alpha_j^2) \frac{P(j, s - N)}{P(j, \alpha_j)} - \frac{-2|N + k|}{-(N + k)^2 + (s - N)^2}
$$
\n
$$
+ \sum_{j=2}^{m} P(-\alpha_j^2) \frac{P(j, s - N)}{P(j, \alpha_j)} - \frac{-2|N + k|}{-(N + k)^2 + (s - N)^2}
$$
\n(76)
\n
$$
- \sum_{j=2}^{m} P(-\alpha_j^2) \frac{P(j, s - N)}{P(j, \alpha_j)} - \frac{-2|N + k|}{-(N + k)^2 + \alpha_j^2}
$$
\n
$$
-P(-(s - N)^2) \pi \tan(\pi(s - N)) + \sum_{j=2}^{m} P(-\alpha_j^2) \frac{P(j, s - N)}{P(j, \alpha_j)} \pi \tan(\pi \alpha_j).
$$
\nThe function W(s) has simple poles in the points $s = N - |N - p|$ and $s = -k$ (k ∈ N) with the residues $2c_W P(-(N - p)^2)$ and $2c_W P(-(N + k)^2)$, respectively. In the other points there is holomorphic behaviour.
\nProof: Clearly we obtain
\n
$$
\frac{P(r^2)}{R(r)} = \sum_{j=1}^{m} \frac{P(-\alpha_j^2)}{Q(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
\n(77)
\nIt follows
\n
$$
\bar{h}(r) = P(-\alpha_1^2) \frac{1
$$

The function $W(s)$ has simple poles in the points $s = N - |N - p|$ and $s = -k$ $(k \in N)$ with the residues $2c_W P(-(N-p)^2)$ and $2c_W P(-(N+k)^2)$, respectively. In the other *points there is holomorphic behaviour. m*

Proof: Clearly we obtain

$$
P(-p)^2
$$
 and $2c_W P(- (N + k)^2)$, respectively. In the other
haviour.

$$
\frac{P(r^2)}{\mathcal{R}(r)} = \sum_{j=1}^m \frac{P(-\alpha_j^2)}{\mathcal{Q}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
 (77)

$$
\frac{P(r^2)}{r^2 + \alpha_1^2} - \sum_{j=2}^m P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
 (78)
-*N*. In order to calculate the integral $\int_{-\infty}^{\infty} r \, \tilde{h}(r) \, \tanh(\pi r) dr$
N. The convergence of the sum $\sum_{i=1}^{\infty} \tilde{h}(r) \, \text{is guaranteed}$

It follows

$$
\frac{P(r^2)}{\mathcal{R}(r)} = \sum_{j=1}^{m} \frac{P(-\alpha_j^2)}{\mathcal{Q}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
(77)

$$
\bar{h}(r) = P(-\alpha_1^2) \frac{1}{r^2 + \alpha_1^2} - \sum_{j=2}^{m} P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
(78)

Thereby we have used $\alpha_1 = s - N$. In order to calculate the integral $\int_{-\infty}^{\infty} r \bar{h}(r) \tanh(\pi r) dr$ Thereby we have used $\alpha_1 = s - N$. In order to calculate the integral $\int_{-\infty}^{\infty} r \bar{h}(r) \tanh(\pi r) dr$ we apply the residue theorem. The convergence of the sum $\sum_{k=1}^{\infty}$ in (76) is guaranteed by the convergence of $\sum_{k=1}^{\infty} (N+k)\bar{h}(|N+k|i) = O(\sum_{k=1}^{\infty} k^{-2})$. Standard calculations (cf. *[9, 14]) prove the lemma. For more details cf. [35].* In order to calculate the integral j

he convergence of the sum $\sum_{k=1}^{\infty}$
 $\frac{1}{h}(|N+k|i) = O(\sum_{k=1}^{\infty} k^{-2})$. Star

re details cf. [35]
 $\in \mathbb{Z}: k \geq N+1\} \cup \{|N-p|\}$
 $\left(\frac{P(-\alpha_1^2)}{\alpha_1 + k} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac$

Lemma 21: *Using* $L = \{k \in \mathbb{Z} : k \geq N + 1\} \cup \{|N - p|\}$ *we get for n even the equation*

$$
\bar{h}(r) = P(-\alpha_1^2) \frac{1}{r^2 + \alpha_1^2} - \sum_{j=2}^m P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{1}{r^2 + \alpha_j^2}.
$$
 (78)
we used $\alpha_1 = s - N$. In order to calculate the integral $\int_{-\infty}^{\infty} r \bar{h}(r) \tanh(\pi r) dr$
residue theorem. The convergence of the sum $\sum_{k=1}^{\infty} \sin(76)$ is guaranteed
gence of $\sum_{k=1}^{\infty} (N+k)\bar{h}(|N+k|i) = O(\sum_{k=1}^{\infty} k^{-2})$. Standard calculations (cf.
the lemma. For more details of. [35] **8**
21: Using $\mathbf{L} = \{k \in \mathbf{Z} : k \ge N+1\} \cup \{|N-p|\}$ we get for n even the

$$
\mathcal{W}(s) = 2c_{\mathbf{W}} \sum_{k \in \mathbf{L}} \left(\frac{P(-\alpha_1^2)}{\alpha_1 + k} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)}\right) + c_{\mathbf{W}} \sum_{\substack{k=1 \\ k \ne l}}^N \left(\frac{P(-\alpha_1^2)}{\alpha_1 + k} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)}\right).
$$
 (79)

Proof: Lemma 21 follows by straightforward calculations starting with (76) using

21 follows by straightforward calculations start
\n
$$
\pi \tan(\pi \alpha) = -\frac{1}{\alpha - \frac{1}{2}} - \sum_{0 \neq k \in \mathbb{Z}} \left(\frac{1}{\alpha - \frac{1}{2} - k} + \frac{1}{k} \right)
$$
\n
$$
-P(-\alpha_1^2) + \sum_{j=2}^m P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} = 0
$$

and

$$
-P(-\alpha_1^2)+\sum_{j=2}^m P(-\alpha_j^2)\frac{\mathcal{P}(j,\alpha_1)}{\mathcal{P}(j,\alpha_j)}=0
$$

Next we consider (73) for *n* odd. Then we get

by straightforward calculations starting with (76) using
\n
$$
= -\frac{1}{\alpha - \frac{1}{2}} - \sum_{0 \neq k \in \mathbb{Z}} \left(\frac{1}{\alpha - \frac{1}{2} - k} + \frac{1}{k} \right)
$$
\n
$$
\alpha_1^2 + \sum_{j=2}^m P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} = 0 \text{ and}
$$
\n
$$
\text{odd. Then we get}
$$
\n
$$
\mathcal{W}(s) = -c_{\mathcal{W}} \int_{-\infty}^{+\infty} \tilde{h}(r) dr \qquad (80)
$$

with

$$
\pi \tan(\pi \alpha) = -\frac{1}{\alpha - \frac{1}{2}} - \sum_{0 \neq k \in \mathbb{Z}} \left(\frac{1}{\alpha - \frac{1}{2} - k} + \frac{1}{k} \right)
$$

\n1
\n1
\n
$$
-P(-\alpha_1^2) + \sum_{j=2}^m P(-\alpha_j^2) \frac{P(j, \alpha_1)}{P(j, \alpha_j)} = 0
$$

\nNext we consider (73) for *n* odd. Then we get
\n
$$
W(s) = -c_W \int_{-\infty}^{+\infty} \tilde{h}(r) dr
$$
\n(80)
\nh
\n
$$
cw = \frac{\binom{n-1}{p} (4\pi)^{-n/2}}{\Gamma(n/2)} \text{ vol } V, \quad \tilde{h}(r) = \tilde{P}(r^2) \frac{Q(1, s - N)}{\mathcal{R}(r)}, \quad \tilde{P}(s) = \prod_{\substack{n=0 \text{odd} \\ n \neq |N-p|}}^N (s + u^2). \quad (81)
$$

\ne polynomial \tilde{P} is of degree $N = m - 1$. Applying (77), we conclude
\n
$$
\tilde{h}(r) = \frac{\tilde{P}(-\alpha_1^2)}{r^2 + \alpha_1^2} - \sum_{j=2}^m \frac{\tilde{P}(-\alpha_j^2)}{r^2 + \alpha_j^2} \frac{P(j, \alpha_1)}{P(j, \alpha_j)}. \quad (82)
$$

\nmg
\n
$$
\int_{-\infty}^{+\infty} \frac{1}{r^2 + \alpha^2} dr = \frac{\pi}{\alpha}
$$
\n(83)
\nallows
\n
$$
\frac{W(s)}{(s)} = -\tilde{P}(-\alpha_1^2) \frac{\pi}{s} - \sum_{j=1}^m \tilde{P}(-\alpha_j^2) \frac{P(j, \alpha_1)}{P(j, \alpha_1)} \frac{\pi}{s}. \quad (84)
$$

The polynomial \tilde{P} is of degree $N=m-1.$ Applying (77), we conclude

$$
\mathcal{R}(r) \qquad \mathcal{R}(r) \qquad \mathcal{L}(r) \qquad \mathcal{L}(r) \qquad \mathcal{L}(r) \qquad (3.7)
$$
\n
$$
\text{degree } N = m - 1. \text{ Applying (77), we conclude}
$$
\n
$$
\tilde{h}(r) = \frac{\tilde{P}(-\alpha_1^2)}{r^2 + \alpha_1^2} - \sum_{j=2}^{m} \frac{\tilde{P}(-\alpha_j^2)}{r^2 + \alpha_j^2} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)}. \qquad (82)
$$
\n
$$
\int_{-\infty}^{+\infty} \frac{1}{r^2 + \alpha^2} dr = \frac{\pi}{\alpha} \qquad (83)
$$
\n
$$
\frac{\tilde{B}}{\tilde{B}} = -\tilde{P}(-\alpha_1^2) \frac{\pi}{\alpha_1} - \sum_{j=2}^{m} \tilde{P}(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{\pi}{\alpha_j}. \qquad (84)
$$
\n
$$
\text{odd the function}
$$

Using

$$
+\alpha_1^2 - \sum_{j=2}^{\infty} r^2 + \alpha_j^2 \overline{\mathcal{P}(j, \alpha_j)}.
$$
\n
$$
\int_{-\infty}^{+\infty} \frac{1}{r^2 + \alpha^2} dr = \frac{\pi}{\alpha}
$$
\n(83)

it follows

$$
\tilde{h}(r) = \frac{\tilde{P}(-\alpha_1^2)}{r^2 + \alpha_1^2} - \sum_{j=2}^m \frac{\tilde{P}(-\alpha_j^2)}{r^2 + \alpha_j^2} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)}.
$$
\n(82)\n
$$
\int_{-\infty}^{+\infty} \frac{1}{r^2 + \alpha^2} dr = \frac{\pi}{\alpha}
$$
\n(83)\n
$$
\frac{\mathcal{W}(s)}{c_{\mathcal{W}}} = -\tilde{P}(-\alpha_1^2) \frac{\pi}{\alpha_1} - \sum_{j=2}^m \tilde{P}(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{\pi}{\alpha_j}.
$$
\n(84)\noved

Thereby we have proved

$$
\tilde{h}(r) = \frac{\tilde{P}(-\alpha_1^2)}{r^2 + \alpha_1^2} - \sum_{j=2}^m \frac{\tilde{P}(-\alpha_j^2)}{r^2 + \alpha_j^2} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)}.
$$
\n
$$
\mathbf{g}
$$
\n
$$
\int_{-\infty}^{+\infty} \frac{1}{r^2 + \alpha^2} dr = \frac{\pi}{\alpha}
$$
\n(82)\n
$$
\text{llows}
$$
\n
$$
\frac{\mathcal{W}(s)}{c_{\mathcal{W}}} = -\tilde{P}(-\alpha_1^2) \frac{\pi}{\alpha_1} - \sum_{j=2}^m \tilde{P}(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \frac{\pi}{\alpha_j}.
$$
\n(84)\n
$$
\text{reby we have proved}
$$
\n
$$
\text{Lemma 22:} \quad \text{For } n \text{ odd the function}
$$
\n
$$
(s - N)\mathcal{W}(s) = \pi c_{\mathcal{W}} \left(-\tilde{P}(-s - N)^2) + \sum_{j=2}^m \tilde{P}(-\alpha_j^2) \frac{\mathcal{P}(j, s - N)}{\mathcal{P}(j, \alpha_j)} \frac{s - N}{\alpha_j} \right)
$$
\n(85)\n
$$
\text{lomorphic in the whole complex plane.}
$$

is holomorphic in the whole complex plane.

From Proposition 18 and the definitions (72) and (73) it follows

for *in the whole complex plane*.

\nm Proposition 18 and the definitions (72) and (73) it follows

\n
$$
\Psi_p(s) = \sum_{j=2}^{m} \frac{\mathcal{P}(j, s - N)}{\mathcal{P}(j, \alpha_j)} \frac{s - N}{\alpha_j} \Psi_p(\alpha_j + N) + \kappa_p \mathcal{A}(s) + 2\kappa_p (s - N) \mathcal{W}(s).
$$
\n(86)

Proposition 23: The logarithmic derivative Ψ_p of the Selberg zeta function Z_p has *a meromorphic* **continuation** *to the whole complex plane. The function 1*P, *has holomorphic behaviour with the exception of the following simple poles with integer residues:*

Proof: The proposition is an immediate consequence of (22), (75), (86), Lemma 19 and Lemma 20. We have chosen $\kappa_p = \Gamma(p + 1)\Gamma(n - p)$ for *n* even in order to guarantee that $2\kappa_p c_W$ is an integer. We remark that κ_p is in general not the smallest number with that property \blacksquare

Let M_p be the number of poles described above, which are real numbers. If we order them by their value, we get s_1, \ldots, s_{M_p} . Gangolli [9] called the poles of type (i) - (iv) *spectral poles* and those of type (v) *topological poles.* The topological poles are determined by the Euler-Poincaré characteristic $\chi(V)$ and the numbers p and n.

As a concequece of Proposition 23 there exists a meromorphic function $Z_p(s)$ with $Z'_p(s)/Z_p(s) = \Psi_p(s)$, which is uniquely determined up to a multiplicative constant. But this constant is determined by our definition (63). *So* we have found an analytic continuation of $Z_p(s)$ and we get Figure 2011

ID: The proposition 23 there exists a meromorphic function $Z_p(s) = \Psi_p(s)$, which is uniquely determined up to a multiplicative constant is determined by our definition (63). So we have found an analytic $\mathcal{P}_$ **a** concequece of Proposition 23 there exists a meromorphic further
 $\mu_p(s) = \Psi_p(s)$, which is uniquely determined up to a multiplication

stant is determined by our definition (63). So we have found as

f $Z_p(s)$ and we get

Theorem 24: The Selberg zeta function $Z_p(s)$ defined by (63) has a meromorphic continuation *to the whole complex plane. We define* $\nu_p = \kappa_p d_p^*(0)$,

 $\kappa_{p} d_{p}^{*}(0) + (-1)^{n/2} 2|N-p|\chi(V) P(-(N-p)^{2})$ for n even

and $\nu_{p}^{k} = (-1)^{n/2} 2(N+k) \chi(V) P(-(N+k)^{2})$ for n even. The function $Z_{p}(s)$ has holo*morphic behaviour with the exception of the following poles and we also state the zeros:*

It follows that the zeros of $Z_p(s)$ are in the intervall $[N - |N - p|, N + |N - p|]$, at the points $s = -k$, $(k \in N)$ or at the line $Res = N$. So we can say that the Selberg zeta function $Z_p(s)$ satisfies a modified Riemann hypothesis.

In order to get estimations *for* the Selberg zeta function we will frequently use a functional equation for $Z_p(s)$. We define

$$
\Phi_p(s) = \begin{cases}\n(-1)^{n/2} 2s \chi(V) P(-s^2) \pi \tan(\pi s) & \text{for n even} \\
-(\binom{n-1}{p} (4\pi)^{-(n-2)/2} \tilde{P}(-s^2) / \Gamma(n/2) & \text{vol } V \text{ for n odd.} \n\end{cases}
$$
\n(87)

Proposition 25: *The logarithmic derivative* Ψ_p *of the Selberg zeta function* Z_p *satisfies the functional equation* $\Psi_p(s) + \Psi_p(2N - s) = \Phi_p(s - N)$.

Proof: From (66) and (72) it follows $A(s) + A(2N - s) = 0$. Using (86) we get $\Psi_p(s) + \Psi_p(2N - s) = 2\kappa_p(s - N)(\mathcal{W}(s) - \mathcal{W}(2N - s)).$ By (76) we obtain for *n* even $W(s) - W(2N - s) = -2c_WP(-(s - N)^2)\pi \tan (\pi (s - N))$. In view of *Lemma 22* we get *for n odd the equation* $(s - N)(W(s) - W(2N - s)) = -2\pi c_W P(-(s - N)^2)$ *. Using (75), (81) and the* definition (*87) the proposition follows at once* ^I

Further *on,* the definition (*87) implies*

Lemma 26: The function Φ_p is meromorphic in the whole complex plane. It satisfies $\Phi_p(s) = \Phi_p(-s)$. For n even the function $\Phi_p(s)$ is holomorphic with the exception of the simple poles $s = \pm (N - p)$, $s = \pm (N + k)$ $(k \in \mathbb{N})$ with integer residues. The residues *at the points* $s = \pm(N - p)$ *are negative for n* $\equiv 2 \mod 4$. The residues at the points $s = \pm (N-p)$ are positive for $n+2p \equiv 2 \mod 4$ for $p < N$ as well as for $n+2p \equiv 0 \mod 4$ *for* $p > N$ *, in the other cases they are negative. The residues at the points* $s = \pm(N + k)$ *are negative for n* \equiv 2 *mod* 4 *and positive for n* \equiv 0 *mod* 4. For *n odd the function* $\Phi_p(s)$ *is holomorphic in the whole complex plane.*

From Lemma 26 it follows that there exists *a merornorphic* function *B(s)* with

$$
\Phi_p(s) = \frac{\mathcal{B}'_p(s)}{\mathcal{B}_p(s)} \quad \text{and} \quad \mathcal{B}_p(0) = 1.
$$

As a consequence of Proposition *25* we get

Proposition 27: The Selberg zeta function Z_p satisfies the functional equation $Z_p(2N - s) = B_p(N - s)Z_p(s).$

Definition 28: We define the open domain

\n
$$
D_{p} = \left\{ s \in C : Re s > 0 \right\}
$$
\n
$$
\left\{ \bigcup_{\substack{\mu \in S_{p}^{4} \\ \mu > (N-p)^{2}}} \left(r_{p}(\mu) i, N + r_{p}(\mu) i \right) \right\}
$$
\n
$$
\left\{ \bigcup_{\substack{\mu \in S_{p}^{4} \\ \mu > (N-p)^{2}}} \left(-r_{p}(\mu) i, N - r_{p}(\mu) i \right) \right\} \left(0, N + |N - p| \right).
$$
\nThe domain D_{p} is simply connected. We have $Z_{p}(s) \neq 0$ for $s \in D_{p}$. As a

\n3) we get

\n
$$
\ln Z_{p}(s) = \kappa_{p} \sum_{\substack{\nu \in \Omega \\ \nu(\nu) = 1}} \sum_{k=1}^{q} \sum_{m \in M} \ln \left(1 - \epsilon_{p,k}(\omega) \beta^{m}(\omega) e^{-l(\omega)(s + |m|)} \right)
$$
\nRes $s > 2N$. This describes the function $\ln Z_{p}(s)$ in terms of the length

The domain D_p is simply connected. We have $Z_p(s) \neq 0$ for $s \in D_p$. As a consequence of *(63)* we get

$$
\ln Z_p(s) = \kappa_p \sum_{\substack{\omega \in \Omega \\ \nu(\omega) = 1}} \sum_{k=1}^q \sum_{m \in \mathbf{M}} \ln \left(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s + |m|)} \right)
$$

for $Res > 2N$. This describes the function $\ln Z_p(s)$ in terms of the length spectrum. The analytic continuation to D_p is meromorphic because of the fact that D_p is simply conected and $Z_p(s)$ has no poles and zeros in that domain. Further on we get by analytic continuation connected. We have $Z_p(s) \neq 0$ for $s \in D_p$. As a consequence
 $\sum_{\omega=1}^{\infty} \sum_{k=1}^{s} \sum_{m \in M} \ln(1 - \epsilon_{p,k}(\omega) \beta^m(\omega) e^{-l(\omega)(s+|m|)})$

bes the function $\ln Z_p(s)$ in terms of the length spectrum.
 D_p is meromorphic because of

$$
Z_p(\bar{s}) = \overline{Z_p(s)}, \quad \Psi_p(\bar{s}) = \overline{\Psi_p(s)}.
$$
\n(88)

8. An estimation for the Selberg zeta function based on the Weyl estimation for the eigenvalue spectrum

We will apply standard methods of prime number theory and function theory. Following Hejhal [14] and Ingham *[16]* we will omit those parts of the proofs which are straigtforward generalizations, for a detailed discussion cf. Schuster [35]. The main result of this section will be

Theorem 29: For all points $s \in \mathbb{C}$, which have a distance not smaller than $1/2$ to *the poles of the Selberg zeta function* $Z_p(s)$, we can state $|Z_p(s)| = \exp (O(|s|^n)).$

We remark that in the classical case $n = 2$, $p = 0$ describt in [14] there are no poles at all. For $\text{Re } s > 2N + \epsilon$ ($0 < \epsilon < 1$) Theorem 29 is a simple conclusion of Lemma 14. Next we will analyze the case $Re s \leq \epsilon$, then it remains to discuss the case $-\epsilon \leq Re s \leq 2N + \epsilon$. For $n \equiv 2 \mod 4$ we define

$$
\mathbf{D}_{p}^{\#} = \left\{ s \in \mathbf{C}: \ \left| s \pm (N-p) \right| > \frac{1}{4}, \ \left| s \pm (N+k) \right| > \frac{1}{4} \ \left(k \in \mathbf{N} \right) \right\}.
$$

For $n \equiv 0 \pmod{4}$ we define

$$
\mathbf{D}_{p}^{\#} = \Big\{s \in \mathbf{C}: \ \ |s \pm (N-p)| > \frac{1}{4}\Big\}.
$$

If *n* is odd we put $D_p^* = C$. We get

Lemma 30: For $\text{Re } s \le -\epsilon$ and $s - N \in D_p^*$, we have $|Z_p(s)| = \exp(O(|s|^n)).$

Proof: Using $B(s) = B(i) \exp(\int_i^s \Phi_p(v) dv)$, where the integration shall be taken along the polygonial connecting the points *i*, *iT* and $s = \sigma^* + iT$ for $Im s \ge 1$ and the points *i, Res* + *i* and s for $0 \leq Im s < 1$, a straightforward generalization of [14, pp. 75] *- 76* applying the maximum modulus principle gives $|B_p(s)| = \exp(O(|s|^n))$ for $s \in D_p^*$. The functional equation described in Proposition 27 and Lemma $14/(\text{iii})$ complete the proof **a**
 Following [14, 16] we use the concept of "good" numbers. For that we define
 $x_k = \sum_{\mu \in S_p \atop \mu = 2 \le r_p(\mu) < k+2} d_p^{\delta}(\mu)$ proof **a**

Following *[14, 16]* we use the concept of "good" numbers. For that we define

$$
x_k = \sum_{\substack{\mu \in S_p \\ k-2 \le r_p(\mu) < k+2}} d_p^{\delta}(\mu)
$$

for $k \in \mathbb{N}$, $k > 2$. The equation (1) implies

We use the concept of "good" numbers. For that we define\n
$$
x_k = \sum_{\substack{\mu \in S_p \\ k-2 \le r_p(\mu) < k+2}} d_p^{\delta}(\mu)
$$
\n
$$
x_{k-2} \le r_p(\mu) < k+2
$$
\n
$$
x_k \approx 4(2^n - 1)n_p L^n \quad \text{for } L \in \mathbb{N}, \ L \to \infty
$$
\n(89)

with n_p given by (2). We put $\bar{n}_p = 4(2^n - 1)n_p$. The number $k \in \mathbb{N}$ is said to be "good" if $x_k \leq \bar{n}_p k^{n-1}$ is satisfied. By a trivial generalization of [14, Proposition 4.21] we see that if $x_k \n\t\leq \bar{n}_p k^{n-1}$ is satisfied. By a trivial generalization of [14, Proposition 4.21] we see that for a fixed $\delta > 0$ there exists a sufficiently large number $L \in \mathbb{N}$ such that the interval $[L, L(1+\delta)]$ contains "good" values. For $L \in \mathbb{N}$, $L \to \infty$ (89)

The number $k \in \mathbb{N}$ is said to be "good"

ion of [14, Proposition 4.21] we see that

number $L \in \mathbb{N}$ such that the interval

oncept of "admissible" numbers used in

the considerations

But a straightforward generalization of the concept of "admissible" numbers used in *[14]* would not be strong enough for the following considerations. Our definition will be different also for the classical case $n = 2$, $p = 0$.

Proposition 31: Let k be a "good" number. Then there exists $T \in [k-1, k+1]$ *with n*, $p = 0$.
 d^{*n*} numbe
 $\frac{d_p^*(\mu)}{-r_p(\mu)}$

Let k be a "good" number. Then there exists
$$
T \in [k-1, k+1]
$$

$$
\sum_{T-1 \le r_p(\mu) \le T+1} \frac{d_p^*(\mu)}{|T - r_p(\mu)|} = O(T^{n-1} \ln T).
$$
 (90)
be called "admissible"

Such a number T shall be called "admissible".

For the proof we need the following

Lemma 32: *There shall be given real numbers* r_i ($i = 1, ..., E$) with $-2 \le r_1 \le$ the proof we need the following
 ama 32: There shall be given real numbers r_i ($i = 1$
 $\leq r_E \leq 2$. Then there exists a number $r_* \in [-1,1]$ with *E*
E
E
E
E
E

use
$$
n = 2
$$
, $p = 0$.
\nbe a "good" number. Then there exists $T \in [k - 1, k + 1]$
\n
$$
\frac{d_p^*(\mu)}{\frac{1}{T - r_p(\mu)}} = O(T^{n-1} \ln T).
$$
\n(90)
\n l "admissible".
\nfollowing
\nbe given real numbers r_i ($i = 1, ..., E$) with $-2 \le r_1 \le r$
\nexists a number $r \in [-1, 1]$ with
\n
$$
\sum_{i=1}^{E} \frac{1}{|r_i - r_*|} = O(E \ln E).
$$
\n(91)
\nis independent of the position of the points r_i .

Thereby the O-term for $E \to \infty$ *is independent of the position of the points r_i.*

Proof: We devide the interval $[-2, 2)$ into 10 E subintervals $I_i = [-2+(i-1)k_0, -2+1]$ *ik*₀) with $k_0 = \frac{2}{5E}$, $i = 1, ..., 10E$. Let *I'* be the union of those intervals I_i in which one of the given r, is situated. Let *I"* be the union of intervals which are already included in *I'* or which have neighbouring intervals (with respect to the intervals introduced above) in *P*. We denote by *I*^{\bullet} the union of the intervals I_i which are contained in $[-1, 1]$ but which are not included in *I"*. By help of the usual Lebesgue measure μ we get $\mu(I'') \leq 6/5$ and $\mu(I^*) \geq 4/5$. We denote the midpoint of the interval *I_j* in which the point r_i is contained by \bar{r}_i . It follows $|r_i - \bar{r}_i| \leq k_0/2$. For $r \in I^*$ and for all r_i, \bar{r}_i we have $|r - r_i| \ge k_0, |r - \bar{r}_i| \ge k_0$. Further on, we get either $r_i>r,\bar{r}_i>r$ or $r_i < r,\bar{r}_i < r$. $|r - r_i| \geq \kappa_0$, $|r - r_i| \geq \kappa_0$. Further on, we get either $r_i > r$, $r_i > r$ or $r_i < r$, $r_i < r$.

Suppose $r_i > r$, $\bar{r}_i > r$. From $\bar{r}_i - r_i \leq k_0/2 < k_0 < r_i - r$ there follows $\frac{1}{|r_i - r|} < 2\frac{1}{|r_i - r|}$. The

same equation same equation follows for the other case $r_i > r$, $\bar{r}_i > r$ using $r_i - \bar{r}_i \le k_0/2 < k_0 \le r - r_i$. We get as an immediate consequence ervals (wi)

f the interport in the midport of the $|\cdot - \bar{r}_i| \leq$
 $|\cdot - \bar{r}_i| \leq k_0/k$

ther case is used the case of the state of the state
 $\frac{1}{\vert -r\vert} < 2\sum_{i=1}^{n}$
 $\frac{1}{\vert -r\vert}$ the minimum of the minimum of the If I' be the union of those in

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the usual Lebesgue measur

midpoint of the interval I_j
 $|S \leq k_0/2$. For $r \in I^*$ are
 $r \in I^*$ on, we ge

$$
\sum_{i=1}^{E} \frac{1}{|r_i - r|} < 2 \sum_{i=1}^{E} \frac{1}{|\bar{r}_i - r|} \quad \text{for } r \in I^*.
$$
\n(92)

We denote by $s_i^*(i = 1, ..., E^*)$ the midpoints of those intervals $I_j(j = 1, ..., E)$ for which we have $I_j \subset I^*$ with $s_i^* < s_j^* < \ldots < s_{E^*}^*$. The inequality $\mu(I^*) \geq 4/5$ implies $E^* \geq 2E$. We obtain *) the mix
 $\langle s_2^* \rangle$
 $\frac{2}{s_0} \left(\frac{1}{2} + \frac{1}{3} \right)$
 $\frac{2}{s_0}$ e that
 $\frac{1}{s_0^*} \leq \frac{2}{k_0}$ *mm_i* E^* + *i* = *i* = *i* + *i* = *i* + *i* = *i* + *i* +

$$
\sum_{j=1}^{E^*} \frac{1}{|\bar{r}_i - s_j^*|} \le \frac{2}{k_0} \Big(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{10E}\Big) \le \frac{2}{k_0} \ln(10E)
$$

for $i = 1, 2, 3, \dots, E$. We deduce that

$$
\sum_{i=1}^{E} \sum_{j=1}^{E^*} \frac{1}{|\bar{r}_i - s_j^*|} \le \frac{2}{k_0} E \ln(10E) = 5E^2 \ln(10E).
$$

It follows

$$
\min_{j=1, \dots, E^*} \sum_{i=1}^{E} \frac{1}{|\bar{r}_i - s_j^*|} \le \frac{5E^2}{E^*} \ln(10E).
$$

The minimum shall be reached for $r_* = s_j^*$. Using $\frac{1}{E^*} \le \frac{1}{2E}$ we get

$$
\sum_{i=1}^{E} \frac{1}{|\bar{r}_i - r_i|} \le \frac{5}{2} E \ln(10E).
$$

for $i = 1, 2, 3, \ldots, E$. We deduce that

$$
\frac{1}{\left|1 - s_j^* \right|} \le \frac{2}{k_0} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{10E} \right) \le \frac{2}{k_0} \ln(
$$
\nWe deduce that

\n
$$
\sum_{i=1}^{E} \sum_{j=1}^{E^*} \frac{1}{|\bar{r}_i - s_j^*|} \le \frac{2}{k_0} E \ln(10E) = 5E^2 \ln(10E).
$$
\n
$$
\min_{j=1,\ldots,E^*} \sum_{i=1}^{E} \frac{1}{|\bar{r}_i - s_j^*|} \le \frac{5E^2}{E^*} \ln(10E).
$$

It follows

$$
\min_{j=1,\dots,E^*}\sum_{i=1}^E\frac{1}{|\bar{r}_i-s_j^*|}\leq \frac{5E^2}{E^*}\ln(10E).
$$

The minimum shall be reached for $r_*=s_j^*$. Using $\frac{1}{E^*}\leq \frac{1}{2E}$ we get

$$
\sum_{i=1}^E \frac{1}{|\bar{r}_i - r_*|} \leq \frac{5}{2} E \ln(10E).
$$

Since (92) we find that

$$
\sum_{i=1}^{E} \frac{1}{|r_i - r_*|} \leq 5E \ln(10E) = O(E \ln E).
$$

Thereby the lemma is proved.

Proof of Proposition 31: By the definition of a "good" number, there are at most $E = \bar{n}_p k^{n-1}$ values $r_p(\mu)$ (counted with multiplicity $d_p^{\delta}(\mu)$) which are contained in the interval $[k-2, k+2]$. Applying Lemma 32 there exists $r_0 = T$ with $T \in [k-1, k+1]$ and By the d
ted with
Lemma 3:
 $d_p^*(\mu)$
 $-r_n(\mu)$

$$
\sum_{k-2 \leq r_p(\mu) \leq k+2} \frac{d_p^*(\mu)}{|T - r_p(\mu)|} = O(k^{n-1} \ln k) = O(T^{n-1} \ln T).
$$

This proves Lemma 31 *^U*

As mentioned above, we need estimates for the Selberg zeta function $Z_p(s)$ for $-\epsilon \leq$ $Re s \leq 2N + \epsilon$ in order to prove Theorem 29. We can state

Lemma 33: *Using* $s = N + \sigma + iT$ we suppose $-\frac{1}{2} - N \leq \sigma \leq N + 1, T \geq 2$ and *that T is "admissible".* Then we *have*

(i) $|\Psi_p(s)| = O(|T|^n)$ (iii) $|Z_p(s)| = \exp(O(|T|^n)).$

Proof: We start with the case of *n* even. As a consequence of (72), (79) and (86) we get

This proves Lemma 31 **Example 21**
\nAs mentioned above, we need estimates for the Selberg zeta function
$$
Z_p(s)
$$

\n $Re s \le 2N + \epsilon$ in order to prove Theorem 29. We can state
\n**Lemma 33:** Using $s = N + \sigma + iT$ we suppose $-\frac{1}{2} - N \le \sigma \le N + 1$,
\nthat *T* is "admissible". Then we have
\n(i) $|\Psi_p(s)| = O(|T|^n)$
\n(ii) $|Z_p(s)| = \exp(O(|T|^n))$.
\nProof: We start with the case of *n* even. As a consequence of (72), (79
\nwe get
\n
$$
\frac{\Psi_p(s)}{s - N} = \sum_{j=2}^{m} \frac{P(j, s - N)}{P(j, \alpha_j)} \frac{\Psi_p(\alpha_j + N)}{\alpha_j} + 2\kappa_p \sum_{\mu \in S_p} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))} + 4c_W \kappa_p \sum_{k \in L} \left(\frac{P(-\alpha_1^2)}{\alpha_1 + k} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \alpha_1)}{P(j, \alpha_j)} \right) + 2c_W \kappa_p \sum_{k=1}^{N} \left(\frac{P(-\alpha_1^2)}{\alpha_1 + k} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \alpha_1)}{P(j, \alpha_j)} \right)
$$
.
\nBy Proposition 23 the poles of $\frac{\Psi_p(s)}{s - N}$ for the supposed domain of *s* are at $T = \pm r_p$
\nfor $\mu \in S_p$, $\mu \ge (N - p)^2 + 4 = \mu_p^0$ with $d_p^*(\mu) \ne 0$. We deduce
\n
$$
\left| \frac{\Psi_p(s)}{s - N} \right| \le O(|T|^{n-2}) + 2\kappa_p \sum_{\mu \in S_p} d_p^*(\mu) \frac{Q(1, \sigma + iT)}{R(r_p(\mu))} - \sum_{\mu \in S_p} \frac{P(-\alpha_j^2)}{R(j, \sigma + iT)} \frac{P(j, \sigma + iT)}{P(j, \sigma + iT)}
$$

By Proposition 23 the poles of $\frac{\Psi_P(s)}{s-N}$ for the supposed domain of *s* are at $T = \pm r_p(\mu)$, $\sigma = 0$

position 23 the poles of
$$
\frac{\Psi_p(s)}{s-N}
$$
 for the supposed domain of s are at $T = \pm r_p(\mu)$, $\sigma = 0$
\n
$$
\left| \frac{\Psi_p(s)}{s-N} \right| \leq (N-p)^2 + 4 = \mu_p^0 \text{ with } d_p^*(\mu) \neq 0. \text{ We deduce}
$$
\n
$$
\left| \frac{\Psi_p(s)}{s-N} \right| \leq O(|T|^{n-2}) + 2\kappa_p \sum_{\substack{\mu \in S_p \\ \mu_p^0 \leq \mu}} d_p^*(\mu) \left| \frac{\mathcal{Q}(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right|
$$
\n
$$
+ 4c_W \kappa_p \sum_{k \in \mathbb{L}} \left| \frac{P(-(\sigma + iT)^2)}{\sigma + iT + k} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{\mathcal{P}(j, \sigma + iT)}{\mathcal{P}(j, \alpha_j)} \right|.
$$
\n(93)

We break up the sum $\sum_{\mu_p^0 \leq \mu \in S_p}$ into contributions

(i) $r_p(\mu) < T-1$, (ii) $T-1 \leq r_p(\mu) \leq T+1$, *(iii)* $T + 1 < r_p(\mu) < 2T$, (iv) $2T \leq r_p(\mu)$. $r_p(\mu) < T - 1$,
 $T - 1 \le r_p(\mu) \le T + 1$,
 $T + 1 < r_p(\mu) < 2T$,
 $2T \le r_p(\mu)$.

ccall that we suppose that T is "*i*

Case $r_p(\mu) < T - 1$: With constan
 $\left| \frac{\mathcal{P}(2, \sigma + iT)}{\mathcal{P}^+(2, r_p(\mu))} \right| \le c_1 \frac{T^{n-2}}{(r_p(\mu))^{n-2}}$,

We recall that we suppose that *T* is "admissible".

Case $r_p(\mu) < T - 1$: With constants c_1 and c'_1 we obtain

e break up the sum
$$
\sum_{\mu_p^o\leq\mu\in S_p}
$$
 into contributions
\n(i) $r_p(\mu) < T - 1$,
\nii) $T - 1 \leq r_p(\mu) \leq T + 1$,
\niii) $T + 1 < r_p(\mu) < 2T$,
\niv) $2T \leq r_p(\mu)$.
\nRecall that we suppose that T is "admissible".
\nCase $r_p(\mu) < T - 1$: With constants c_1 and c'_1 we obtain
\n
$$
\left|\frac{\mathcal{P}(2, \sigma + iT)}{\mathcal{P}^+(2, r_p(\mu))}\right| \leq c_1 \frac{T^{n-2}}{(r_p(\mu))^{n-2}}, \quad \left|\frac{1}{(r_p(\mu))^2 + \alpha_2^2} \frac{\mathcal{P}(2, \sigma + iT)}{\mathcal{P}^+(2, r_p(\mu))}\right| \leq c'_1 \frac{T^{n-2}}{(r_p(\mu))^n}.
$$
\n(94)

Using $|T - r_p(\mu)| > 1$, we see that

c: up the sum
$$
\sum_{\mu_p^0 \le \mu \in S_p}
$$
 into contributions
\n
$$
(\mu) < T - 1,
$$
\n
$$
-1 \le r_p(\mu) \le T + 1,
$$
\n
$$
+1 < r_p(\mu) < 2T,
$$
\n
$$
T \le r_p(\mu).
$$
\nthat we suppose that T is "admissible".
\n
$$
r_p(\mu) < T - 1
$$
: With constants c_1 and c'_1 we obtain
\n
$$
\left| \frac{\partial \sigma + iT}{\partial (r_p(\mu))} \right| \le c_1 \frac{T^{n-2}}{(r_p(\mu))^{n-2}}, \quad \left| \frac{1}{(r_p(\mu))^2 + \alpha_2^2} \frac{\mathcal{P}(2, \sigma + iT)}{\mathcal{P}^+(2, r_p(\mu))} \right| \le c'_1 \frac{T^{n-2}}{(r_p(\mu))^n}
$$
\n
$$
-r_p(\mu) > 1, \text{ we see that}
$$
\n
$$
\left| \frac{1}{(r_p(\mu))^2 + (\sigma + iT)^2} \right| = \left| \frac{1}{(r_p(\mu) + i\sigma - T)(r_p(\mu) - i\sigma + T)} \right| \le \frac{1}{r_p(\mu)}.
$$

The spectral estimate (24) implies

Spectral Estimates for Compact Hyperbolic Space Forms
\ntimate (24) implies
\n
$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) < T-1}} \frac{d_p^*(\mu)}{(r_p(\mu))^{n-1}} = O(T), \qquad \sum_{\substack{\mu \in S_p \\ r_p(\mu) < T-1}} \frac{d_p^*(\mu)}{(r_p(\mu))^n} = O(\ln T). \tag{95}
$$

We obtain

$$
\text{Spectral Estimates for Compact Hyperbolic Space Forms}
$$
\n
$$
\text{pectral estimate (24) implies}
$$
\n
$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) < T-1}} \frac{d_p^*(\mu)}{(r_p(\mu))^{n-1}} = O(T), \sum_{\substack{\mu \in S_p \\ r_p(\mu) < T-1}} \frac{d_p^*(\mu)}{(r_p(\mu))^n} = O(\ln T).
$$
\nbtain

\n
$$
\sum_{\substack{\mu \in S_p \\ \mu \in S_p \\ \mu_p^0 < \mu, r_p(\mu) < T-1}} d_p^*(\mu) \left| \frac{Q(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right|
$$
\n
$$
\leq \sum_{\substack{\mu \in S_p \\ \mu_p^0 < \mu, r_p(\mu) < T-1}} d_p^*(\mu) \left| \left(\frac{1}{(r_p(\mu))^2 + (\sigma + iT)^2} - \frac{1}{(r_p(\mu))^2 + \alpha_2^2} \right) \frac{\mathcal{P}(2, \sigma + iT)}{\mathcal{P}^+(2, r_p(\mu))} \right|
$$
\n
$$
= O(T^{n-1}).
$$

Case $T - 1 \leq r_p (\mu) \leq T + 1$: Here we have to use that T is "admissible". We get

$$
\leq \sum_{\substack{\mu \in S_p \\ \mu_1 \in S_p}} d_p^*(\mu) \Big| \Big(\frac{1}{(r_p(\mu))^2 + (\sigma + iT)^2} - \frac{1}{(r_p(\mu))^2 + \alpha_2^2} \Big) \frac{\mu_2(\sigma_1 + iT)}{\mu_1(\sigma_2, r_p(\mu))} \Big|
$$

\n
$$
= O(T^{n-1}).
$$

\n
$$
\text{as } T - 1 \leq r_p(\mu) \leq T + 1: \text{ Here we have to use that } T \text{ is "admissible". We}
$$

\n
$$
\sum_{\substack{\mu \in S_p \\ \mu_1 \in S_p \\ \mu_2 \in S_p}} d_p^*(\mu) \Big| \frac{Q(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \Big| = \Big(\sum_{\substack{\mu \in S_p \\ \mu_1 \in S_p \\ \mu_2 \in S_p \\ \mu_3 \in S_p}} d_p^*(\mu) \frac{Q(1, \sigma + iT)}{|\sigma_p(\mu) - T|} \Big) O(\frac{1}{T})
$$

\n
$$
= O(T^{n-1}).
$$

\n
$$
\text{as } T + 1 < r_p(\mu) < 2T: \text{ We get the same result as for the case } r_p(\mu) < \text{sech } T \text{ as } T \leq r_p(\mu): \text{We clearly obtain}
$$

\n
$$
\frac{1}{(r_p(\mu))^2 + (\sigma + iT)^2} = \frac{1}{|r_p(\mu) + i\sigma - T| |r_p(\mu) - i\sigma + T|} \leq \frac{2}{(r_p(\mu))^2}.
$$

\n
$$
\text{timation } (25) \text{ implies}
$$

Case $T + 1 < r_p(\mu) < 2T$: We get the same result as for the case $r_p(\mu) < T - 1$ because of the fact that we have $|T - r_p(\mu)| > 1$ again.

Case $2T < r_p(\mu)$: We clearly obtain

$$
\left|\frac{1}{(r_p(\mu))^2 + (\sigma + iT)^2}\right| = \frac{1}{|r_p(\mu) + i\sigma - T| |r_p(\mu) - i\sigma + T|} \le \frac{2}{(r_p(\mu))^2}.
$$

The estimation (25) implies

$$
\overline{(r)} = \frac{1}{|r_p(\mu) + i\sigma - T| |r_p(\mu) - i\sigma|}
$$
\n
$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) \ge 2T}} d_p^*(\mu) \frac{1}{(r_p(\mu))^{n+2}} = O(T^{-2})
$$

and thereby

 \cdot

$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) \geq 2T}} d_p^*(\mu) \left| \frac{\mathcal{Q}(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right| = O(T^{n-2}).
$$

Summarizing the 4 cases considered above, we get

$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) \ge 2T}} d_p^{\circ}(\mu) \frac{1}{(r_p(\mu))^{n+2}} = O(T^{-2})
$$
\n
$$
\sum_{\substack{\mu \in S_p \\ r_p(\mu) \ge 2T}} d_p^{\circ}(\mu) \left| \frac{Q(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right| = O(T^{n-2}).
$$
\nis considered above, we get

\n
$$
2\kappa_p \sum_{\substack{\mu \in S_p \\ \mu_p^{\circ} \le \mu}} d_p^{\circ}(\mu) \left| \frac{Q(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right| = O(T^{n-1}). \tag{96}
$$

Next we consider the sum $\sum_{k\in\mathbf{L}}$ in (93). We break up the sum into two parts with

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\nNext we consider the sum
$$
\sum_{k\in L}
$$
 in (93). We break up the sum into two pa
\n $k < 2T$ and $k \ge 2T$. Using $T > 2$ we get
\n
$$
4c_W \sum_{k=N+1}^{[2T]} \left(\frac{P(-(s-N)^2)}{s-N+k} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j,\alpha_1)}{P(j,\alpha_j)} \right) + 4c_W \left(\frac{P(-(s-N)^2)}{s-N+|p-N|} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + |p-N|} \frac{P(j,\alpha_1)}{P(j,\alpha_j)} \right)
$$
\n
$$
= O(T^{n-2}) \sum_{k=N+1}^{[2T]} \left(\frac{1}{T} + \frac{1}{k+1} \right) + O(T^{n-2}) \left(\frac{1}{T} + \frac{1}{|p-N|+1} \right)
$$
\n
$$
= O(T^{n-2} \ln T) + O(T^{n-2})
$$
\n
$$
= O(T^{n-1}).
$$
\nIt is easily seen that
\n
$$
P(-(s-N)^2) - \sum_{j=2}^{m} P(-\alpha_j^2) \frac{P(j,\alpha_1)}{P(j,\alpha_j)} = 0.
$$
\nIt follows that

It is easily seen that

$$
P(-(s-N)^{2})-\sum_{j=2}^{m}P(-\alpha_{j}^{2})\frac{\mathcal{P}(j,\alpha_{1})}{\mathcal{P}(j,\alpha_{j})}=0.
$$

It follows that

$$
P(-(s - N)^2) - \sum_{j=2}^{m} P(-\alpha_j^2) \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} = 0.
$$

\n
$$
4c_W \sum_{k=[2T+1]}^{\infty} \left(\frac{P(-(s - N)^2)}{s - N + k} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{\mathcal{P}(j, \alpha_1)}{\mathcal{P}(j, \alpha_j)} \right)
$$

\n
$$
= \sum_{k=[2T+1]}^{\infty} \left(\frac{O(k^{m-2})T^{n-2}}{(s - N + k) \prod_{j=2}^{m} (\alpha_j + k)} \right)
$$

\n
$$
= O(T^{n-2}).
$$

\n(97)
\n98

This comletes the proof of the first part of Lemma 33. In order to prove (ii) we integrate $\Psi_p(s) = \frac{Z_p'(s)}{Z_p(s)}$ along the straight line from $2N + 1 + iT$ to $\sigma + N + iT$. For n odd we get a similar estimation **a**

Proof of Theorem 29: The theorem is an imediate consequence of (88), Lemma 30, Lemma 33 and the maximum modulus principle.

9. Spectral estimations based on estimations for the zeta function

In this section we prove a weaker version of Theorem B. We need this result in order to get estimations for the logarithmic derivative of the Selberg zeta function (cf. Section 10) as a tool for the proof of Theorem B (cf. Section 11). As we remarked in the Introduction (Part I), this weaker version is well known in the literature. The proof is reached easily generalizing [14]. We mainly want to point out, which tools are necessary for our approach, and we give those details which we will need later on. For more details cf. Schuster [35]. Proposition 34: *The error term R..(T) defined by*

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\n:
\nThe error term
$$
\mathcal{R}_p(T)
$$
 defined by
\n
$$
\sum_{\mu \in S_p} d_p^{\delta}(\mu) = \frac{\binom{n-1}{p} vol V}{(4\pi)^{n/2} \Gamma(\frac{n+2}{2})} T^n + \mathcal{R}_p(T)
$$
\n(12.1)
\n(21.1)
\n(31.1)
\n(42.1)
\n(53.1)
\n(7ⁿ⁻¹).
\n(7ⁿ⁻¹).
\n(98)
\n(99)
\n(99)
\n101.1
\n(99)
\n(90)
\n(91)
\n(91)
\n(92)
\n(93)
\n(95)
\n(96)
\n(99)
\n(90)
\n(91)
\n(92)
\n(95)
\n(96)
\n(97)
\n(98)
\n(99)
\n(99)
\n(90)
\n(91)
\n(91)
\n(91)
\n(92)
\n(93)
\n(95)
\n(96)
\n(97)
\n(98)
\n(99)
\n(99)
\n(90)
\n(91)
\n(91)
\n(

satisfies $|\mathcal{R}_p(T)| = O(T^{n-1}).$

We will prove Proposition 34 after Proposition 38. For $s \in D_p$ (cf. Definition 28) we define

$$
arg Z_p(s) = Im ln Z_p(s). \qquad (98)
$$

The following proposition states a connection between $arg Z_p(s)$ and the p-eigenvalue spectrum.

Proposition 35: *For* $T \geq 1$, $T \notin \{r_p(\mu) : \mu \in S_p\}$ *we obtain*

$$
\mathcal{N}_{\mathbf{p}}(T) = n_{\mathbf{p}}T^{n} + \mathcal{E}_{\mathbf{p}}(T) + \mathcal{H}_{\mathbf{p}}(T)
$$

with

$$
\arg Z_p(s) = Im \ln Z_p(s).
$$

llowing proposition states a connection between $\arg Z_p(s)$ and the p-eig
rm.
toposition 35: For $T \ge 1$, $T \notin \{r_p(\mu) : \mu \in S_p\}$ we obtain

$$
\mathcal{N}_p(T) = n_p T^n + \mathcal{E}_p(T) + \mathcal{H}_p(T)
$$

$$
\mathcal{E}_p(T) = \begin{cases} (-1)^{n/2} 2\chi(V) \int_0^T \left[t P(t^2) \tanh t - t^{n-1} \right] dt + \frac{m_p^2}{2\kappa_p} & \text{for } n \text{ even} \\ \frac{4\binom{n-1}{p} \nu o l V}{(4\pi)^{n/2} \Gamma(\frac{n}{2})} \int_0^T \left[\tilde{P}(t^2) - t^{n-1} \right] dt + \frac{m_p^2}{2} & \text{for } n \text{ odd}, \end{cases}
$$

$$
\mathcal{H}_p(T) = \frac{1}{\pi \kappa_p} \arg Z_p(N + iT)
$$

with an integer m_p^* depending only on p and the space form V .

Proof: Proposition *35* is a generalization of *[14,* Theorem *7.11.* For the straightforward generalization of the proof one has to use the Cauchy residue theorem, Proposition *23,* Lemma *26,* Proposition *27* and (88). **Proof:** Proposition 35
ward generalization of the p
23, Lemma 26, Proposition
An easy computation s
Lemma 36: We obse
Proposition 37: For
 $|Z_p$
with
 a_p

An easy computation shows

Lemma 36: We observe that $\mathcal{E}_p(T) = O(T^{n-2})$.

An easy computation shows

Lemma 36: *We observe that* $\mathcal{E}_p(T) = O(T^{n-2})$.

Proposition 37: *For* $\sigma^* \geq -\frac{1}{2}$ and $T \geq 1$ we get

$$
|Z_p(\sigma^* + iT)| \le \exp\left(a_p T^{n-1} + O(T^{n-2})\right)
$$

$$
Z_p(\sigma^* + iT)| \le \exp\left(a_p T^{n-1} + O(T^{n-2})\right)
$$

$$
a_p = \begin{cases} (-1)^{n/2} \pi n \chi(V) & \text{for } n \text{ even} \\ -\frac{n^{(n-1)} \text{ vol } V}{2(4\pi)^{(n-2)/2} \Gamma(n/2)} & \text{for } n \text{ odd.} \end{cases}
$$

Proof: The related proof of $[14]$ for $n = 2$, $p = 0$ as well as the straightforward generalization to our situation is based on the Phragmén-Lindelöf principle (cf. Boas [3, Theorem 1.4.2]) and the functional equation of Proposition *27.*

Using an estimation of Titchmarsh [26, Lemma 9.41 and Jensens' Theorem (Boas [3, Theorem 1.2.1]), Proposition 37 implies the following proposition (cf. Hejhal [14] for $n = 2, p = 0$.

Proposition 38: *For* $T \ge 1$, $T \notin \{r_p(\mu), \mu \in S_p\}$, we have $|\arg Z_p(N + iT)| = O(T^{n-1})$, $|\mathcal{H}_p(T)| = O(T^{n-1})$.

Proof of **Proposition 34:** By virtue of Proposition 35, Lemma 36, Proposition 38 and a usual continuity argument we get the assertion

10. Estimations for the logarithmic derivative of the Selberg zeta **Proof of Proposition 34:** By virtue of Proposition
and a usual continuity argument we get the assertic
10. Estimations for the logarithmic defunction using spectral estimations

As we have already remarked, we want to use Proposition 34 in order to get estimations of $\Psi_p(s)$.

Proposition 39: *Supposing* $s = N + \sigma + iT$ *,* $-\frac{1}{2} - N \le \sigma \le N + 1$, $T \ge 100$, $T \notin \{r_p(\mu) : \mu \in S_p\}$ *we get*

Proposition 34: By virtue of Proposition 35, Lemma 36, Proposition 38
\ntinuity argument we get the assertion
\nions for the logarithmic derivative of the Selberg zeta
\nusing spectral estimations
\n
$$
y_{\text{r}} = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \, dx
$$
\n
$$
y_{\text{r}} = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, dx
$$
\n
$$
y_{\text{r}} = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, dx
$$
\n
$$
y_{\text{r}}(s) = O(|T|^{n-2}) + 2\kappa_{\text{r}} \sum_{\mu \in S_{\text{r}} \atop T^{-1} \leq \tau_{\text{r}}(\mu) \leq T+1} d_{\text{r}}^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_{\text{r}}(\mu))}.
$$
\n(99)

Proof: We suppose $\alpha_m > \alpha_{m-1} > ... > \alpha_2 > n$ with $m = \frac{n}{2} + 1$ for n even and $m = \frac{n+1}{2}$ for *n* odd (as above, cf. Assumption 16). We start with the case of *n* even. As a consequence of (72), (79) and (86) we get in analogy to (93) (cf. the proof of Lemma 33) the equation

$$
\frac{\Psi_p(s)}{s - N} = O(|T|^{n-2}) + 2\kappa_p \sum_{\substack{\mu \in S_p \\ T-1 \le r_p(\mu) \le T+1}} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))}. \tag{99}
$$
\nof: We suppose $\alpha_m > \alpha_{m-1} > ... > \alpha_2 > n$ with $m = \frac{n}{2} + 1$ for n even and
\nfor n odd (as above, cf. Assumption 16). We start with the case of n even. As
\nuence of (72), (79) and (86) we get in analogy to (93) (cf. the proof of Lemma
\nquation\n
$$
\frac{\Psi_p(s)}{s - N} = O(|T|^{n-2}) + 2\kappa_p \sum_{\substack{\mu \in S_p \\ r_p(\mu) \ge 0}} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))} \tag{100}
$$
\n
$$
+ 4c_W \kappa_p \sum_{k \in \mathbf{L}} \left(\frac{P(-(\sigma + iT)^2)}{\sigma + iT + k} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{\mathcal{P}(j, \sigma + iT)}{\mathcal{P}(j, \alpha_j)} \right).
$$

Thereby the assumption $r_p(\mu) > 0$ shall include the assumption $r_p(\mu) \in \mathbf{R}$. We break up the sum $\sum_{\substack{p \in S_p \\ p \nmid k \leq \mu}}$ in the same way as we have done it with the sum $\sum_{\substack{p \in S_p \\ p \mid k \leq \mu}}$ in the proof of Lemma 33.

Case $r_p(\mu) < T - 1$: We choose a number δ with $2N < \delta < 2N + 1$ and $\delta \notin \{r_p(\mu)$: $\mu \in S_p$. Then we get

$$
S_1 = 2\kappa_p \sum_{\substack{\mu \in S_p \\ r_p(\mu) < T-1}} d_p^*(\mu) \frac{\mathcal{Q}(1, s - N)}{\mathcal{R}(r_p(\mu))}
$$

$$
\begin{aligned}\n &\text{Spectral Estimates for Compact Hyperbolic Space} \\
 &= O(T^{n-2}) + 2\kappa_p \int_{\delta}^{T-1} \frac{Q(1, s - N)}{\mathcal{R}(r)} \, d\mathcal{N}_p(r) \\
 &= O(T^{n-2}) + 2\kappa_p \int_{\delta}^{T-1} \left(\frac{1}{2(\sigma + iT)} \left[\frac{1}{\sigma + i(T + r)} \right. \right. \\
 &\left. + \frac{1}{\sigma + i(T - r)} \right] - \frac{1}{r^2 + \alpha_2^2} \int \frac{P(2, s - N)}{\mathcal{P}^+(2, r)} \, d\mathcal{N}_p(r).\n \end{aligned}
$$
\n
$$
I_1 = \frac{\kappa_p}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T + r)} \frac{P(2, s - N)}{\mathcal{P}^+(2, r)} \, d\mathcal{N}_p(r).
$$

We define

$$
+\frac{1}{\sigma+i(T-r)}\bigg]-\frac{1}{r^2+\alpha_2^2}\bigg)\frac{\mathcal{P}(2,s-N)}{\mathcal{P}^+(2,r)}d\mathcal{N}
$$

$$
I_1=\frac{\kappa_p}{\sigma+iT}\int\limits_{\delta}^{T-1}\frac{1}{\sigma+i(T+r)}\frac{\mathcal{P}(2,s-N)}{\mathcal{P}^+(2,r)}d\mathcal{N}_p(r).
$$

It follows

$$
|I_{1}| \leq \frac{\kappa_{p}}{T} |\mathcal{P}(2, s - N)| \int_{\delta}^{T-1} \frac{1}{(T+r)r^{n-2}} d\mathcal{N}_{p}(r) \leq O(T^{n-4}) \int_{\delta}^{T-1} \frac{1}{r^{n-2}} d\mathcal{N}_{p}(r).
$$

\n
$$
[e\mathbf{y}] \text{ asymptotic estimate (1) states } \mathcal{N}_{p}(r) = O(r^{n}) \text{ and thereby we get by}
$$

\n
$$
I_{1}| = O(T^{n-2}). \text{ Next we consider}
$$

\n
$$
I_{2} = \frac{\kappa_{p}}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^{+}(2, r)} d\mathcal{N}_{p}(r).
$$

\ne use Proposition 34. By help of $\mathcal{N}_{p}(T) = n_{p}T^{n} + \mathcal{R}_{p}(T)$ we break up

The Weyl asymptotic estimate (1) states $\mathcal{N}_p(r) = O(r^n)$ and thereby we get by partial integration $|I_1| = O(T^{n-2})$. Next we consider

$$
I_2 = \frac{\kappa_p}{\sigma + iT} \int\limits_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} dN_p(r).
$$

Now we use Proposition 34. By help of $\mathcal{N}_p(T) = n_p T^n + \mathcal{R}_p(T)$ we break up I_2 into $I_2 = I'_2 + I''_2$ with

$$
I_2 = \frac{\kappa_p}{\sigma + iT} \int\limits_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} dN_p(r).
$$

sition 34. By help of $\mathcal{N}_p(T) = n_p T^n + \mathcal{R}_p(T)$ v

$$
I_2' = \frac{\kappa_p}{\sigma + iT} \int\limits_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} d(n_p r^n).
$$

We obtain

$$
I_2 = \frac{V_1}{\sigma + iT} \int_{\delta} \frac{V_2}{\sigma + i(T - r)} \frac{V_1}{\rho + i(2, r)} dV_p(r).
$$

\nProposition 34. By help of $N_p(T) = n_p T^n + R_p(T)$ we be
\nwith
\n
$$
I_2' = \frac{\kappa_p}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T - r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} d(n_p r^n).
$$
\n
$$
I_2' = \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{\delta}^{T-1} \frac{n n_p r^{n-1}}{(\sigma + i(T - r)) \mathcal{P}^+(2, r)} dr
$$
\n
$$
= \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{\delta}^{T-1} \frac{n n_p r \mathcal{P}^+(2, r) + P_{n-3}(r)}{(\sigma + i(T - r)) \mathcal{P}^+(2, r)} dr
$$
\n
$$
= \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) n n_p \left(\int_{\delta}^{T-1} \frac{r}{\sigma + i(T - r)} dr + O(1) \right)
$$
\n
$$
mial P_{n-3}(r) of degree n - 3. Further on we use
$$
\n
$$
\frac{r}{r + i(T - r)} dr = -i \int_{\delta}^{T-1} \frac{i(r - T) - \sigma}{\sigma + i(T - r)} dr + i \int_{\delta}^{T-1} \frac{-iT - \sigma}{\sigma + i(T - r)} dr
$$
\n
$$
= i(T - 1 - \delta) + (iT + \sigma) \int_{\delta}^{T-1} \frac{1}{\sigma i + (r - T)}
$$

with a polynomial $P_{n-3}(r)$ of degree $n-3$. Further on we use

$$
\int_{\delta}^{T-1} \frac{r}{\sigma + i(T-r)} dr = -i \int_{\delta}^{T-1} \frac{i(r-T) - \sigma}{\sigma + i(T-r)} dr + i \int_{\delta}^{T-1} \frac{-iT - \sigma}{\sigma + i(T-r)} dr
$$

$$
= i(T-1-\delta) + (iT+\sigma) \int_{\delta}^{T-1} \frac{1}{\sigma i + (r-T)} dr
$$

$$
= i(T-1-\delta) + (iT+\sigma) \ln(\sigma i + r - T) \Big|_{\delta}^{T-1}.
$$

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It follows

$$
\frac{1}{\sigma + iT} \int_{\delta}^{T-1} \frac{r}{\sigma + i(T-r)} dr = -\ln T + O(1)
$$

$$
I_2' = -\kappa_p n n_p \mathcal{P}(2, s - N) \ln T + O(T^{n-2}).
$$

$$
= \frac{\kappa_p}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} d\mathcal{R}_p dr
$$

and thereby

$$
I_2'=-\kappa_pnn_p\mathcal{P}(2,s-N)\ln T+O(T^{n-2}).
$$

Further on we get for

$$
\sigma + iT \frac{J}{\delta} \quad \sigma + i(T - r)
$$

\n
$$
I'_2 = -\kappa_p n n_p \mathcal{P}(2, s - N) \ln T + O(T^{n-2}).
$$

\nor
\n
$$
I''_2 = \frac{\kappa_p}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T - r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} d\mathcal{R}_p(r)
$$

the estimation

Further on we get for
\n
$$
I_2'' = \frac{\kappa_p}{\sigma + iT} \int_{\delta}^{T-1} \frac{1}{\sigma + i(T-r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} d \mathcal{R}_p(r)
$$
\nthe estimation
\n
$$
|I_2''| \leq \frac{\kappa_p}{T} |\mathcal{P}(2, s - N)| \int_{\delta}^{T-1} \frac{1}{(T-r)r^{n-2}} d |\mathcal{R}_p(r)|
$$
\n
$$
\leq \frac{\kappa_p}{T} |\mathcal{P}(2, s - N)| \left(\frac{|\mathcal{R}_p(r)|}{(T-r)r^{n-2}} \Big|_{\delta}^{T-1} + \int_{\delta}^{T-1} |\mathcal{R}_p(r)| \frac{|(n-1)r^{n-2} - T(n-2)r^{n-3}|}{(T-r)^2r^{2n-4}} dr \right)
$$
\n
$$
\leq O(T^{n-3}) \left(O(T) + \int_{\delta}^{T-1} |O(r^{n-1})| \frac{|(n-1)r - T(n-2)|}{(T-r)^2r^{n-1}} dr \right)
$$
\n
$$
\leq O(T^{n-3}) \left(O(T) + \int_{\delta}^{T-1} |O(1)| \frac{|(n-1)r - T(n-2)|}{(T-r)^2} dr \right)
$$
\nand thereby $|I''| = O(T^{n-2})$. We conclude

and therby $|I''| = O(T^{n-2})$. We conclude

$$
I_2=-\kappa_pnn_p\mathcal{P}(2,s-N)\ln T+O(T^{n-2}).
$$

For the estimation of S_1 it remains to consider

$$
I_3 = -2\kappa_p \int\limits_{\delta}^{T-1} \frac{\mathcal{P}(2, s - N)}{\mathcal{Q}^+(1, r)} d\mathcal{N}_p(r).
$$

We find that

$$
I_2 = -\kappa_p n n_p \mathcal{P}(2, s - N) \ln T + O(T^{n-2}).
$$

the estimation of S_1 it remains to consider

$$
I_3 = -2\kappa_p \int_s^T \frac{\mathcal{P}(2, s - N)}{\mathcal{Q}^+(1, r)} d\mathcal{N}_p(r).
$$

find that

$$
I_3 = -2\kappa_p n n_p \mathcal{P}(2, s - N) \int_s^T \frac{r^{n-1}}{\mathcal{Q}^+(1, r)} dr - 2\kappa_p \mathcal{P}(2, s - N) \int_s^T \frac{1}{\mathcal{Q}^+(1, r)} d\mathcal{R}_p(r).
$$

nalogy to the estimation of I'_2 and I''_2 we obtain

$$
I_3 = -2\kappa_p n n_p \mathcal{P}(2, s - N) \ln T T^{n-2} + O(T^{n-2}).
$$

easy calculation shows $\mathcal{P}(2, s - N) = T^{n-2} + O(T^{n-3})$, and we immediately obtain

$$
S_1 = -3\kappa_p n n_p \ln T T^{n-2} + O(T^{n-2}).
$$
 (

In analogy to the estimation of I'_2 and I''_2 we obtain

$$
I_3 = -2\kappa_p n n_p \mathcal{P}(2, s - N) \ln T T^{n-2} + O(T^{n-2})
$$

An easy calculation shows $\mathcal{P}(2,s-N) = T^{n-2} + O(T^{n-3}),$ and we immediately obtain

$$
S_1 = -3\kappa_p n n_p \ln T T^{n-2} + O(T^{n-2}).
$$
\n(101)

 \sim

In the case $T-1 \leq r_{p}(\mu) \leq T+1$ we have to do no changes for the assertion of the proof.

Next we consider the case $T + 1 < r_p(\mu) < 2T$. We then have to estimate

$$
r_p(\mu) \le T + 1
$$
 we have to do no change
le case $T + 1 < r_p(\mu) < 2T$. We then h

$$
S_3 = 2\kappa_p \sum_{\substack{\mu \in S_p \\ r + 1 < r_p(\mu) < 2T}} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))}
$$

If $r_p(\mu) = T + 1$ or $r_p(\mu) = 2T$, then Proposition 34 implies $d_p^*(\mu) = O(T^{n-1})$. We conclude

$$
S_3 = 2\kappa_p \sum_{\substack{\mu \in S_p \\ T+1 \le r_p(\mu) < 2T}} d_p^*(\mu) \frac{\mathbb{E}(1, 0, 1)}{\mathcal{R}(r_p(\mu))}.
$$
\n
$$
p(\mu) = 2T, \text{ then Proposition 34 implies } c
$$
\n
$$
S_3 = O(T^{n-2}) + 2\kappa_p \int_{T+1}^{2T} \frac{\mathcal{Q}(1, s - N)}{\mathcal{R}(r_p(\mu))} \mathcal{N}_p(r).
$$

We use the decomposition

of.
\nNext we consider the case
$$
T + 1 < r_p(\mu) < 2T
$$
. We then have to estimate
\n
$$
S_3 = 2\kappa_p \sum_{\substack{\mu \in S_p \\ T+1 \le r_p(\mu) < 2T}} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))}.
$$
\n
$$
p(\mu) = T + 1 \text{ or } r_p(\mu) = 2T, \text{ then Proposition 34 implies } d_p^*(\mu) = O(T^{n-1}).
$$
\n
$$
S_3 = O(T^{n-2}) + 2\kappa_p \int_{T+1}^{2T} \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))} \mathcal{N}_p(r).
$$
\nuse the decomposition
\n
$$
\frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))} = \left(\frac{1}{2(\sigma + iT)}\left(\frac{1}{\sigma + i(T + r)} + \frac{1}{\sigma + i(T - r)}\right) - \frac{1}{r^2 + \alpha_2^2}\right) \frac{P(2, s - N)}{P^+(2, r)}
$$
\nabove. First we consider
\n
$$
J_1 = \frac{\kappa_p}{\sigma + iT} \int_{T}^{2T} \frac{1}{\sigma + i(T + r)} \frac{P(2, s - N)}{P^+(2, r)} d\mathcal{N}_p(r).
$$

as above. First we consider

 $\sim 10^{-1}$

 \sim \sim

$$
T+1 \longrightarrow V+V+V
$$
\n
$$
\text{position}
$$
\n
$$
\left(\frac{1}{2(\sigma+iT)}\left(\frac{1}{\sigma+i(T+r)}+\frac{1}{\sigma+i(T-r)}\right)-\frac{1}{r^2+\sigma}\right)
$$
\n
$$
\text{consider}
$$
\n
$$
J_1 = \frac{\kappa_p}{\sigma+iT} \int_{T+1}^{2T} \frac{1}{\sigma+i(T+r)} \frac{\mathcal{P}(2,s-N)}{\mathcal{P}^+(2,r)} \, d\mathcal{N}_p(r).
$$
\n
$$
|J_1| \leq \frac{\kappa_p}{T^2} |\mathcal{P}(2,s-N)| \int_{T+1}^{2T} \frac{1}{r^{n-2}} \, d\mathcal{N}_p(r)
$$
\n
$$
(T^{n-2}). \text{ In the same way we get for}
$$
\n
$$
J_3 = 2\kappa_p \int_{T+1}^{2T} \frac{\mathcal{P}(2,s-N)}{\mathcal{Q}^+(1,r)} \, d\mathcal{N}_p(r)
$$
\n
$$
= O(T^{n-2}). \text{ It remains to consider}
$$

We get

$$
\frac{\kappa_p}{\sigma + iT} \int\limits_{T+1}^{\infty} \frac{1}{\sigma + i(T+r)} \frac{\mathcal{P}(2, s - N)}{\mathcal{P}^+(2, r)} dV
$$

$$
|J_1| \le \frac{\kappa_p}{T^2} |\mathcal{P}(2, s - N)| \int\limits_{T+1}^{2T} \frac{1}{r^{n-2}} d\mathcal{N}_p(r)
$$

and therby $J_1 = O(T^{n-2})$. In the same way we get for

$$
T^{2^{1}} \sum_{T+1}^{r-2} r^{n-2}
$$

\nthe same way we get for
\n
$$
J_3 = 2\kappa_p \int_{T+1}^{2T} \frac{\mathcal{P}(2, s - N)}{\mathcal{Q}^+(1, r)} d\mathcal{N}_p(r)
$$

the estimation $J_3 = O(T^{n-2})$. It remains to consider

$$
|J_1| \leq \frac{\kappa_p}{T^2} |\mathcal{P}(2, s - N)| \int_{T+1}^{2T} \frac{1}{r^{n-2}} dN_p(r)
$$

\n
$$
O(T^{n-2}). \text{ In the same way we get for}
$$

\n
$$
J_3 = 2\kappa_p \int_{T+1}^{2T} \frac{\mathcal{P}(2, s - N)}{\mathcal{Q}^+(1, r)} dN_p(r)
$$

\n
$$
J_3 = O(T^{n-2}). \text{ It remains to consider}
$$

\n
$$
J_2 = \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{T+1}^{2T} \frac{1}{\sigma + i(T - r)} \frac{1}{\mathcal{P}^+(2, r)} \mathcal{N}_p(r).
$$

\n
$$
\text{de decomposition } \mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p(r), \text{ we study}
$$

According to the decomposition $\mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p(r)$, we study

$$
= O(T^{n-2}). \text{ It remains to consider}
$$
\n
$$
a = \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{T+1}^{2T} \frac{1}{\sigma + i(T-r)} \frac{1}{\mathcal{P}^+(2, r)} \mathcal{N}_p(r)
$$
\ndecomposition\n
$$
\mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p(r), \text{ we study}
$$
\n
$$
J'_2 = \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{T+1}^{2T} \frac{1}{\sigma + i(T-r)} \frac{n n_p r^{n-1}}{\mathcal{P}^+(2, r)} dr.
$$

We have

$$
\int\limits_{T+1}^{2T}\frac{1}{\sigma+i(T-r)}\frac{r^{n-1}}{\mathcal{P}^+(2,r)}\,dr
$$

 $\ddot{}$

 $\ddot{}$

$$
\begin{split}\n\text{B}} \text{where} \\
&= \int_{T+1}^{2T} \frac{1}{\sigma + i(T-r)} \left(r + O(\frac{1}{r}) \right) dr \\
&= \int_{T+1}^{2T} \left(\frac{(r-T) + i\sigma}{\sigma + i(T-r)} + \frac{T-i\sigma}{\sigma + i(T-r)} + \frac{O(1/r)}{\sigma + i(T-r)} \right) dr \\
&= O(T) + (\sigma + iT) \int_{T+1}^{2T} \frac{1}{\sigma i - T + r} dr \\
&= O(T) + (\sigma + iT) \ln T. \\
\kappa_p n n_p T^{n-2} \ln T + O(T^{n-2}). \text{ In analogy to the calculations above} \\
&\frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{T+1}^{2T} \frac{1}{\sigma + i(T-r)} \frac{1}{\mathcal{P}^+(2,r)} dR_p(r) = O(T^{n-2}) \\
S_3 &= \kappa_p n n_p T^{n-2} \ln T + O(T^{n-2}). \\
\text{that for the case } 2T < r_p(\mu) \text{ we have shown in Section 8 the est} \\
S_4 &= 2r \sum_{r \text{min}} \mathcal{P}(\mu) \left(\mathcal{Q}(1, \sigma + iT) \right) \qquad O(T^{n-2}).\n\end{split}
$$

It follows $J'_2 = \kappa_p n n_p T^{n-2} \ln T + O(T^{n-2})$. In analogy to the calculations above, we get

$$
J_2'' = \frac{\kappa_p}{\sigma + iT} \mathcal{P}(2, s - N) \int_{T+1}^{2T} \frac{1}{\sigma + i(T-r)} \frac{1}{\mathcal{P}^+(2, r)} d\mathcal{R}_p(r) = O(T^{n-2}).
$$

It follows

(102)

We recall that for the case $2T < r_p(\mu)$ we have shown in Section 8 the estimation

$$
\frac{1}{T} \mathcal{F}(2, s - N) \int_{T+1}^{\infty} \frac{1}{\sigma + i(T-r)} \overline{\mathcal{P}^+(2, r)} d\mathcal{K}_p(r) =
$$
\n
$$
S_3 = \kappa_p n n_p T^{n-2} \ln T + O(T^{n-2}).
$$
\nfor the case $2T < r_p(\mu)$ we have shown in Section

\n
$$
S_4 = 2\kappa_p \sum_{\substack{\mu \in S_p \\ aT < r_p(\mu)}} d_p^*(\mu) \left| \frac{\mathcal{Q}(1, \sigma + iT)}{\mathcal{R}(r_p(\mu))} \right| = O(T^{n-2}).
$$
\nthen, the definition of the estimate of the sum $\sum_{\mu \in S_p} \mathcal{F}^+(1, \mu) = O(T^{n-2}).$

We now turn our attention to the estimation of the sum $\sum_{k\in L}$ in the right-hand side of (100) using an argument of the proof of Lemma 33. We define

$$
r \in S_p
$$

\n1 our attention to the estimation of the sum $\sum_{k \in L}$ in the right-
\nin argument of the proof of Lemma 33. We define
\n
$$
S_5 = 4c_W \kappa_p \sum_{k \in L_T} \left(\frac{P(-(\sigma + iT)^2)}{\sigma + k + iT} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \sigma + iT)}{P(j, \alpha_j)} \right)
$$

with $L_T = \{k \in L : k \leq 2T\}$. We apply

$$
S_5 = 4c_W \kappa_p \sum_{k \in L_T} \left(\frac{P(-(\sigma + iT)^2)}{\sigma + k + iT} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \sigma + iT)}{P(j, \alpha_j)} \right)
$$

= {k \in L : k \le 2T}. We apply

$$
\ln(\alpha + B + 1) - \ln(\alpha + A) < \sum_{k=4}^B \frac{1}{\alpha + k} < \ln(\alpha + B) - \ln(\alpha + A + 1)
$$

with $A, B \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha > 0$ and get $\sum_{k \in \mathbf{L}_T} \frac{1}{\alpha + k} = \ln T + O(1)$. We immediately see that

$$
s = 4c_W \kappa_p \sum_{k \in L_T} \left(\frac{P(-(\sigma + iT)^2)}{\sigma + k + iT} - \sum_{j=2}^m \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \sigma + iT)}{P(j, \alpha_j)} \right)
$$

\n
$$
\in \mathbf{L}: k \le 2T\}.
$$
 We apply
\n
$$
+ B + 1) - \ln(\alpha + A) < \sum_{k=A}^B \frac{1}{\alpha + k} < \ln(\alpha + B) - \ln(\alpha + A)
$$

\nI, $\alpha \in \mathbf{R}, \alpha > 0$ and get $\sum_{k \in L_T} \frac{1}{\alpha + k} = \ln T + O(1)$. We in
\n
$$
S_5 = 4c_W \kappa_p \left(-\sum_{j=2}^m P(-\alpha_j^2) \frac{P(j, \sigma + iT)}{P(j, \alpha_j)} \ln T \right) + O(T^{n-2})
$$

\n
$$
= -4c_W \kappa_p P(- (s - N)^2) \ln T + O(T^{n-2})
$$

\n
$$
= -4c_W \kappa_p T^{n-2} \ln T + O(T^{n-2}).
$$

Equation (97) states

$$
= -4c_W\kappa_p T^{-2} \ln T + O(T^{n-2}).
$$

\n
$$
= -4c_W\kappa_p T^{n-2} \ln T + O(T^{n-2}).
$$

\n
$$
S_5 = 4c_W\kappa_p \sum_{k=[2T]+1}^{\infty} \left(\frac{P(-(\sigma + iT)^2)}{\sigma + k + iT} - \sum_{j=2}^{m} \frac{P(-\alpha_j^2)}{\alpha_j + k} \frac{P(j, \sigma + iT)}{P(j, \alpha_j)} \right) = O(T^{n-2}).
$$
 (103)

Summarizing (100) - (103) it follows

$$
\frac{\Psi_p(s)}{s-N} = O(T^{n-2}) - 2\kappa_p n n_p \ln T \ T^{n-2} - 4c_W \kappa_p \ln T \ T^{n-2}.
$$

Using

Spectral Estimates for Compact Hyperbolic Space For
\ng (100) - (103) it follows
\n
$$
\frac{\Psi_p(s)}{s - N} = O(T^{n-2}) - 2\kappa_p n n_p \ln T T^{n-2} - 4c_W \kappa_p \ln T T^{n-2}.
$$
\n
$$
n_p = (-1)^{n/2} \frac{\binom{n-1}{p} \chi(V)}{\Gamma(n+1)}
$$
 and $c_W = (-1)^{\frac{n+2}{2}} \frac{\chi(V)}{2\Gamma(p+1)\Gamma(n-p)}$
\nssection of the proposition for *n* even.
\ne consider the case of *n* odd. Instead of (100) we get
\n
$$
\Psi(s) = O(1.8 - N).
$$

we get the assertion of the proposition for n even.

Next we consider the case of *n* odd. Instead of (100) we get

Spectral Estimates for Compact Hyperbolic Space Forms 285
\n(103) it follows
\n
$$
\frac{1}{\sqrt{N}} = O(T^{n-2}) - 2\kappa_p n n_p \ln T T^{n-2} - 4c_W \kappa_p \ln T T^{n-2}.
$$
\n
$$
1)^{n/2} \frac{{n-1 \choose p} \chi(V)}{\Gamma(n+1)}
$$
 and $c_W = (-1)^{\frac{n+2}{2}} \frac{\chi(V)}{2\Gamma(p+1)\Gamma(n-p)}$
\nof the proposition for *n* even.
\nFor the case of *n* odd. Instead of (100) we get
\n
$$
\frac{\Psi_p(s)}{s - N} = O(|T|^{n-2}) + 2 \sum_{\substack{\mu \in S_p \\ \eta_p(\mu) \ge 0}} d_p^*(\mu) \frac{Q(1, s - N)}{\mathcal{R}(r_p(\mu))}.
$$
 (104)
\ncomposition of the sum on the right-hand side as in the case of *n* even.
\nase we get

We use the same decomposition of the sum on the right-hand side as in the case of n even. In contrast to this case we get

$$
(-1)^{n/2} \frac{{n-1 \choose p} \chi(V)}{\Gamma(n+1)}
$$
 and $c_W = (-1)^{\frac{n+2}{2}} \frac{\chi(V)}{2\Gamma(p+1)\Gamma(}$
on of the proposition for *n* even.
sideer the case of *n* odd. Instead of (100) we get

$$
\frac{\Psi_p(s)}{s-N} = O(|T|^{n-2}) + 2 \sum_{\substack{\mu \in S_p \\ r_p(\mu) \ge 0}} d_p^*(\mu) \frac{Q(1, s - N)}{R(r_p(\mu))}.
$$

decomposition of the sum on the right-hand side as in
s case we get

$$
I'_2 = \frac{\mathcal{P}(2, s - N)}{\sigma + iT} \int_s^{T-1} \frac{n n_p r^{n-1}}{(\sigma + i(T - r))\mathcal{P}^+(2, r)} dr
$$

$$
= \frac{\mathcal{P}(2, s - N)}{\sigma + iT} \int_s^{T-1} \frac{n n_p r^2 \mathcal{P}^+(2, r) + P_{n-3}(r)}{(\sigma + i(T - r))\mathcal{P}^+(2, r)} dr
$$

and $P_{n-3}(r)$ of degree $n - 3$ and consequently

$$
\frac{n n_p \mathcal{P}(2, s - N)}{r + iT} \left(\int_s^{T-1} \frac{r^2}{\sigma + i(T - r)} + \int_s^{T-1} \frac{O(1)}{\sigma + i(T - r)} dr \right)
$$

$$
\frac{r^2}{(T - r)} dr = i \int_s^{T-1} r dr - i(\sigma + iT) \int_s^{T-1} \frac{r}{\sigma + i(T - r)} dr
$$

$$
\frac{r^{n-2}}{(T - r)}.
$$

with a polynomial $P_{n-3}(r)$ of degree $n-3$ and consequently

$$
= \frac{}{\sigma + iT} \int_{\delta} \frac{}{\sigma + i(T-r))} \frac{p+2(r)}{(r+i(T-r))} dr
$$
\n
$$
I_2' = \frac{n n_p \mathcal{P}(2, s-N)}{\sigma + iT} \left(\int_{\delta}^{T-1} \frac{r^2}{\sigma + i(T-r)} + \int_{\delta}^{T-1} \frac{O(1)}{\sigma + i(T-r)} dr \right).
$$
\n
$$
I_3' = \frac{n n_p \mathcal{P}(2, s-N)}{\sigma + i(T-r)} \left(\int_{\delta}^{T-1} \frac{r^2}{\sigma + i(T-r)} dr - i(\sigma + iT) \int_{\delta}^{T-1} \frac{r}{\sigma + i(T-r)} dr \right) = O(n)
$$

We apply

Spectral Estimates for Compact Hyperbolic Space Forms 285
\nSummarizing (100) - (103) it follows
\n
$$
\frac{\Psi_p(s)}{s - N} = O(T^{n-2}) - 2\kappa_p n n_p \ln T T^{n-2} - 4c_W \kappa_p \ln T T^{n-2}.
$$
\nUsing
\n
$$
n_p = (-1)^{n/2} \frac{\binom{n-1}{p} \chi(V)}{\Gamma(n+1)} \text{ and } c_W = (-1)^{\frac{n+2}{2}} \frac{\chi(V)}{2\Gamma(p+1)\Gamma(n-p)}
$$
\nwe get the assertion of the proposition for n even.
\nNext we consider the case of n odd. Instead of (100) we get
\n
$$
\frac{\Psi_p(s)}{s - N} = O(|T|^{n-2}) + 2 \sum_{\substack{r \neq 0, 2 \text{ odd} \\ r \neq 0}} d_p^r(\mu) \frac{Q(1, s - N)}{R(r_p(\mu))}.
$$
\n(104)
\nWe use the same decomposition of the sum on the right-hand side as in the case of n even.
\nIn contrast to this case we get
\n
$$
I'_1 = \frac{P(2, s - N)}{\sigma + iT} \int_s^{T-1} \frac{n n_p r^{n-1}}{(\sigma + i(T - r))P^+(2, r)} dr
$$
\n
$$
= \frac{P(2, s - N)}{\sigma + iT} \int_s^{T-1} \frac{n n_p r^{2} P^+(2, r) + P_{n-3}(r)}{(\sigma + i(T - r))P^+(2, r)} dr
$$
\nwith a polynomial $P_{n-3}(r)$ of degree $n - 3$ and consequently
\n
$$
I'_2 = \frac{n n_p P(2, s - N)}{\sigma + iT} \left(\int_s^{T-1} \frac{r^2}{\sigma + i(T - r)} + \int_s^{T-1} \frac{O(1)}{\sigma + i(T - r)} dr \right).
$$
\nWe apply
\n
$$
\int_{s}^{T-1} \frac{r^2}{\sigma + i(T - r)} dr = i \int_s^T r dr - i(\sigma + iT) \int_s^T \frac{1}{\sigma + i(T - r)} dr = O(T^2)
$$
\nand get $I'_2 = O(T^{n-2})$. In the same way we get $I_3 = O(T^{n-2})$. It follows $I_1 = O(T^{n-2})$.
\nA similar calculation gives $S_3 = O(T^{n$

A similar calculation gives $S_3 = O(T^{n-2})$. This completes the proof **i**

Proposition 40: $For -\frac{1}{2} \leq Res \leq n$, $Im s \geq 100$, $s = N + \sigma + iT$ and $T \notin$ ${r_p(\mu): \mu \in S_p}$ we obtain

$$
\Psi_p(s) = O(T^{n-1}) + \kappa_p \sum_{\substack{\mu \in S_p \\ |r_p(\mu) - T| \le 1}} \frac{d_p^{\star}(\mu)}{s - N - ir_p(\mu)}.
$$
 (105)

Proof: Proposition 39 implies

general in

$$
\Psi_p(s) = O(T^{n-1}) + \kappa_p \sum_{\substack{\mu \in S_p \\ |\tau_p(\mu) - T| \le 1}} \frac{d_p^*(\mu)}{s - N - ir_p(\mu)}.
$$
\nproposition 39 implies

\n
$$
\Psi_p(s) = O(T^{n-1}) + \kappa_p \sum_{\substack{\mu \in S_p \\ |\tau_p(\mu) - T| \le 1}} \frac{d_p^*(\mu)}{s - N - ir_p(\mu)}.
$$
\n
$$
\Psi_p(s) = O(T^{n-1}) + \kappa_p \sum_{\substack{\mu \in S_p \\ |\tau_p(\mu) - T| \le 1}} \frac{d_p^*(\mu)}{\mu_p^*(\mu) - \mu_p^*(\mu)} \left(\frac{1}{s - N - ir_p(\mu)} \right) + \frac{1}{s - N + ir_p(\mu)} - \frac{2(s - N)}{r_p^2(\mu) + \alpha_2^2} \frac{P(2, s - N)}{P^+(2, r_p(\mu))}.
$$

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Using $-\frac{1}{2} \leq Re s \leq n$ and $|r_p(\mu) - T| \leq 1$ we conclude

$$
\frac{\mathcal{P}(2,s-N}{\mathcal{P}^+(2,r)}=(-1)^{m-2}+O\Big(\frac{1}{T^2}\Big).
$$

As a consequence of Proposition 34 we get $\mathcal{N}_p(T+2) - \mathcal{N}_p(T-2) = O(T^{n-1})$ and thereby

huster

\n
$$
Re\ s \leq n \text{ and } |r_{p}(\mu) - T| \leq 1 \text{ we conclude}
$$
\n
$$
\frac{\mathcal{P}(2, s - N)}{\mathcal{P}^{+}(2, r)} = (-1)^{m-2} + O\left(\frac{1}{T^{2}}\right).
$$
\nence of Proposition 34 we get

\n
$$
\mathcal{N}_{p}(T+2) - \mathcal{N}_{p}(T-2) = O(T^{n-1})
$$
\n
$$
\sum_{\substack{\mu \in S_{p} \\ \mu_{p}(\mu) - T| \leq 1}} \frac{d_{p}^{*}(\mu)}{|s - N + ir_{p}(\mu)|} \leq \sum_{\substack{\mu \in S_{p} \\ \mu_{p}(\mu) - T| \leq 1}} \frac{d_{p}^{*}(\mu)}{r_{p}(\mu) + T} = O(T^{n-2})
$$
\n
$$
\sum_{\substack{\mu \in S_{p} \\ \mu_{p}(\mu) - T| \leq 1}} \frac{d_{p}^{*}(\mu)}{(r_{p}(\mu))^{2} + \alpha_{2}^{2}} \leq \sum_{\substack{\mu \in S_{p} \\ \mu_{p}(\mu) - T| \leq 1}} \frac{d_{p}^{*}(\mu)}{(T-1)^{2}} = O(T^{n-3}).
$$
\nes the proof

This completes the proof \blacksquare

To compare the analogous considerations for the Riemann zeta function, we refer to Landau [19] and Titchmarsh [26], for the case $n = 2$, $p = 0$ cf. Hejhal [14]. We easily get the following conclusion.

Proposition 41: *For* $0 < \epsilon < 1$, $s = N + \sigma + iT$, $\sigma \ge \epsilon$ and $T \ge 100$, we have $\Psi_p(s) = O\left(\frac{T^{n-1}}{s}\right).$

Proof: In the case $\sigma \leq N + 1$ the assertion is a consequence of $\frac{1}{|s - N + ir_p(\mu)|} \leq \frac{1}{\epsilon}$, the estimation $\mathcal{N}_p(T + 1) - \mathcal{N}_p(T - 1) = O(T^{n-1})$ (which follows from Proposition 34) and Proposition 40. In the case $\sigma \geq N + 1$ we have $\Psi_p(s) = O(1)$ by Lemma 15.

We need the following Phragmén-Lindelöf principle, stated in Landau [19, Satz 405].

Proposition 42: *Suppose* $\beta > \alpha$, $T_0 > 0$ *and let the function* $f(s)$ *be holomorphic in the half strip* $\alpha \leq \sigma^* \leq \beta$, $T \geq T_0$ with $s = \sigma^* + iT$. Further on, we suppose $f(s) = O(T^{\alpha})$ for $\sigma^* = \alpha$ and $f(s) = O(T^{\beta})$ for $\sigma^* = \beta$ and $|f(s)| = O(T^{\alpha})$ in the used *half strip. Then it follows*

$$
|f(s)| = O\left(T^{a\frac{\beta-\sigma^*}{\beta-\alpha}+b\frac{\sigma^*-\alpha}{\beta-\alpha}}\right)
$$

in the introduced half strip.

Proposition 43: *Suppose* $0 < \epsilon < 1$, $s = N + \sigma + iT$, $\sigma \ge \epsilon$ and $T \ge 100$. Then *we can state* $\Psi_p(s) = O(\frac{1}{2}T^2 \max[0, -\sigma + N + \epsilon])$.

Proof: We use Proposition 42 with $f(s) = e\Psi_p(s + \epsilon)$ and the half strip $N \leq Res \leq$ $2N$, $Im s \ge 100$. The function $f(s)$ is holomorphic in this half strip by Proposition 23. As a consequense of Proposition 41 we have with $f(s) = \epsilon \Psi_p(s + \epsilon)$ and the half s

is holomorphic in this half strip by

we have

) and $f(N + iT) = O(T^{n-1})$

$$
f(s) = O(\frac{1}{\epsilon}|Im s|^{n-1}) \quad \text{and} \quad f(N + iT) = O(T^{n-1}).
$$

Lemma 15 implies $f(2N + iT) = O(1)$. We get the assertion for $Re s \le 2N$. In the case $Re s \geq 2N$ we only have to apply Lemma 15 again.

11. Proof of Theorem B

We are generalizing the proof for $n = 2$, $p = 0$ given by Hejhal [14], but we have much more technical problems. Using the length spectrum of the considered compact space form *V* we introduce Estimates for Compact Hyperbolic Space Forms 287
 = 2, p = 0 given by Hejhal [14], but we have much

elength spectrum of the considered compact space
 $= l(\omega)\sigma(\omega)e_p(\omega)e^{l(\omega)N}$. (106)
 $e \cdot s > 2N$) in the form
 $- r \sum A(\omega)e^{-l(\$ pectral Estimates for Compact Hyperbolic Space Forms 287

B

for $n = 2$, $p = 0$ given by Hejhal [14], but we have much

ng the length spectrum of the considered compact space
 $\mu_p(\omega) = l(\omega)\sigma(\omega)e_p(\omega)e^{l(\omega)N}$. (106)

for $Re s >$ *p* = 0 given by Hejhal [14], but we have much

gth spectrum of the considered compact space
 y) $\sigma(\omega)e_p(\omega)e^{i(\omega)N}$. (106)

2*N*) in the form
 $\sum_{\omega \in \Omega} \Lambda_p(\omega)e^{-i(\omega)s}$. (107)
 $t \ge 20$ (108)
 $\int_{Re \xi = \alpha} X(\xi, s, t) \Psi_p(\xi) d\xi$ given by Hejhal [14], but we have much
ctrum of the considered compact space
 $\frac{1}{p}(\omega)e^{i(\omega)N}$. (106)
the form
 $(\omega)e^{-i(\omega)s}$. (107)
 $X(\xi, s, t)\Psi_p(\xi) d\xi$ (109)
 $\frac{1}{z}e^{2(\xi-s)t}y^{n-1}$ (110) Spectral Estimates for Compact Hyperbolic Space Forms
 XIII B

of for $n = 2$, $p = 0$ given by Hejhal [14], but we have 1

sling the length spectrum of the considered compact $\lambda_p(\omega) = l(\omega)\sigma(\omega)e_p(\omega)e^{l(\omega)N}$.

id for $Re s >$

$$
\Lambda_p(\omega) = l(\omega)\sigma(\omega)e_p(\omega)e^{l(\omega)N}.\tag{106}
$$

Then we can write (62) (valid for $Re s > 2N$) in the form

$$
\Psi_p(s) = \kappa_p \sum_{\omega \in \Omega} \Lambda_p(\omega) e^{-l(\omega)s}.
$$
 (107)

Using a cut-off number *t* with

$$
\geq 20\tag{108}
$$

we define

$$
t \text{ with}
$$
\n
$$
t \ge 20 \tag{108}
$$
\n
$$
\Psi_p^{[t]}(s) = \frac{1}{2\pi i} t^{1-n} \int_{Re\{\xi = \alpha\}} X(\xi, s, t) \Psi_p(\xi) d\xi \tag{109}
$$

with

$$
\Psi_p(s) = \kappa_p \sum_{\omega \in \Omega} \Lambda_p(\omega) e^{-i(\omega)t}.
$$
\nnumber *t* with

\n
$$
t \ge 20
$$
\n(108)

\n
$$
\Psi_p^{[t]}(s) = \frac{1}{2\pi i} t^{1-n} \int_{Re\{\epsilon = \alpha}} X(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n(109)

\n
$$
X(\xi, s, t) = \frac{(e^{(\xi - s)t} - e^{2(\xi - s)t})^{n-1}}{(\xi - s)^n}
$$
\n(110)

\n
$$
\alpha = \max(4N, 2N + Re\, s).
$$
\n(111)

\nwhere $\Psi_p(\xi) = O(1)$ by Lemma 15 and thereby the integral in (109) exists.

\n(as above) we suppose

\n
$$
0 \le \sigma^* \le 4N
$$
\n(112)

\n
$$
T \ge 50, \quad T + k \notin \{r_p(\mu) : \mu \in S_p\} \quad \text{for } k \in \mathbb{Z}, |k| \le 15.
$$
\n(113)

\nsiderations we use numbers

\n
$$
\sigma, \quad \in \mathbb{R} \text{ with } N < \sigma_1 < 4N/3
$$
\n(114)

and

$$
\alpha = \max(4N, 2N + Re s). \tag{111}
$$

For $Re \xi = \alpha$ we have $\Psi_p(\xi) = O(1)$ by Lemma 15 and thereby the integral in (109) exists. For $s = \sigma^* + iT$ (as above) we suppose $X(\xi, s, t) = \frac{(c - \xi)^n}{(\xi - s)^n}$ (110)
 $\alpha = \max(4N, 2N + Re s).$ (111)
 $= O(1)$ by Lemma 15 and thereby the integral in (109) exists.

we suppose
 $0 \le \sigma^* \le 4N$ (112)
 $+ k \notin \{r_p(\mu) : \mu \in S_p\}$ for $k \in \mathbb{Z}, |k| \le 15.$ (113)

we use $\alpha = \max(4N, 2N + Re s)$. (111)
 $= O(1)$ by Lemma 15 and thereby the integral in (109) exists.

we suppose
 $0 \le \sigma^* \le 4N$ (112)
 $+ k \notin \{r_p(\mu) : \mu \in S_p\}$ for $k \in \mathbb{Z}, |k| \le 15$. (113)

re use numbers.
 $r_1 \in \mathbb{R}$ with $N < \sigma_$ $\alpha = \max(4N, 2N + Re s).$ (111)
 $= O(1)$ by Lemma 15 and thereby the integral in (109) exists.

we suppose
 $0 \le \sigma^* \le 4N$ (112)
 $+ k \notin \{r_p(\mu) : \mu \in S_p\}$ for $k \in \mathbb{Z}, |k| \le 15.$ (113)

re use numbers
 $\sigma_1 \in \mathbb{R}$ with $N < \sigma_1$

$$
0 \leq \sigma^* \leq 4N \tag{112}
$$

$$
T \ge 50, \quad T + k \notin \{r_p(\mu) : \mu \in S_p\} \quad \text{for } k \in \mathbb{Z}, \, |k| \le 15. \tag{113}
$$

For further considerations we use numbers

$$
\sigma_1 \in \mathbf{R} \text{ with } N < \sigma_1 < 4N/3 \tag{114}
$$

$$
F = T + 10, \tag{115}
$$

$$
G = T - 10 \tag{116}
$$

$$
A \in \{k+1/2 : k \in \mathbb{Z}, k \ge N\}.
$$
 (117)

*^A*E *{k+1/2: kEZ,k>_N}. (* 117) We denote by $R(A, s)$ the rectangle defined by the points $-A-iF$, $\alpha-iF$, $\alpha+iF$ and $-A + iF$. We remark that *F*, *G* and α are depending on *s*. An interesting background of the definition of $\Psi_{\mathbf{p}}^{[t]}(\omega)$ is the fact that it will turn out to be a sum which is similar to *(107),* but finite. More precisely, we get with $N < \sigma_1 < 4N/3$ (114)
 $+ 10,$ (115)
 $- 10$ (116)
 $\colon + 1/2 : k \in \mathbb{Z}, k \ge N$. (117)

defined by the points $-A - iF, \alpha - iF, \alpha + iF$ and

x are depending on s. An interesting background of

nat it will turn out to be a sum wh that F, G
 ω) is the 1

e precisely

: For Re
 Ψ
 $=\sum_{k=0}^{n-1} \frac{1}{k!(n+1)!}$ $k + 1/2 : k$
defined by
 α are depented it will the
response N we obtain
 $= \kappa_p \sum_{\omega \in \Omega} \Lambda_p^1$
 $= k$. nts $-A - iF$

1 s. An inter

t to be a sure
 ω)
 ω)
 $-\frac{l(\omega)}{t}\Bigg\}_{+}^{n-1}$ *A*, $\alpha - iF$, $\alpha + iF$ and
 A esting background of
 A which is similar to

(118)
 $\Lambda_p(\omega)$. (119)

$$
\Psi_{p}^{[t]}(s) = \kappa_{p} \sum_{\omega \in \Omega} \Lambda_{p}^{t}(\omega) e^{-l(\omega)s}
$$
\n(118)

with

$$
\begin{aligned}\n\text{Proposition 44:} \quad & \text{More precisely, we get} \\
\text{Proposition 44:} \quad & \text{For } \text{Re } s > 2N \text{ we obtain} \\
& \Psi_p^{[t]}(s) = \kappa_p \sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-l(\omega)s} \\
& \Lambda_p^t(\omega) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!(n-1-k)!} \left\{ (n-1+k) - \frac{l(\omega)}{t} \right\}_+^{n-1} \Lambda_p(\omega).\n\end{aligned}\n\tag{119}
$$

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 Thereby we use $\nu_+^{n-1} = \begin{cases} \nu^{n-1} & \text{for } \nu > 0 \\ 0 & \text{for } \nu \leq 0. \end{cases}$
 Proof: The definitions imply

Proof: The definitions imply

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\nby we use
$$
\nu_+^{n-1} = \begin{cases} \nu^{n-1} & \text{for } \nu > 0 \\ 0 & \text{for } \nu \le 0. \end{cases}
$$

\n**Proof:** The definitions imply
\n
$$
\Psi_p^{[t]}(s) = \frac{\kappa_p}{2\pi i} t^{1-n} \sum_{\omega \in \Omega} \int_{Re \epsilon = \alpha} X(\xi, s, t) e^{-i(\omega)\epsilon} d\xi \Lambda_p(\omega) \qquad (120)
$$
\n
$$
= \frac{\kappa_p}{2\pi i} t^{1-n} \sum_{\omega \in \Omega} \int_{Re \mu = \alpha - \sigma^*} (e^{ut} - e^{2ut})^{n-1} u^{-n} e^{-i(\omega)u} du \Lambda_p(\omega) e^{-i(\omega)s}.
$$
\n
\ner to calculate the integral we use the Cauchy residue theorem. Let K_p^0 denote the
\nthe value from $\alpha - \sigma^* - i\rho$ to $\alpha - \sigma^* + i\rho$. We denote the right circular arc of the

In order to calculate the integral we use the Cauchy residue theorem. Let K^0_ρ denote the straight line from $\alpha - \sigma^* - i\rho$ to $\alpha - \sigma^* + i\rho$. We denote the right curcular arc of the circle with center 0, radius ρ given by the chord K^0_ρ by K^R_ρ . Analogously we define the left circular arc K_p^L . We can state $\sum_{\omega \in \Omega} \int_{Re u = \alpha - \sigma^*} (e^{ut} - e^{2ut})$
tegral we use the Cauchy
 $-i \rho$ to $\alpha - \sigma^* + i \rho$. We
 $i \rho$ given by the chord K
in state
 $\mu^{n-1} u^{-n} = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}$
 $\mu^{n-1} u^{-n} = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}$
the integrand $e^{(n$

L. We can state
\n
$$
(e^{ut} - e^{2ut})^{n-1} u^{-n} = \sum_{k=0}^{n-1} (-1)^k {n-1 \choose k} e^{(n-1+k)ut} u^{-n}.
$$
\nttegrals
\n
$$
I_{k,l(\omega)}^* = \int_{Re u = \alpha - \sigma^*} e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u} du.
$$
\n
$$
l \le l(\omega) \text{ the integrand } e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u} \text{ is holomo}
$$
\n
$$
|e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u}| \le \rho^{-n}
$$

We consider the integrals

$$
I_{k,l(\omega)}^* = \int_{Re\,u=\alpha-\sigma^*} e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u} \, du.
$$

For $0 < (n-1+k)t < l(\omega)$ the integrand $e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}$ is holomorphic with respect to *n* in the domain bounded by K_{ρ}^{0} and K_{ρ}^{R} . For $u \in K_{\rho}^{R}$ we obtain

$$
|e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}|\leq \rho^{-n}
$$

and the arc length of K_p^R is smaller then $\pi \rho$. It follows

$$
\lim_{\rho\to\infty}\int_{K_{\rho}^R}e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}\,du=0.
$$

Now the residue theorem implies

$$
\int_{Re u=\alpha-\sigma^*} e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u} du = 0.
$$

For $0 < (n-1+k)t < l(\omega)$ the integrand $e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}$ is holomorphic with respect

to *n* in the domain bounded by K_{ρ}^0 and K_{ρ}^R . For $u \in K_{\rho}^R$ we obtain
 $|e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}| \le \rho^{-n}$

and the arc le bounded by K^0_ρ and K^L_ρ with the exception of the pole in $u = 0$. For $u \in K^L_\rho$ we can 100 km estimate JReu=a-o*

k)t the integrand $e^{(n-1)}$
 K_{ρ}^{L} with the exception
 $|e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}| \leq$

is smaller then $2\pi a$ and with the exception of the pole in $u = 0$. For $u \in$
 $e^{1+k}u^u u^{-n}e^{-l(\omega)u}| \leq e^{[(n-1+k)t-l(\omega)](\alpha-\sigma^*)}\rho^{-n}$.

Simples then $2\pi\rho$ and thereby we get
 $\lim_{\rho \to \infty} \int_{K^L_\rho} e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u} du = 0.$

and in $u = 0$ follows fro

$$
|e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}|\leq e^{[(n-1+k)t-l(\omega)](\alpha-\sigma^*)}\rho^{-n}.
$$

The arc length of K_{ρ}^{L} is smaller then $2\pi\rho$ and thereby we get

which the exception of the pole in
$$
u^{1+k}u^1u^{-n}e^{-l(\omega)u}| \leq e^{[(n-1+k)t-l(\omega)](\alpha-\sigma)}
$$

smaller then $2\pi\rho$ and thereby we get

$$
\lim_{\rho\to\infty}\int_{K^L_{\rho}}e^{(n-1+k)ut}u^{-n}e^{-l(\omega)u}du=0.
$$

The residue of the integrand in $u = 0$ follows from the equation

$$
e^{[(n-1+k)t-l(\omega)]u}u^{-n}=u^{-n}+\ldots+\frac{1}{(n-1)!}\left[(n-1+k)t-l(\omega)\right]^{n-1}u^{-1}+\ldots.
$$

For $l(\omega) \leq (n - 1 + k)t$ the application of the Cauchy residue theorem gives For $l(\omega) \leq 0$

Spectral Estimates for Compact Hyperbolic Space Form
\n
$$
\leq (n-1+k)t
$$
 the application of the Cauchy residue theorem gives
\n
$$
\int_{Re u = \alpha - \sigma^*} e^{(n-1+k)ut} u^{-n} e^{-i(\omega)u} du = \frac{2\pi i}{(n-1)!} ((n-1+k)t - i(\omega))^{n-1}
$$

\n
$$
\lim_{n \to +\infty} \text{ the results gives}
$$

Summarizing the results gives

$$
\int_{Re u = \alpha - \sigma^*} e^{(n-1+k)ut} u^{-n} e^{-l(\omega)u} du
$$
\n
$$
= \sum_{k=0}^{n-1} (-1)^k {n-1 \choose k} \frac{2\pi i}{(n-1)!} \{(n-1+k)t - l(\omega)\}_+^{n-1}.
$$

Now the assertion follows from equation (120) **^u**

We have

$$
\Lambda_p^t(\omega) = 0 \text{ for } l(\omega) \ge 2(n-1)t. \tag{121}
$$

It follows that the sum in (118) is finite in contrast to the sum in (107). We use the fact that we have found an analytic continuation of $\Psi_p(\xi)$ (cf. Proposition 23) and apply the Cauchy integral theorem to the function $X(\xi,s,t)\Psi_p(\xi)$ and the domain $R(A,s)$:

$$
= \sum_{k=0}^{n-1} (-1)^k {n-1 \choose k} \frac{2\pi i}{(n-1)!} \{(n-1+k)t - l(\omega)\}_+^{n-1}.
$$

extition follows from equation (120)

we

$$
\Lambda_p^t(\omega) = 0 \text{ for } l(\omega) \ge 2(n-1)t. \tag{121}
$$

that the sum in (118) is finite in contrast to the sum in (107). We use the fact
the found an analytic continuation of $\Psi_p(\xi)$ (cf. Proposition 23) and apply the
gral theorem to the function $X(\xi, s, t)\Psi_p(\xi)$ and the domain $R(A, s)$:

$$
\frac{1}{2\pi i} \int_{-\pi - iF}^{\pi + iF} X(\xi, s, t)\Psi_p(\xi) d\xi
$$

$$
= \frac{1}{2\pi i} \int_{-A - iF}^{-A + iF} X(\xi, s, t)\Psi_p(\xi) d\xi + \frac{1}{2\pi i} \int_{-A + iF}^{\pi + iF} X(\xi, s, t)\Psi_p(\xi) d\xi \tag{122}
$$

$$
= \frac{1}{2\pi i} \int_{-A - iF}^{\infty - iF} X(\xi, s, t)\Psi_p(\xi) d\xi + \sum_{\xi \in Pol_{A,t}} Res(X(\xi, s, t)\Psi_p(\xi)).
$$

Thereby $Pol_{A,s}$ denotes the set of simple poles of the function $X(\xi, s, t)\Psi_p(\xi)$ with respect to the variable ξ within the rectangle $R(A,s)$. Because of (113), (117) and Proposition 23 there are no poles situated on the boundary of $R(A, s)$. Our next task will be the estimation of the terms on the right-hand side of (122).

Proposition 45: If we *suppose (108) - (117)* we *get*

$$
\frac{1}{2\pi i}\int_{-A-iF}^{-A+iF} X(\xi,s,t)\Psi_p(\xi)\,d\xi = O\left(\left[1+(F/A)^{n-1}\right]e^{-(n-1)(A+\sigma^*)t}\right).
$$

Proof: We consider the integrals

 λ

e integrals
\n
$$
L_1 = \int_{-A-iF}^{-A-i} Y(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n
$$
L_2 = \int_{-A-i}^{-A+i} Y(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n
$$
L_3 = \int_{-A+i}^{-A+iF} Y(\xi, s, t) \Psi_p(\xi) d\xi
$$

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\nwith
\n
$$
Y(\xi, s, t) = \frac{e^{\xi - s)t}}{(\xi - s)^n}.
$$
\n
$$
Using the functional equation of Proposition 25 we obtain
$$
\n
$$
-A-i
$$
\n(123)

Using the functional equation of Proposition 25 we obtain

$$
Y(\xi, s, t) = \frac{e^{\xi - s)t}}{(\xi - s)^n}.
$$
\n(123)

\nnal equation of Proposition 25 we obtain

\n
$$
L_1 = \int_{-A - iF}^{-A - i} Y(\xi, s, t) [-\Psi_p(2N - \xi) + \Phi_p(\xi - N)] d\xi.
$$
\n(124)

\n(125)

\n(126)

\n(127)

\n(128)

\n(129)

\n(129)

\n(124)

\n
$$
L_1 = \int_{-A - iF}^{-A - iF} Y(\xi, s, t) [-\Psi_p(2N - \xi) + \Phi_p(\xi - N)] d\xi.
$$
\n(125)

\n(126)

\n
$$
L_2 = \int_{0}^{2A - i} \frac{O(s^{n-1} \tan(\pi s))}{O(s^{n-1})} \text{ for } n \text{ even for } n \text{ odd.}
$$

If we suppose (117) and apply Lemma 15 we get the estimation $|\Psi_p(2N - s)| = O(1)$ for $Re s = -A$, $A \ge \frac{3}{2}$. Applying (87) we obtain

e (117) and apply Lemma 15 we get the estimation
$$
|\Psi_p(2N - s)| = O(1)
$$
 for $A \ge \frac{3}{2}$. Applying (87) we obtain
\n
$$
|-\Psi_p(2N - s) + \Phi_p(s - N)| = \begin{cases} O(s^{n-1} \tan(\pi s)) & \text{for } n \text{ even} \\ O(s^{n-1}) & \text{for } n \text{ odd.} \end{cases}
$$
\n
$$
\frac{1}{2} + iT, \quad k \in \mathbb{Z}, \quad T \ge 0 \text{ we have}
$$
\n
$$
= \tan(\pi(k + \frac{1}{2} + iT)) = -\cot(\pi i) = i\cot(\pi T) = i\frac{1 + e^{-2\pi T}}{1 - e^{-2\pi T}} = O(1)
$$
\n
$$
|-\Psi_p(2N - \xi) + \Phi_p(\xi - N)| = O(\xi^{n-1})
$$
\n
$$
-iF, -A - i]. \text{ It follows } L_1 = O(L'_1) \text{ with}
$$

For $s=k+\frac{1}{2}+iT$, $k\in\mathbb{Z}$, $T\geq 0$ we have

For
$$
s = k + \frac{1}{2} + iT
$$
, $k \in \mathbb{Z}$, $T \ge 0$ we have
\n
$$
\tan(\pi s) = \tan(\pi (k + \frac{1}{2} + iT)) = -\cot(\pi iT) = i \cot(\pi T) = i \frac{1 + e^{-2\pi T}}{1 - e^{-2\pi T}} = O(1)
$$
\nand thereby
\n
$$
|-\Psi_p(2N - \xi) + \Phi_p(\xi - N)| = O(\xi^{n-1})
$$
\nfor $\xi \in [-A - iF, -A - i]$. It follows $L_1 = O(L'_1)$ with

and thereby

$$
-\Psi_p(2N-\xi)+\Phi_p(\xi-N)|=O(\xi^{n-1})
$$
\n(126)

$$
| - \Psi_p(2N - s) + \Phi_p(s - N) | = \begin{cases} O(s^{n-1} \tan(\pi s)) & \text{for } n \text{ even} \\ O(s^{n-1}) & \text{for } n \text{ odd.} \end{cases}
$$

= $k + \frac{1}{2} + iT$, $k \in \mathbb{Z}$, $T \ge 0$ we have

$$
\tan(\pi s) = \tan(\pi (k + \frac{1}{2} + iT)) = -\cot(\pi i) = i \cot(\pi T) = i \frac{1 + e^{-2\pi T}}{1 - e^{-2\pi T}} = O(1)
$$

hereby
$$
| - \Psi_p(2N - \xi) + \Phi_p(\xi - N) | = O(\xi^{n-1})
$$

$$
\in [-A - iF, -A - i]. \text{ It follows } L_1 = O(L'_1) \text{ with}
$$

$$
L'_1 = \int_{-A - iF}^{-A - i} |Y(\xi, s, t)| |\xi|^{n-1} d\xi = \int_{-F}^{-1} \frac{e^{-(A + \sigma^*)t}}{((A + \sigma^*)^2 + (\eta + T)^2)^{n/2}} (A^2 + \eta^2)^{\frac{n-1}{2}} d\eta.
$$

 $f(r) = r^{(n-1)/2}$ is a convex function for $n > 3$, we get $(A^2 + n^2)^{(n-1)}$

Since $f(r) = r^{(n-1)/2}$ is a convex function for $n \geq 3$, we get $(A^2 + \eta^2)^{(n-1)/2} \leq$ $2^{(n-3)/2}(A^{n-1} + \eta^{n-1})$. For $n = 2$ we use $(A^2 + \eta^2)^{1/2} \le A + \eta$. We conclude $L'_1 = O(L''_1)$ with

$$
|\psi_{p}(2N - \xi) + \Phi_{p}(\xi - N)| = O(\xi^{n-1})
$$

\n
$$
-iF, -A - i]. \text{ It follows } L_{1} = O(L'_{1}) \text{ with}
$$

\n
$$
\int_{4-iF}^{-A-i} |Y(\xi, s, t)| |\xi|^{n-1} d\xi = \int_{-F}^{-1} \frac{e^{-(A+\sigma^{*})t}}{((A+\sigma^{*})^{2} + (\eta + T)^{2})^{n/2}} (A^{2} + \eta^{2})
$$

\n
$$
= r^{(n-1)/2} \text{ is a convex function for } n \geq 3, \text{ we get } (A^{2} + 1 + \eta^{n-1}). \text{ For } n = 2 \text{ we use } (A^{2} + \eta^{2})^{1/2} \leq A + \eta. \text{ We conclude}
$$

\n
$$
L''_{1} = \int_{1}^{F} \frac{e^{-(A+\sigma^{*})t}}{[(A+\sigma^{*})^{2} + (\eta + T)^{2}]^{n/2}} (A^{n-1} + \eta^{n-1}) d\eta
$$

\n
$$
\leq (A^{n-1} + F^{n-1}) e^{-(A+\sigma^{*})t} \int_{1}^{F} \frac{1}{[(A+\sigma^{*})^{2} + (\eta + T)^{2}]^{n/2}} d\eta
$$

\n
$$
\leq \frac{A^{n-1} + F^{n-1}}{A^{n-2}} e^{-(A+\sigma^{*})t} \int_{1}^{F} \frac{1}{(A+\sigma^{*})^{2} + (\eta + T)^{2}} d\eta,
$$

\n
$$
L''_{1} = O\left(\left[1 + \left(\frac{F}{A}\right)^{n-1}\right] e^{-(A+\sigma^{*})t}\right).
$$

In order to get the last equation, we have used $\int_1^F \frac{1}{(A+\sigma^*)^2+(n+T)^2} d\eta = O(\frac{1}{A})$. It follows that **n**- **¹**]

$$
L_1 = O\left(\left[1 + \left(\frac{F}{A}\right)^{n-1}\right]e^{-(A+\sigma^*)t}\right).
$$

We get the same estimation for *L3.* We get the

It remains to estimate *L2 .* For that purpose we apply again the functional equation of Proposition *25* and the estimation *(126):*

Spectral Estimates for Compact Hyperbolic Sp-
\nestimation for
$$
L_3
$$
.
\n ν estimate L_2 . For that purpose we apply again the fu
\nand the estimation (126):
\n
$$
L_2 = O\left(\int_{-A-i}^{-A+i} |Y(\xi, s, t)|\xi^{n-1} d\xi\right)
$$
\n
$$
= e^{-(A+\sigma^*)t} O\left(\int_{-1}^{1} \frac{(A^2 + \eta^2)^{\frac{n-1}{2}}}{[(A+\sigma^*)^2 + (\eta + T)^2]^{n/2}} d\eta\right)
$$
\n
$$
= e^{-(A+\sigma^*)t} O\left(\frac{A^{n-1}}{A^n}\right).
$$

So we have proved

$$
\int_{-A-iF}^{-A+iF} Y(\xi,s,t)\Psi_p(\xi)\,d\xi = O\left(\left[1+\left(\frac{F}{A}\right)^{n-1}\right]e^{-(A+\sigma^*)t}\right).
$$

We use this estimation for every term of the binomial expansion of $(e^{(\xi - s)t} - e^{2(\xi - s)t})^{n-1}$ to complete the proof \blacksquare

Proposition 46: If we suppose (108) - (117), we get
\n(i)
$$
\int_{-A+iF}^{-2N+iF} |X(\xi, s, t)\Psi_p(\xi)| |d\xi| = O\left(\frac{F^{n-1}}{t} e^{-(2N+\sigma^*)t}\right)
$$

(ii) $\int_{4N+iF}^{\alpha+iF} |X(\xi, s, t)\Psi_p(\xi)| |d\xi| = O\left(e^{2(n-1)(\alpha-\sigma^*)t}\right).$

Proof: We consider

$$
L_4 = \int_{-A+iF}^{-2N+iF} |Y(\xi, s, t)\Psi_p(\xi)| \, |d\xi|.
$$

 $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$

 $\mathcal{L}^{\text{max}}_{\text{max}}$, where $\mathcal{L}^{\text{max}}_{\text{max}}$

 $\hat{\boldsymbol{\epsilon}}$

The functional equation of Proposition *25* implies

uation of Proposition 25 implies
\n
$$
L_4 = \int_{-A+iF}^{-2N+iF} |Y(\xi, s, t)| \Psi_p(2N - \xi) + \Phi_p(\xi - N)| |d\xi|.
$$

We will apply (125) and suppose $-A - N \le r \le -3N$ for $s = r + iF$. Then we get

$$
\tan(\pi s) = \tan(\pi(r + iF)) = -i \tanh(\pi(ri - F)) = -i \frac{e^{2\pi(ri - F)} - 1}{e^{2\pi(ri - F)} + 1}.
$$

 $L_4 = \int_{-A+iF} |Y(\xi, s, t)[\Psi_p(2N - \xi) + \Phi_p(\xi - N)]| |d\xi|.$

We will apply (125) and suppose $-A - N \le r \le -3N$ for $s = r + iF$. Then we get
 $\tan(\pi s) = \tan(\pi(r + iF)) = -i \tanh(\pi(ri - F)) = -i \frac{e^{2\pi(ri - F)} - 1}{e^{2\pi(ri - F)} + 1}$.

Using (113) and (115) we obtai Because of Using (113) and (115) we obtain $|\tan(\pi s)| < 2$ and thereby $|\Phi_p(\xi - N)| = O(\xi^{n-1})$.

$$
Re(2N - \xi) > 2N + \frac{1}{2} \text{ for } \xi \in [-A + iF, -2N + iF]
$$

we get $\Psi_p(2N - \xi) = O(1)$. It follows

$$
L_4 = O\left(\int_{-A+iF}^{-2N+iF} |Y(\xi,s,t)| |\xi|^{n-1} |d\xi|\right).
$$

By (108) and (115) we get

$$
(-A+iF)
$$
\nand (115) we get\n
$$
L_4 = O\left(F^{n-1} \int_{-A}^{-2N} e^{(\eta-\sigma^*)t} |d\eta| \right)
$$
 and
$$
L_4 = O\left(\frac{F^{n-1}}{t} e^{(-2N-\sigma^*)t} \right).
$$
\nthe binomial expansion of $(e^{(\xi-\rho)t} - e^{2(\xi-\rho)t})^{n-1}$, we get the assertion in the image. 2*N* we have $\alpha = 4N$ and consequently the upper and lower bound in the image. So let us suppose $2N \le \sigma^* \le 4N$. Then we get\n
$$
\int_{4N+iF}^{\alpha+iF} |X(\xi, s, t)\Psi_p(\xi)| |d\xi| \le \int_{4N}^{\alpha} \frac{e^{2(n-1)(\eta-\sigma^*)t}}{10^n} |O(1)| d\eta = O\left(e^{2(n-1)(\alpha-\mu)}\right).
$$
\nwe have used Lemma 15.

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we get $\Psi_p(2N - \xi) = O(1)$. It follows
 $L_4 = O\left(\int_{-A+iF}^{-2N+iF} |Y(\xi, s, t)| |\xi|^{n-1} \mid \right)$

By (108) and (115) we get
 $L_4 = O\left(F^{n-1} \int_{-A}^{-2N} e^{(\eta - \sigma^*)t} |d\eta| \right)$ and $L_4 = O$

If we use the binomial expansion o we get the assertion (i). For $0 \le \sigma^* \le 2N$ we have $\alpha = 4N$ and consequently the upper and lower bounds of the integral in (ii) are the same. So let us suppose $2N \le \sigma^* \le 4N$. Then we get If we use the binomial expansion of $(e^{(\xi - s)t} - e^{2(\xi - s)t})^{n-1}$, we get the asser
 $0 \le \sigma^* \le 2N$ we have $\alpha = 4N$ and consequently the upper and lower bintegral in (ii) are the same. So let us suppose $2N \le \sigma^* \le 4N$. Then

$$
L_5=\int_{4N+iF}^{\alpha+iF} |X(\xi,s,t)\Psi_p(\xi)|\,|d\xi|\leq \int_{4N}^{\alpha}\frac{e^{2(n-1)(\eta-\sigma^*)t}}{10^n}|O(1)|\,d\eta=O\Big(e^{2(n-1)(\alpha-\sigma^*)t}\Big).
$$

Thereby we have used Lemma 15.

Assumption *47: We suppose that the cutting-off parameter I and the* imaginary part $T = Im s$ of the considered complex parameter s satisfy the inequality $e^t \leq T^2$.

Figure 13. The suppose that the cutting-off parameter t and the imaginary
 $T = Im s$ of the considered complex parameter s satisfy the inequality $e^t \leq T^2$.

Proposition 48: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, (108) - (1 *and Assumption 47. Then it holds*

(i) $\int_{N+\epsilon+iF}^{4N+iF} |Y(\xi,s,t)\Psi_p(\xi)| \, |d\xi| = O\left(\frac{F^{2N}}{\epsilon}e^{(\sigma_1-\sigma^*)t}\right) + O\left(\frac{1}{\epsilon}e^{(4N-\sigma^*)t}\right)$ (ii) $\int_{0}^{N-\epsilon+iF}$ | 1

$$
\text{ii)} \ \int_{-2N+iF}^{N-\epsilon+iF} |Y(\xi,s,t)\Psi_p(\xi)|\,|d\xi| = O\left(\frac{F^{2N}}{\epsilon}e^{(N-\epsilon-\sigma^*)t}\right).
$$

Proof: For $\xi = \eta + iF$, $s = \sigma^* + iT$ we define

$$
s = \sigma^* + iT \text{ we define}
$$

$$
L_6 = \int_{N + i + iF}^{4N + iF} |Y(\xi, s, t)\Psi_p(\xi)| |d\xi|
$$

and get

$$
N+\epsilon+iF
$$
\n
$$
L_6 \leq 10^{-n} \int_{N+\epsilon}^{4N} e^{(\eta-\sigma^*)t} |\Psi_p(\eta+iF)| d\eta.
$$

By using Proposition 43, it follows

$$
N + \epsilon + iF
$$
\n
$$
L_6 \le 10^{-n} \int_{N+\epsilon}^{4N} e^{(\eta - \sigma^*)t} |\Psi_p(\eta + iF)| d\eta.
$$
\nposition 43, it follows

\n
$$
|\Psi_p(\eta + iF)| = O(\frac{1}{\epsilon} F^{2 \max(0, -\eta + 2N + \epsilon)}) \quad \text{for } \eta \in [N + \epsilon, 4N].
$$
\nuce that

We easily deduce that

$$
L_6 \le 10^{-n} \int_{N+\epsilon} e^{(\eta-\sigma^*)t} |\Psi_p(\eta + i\theta)|
$$

Proposition 43, it follows

$$
|\Psi_p(\eta + iF)| = O(\frac{1}{\epsilon} F^{2 \max(0, -\eta + 2N + \epsilon)})
$$
 for
deduce that

$$
L_6 = O\left(\frac{1}{\epsilon} \int_{N+\epsilon}^{4N} e^{(\eta-\sigma^*)t} F^{2 \max(0, -\eta + 2N + \epsilon)} d\eta\right)
$$

$$
\begin{aligned}\n\text{Spectral Estimates for Compact Hyperbolic Space For} \\
&= O\left(\frac{1}{\epsilon} \int_{N+\epsilon}^{2N+\epsilon} e^{(\eta-\sigma^*)t} F^{2(-\eta+2N+\epsilon)} d\eta \right) + O\left(\frac{1}{\epsilon} \int_{2N+\epsilon}^{4N} e^{(\eta-\sigma^*)t} d\eta \right) \\
&= O\left(\frac{1}{\epsilon} e^{-\sigma^* t} F^{2(2N+\epsilon)} \int_{N+\epsilon}^{2N+\epsilon} (e^t/F^2)^{\eta} d\eta \right) + O\left(\frac{1}{\epsilon} e^{(4N-\sigma^*)t} \right).\n\end{aligned}
$$
\nUsing (115) and Assumption 47, we obtain $e^t \leq T^2 \leq F^2$. Accordingly we get

\n
$$
\int_{N+\epsilon} \left(1 - e^{(4N+\epsilon)} \right) e^{\int_{N+\epsilon}^{2N+\epsilon} (e^t/F^2)^{N+\epsilon}} d\eta \leq (1 - e^{(4N+\epsilon)})
$$

$$
= O\left(\frac{1}{\epsilon}e^{-\sigma^*t}F^{2(2N+\epsilon)}\int_{N+\epsilon} (e^t/F^2)^{\eta} d\eta\right) + O\left(\frac{1}{\epsilon}e^{(4N-\sigma^*)t}\right)
$$

Assumption 47, we obtain $e^t \leq T^2 \leq F^2$. Accordingly w.

$$
L_6 = O\left(\frac{1}{\epsilon}e^{-\sigma^*t}F^{2(2N+\epsilon)}\left(\frac{e^t}{F^2}\right)^{N+\epsilon}\right) + O\left(\frac{1}{\epsilon}e^{(4N-\sigma^*)t}\right)
$$

$$
= O\left(\frac{1}{\epsilon}F^{2N}e^{(\sigma_1-\sigma^*)t}\right) + O\left(\frac{1}{\epsilon}e^{(4N-\sigma^*)t}\right).
$$

This proves (i).

In order to study

$$
L_7 = \int_{-2N+iF}^{N-\epsilon+iF} |Y(\xi,s,t)\Psi_p(\xi)| \, |d\xi|
$$

we use Proposition *25* and get

$$
L_{7} = \int_{-2N+iF}^{N-\epsilon+iF} |Y(\xi, s, t)\Psi_{p}(\xi)||d\xi|
$$

tion 25 and get

$$
L_{7} = O\left(\int_{-2N+iF}^{N-\epsilon+iF} e^{(\eta-\sigma^{*})t} |-\Psi_{p}(2N-\xi)+\Phi_{p}(\xi-N)||d\xi| \right).
$$

By Proposition 41 we obtain

$$
|\Psi_p(2N-\xi)|=O\left(\frac{1}{\epsilon}F^{n-1}\right) \quad \text{for } \xi \in [-2N+iF, N-\epsilon+iF].
$$

Using $|\Phi_p(\xi - N)| = O(F^{n-1})$, it follows

$$
L_7 = O\left(\left(\frac{1}{\epsilon}F^{n-1} + F^{n-1}\right) \int\limits_{-2N}^{N-\epsilon} e^{(n-\sigma^*)t} d\eta\right) = O\left(\frac{F^{2N}}{\epsilon}e^{(N-\epsilon-\sigma^*)t}\right).
$$

Thus also statement (ii) is proved \blacksquare

Using Proposition 48 a quick calculation shows

Proposition 49: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \leq \sigma^*$, (108) - (117) *and Assumption 47. Then it holds* Using Propos
 Proposition
 Assumption

(*i*) \int_{N+iF}^{4N+iF}

(*ii*) \int_{-2N+iF}^{N+iF}
 Proposition

(i) $\int_{N+\epsilon+iF}^{4N+iF} |X(\xi,s,t)\Psi_p(\xi)| \, |d\xi| = O\left(\frac{F^{2N}}{\epsilon}e^{(\sigma_1-\sigma^*)t}\right) + O\left(\frac{1}{\epsilon}e^{(4N-\sigma^*)t}\right)$

$$
(ii) \int_{-2N+iF}^{N-\epsilon+iF} |X(\xi,s,t)\Psi_p(\xi)| \, |d\xi| = O\left(\frac{F^{2N}}{\epsilon}e^{(N-\epsilon-\sigma^*)t}\right).
$$

Proposition 50: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \le \sigma^*$, (108) - (117) *and Assumption 47. Then it holds*

1) is proved
\n1 48 a quick calculation shows
\n: We suppose
$$
\sigma_1 = N + \epsilon
$$
, $0 < \epsilon < 1/10$, $\sigma_1 \le \sigma^*$, (10
\nThen it holds
\n
$$
s, t) \Psi_p(\xi) |d\xi| = O\left(\frac{F^{2N}}{\epsilon} e^{(\sigma_1 - \sigma^*)t}\right) + O\left(\frac{1}{\epsilon} e^{(4N - \sigma^*)t}\right)
$$
\n
$$
s, s, t) \Psi_p(\xi) |d\xi| = O\left(\frac{F^{2N}}{\epsilon} e^{(N - \epsilon - \sigma^*)t}\right).
$$
\n: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \le \sigma^*$, (10
\nThen it holds
\n
$$
N + \epsilon + iF
$$
\n
$$
\int_{N - \epsilon + iF} Y(\xi, s, t) \Psi_p(\xi) d\xi = O(F^{2N} e^{(\sigma_1 - \sigma^*)t}).
$$

Proof: Proposition 40 with $\xi = \eta + iF$ implies

$$
\begin{aligned}\n\text{e} & \text{or} \\
\text{a} & \text{or} \\
\psi_p(\xi) = O(F^{n-1}) + \kappa_p \sum_{\mu \in S_p, \atop |r_p(\mu) - F| \le 1} d_p^*(\mu) \frac{1}{\xi - N - ir_p(\mu)}\n\end{aligned}
$$

Proof: Proposition 40 with
$$
\xi = \eta + iF
$$
 implies
\n
$$
\Psi_p(\xi) = O(F^{n-1}) + \kappa_p \sum_{\substack{\mu \in S_p \\ |\nu_p(\mu) - \bar{\nu}| \le 1}} d_p^*(\mu) \frac{1}{\xi - N - ir_p(\mu)}
$$
\nfor $\xi \in [N - \epsilon + iF, N + \epsilon + iF]$. We get
\n
$$
L_8 = \int_{N - \epsilon + iF}^{N + \epsilon + iF} Y(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n
$$
= \int_{N - \epsilon + iF}^{N + \epsilon + iF} Y(\xi, s, t) O(F^{n-1}) d\xi
$$
\n
$$
= \int_{r_p(\mu) - F[\le 1]}^{N + \epsilon + iF} d_p^*(\mu) \int_{N - \epsilon + iF}^{N + \epsilon + iF} Y(\xi, s, t) \frac{1}{\xi - N - ir_p(\mu)} d\xi.
$$
\nIt holds $|\xi - s| \ge 10$ and thereby we get
\n
$$
\sum_{\substack{\mu \in S_p \\ |\nu_p(\mu) - F[\le 1] \\ |\mu| = \epsilon + iF}}^{N + \epsilon + iF} Y(\xi, s, t) O(F^{n-1}) d\xi = O(\epsilon e^{(\sigma_1 - \sigma^*)t} F^{n-1}).
$$
\n(128)

It holds $|\xi - s| \geq 10$ and thereby we get

$$
\sum_{\substack{|\mathbf{r}_{\mathbf{p}}(\mu) = F|\leq 1 \\ |\mathbf{r}_{\mathbf{p}}(\mu) = F|\leq 1}} a_{\mathbf{p}}(\mu) \int_{N - \epsilon + iF} I(\zeta, s, t) \frac{1}{\xi - N - ir_{\mathbf{p}}(\mu)} d\zeta.
$$
\nand thereby we get

\n
$$
\int_{N - \epsilon + iF}^{N + \epsilon + iF} Y(\xi, s, t) O(F^{n-1}) d\xi = O(\epsilon e^{(\sigma_1 - \sigma^*)t} F^{n-1}). \tag{128}
$$
\n
$$
L_{8, \mu} = \int_{N - \epsilon + iF}^{N + \epsilon + iF} Y(\xi, s, t) \frac{1}{\xi - N - ir_{\mathbf{p}}(\mu)} d\xi
$$
\nintegral theorem transforming the path of integration into a semi-

In order to analyze

$$
L_{8,\mu} = \int\limits_{N-\epsilon+iF}^{N+\epsilon+iF} Y(\xi,s,t) \frac{1}{\xi - N - ir_p(\mu)} d\xi
$$

we use the Cauchy integral theorem transforming the path of integration into a semicircular. We denote the upper semi-circular above $[N - \epsilon + iF, N + \epsilon + iF]$ by H^+ and the corresponding lower semi-circular by H^- . Since (113) we have $r_p(\mu) \neq F$. Then the function $\frac{Y(\xi, s, t)}{\xi - N - ir_p(\mu)}$ is holomorphic with respect to ξ in the domain bounded by H^+ and *H* $^-$ because of $\xi \neq s$. For $r_p(\mu) > F$ and $r_p(\mu) < F$ we use H^- and H^+ , respectively as the new path of integration. Using $\epsilon < 1/10$ we get $|\xi - s| > 9$ and $|\xi - N - ir_p(\mu)| > \epsilon$. It follows Fⁿ⁻¹) $d\xi = O(\epsilon e^{(\sigma_1-\sigma^*)t}F^{n-1}).$
 $\left(\xi, s, t\right) \frac{1}{\xi - N - ir_p(\mu)} d\xi$

ransforming the path of integrat

cular above $[N - \epsilon + iF, N + \epsilon + \gamma H^-.$ Since (113) we have $r_p(\mu)$

respect to ξ in the domain bour

nd $r_p(\mu) < F$ we use *Of* $\left| \frac{1}{2} \right|$ is the parallel point of integration into a semi-
 O($\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$,

$$
L_{8,\mu} = O(e^{(\sigma_1 - \sigma^*)t}). \tag{129}
$$

Thereby the O-term is independent of μ and *T*. From (127) - (129) we get

$$
L_{8,\mu} = O(e^{(\sigma_1 - \sigma^*)t}).
$$
\n(129)

\nerm is independent of μ and T . From (127) - (129) we get

\n
$$
L_8 = O\left(e^{(\sigma_1 - \sigma^*)t} F^{n-1} \epsilon\right) + \sum_{|r_p(\mu) - F| \le 1} O\left(d_p^*(\mu) e^{(\sigma_1 - \sigma^*)t}\right).
$$

Proposition 34 implies $\sum_{|r_p(\mu)-F|\leq 1} d_p^*(\mu) = O(F^{n-1})$ and we easily obtain the assertion

Of course, we again get the corresponding result using $X(\xi, s, t)$ instead of $Y(\xi, s, t)$.

Proposition 51: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \leq \sigma^*$, (108) - (117) *and Assumption 47. Then it holds*

$$
\int_{N-\epsilon+iF}^{N+\epsilon+iF} X(\xi,s,t)\Psi_p(\xi)\,d\xi=O(F^{2N}e^{(\sigma_1-\sigma^*)t}).
$$

We have described the integrals on the right-hand side of (122) and get information about the term on the left-hand side of (122).

and Assumption 47. Then it holds

Proposition 52: We suppose
$$
\sigma_1 = N + \epsilon
$$
, $0 < \epsilon < 1/10$, $\sigma_1 \le \sigma^*$, (108) - (117)
\nAssumption 47. Then it holds
\n
$$
\frac{1}{2\pi i} \int_{\alpha - iF}^{a + iF} X(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n
$$
= O\left(\left[1 + \left(\frac{F}{A}\right)^{2N}\right] e^{-2N(A + \sigma^*)t}\right) + O\left(\frac{F^{2N}}{t} e^{-(2N + \sigma^*)t}\right)
$$
\n
$$
+ O\left(e^{4N(\alpha - \sigma^*)t}\right) + O\left(\frac{1}{\sigma_1 - N} e^{4N(4N - \sigma^*)t}\right)
$$
\n
$$
+ O\left(\frac{F^{2N}}{\sigma_1 - N} e^{(\sigma_1 - \sigma^*)t}\right) + \sum_{s \in Pol_{A,s}} Res\left(X(\xi, s, t)\Psi_p(\xi)\right).
$$

Proof: Proposition 45 states

$$
\frac{1}{2\pi i}\int_{-A-iF}^{-A+iF} X(\xi,s,t)\Psi_p(\xi)\,d\xi=O\left(\left[1+(\frac{F}{A})^{2N}\right]e^{-2N(A+\sigma^*)t}\right).
$$

Further on, we apply Propositions 46, 49 and 51:

Proof: Proposition 45 states
\n
$$
\frac{1}{2\pi i} \int_{-\mathbf{A}-i\mathbf{F}}^{\mathbf{A}+i\mathbf{F}} X(\xi, s, t) \Psi_p(\xi) d\xi = O\left(\left[1 + \left(\frac{F}{A}\right)^{2N}\right] e^{-2N(A+\sigma^*)t}\right).
$$
\nFurther on, we apply Propositions 46, 49 and 51:
\n
$$
\frac{1}{2\pi i} \int_{-\mathbf{A}+i\mathbf{F}}^{\mathbf{a}+i\mathbf{F}} X(\xi, s, t) \Psi_p(\xi) d\xi
$$
\n
$$
= O\left(\frac{F^{2N}}{t} e^{-(2N+\sigma^*)t}\right) + O\left(e^{4N(\alpha-\sigma^*)t}\right)
$$
\n
$$
+ O\left(\frac{1}{\sigma_1 - N} e^{4N(4N-\sigma^*)t}\right) + O\left(\frac{F^{2N}}{\sigma_1 - N} e^{(\sigma_1 - \sigma^*)t}\right).
$$
\nThere are similar calculations for $\int_{-\mathbf{A}-i\mathbf{F}}^{\mathbf{a}-i\mathbf{F}} \text{ instead of } \int_{-\mathbf{A}+i\mathbf{F}}^{\mathbf{a}+i\mathbf{F}}.$ The proposition is completed by a simple application of the Cauchy integral theorem \blacksquare

÷.

by a simple application of the Cauchy integral theorem **I**
 Proposition 53: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon$

and Assumption 47. Then it holds
 $\frac{1}{2\pi i}t^{1-n}\int_{\alpha-iF}^{\alpha+iF}X(\xi,s,t)\Psi_p(\xi) d\xi$ **Proposition 53:** We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \leq \sigma^*$, (108) - (117) *and Assumption 47. Then it holds*

application of the Cauchy integral theorem 1
ition 53: We suppose
$$
\sigma_1 = N + \epsilon
$$
, $0 < \epsilon < 1/10$, $\sigma_1 \le \sigma^*$, (108) - (117)
tion 47. Then it holds

$$
\frac{1}{2\pi i}t^{1-n}\int_{\alpha-iF}^{\alpha+iF}X(\xi,s,t)\Psi_p(\xi)d\xi
$$

$$
= O\left(t^{1-n}e^{4N(\alpha-\sigma^*)t}\right) + O\left(\frac{t^{1-n}}{\sigma_1 - N}e^{4N(4N-\sigma^*)t}\right) + O\left(F^{2N}t^{-n}e^{-(2N+\sigma^*)t}\right) + O\left(\frac{F^{2N}t^{1-n}}{\sigma_1 - N}e^{4N(\sigma_1 - \sigma^*)t}\right) + t^{1-n} \sum_{k=1}^{\infty} v_k X(-k, s, t) + t^{1-n} \sum_{k=1}^{M_p} r_k X(s_k, s, t) + t^{1-n} \sum_{\substack{\rho \in Sp \\ \sigma \le r_p(\mu) \le F}} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t) + t^{1-n} \sum_{\substack{\rho \in Sp \\ \sigma \le r_p(\mu) \le F}} 2\kappa_p d_p^*(\mu) X(s^-(\mu), s, t) + (-1)^{n-1} \Psi_p(s).
$$

Thereby we have used the residues of $\Psi_p(s)$ in the poles $\xi = -k$ given by $v_k = (-1)^{n/2}2(N+k)$ k) χ (V)P(-(N + k)²) for *n even and v*_k = 0 for *n odd. The residues in the poles* ξ = s_1, \ldots, s_{M_p} are denoted by r_k ($k = 1, \ldots, M_p$).

Proof: We use Proposition 52 and take the limit $A \rightarrow \infty$. We obtain

$$
\lim_{A\to\infty} O\left(\left[1+(\frac{F}{A})^{2N}\right]e^{-2N(A+\sigma^*)t}\right)=0.
$$

For the calculation of the residues we use Proposition 23. From the equation

$$
[e^{xt} - e^{2xt}]^{n-1}x^{-n} = e^{(n-1)xt}[1 - e^{xt}]^{n-1}x^{-n}
$$

= $(1 + ...)(-xt + ...)^{n-1}x^{-n}$
= $(-t)^{n-1}x^{-1} + ...$

we conclude $Res_{\xi=s}X(\xi,s,t) = (-t)^{n-1}$. We remark, that (112) and (113) ensure that $\xi = s$ is no pole of $\Psi_p(\xi)$

Proposition 54: *We suppose (108) - (117) and Assumption 47. Then it holds* (i) $t^{1-n} \int_{\alpha + i\infty}^{\alpha + i\infty} |X(\xi, s, t)\Psi_p(\xi)| |d\xi| = O(t^{1-n}e^{4N(\alpha - \sigma^*)t})$

(ii) $t^{1-n} \int_{\alpha - i\infty}^{\alpha - iF} |X(\xi, s, t)\Psi_p(\xi)| |d\xi| = O(t^{1-n}e^{4N(\alpha - \sigma^*)t})$ (iii) $t^{1-n} \int_{\alpha - i\infty}^{\alpha - iF} |X(\xi, s, t)\Psi_p(\xi)| \, |d\xi| = O(t^{1-n} e^{4N(\alpha - \sigma^*)t}).$ oposition 54
 0 $t^{1-n} \int_{\alpha+iF}^{\alpha+i\infty}$
 0 $t^{1-n} \int_{\alpha-i\infty}^{\alpha-iF}$
 0 of: Since (1
 $[\alpha+iF,\alpha+iF]$
 $\alpha+i\infty$
 $\int_{-\pi}^{\alpha+i\infty}$
 $|X(\xi,s,t)|$

Proof: Since (111) and (112) we have $4N \le \alpha \le 6N$ and $Re(\xi - s) = \alpha - \sigma^* \ge 2N$ for $\xi \in [\alpha + iF, \alpha + i\infty)$. It follows

is no pole of
$$
\Psi_p(\xi)
$$
 is
\n**proposition** 54: We suppose (108) - (117) and Assumption 47. Then it ho
\n*i*) $t^{1-n} \int_{\alpha + iF}^{\alpha + i\infty} |X(\xi, s, t)\Psi_p(\xi)||d\xi| = O(t^{1-n}e^{4N(\alpha - \sigma^*)t})$
\n*j*) $t^{1-n} \int_{\alpha - i\infty}^{\alpha - iF} |X(\xi, s, t)\Psi_p(\xi)||d\xi| = O(t^{1-n}e^{4N(\alpha - \sigma^*)t})$.
\n**proof:** Since (111) and (112) we have $4N \leq \alpha \leq 6N$ and $Re(\xi - s) = \alpha - \sigma^*$
\n $[\alpha + iF, \alpha + i\infty)$. It follows
\n
$$
\int_{\alpha + iF}^{\alpha + i\infty} |X(\xi, s, t)\Psi_p(\xi)||d\xi| \leq \int_{\alpha + iF}^{\alpha + i\infty} \frac{e^{4N(\alpha - \sigma^*)t}}{|\xi - s|^n} |d\xi| O(1)
$$
\n
$$
\leq O(e^{4N(\alpha - \sigma^*)t}) \int_{F}^{\infty} \frac{1}{[(\alpha - \sigma^*)^2 + (\eta - T)^2]^{n/2}} d\eta
$$
\n
$$
\leq O(e^{4N(\alpha - \sigma^*)t}).
$$

This proves (i), the proof of (ii) is similar **u**

Proposition 55: We suppose $\sigma_1 = N + \epsilon$, $0 < \epsilon < 1/10$, $\sigma_1 \leq \sigma^*$, (108) - (117) *and Assumption 47. Then it holds*

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\nposition 55: We suppose
$$
\sigma_1 = N + \epsilon
$$
, $0 < \epsilon < 1/10$, $\sigma_1 \leq \sigma^*$, (108)

\numption 47. Then it holds

\n
$$
\Psi_p^{[t]}(s) = (-1)^{n-1}\Psi_p(s) + O\left(t^{1-n}e^{4N(\alpha-\sigma^*)t}\right) + O\left(\frac{t^{1-n}}{\sigma_1 - N}e^{4N(4N-\sigma^*)t}\right)
$$
\n
$$
+ O\left(F^{2N}t^{-n}e^{-(2N+\sigma^*)t}\right) + O\left(\frac{F^{2N}t^{1-n}}{\sigma_1 - N}e^{(\sigma_1 - \sigma^*)t}\right)
$$
\n
$$
+ t^{1-n} \sum_{k=1}^{\infty} v_k X(-k, s, t) + t^{1-n} \sum_{k=1}^{\frac{M_p}{\sigma_1}} r_k X(s_k, s, t)
$$
\n
$$
+ t^{1-n} \sum_{\substack{\rho \in S_p \\ 0 \leq r_p(\mu) \leq F}} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t)
$$
\n
$$
+ t^{1-n} \sum_{\substack{\rho \in S_p \\ \rho \leq r_p(\mu) \leq F}} 2\kappa_p d_p^*(\mu) X(s^-(\mu), s, t).
$$

Proof: The assertion is an immediate consequence of definition (109), Proposition *53* and Proposition *54.*

Our next aim is a simplification of the estimation of Proposition *55.* It will turn out to be useful to suppose the following

Assumption 56: *From now on we suppose* $e^t \leq T^{\frac{1}{1N}}$ (i) e^t (i) e \geq 1^{n}
 (ii) σ_1 $=$ $N+$ (iii) $\sigma_1 \leq \sigma^* \leq 4N$. **Proof:** The assertion is an immediate consequence of definition (109), Proposition

our next aim is a simplification of the estimation of Proposition 55. It will turn ou

useful to suppose the following

Assumption 56: F

Proposition 57: *We suppose (108) - (117), Assumption 47 and 56. Then it holds*

of Assumption 56 if we minimize
$$
\frac{1}{\sigma_1 - N} e^{(\sigma_1 - \sigma_1)}
$$

7: We suppose (108) - (117), Assumption 47

$$
(-1)^{n-1} \Psi_p(s) = \Psi_p^{[t]}(s) + S^{\#} + O(F^{2N} e^{(N-\sigma^*)t})
$$

with

\n- \n simplification of the estimation of *I* toposition 55.\n
\n- \n the following\n
	\n- From now on we suppose
	\n- $$
	4N
	$$
	.
	\n\n
\n- \n Assumption 56 if we minimize\n
$$
\frac{1}{\sigma_1 - N} e^{(\sigma_1 - \sigma^*)t}
$$
\n with\n
$$
We suppose (108) - (117)
$$
,\n
$$
Assumption 47 and 5
$$
\n
$$
-1)^{n-1} \Psi_p(s) = \Psi_p^{[t]}(s) + S^* + O(F^{2N}e^{(N-\sigma^*)t})
$$
\n
$$
S^* = t^{1-n} \sum_{\substack{\mu \in S_p \\ |\tau_p(\mu) - T| \le 10}} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t).
$$
\n
\n- \n
$$
\tilde{S}^* = t^{1-n} \sum_{\substack{\mu \in S_p \\ \sigma \le \tau_p(\mu) \le F}} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t)
$$
\n
\n- \n
$$
\tilde{S}^* = t^{1-n} \sum_{\substack{\mu \in S_p \\ \sigma \le \tau_p(\mu) \le F}} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t)
$$
\n
\n- \n Here to reach this goal we want to show that the luded in the *O*-estimate of Proposition 57.\n
$$
-n e^{4N(\alpha - \sigma^*)t}
$$
. If we have\n
$$
0 \leq \sigma^* \leq 2N
$$
, then we\n summation 56/(i) gives\n
$$
e^{4N(\alpha - \sigma^*)t} \leq e^{2(N - \sigma^*)Nt} T^{2N}
$$
\n
\n

Proof: We first want to prove the assertion of Proposition 57 with
\n
$$
\tilde{S}^{\#} = t^{1-n} \sum_{\mu \in S_p \atop 0 \leq r_p(\mu) \leq F} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t)
$$

instead of S^* . In order to reach this goal we want to show that the O-estimates of Proposition 55 are included in the O-estimate of Proposition 57.

1. We consider $t^{1-n}e^{4N(a-\sigma^*)t}$. If we have $0 \le \sigma^* \le 2N$, then we get $\alpha = 4N$ by (Ill) and therefore Assumption *56/(i)* gives

$$
e^{(2\alpha-\sigma^* - N)2Nt} \leq T^{2N} \quad \text{and} \quad e^{4N(\alpha-\sigma^*)t} \leq e^{2(N-\sigma^*)Nt} T^{2N}
$$

and consequently

$$
t^{-2N} e^{4N(\alpha-\sigma^*)t} \leq T^{2N} e^{2N(N-\sigma^*)t}.
$$

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and consequently
 $t^{-2N} e^{4N(\alpha-\sigma^*)t} \leq T^{2N} e^{2Nt}$

For $2N \leq \sigma^* \leq 4N$ it holds $\alpha = 2N + \sigma^*$ and thereby
 $e^{(3N+\sigma^*)t} \leq T$, $e^{(2\alpha-2\sigma^*+\sigma^*-N)} \leq T$, $t^{-2N} e^{t}$

R. Schuster\nequently\n
$$
t^{-2N} e^{4N(\alpha-\sigma^*)t} \leq T^{2N} e^{2N(N-\sigma^*)t}.
$$
\n
$$
\leq \sigma^* \leq 4N \text{ it holds } \alpha = 2N + \sigma^* \text{ and thereby}
$$
\n
$$
e^{(3N+\sigma^*)t} \leq T, \quad e^{(2\alpha-2\sigma^*+\sigma^* - N)} \leq T, \quad t^{-2N} e^{4N(\alpha-\sigma^*)t} \leq T^{2N} e^{2N(N-\sigma^*)t}.
$$
\n
$$
e^{4N(\alpha-\sigma^*)t} \leq C(T^{2N} e^{2N(N-\sigma^*)t}) \text{ and therefore}
$$

It follows $t^{-2N} e^{4N(a-\sigma^*)t} \leq O(T^{2N} e^{2N(N-\sigma^*)t})$ and thereby

$$
t^{-2N} e^{4N(\alpha - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t}).
$$
\n(132)

 $t^{-2N} e^{4N(\alpha - \sigma^*)t} \leq T^{2N} e^{2N(N - \sigma^*)t}$
 $\alpha = 2N + \sigma^*$ and thereby
 $t^{\alpha - 2\sigma^* + \sigma^* - N} \leq T$, $t^{-2N} e^{4N(\alpha - \sigma^*)t} \leq T^{2N} e^{2N(T - \sigma^*)t}$
 $t^{-2N} e^{4N(\alpha - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t})$.
 $t^{-2N} e^{4N(4N - \sigma^*)t}$. According to Assump 2. We consider $\frac{1}{\sigma_1-N}t^{-2N}e^{4N(4N-\sigma^*)t}$. According to Assumption 56/(ii) we have $(\sigma_1 - N)t = 1$ and therefore $\frac{1}{\sigma_1 - N}t^{1-n} \leq 1$. Furthermore Assumption 56/(i) implies $e^{4N(4N-\sigma^*)t} \leq T^{2N}e^{2N(N-\sigma^*)t}$. Consequently we get $\frac{1}{N}t^{-2l}$
efore
 $\frac{1}{N}$, ($t^{-2N} e^{4N(\alpha-\sigma^*)t} = O(T^{2N} e^{(N-\sigma^*)t}).$
 $\overline{N}t^{-2N} e^{4N(4N-\sigma^*)t}$. According to Assumption 56,

fore $\frac{1}{\sigma_1-N}t^{1-n} \leq 1$. Furthermore Assumption \overline{N}
 $\overline{N}t^{1-n} e^{4N(4N-\sigma^*)t} = O(T^{2N} e^{2N(N-\sigma^*)t})$
 $\frac{1}{-N}t^{1-n} e^{4$ Furthermore Assumption
et
 $O(T^{2N} e^{2N(N-\sigma^*)t})$
 $O(T^{2N} e^{(N-\sigma^*)t}).$ ediately obtain
 $(T^{2N} e^{(N-\sigma^*)t}).$
 $N-\sigma^*)t$ e and consequently

$$
\frac{1}{\sigma_1 - N} t^{1-n} e^{4N(4N - \sigma^*)t} = O(T^{2N} e^{2N(N - \sigma^*)t})
$$

and

$$
\frac{1}{\sigma_1 - N} t^{1-n} e^{4N(4N - \sigma^*)t} = O(T^{2N} e^{2N(N - \sigma^*)t})
$$
\n
$$
\frac{1}{\sigma_1 - N} t^{1-n} e^{4N(4N - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t})
$$
\n
$$
\frac{1}{\sigma_1 - N} t^{1-n} e^{4N(4N - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t}). \tag{133}
$$
\n3. We consider $F^{2N} t^{-n} e^{-(2N + \sigma^*)t}$. We immediately obtain

\n
$$
F^{2N} t^{-n} e^{-(2N + \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t}). \tag{134}
$$
\n4. Assumption 56/(ii) implies $e^{(\sigma_1 - \sigma^*)t} = e^{(N - \sigma^*)t} e$ and consequently we get

\n
$$
\frac{1}{\sigma_1 - N} t^{1-n} F^{2N} e^{(\sigma_1 - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t}). \tag{135}
$$
\n5. Since $v_k = O(k^{n-1})$ by Proposition 23, we obtain

3. We consider $F^{2N}t^{-n}e^{-(2N+\sigma^*)t}$. We immediately obtain

$$
F^{2N} t^{-n} e^{-(2N+\sigma^*)t} = O(T^{2N} e^{(N-\sigma^*)t}).
$$
\n(134)

4. Assumption 56/(ii) implies $e^{(\sigma_1-\sigma^*)t} = e^{(N-\sigma^*)t}e$ and consequently we get

$$
\frac{1}{\sigma_1 - N} t^{1-n} F^{2N} e^{(\sigma_1 - \sigma^*)t} = O(T^{2N} e^{(N - \sigma^*)t}). \tag{135}
$$

3. We consider
$$
F^{2N}t^{-n}e^{-(2N+\sigma^*)t}
$$
. We immediately obtain
\n
$$
F^{2N}t^{-n}e^{-(2N+\sigma^*)t} = O(T^{2N}e^{(N-\sigma^*)t}). \qquad (134)
$$
\n4. Assumption 56/(ii) implies $e^{(\sigma_1-\sigma^*)t} = e^{(N-\sigma^*)t}e$ and consequently we get
\n
$$
\frac{1}{\sigma_1 - N}t^{1-n}F^{2N}e^{(\sigma_1-\sigma^*)t} = O(T^{2N}e^{(N-\sigma^*)t}). \qquad (135)
$$
\n5. Since $v_k = O(k^{n-1})$ by Proposition 23, we obtain
\n
$$
\left|t^{1-n}\sum_{k=1}^{\infty} v_kX(-k, s, t)\right| = O\left(t^{1-n}\sum_{k=1}^{\infty}\frac{e^{-(n-1)(k+\sigma^*)t}}{|k+s|^{n}}k^{n-1}\right) \qquad (136)
$$
\n
$$
= O\left(t^{1-n}e^{-(n-1)\sigma^*}\sum_{k=1}^{\infty}T^{-n}e^{-(n-1)kt}\right) \qquad (137)
$$
\n
$$
= O\left(t^{1-n}T^{-n}e^{-(n-1)\sigma^*t}). \qquad (138)
$$
\n
$$
t^{1-n}T^{-n}e^{-(n-1)\sigma^*t} = O(T^{2N}e^{2N(N-\sigma^*)t})
$$
\nget
\n
$$
t^{1-n}\sum_{k=1}^{\infty} v_kX(-k, s, t) = O(T^{2N}e^{(N-\sigma^*)t}). \qquad (139)
$$
\n6. We consider the poles s_k ($k = 1, ..., M_p$) of $\Psi_p(s)$ which have the imaginary part.
\nProposition 23. Then it holds $0 \leq R_2 e_i \leq 2N$ and we have $0 \leq R_2 e_1(t) \leq 2N$

$$
= O\left(t^{1-n} e^{-(n-1)\sigma^*} \sum_{k=1}^{\infty} T^{-n} e^{-(n-1)kt}\right) \qquad (137)
$$

$$
= O\left(t^{1-n}T^{-n}e^{-(n-1)\sigma^*t}\right).
$$
 (138)

We apply

 $\mathcal{F}_{\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}}$

$$
t^{1-n}T^{-n}e^{-(n-1)\sigma^*t}=O(T^{2N}e^{2N(N-\sigma^*)t})
$$

and get

 $\sim 10^{11}$

$$
t^{1-n} \sum_{k=1}^{\infty} v_k X(-k, s, t) = O(T^{2N} e^{(N-\sigma^*)t}).
$$
\n(139)

0 (cf. Proposition 23). Then it holds $0 \leq Re s_k \leq 2N$ and we have $0 \leq Re s^+(\mu) \leq 2N$ and

$$
|X(s_k, s, t)| \leq 2^{n-1} T^{-n} [e^{(2N-\sigma^*)t} + e^{4N(2N-\sigma^*)t}].
$$

From Assumption 56/(i) we get

$$
T^{-n} e^{(2N-\sigma^*)t} = O(T^{2N} e^{(N-\sigma^*)t}) \quad \text{and} \quad T^{-n} e^{4N(2N-\sigma^*)t} = O(T^{2N} e^{(N-\sigma^*)t}).
$$

Consequently we obtain

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\n
$$
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$$
\non 56/(i) we get

\n
$$
-e^{x}t = O(T^{2N}e^{(N-\sigma^*)t})
$$
\nand

\n
$$
T^{-n}e^{4N(2N-\sigma^*)t} = O(T^{2N}e^{(N-\sigma^*)t}).
$$
\nis obtain

\n
$$
\left|T^{1-n}\sum_{k=1}^{M_p}r_k\frac{[e^{-(s_k-s)t}-e^{-2(s_k-s)t}]^{n-1}}{(s_k-s)^n}\right| = O(T^{2N}e^{2N(N-\sigma^*)t}).
$$
\n(140)

\non we get

7. Further on we get

$$
\int \int \int_{\mathbb{R}^n} \int_{\mathbb{R}^n}
$$

The condition $0 \leq r_p(\mu) \leq F$ shall be an abbreviation for $\mu \in S_p$, $Im r_p(\mu) = 0$ and $\leq r_p(\mu) \leq F$. The Weyl asymptotic estimate (1) implies $\sum_{0 \leq r_p(\mu) \leq F} d_p^*(\mu) = O(F^n) = O(T^n)$ and thereby $O(T^n)$ and thereby \leq *F* shall be an
symptotic estin
= $O(t^{1-n}e^{(N-\sigma^*)})$
ion we have use
tion for $s^+(\mu)$ in
 $2\kappa_p d_p^*(\mu)X(s)$
 $\leq F$
for \tilde{S}^* instead c
 $+ O(T^{2N}e^{(N-\sigma^*)})$
= t^{1-n}
 $\sum_{0 \leq r_p(\mu) \leq G}$

$$
L_9 = O(t^{1-n}e^{(N-\sigma^*)t}) = O(T^{2N}e^{(N-\sigma^*)t}).
$$

not use a similar consideration for $s^+(\mu)$ instead of $s^-(\mu)$. We get and the control of the con-

In order to get this estimation we have used
$$
Im s^-(\mu) \leq 0
$$
 and $|s^-(\mu) - s| \geq T$. We can
not use a similar consideration for $s^+(\mu)$ instead of $s^-(\mu)$. We get

$$
\left| t^{1-n} \sum_{0 \leq r_p(\mu) \leq F} 2\kappa_p d_p^*(\mu) X(s^-(\mu), s, t) \right| = O(T^{2N} e^{2N(N-\sigma^*)t}) \qquad (141)
$$

and the theorem is proved for $\tilde{S}^{\#}$ instead of $S^{\#}$. We have shown
 $(-1)^{n-1} \Psi_p(s) = \Psi_p^{[t]}(s) + O(T^{2N} e^{(N-\sigma^*)t}) + t^{1-n} \sum_{0 \leq r_p(\mu) \leq F} 2\kappa_p d_p^*(\mu) X(s^+(\mu), s, t)$

and the theorem is proved for $\tilde{S}^{\#}$ instead of $S^{\#}$. We have shown

$$
(-1)^{n-1}\Psi_p(s)=\Psi_p^{[t]}(s)+O(T^{2N}e^{(N-\sigma^*)t})+t^{1-n}\sum_{0\leq r_p(\mu)\leq F}2\kappa_p d_p^*(\mu)X(s^+(\mu),s,t).
$$

Next we consider

$$
L_{10}=t^{1-n}\sum_{0\leq r_p(\mu)\leq G}2\kappa_p d_p^*(\mu)X(s^+(\mu),s,t).
$$

We easily see

$$
|L_{10}| = O\left(t^{1-n} \sum_{0 \le r_p(\mu) \le G} d_p^*(\mu) \frac{e^{(N-\sigma^*)(n-1)t}}{\left[(\sigma^* - N)^2 + (T - r_p(\mu)^2\right]^{n/2}}\right).
$$

 $\mathcal{O}(\mathcal{O}_\mathcal{A})$, and $\mathcal{O}(\mathcal{O}_\mathcal{A})$, and $\mathcal{O}(\mathcal{O}_\mathcal{A})$

It follows

$$
|L_{10}| = O\left(t - \sum_{0 \le r_p(\mu) \le G} a_p(\mu) \left[(\sigma^* - N)^2 + (T - r_p(\mu))^2 \right]^{n/2} \right)
$$

lows

$$
|L_{10}| = O\left(t^{1-n} e^{(N-\sigma^*)(n-1)t} \left[1 + \int_{1}^{G} \frac{1}{\left[(\sigma^* - N)^2 + (T - r)^2 \right]^{n/2}} dN_p(r) \right] \right).
$$
 (142)

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We apply Proposition 34. It holds $N_p(r) = n_p r^n + \mathcal{R}_p(r)$ with $|\mathcal{R}_p(r)| = O(r^{n-1})$. We

split up the integral in (142). We first consider
 $L_{11} = t^{1-n} \int_{1}^{G} \frac{1}{[(\sigma^2 - N)^2 + (T - r)^2]^{n/2}} d(n_p r^n)$ split up the integral in (142). We first consider

1.
$$
34. \text{ It holds } \mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p(r) \text{ with } |\mathcal{R}_p
$$

\n1.
$$
L_{11} = t^{1-n} \int_{1}^{G} \frac{1}{[(\sigma^n - N)^2 + (T - r)^2]^{n/2}} d(n_p r^n)
$$

\n2.
$$
= t^{1-n} \int_{1}^{G} \frac{n n_p r^{n-1}}{[(\sigma^n - N)^2 + (T - r)^2]^{n/2}} dr.
$$

It follows

position 34. It holds
$$
N_p(r) = n_p r^n + R_p(r)
$$
 with $|R_p(r)|$
\ntegral in (142). We first consider
\n
$$
L_{11} = t^{1-n} \int_{1}^{G} \frac{1}{[(\sigma^* - N)^2 + (T - r)^2]^{n/2}} d(n_p r^n)
$$
\n
$$
= t^{1-n} \int_{1}^{G} \frac{n n_p r^{n-1}}{[(\sigma^* - N)^2 + (T - r)^2]^{n/2}} dr.
$$
\n
$$
|L_{11}| = O\left(\frac{G^{n-1}}{t^{n-1}} \int_{1}^{G} \frac{1}{[(\sigma^* - N)^2 + (T - r)^2]^{n/2}} dr\right)
$$
\n
$$
= O\left(\frac{G^{n-1}}{t^{n-1}} \frac{1}{(\sigma^* - N)^n} \int_{1}^{G} \left[1 + \left(\frac{T - r}{\sigma^* - N}\right)^2\right]^{-n/2} dr\right)
$$
\n
$$
= O\left(\frac{G^{n-1}}{t^{n-1}} \frac{1}{(\sigma^* - N)^n} \int_{1}^{G} \left[1 + \left(\frac{T - r}{\sigma^* - N}\right)^2\right]^{-1} dr\right)
$$
\n
$$
= O\left(\frac{G^{n-1}}{t^{n-1}} \frac{1}{(\sigma^* - N)^n} \int_{\sigma^2 = N}^{T - 1} \frac{1}{1 + \tilde{r}^2} d\tilde{r}\right).
$$
\n156/(ii) and (iii) we get $t(\sigma^* - N) \ge 1$ and thereby $|L_{11}$ there term resulting from the decomposition $N_p(r) = n_p r^n$

By Assumption 56/(ii) and (iii) we get $t(\sigma^* - N) \ge 1$ and thereby $|L_{11}| = O(G^{n-1}) =$ $O(T^{2N})$. The other term resulting from the decomposition $\mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p(r)$ is

$$
L_{12}=t^{1-n}\int\limits_{1}^{G}[(\sigma^*-N)^2+(T-r)^2]^{-n/2}d\mathcal{R}_p(r).
$$

The integrand $[(\sigma^* - N)^2 + (T - r)^2]^{-n/2}$ is monotonically encreasing for $0 \le r \le G$. We obtain $L_{12} = O(t^{1-n} \int_1^G dR_p(r))$. Using partial integration, it follows $L_{12} = O(G^{n-1}/t^{n-1})$ and thereby $L_{12} = O(T^{2N})$. Using (142), w *thereof* $\lim_{r \to \infty} \frac{G}{r}$
 tin existing from the decomposition $\mathcal{N}_p(r) = n_p r^n + \mathcal{R}_p$
 $L_{12} = t^{1-n} \int_{1}^{G} \left[(\sigma^* - N)^2 + (T - r)^2 \right]^{-n/2} d\mathcal{R}_p(r)$.
 $N)^2 + (T - r)^2 \int_{1}^{-n/2}$ is monotonically encreasing for $0 \le r^G$

$$
\left| t^{1-n} \sum_{0 \le r_p(\mu) \le G} X(s^+(\mu), s, t) \right| = O\big(T^{2N} e^{(N-\sigma^*)t}\big).
$$
 (143)

Thereby the proof of Proposition 57 is completed

Proposition 58: *We suppose (108) - (117), Assumption 47 and 56. Then it holds* **i**
 $\begin{vmatrix} t^{1-n} & \sum_{0 \le r_p(\mu) \le G} X(s^+(\mu), s, t) \end{vmatrix} = O(T^{2N}e^{(N-\sigma^*)t}).$
 reby the proof of Proposition 57 is completed a
 Proposition 58: We suppose (108) - (117), Assumption 47 and 56. Then it
 $\Psi_p(\sigma^* + iT) = (-1)^{n-1} \Psi_p^{[$ (144) sition 57 is completed **a**

sition 57 is completed **a**
 x suppose (108) - (117), A
 $\Psi_{p}^{[i]}(s) + O(T^{2N}e^{(N-\sigma^{*})t}) +$
 $|e^{k(s^{+}(\mu)-s)t}| = e^{k(N-\sigma^{*})t} \le$

Proof: It holds

$$
|e^{k(s^+(\mu)-s)t}|=e^{k(N-\sigma^*)t}
$$

with $k \in \mathbb{N}$. It follows

$$
|e^{(s^+(\mu)-s)t}-e^{2(s^+(\mu)-s)t}]^{n-1}| \leq 2^{n-1}e^{(N-\sigma^*)t}
$$

and thereby

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\nwith
$$
k \in \mathbb{N}
$$
. It follows
\n
$$
\left| [e^{(s^+(\mu)-s)t} - e^{2(s^+(\mu)-s)t}]^{n-1} \right| \le 2^{n-1}e^{(N-\sigma^*)t}
$$
\nand thereby
\n
$$
t^{1-n} \sum_{|r_p(\mu)-T| \le 10} d_p^*(\mu) X(s^+(\mu), s, t)
$$
\n
$$
= t^{1-n} 2^{n-1} \vartheta e^{(N-\sigma^*)t} \sum_{|r_p(\mu)-T| \le 10} \frac{d_p^*(\mu)}{[(\sigma^* - N)^2 + (T - r_p(\mu))^2]^{n/2}} \qquad (145)
$$
\nwith $|\vartheta| \le 1$. Assumption 56/(ii) implies
\n
$$
\frac{1}{t^{n-1}} \frac{1}{[(\sigma^* - N)^2 + (T - r_p(\mu))^2]^{n/2}} \le \frac{(\sigma_1 - N)^{n-1}}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]^{n/2}}
$$
\n
$$
\le \frac{\sigma_1 - N}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]}.
$$

$$
= t^{1-n} 2^{n-1} \vartheta e^{(N-\sigma^*)t} \sum_{|r_p(\mu)-T| \le 10} \frac{e_p(\mu)}{[(\sigma^* - N)^2 + (T - r_p(\mu))^2]^{n/2}}
$$

\n1. Assumption 56/(ii) implies
\n
$$
\frac{1}{t^{n-1}} \frac{1}{[(\sigma^* - N)^2 + (T - r_p(\mu))^2]^{n/2}} \le \frac{(\sigma_1 - N)^{n-1}}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]^{n/2}}
$$
\n
$$
\le \frac{\sigma_1 - N}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]}.
$$
\nProposition 57 and we get with a number ϑ_1 with $|\vartheta_1| \le 1$ the equation
\n $(-1)^{n-1} \Psi_p(s) = \Psi_p^{[t]}(s) + O(T^{2N} e^{(N-\sigma^*)t}) + 2^{n-1} \vartheta_1 e^{(N-\sigma^*)t}$

We apply Proposition 57 and we get with a number ϑ_1 with $|\vartheta_1| \leq 1$ the equation

$$
\leq \frac{\sigma_1 - N}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]}.
$$
\nWe apply Proposition 57 and we get with a number ϑ_1 with $|\vartheta_1| \leq 1$ the equation
\n
$$
(-1)^{n-1}\Psi_p(s) = \Psi_p^{[t]}(s) + O(T^{2N}e^{(N-\sigma^*)t})
$$
\n
$$
+ 2^{n-1}\vartheta_1e^{(N-\sigma^*)t}
$$
\n
$$
\times \sum_{|r_p(\mu)-T|\leq 10} 2\kappa_p d_p^*(\mu) \frac{\sigma_1 - N}{[(\sigma_1 - N)^2 + (T - r_p(\mu))^2]}
$$
\nfor $\sigma_1 \leq \sigma^* \leq 4N$. We 'remark that by Proposition 34 we can use $\sum_{|r_p(\mu)-T|\leq 10} \exp\left(-\frac{\sigma_1}{2}\right)$ as well as
\n
$$
\sum_{|r_p(\mu)-T|\leq 10} \ln \text{ the equations above. Further on, by Proposition 40 we obtain}
$$
\n
$$
\Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|r_p(\mu)-T|\leq 10} d_p^*(\mu) \frac{1}{(\sigma_1 - N) + i(T - r_p(\mu))}.
$$
\nWe get
\n
$$
\text{Re } \Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|r_p(\mu)-T|\leq 10} d_p^*(\mu) \frac{\sigma_1 - N}{(\sigma_1 - N)^2 + (T - r_p(\mu))^2}.
$$
\n(147)
\nEquation (146) implies

$$
\Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|r_p(\mu) - T| \leq 10} d_p^*(\mu) \frac{1}{(\sigma_1 - N) + i(T - r_p(\mu))}
$$

We get

$$
\Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|r_p(\mu) - T| \le 10} d_p^*(\mu) \frac{1}{(\sigma_1 - N) + i(T - r_p(\mu))}.
$$

get

$$
Re \Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|r_p(\mu) - T| \le 10} d_p^*(\mu) \frac{\sigma_1 - N}{(\sigma_1 - N)^2 + (T - r_p(\mu))^2}.
$$
 (147)
ation (146) implies

Equation *(146)* implies

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\nwith
$$
k \in \mathbb{N}
$$
. It follows
\n
$$
\left| [e^{(x^*(\mu)-r)t} - e^{2(x^*(\mu)-r)t}]^{n-1} \right| \leq 2^{n-1}e^{(N-r^*)t}
$$
\nand thereby
\n
$$
t^{1-n} \sum_{|\tau_{p}(x)-T| \leq 10} d_{\tau}^*(\mu) X(s^+(\mu), s, t)
$$
\n
$$
= t^{1-n} 2^{n-1} e^{(N-r^*)t} \sum_{|\tau_{p}(x)-T| \leq 10} \frac{d_{\tau}^*(\mu)}{((\sigma^* - N)^2 + (T - \tau_p(\mu)))^{2|n/2}}
$$
(145)
\nwith $|\theta| \leq 1$. Assumption 56/(ii) implies
\n
$$
\frac{1}{t^{n-1}} \frac{1}{[(\sigma^* - N)^2 + (T - \tau_p(\mu)))^2|^{n/2}} \leq \frac{(\sigma_1 - N)^{n-1}}{[(\sigma_1 - N)^2 + (T - \tau_p(\mu)))^2|^{n/2}}
$$
\nWe apply Proposition 57 and we get with a number ϑ_1 with $|\vartheta_1| \leq 1$ the equation
\n $(-1)^{n-1} \Psi_p(s) = \Psi_p^{[l]}(s) + O(T^{2N}e^{(N-r^*)t})$ (146)
\n $+ 2^{n-1} \vartheta_1 e^{(N-r^*)t}$
\n $\times \sum_{|\tau_p(x)-T| \leq 10} 2\kappa_p d_{\tau}^*(\mu) \frac{\sigma_1 - N}{[(\sigma_1 - N)^2 + (T - \tau_p(\mu))^2]}$
\nfor $\sigma_1 \leq \sigma^* \leq 4N$. We remark that by Proposition 40 we can use $\sum_{|\tau_p(x)-T| \leq 10} \frac{\sigma_1 - N}{\kappa_p(\mu) - T| \leq 10}$
\n $\Psi_p(\sigma_1 + iT) = O(T^{n-1}) + \kappa_p \sum_{|\tau_p(x)-T| \leq 10} d_{\tau}^*(\mu) \frac{\sigma_1 - N}{(\sigma_1 - N)^2 + (T - \tau_p(\mu))^2}$.
\nWe get
\n $Re \Psi_p(\sigma_1$

with $|\vartheta_1| \leq 2$. Using (147) and (148) it follows

$$
\times \sum_{|r_p(\mu)-T| \le 10} 2\kappa_p d_p^*(\mu) \frac{\sigma_1 - N}{(\sigma_1 - N)^2 + (T - r_p(\mu))^2}
$$

\n
$$
\le 2. Using (147) and (148) it follows
$$

\n
$$
Re \Psi_p^{[t]}(s) = \left[1 + (-1)^{n-1} 2^{n-1} \vartheta_1 \frac{1}{e} \kappa_p \right]
$$

\n
$$
\times \sum_{|r_p(\mu)-T| \le 10} \frac{d_p^*(\mu) (\sigma_1 - N)}{(\sigma_1 - N)^2 + (T - r_p(\mu))^2} + O(T^{2N}).
$$
 (149)

We get

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et

$$
\sum_{|\mathbf{r}_{p}(\mu)-T|\leq 10} d_{p}^{*}(\mu) \frac{\sigma_{1}-N}{(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}} = O\left(|\Psi_{p}^{[t]}(\sigma_{1}+iT)|\right) + O(T^{2N}). \tag{150}
$$

The combination of the last equation and equation(146) proves the assertion of the theo**remi**

Proposition 59: *We suppose (108) - (117), Assumption 47 and 56. Then it holds*

$$
\arg Z_p(N + iT) = -Im \left[\sum_{\omega \in \Omega} \Lambda_p^t(\omega) \frac{1}{l(\omega)} e^{-l(\omega)(\sigma_1 + iT)} \right] + O \left(\frac{T^{n-1}}{t} \right) + O \left(\frac{1}{t} \Big| \sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-l(\omega)(\sigma_1 + iT)} \Big| \right).
$$

Proof. We have

position 59: We suppose (108) - (117), Assumption 47 and 56. Then it holds
\n
$$
\arg Z_p(N + iT) = -Im \left[\sum_{\omega \in \Omega} \Lambda_p^i(\omega) \frac{1}{l(\omega)} e^{-l(\omega)(\sigma_1 + iT)} \right]
$$
\n
$$
+ O\left(\frac{T^{n-1}}{t}\right) + O\left(\frac{1}{t} \left| \sum_{\omega \in \Omega} \Lambda_p^i(\omega) e^{-l(\omega)(\sigma_1 + iT)} \right| \right).
$$
\nof: We have
\n
$$
\arg Z_p(N + iT) = -\int_{N}^{4N} Im \Psi_p(\sigma + iT) d\sigma + \arg Z_p(4N + iT)
$$
\n
$$
= -\int_{N}^{4N} Im \Psi_p(\sigma + iT) d\sigma + O(1) \qquad (151)
$$
\n
$$
= -\int_{\sigma_1}^{4N} Im \Psi_p(\sigma + iT) d\sigma - (\sigma_1 - N) Im \Psi_p(\sigma_1 + iT)
$$
\n
$$
+ \int_{N}^{4N} Im [\Psi_p(\sigma_1 + iT) - \Psi_p(\sigma + iT)] d\sigma + O(1).
$$
\n
$$
\text{where on the right hand side.}
$$

We consider the terms on the right-hand side.

1) We first consider $J_1 = \int_{\sigma_1}^{\sigma_1} Im \Psi_p(\sigma + iT) d\sigma$. Proposition 58 gives

$$
= - \int_{\sigma_1} Im \Psi_p(\sigma + iT) d\sigma - (\sigma_1 - N) Im \Psi_p(\sigma_1 + iT)
$$

\n
$$
+ \int_{\sigma_1} Im [\Psi_p(\sigma_1 + iT) - \Psi_p(\sigma + iT)] d\sigma + O(1).
$$

\nder the terms on the right-hand side.
\nWe first consider $J_1 = \int_{\sigma_1}^{\sigma_1} Im \Psi_p(\sigma + iT) d\sigma$. Proposition 58 gives
\n
$$
J_1 = (-1)^{n-1} \int_{\sigma_1}^{\sigma_1} Im \Psi_p^{[i]}(\sigma + iT) d\sigma
$$

\n
$$
+ O(|\Psi_p^{[i]}(\sigma_1 + iT)|) \int_{\sigma_1}^{\sigma_1} e^{(N-\sigma)t} d\sigma + O(T^{2N} \int_{\sigma_1}^{\sigma_1} e^{(N-\sigma)t} d\sigma)
$$

\n
$$
= (-1)^n \kappa_p Im \sum_{\omega \in \Omega} \Lambda_p^i(\omega) \frac{1}{i(\omega)} e^{-i(\omega)(\sigma_1 + iT)} + O(1)
$$

\n
$$
+ O(|\sum_{\omega \in \Omega} \Lambda_p^i(\omega) e^{-i(\omega)(\sigma_1 + iT)}|_t^1 e^{(N-\sigma_1)t} + O(T^{2N} \frac{1}{t} e^{(N-\sigma_1)t}).
$$

\n3
\n
$$
J_1 = (-1)^n \kappa_p Im \sum_{\omega \in \Omega} \Lambda_p^i(\omega) \frac{1}{i(\omega)} e^{-i(\omega)(\sigma_1 + iT)} + O(1)
$$

\n
$$
+ O(\frac{1}{t} |\sum_{\omega \in \Omega} \Lambda_p^i(\omega) e^{-i(\omega)(\sigma_1 + iT)}|) + O(T^{2N} \frac{1}{t}).
$$

\n(152)

It follows

 \bar{z}

ų.

$$
+ O\left(\left|\sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-l(\omega)(\sigma_1 + iT)}\right|_t^1 e^{(N-\sigma_1)t}\right) + O\left(T^{2N}\frac{1}{t} e^{(N-\sigma_1)t}\right).
$$

$$
J_1 = (-1)^n \kappa_p Im \sum_{\omega \in \Omega} \Lambda_p^t(\omega) \frac{1}{l(\omega)} e^{-l(\omega)(\sigma_1 + iT)} + O(1)
$$

$$
+ O\left(\frac{1}{t}\left|\sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-l(\omega)(\sigma_1 + iT)}\right|\right) + O\left(T^{2N}\frac{1}{t}\right).
$$
 (152)

2) We consider $J_2 = -(\sigma_1 - N)Im \Psi_p(\sigma_1 + iT)$. By Proposition 58 we obtain

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\nr
$$
J_2 = -(\sigma_1 - N)Im \Psi_p(\sigma_1 + iT)
$$
. By Proposition 58 we obtain
\n
$$
|J_2| = O((\sigma_1 + iT)|\Psi_p(\sigma_1 + iT)|)
$$
\n
$$
= O(\frac{1}{t} \Big| \sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-t(\omega)(\sigma_1 + iT)} \Big|) + O(T^{2N} \frac{1}{t}).
$$
\n(153)

3) We consider $J_3 = \int_{N}^{\sigma_1} Im \left[\Psi_p(\sigma_1 + iT) - \Psi_p(\sigma + iT) \right] d\sigma$. Proposition 40 yields

IM + iT) - 'I¹ (cr + *iT)] - O(T-)* 1 1 = **+K** *()Im* fr (p)-TI<lO *—(T -* rp (j)) + *T - r() (1 - N + i(T - r()) - a - N + i(T - rp(p))) = KPE di)(* fr(u)-TI<lO (aj - *N)2 + (T -* r(&))2 (a - *N)2 + (T - rp(ji))2)* = [T - r(i)][(ai - *N) 2 - (a - N)2) - N)2 + (T -* r9(L))2][(a - *N)2 + (T -* r(/L))2] IT - rp()I(o i - *N)2 d(1)* **[(Cl -** Ir(u)-TI<lO *N)2 + (T -* rp(jz))2][(a - *N)2* **+** *(T -*

It follows

$$
= \kappa_{p} \sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} d_{p}^{*}(\mu) \frac{1}{[(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}][((\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}])}
$$
\n
$$
\leq \kappa_{p} \sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} d_{p}^{*}(\mu) \frac{|T-r_{p}(\mu)|(\sigma_{1}-N)^{2}}{[(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}][((\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}]}.
$$
\nt follows\n
$$
|J_{3}| \leq \int_{N}^{\sigma_{1}} O(T^{n-1})
$$
\n
$$
+ \kappa_{p} \sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} \frac{d_{p}^{*}(\mu)|T-r_{p}(\mu)|(\sigma_{1}-N)^{2}}{[(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}][((\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}]} d\sigma
$$
\n
$$
\leq O\Big(T^{n-1}(\sigma_{1}-N)\Big)
$$
\n
$$
+ \kappa_{p} \sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} d_{p}^{*}(\mu) \frac{(\sigma_{1}-N)^{2}}{(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}} \int_{N}^{\sigma_{j}} \frac{|T-r_{p}(\mu)|}{(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}} d\sigma
$$
\n
$$
= O\Big(T^{n-1}(\sigma_{1}-N)\Big) + O\left(\sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} d_{p}^{*}(\mu) \frac{(\sigma_{1}-N)^{2}}{(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}}\right)
$$
\n
$$
= O\Big(T^{n-1}\frac{1}{t}\Big) + O\left(\frac{1}{t} \sum_{|\mathbf{r}_{p}(\mu)-T| \leq 10} d_{p}^{*}(\mu) \frac{\sigma_{1}-N}{(\sigma_{1}-N)^{2}+(T-r_{p}(\mu))^{2}}\right).
$$
\nWe apply (151) and get\n<

We apply (151) and get

$$
J_3 = O\left(T^{n-1}\frac{1}{t}\right) + O\left(\frac{1}{t}\Big|\sum_{\omega \in \Omega} \Lambda_p^t(\omega) e^{-l(\omega)s}\Big|\right). \tag{154}
$$

Summarizing (152) - (155) we obtain the assertion **^u**

Now we can give the

 $\mathcal{L}_{\mathcal{A}}$

 \mathcal{L}_{max}

Proof of Theorem B: We first suppose (113). Proposition 59 implies

arg Z(N + iT) = (-1) A(w)--e'' '' sin (l(w)T) *wEO F(W)* +0 (11F *A(w)e_*1(1d'4iT) I) + *o(T__'_) =* 0 (+ o(_). wEfi *ary ZP (N + iT) = 0 (*

Equation (119) implies $\Lambda_p^{\mathfrak{t}}(\omega) = O(\Lambda_p(\omega))$ with an O-term not depending on $\omega \in \Omega$. Using (122), it follows

$$
arg Z_p(N+iT) = O\left(\sum_{\substack{\omega \in \Omega \\ t(\omega) \le 2(n-1)t}} \frac{1}{t} \Lambda_p(\omega) e^{-t(\omega)N}\right) + O\left(\frac{T^{n-1}}{t}\right).
$$

Proposition 5 implies

$$
\langle u_{\omega} \rangle \le 2(n-1)t
$$
\n
$$
\sum_{\substack{u \in \Omega \\ u_{(\omega)} \le 2(n-1)t}} \frac{1}{t} \Lambda_p(\omega) e^{-l(\omega)N} = O(e^{N2(n-1)t}).
$$
\n(cf. Assumption 56/(i)) and get

\n
$$
\arg Z_p(N + iT) = O\left(\frac{T^{4N/7}}{t}\right) + O\left(\frac{T^{n-1}}{t}\right).
$$

We now use $e^t = T^{1/7N}$ (cf. Assumption 56/(i)) and get

$$
arg Z_p(N+iT) = O\left(\frac{T^{4N/7}}{t}\right) + O\left(\frac{T^{n-1}}{t}\right).
$$

It follows

$$
arg Z_p(N+iT) = O\left(\frac{T^{n-1}}{\ln T}\right)
$$

and thereby

$$
|\mathcal{R}_p(T)| = O\Big(\frac{T^{n-1}}{\ln T}\Big).
$$

By right continuity this equation is valid for all *T* and Theorem B is proved.

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Received 19.09.1993