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On the Existence of Holomorphic Functions Having Prescribed Asymptotic Expansions

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Abstract. A generalization of some results of T. Carleman in [1] is developped. The practical form of it states that if the non-empty subset D of the boundary $\partial\Omega$ of a domain Ω of $\mathbb{C}$ has no accumulation point and if the connected component in $\partial\Omega$ of every $u \in D$ has more than one point, then D is regularly asymptotic for Ω, i.e. for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$ of complex numbers, there is a holomorphic function f on Ω which at every $u \in D$ has $\sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ as asymptotic expansion at u.

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1. Generalities

a) About the vector spaces. All the vector spaces we consider are over the field $\mathbb{C}$ of the complex numbers. If A is a subset of a vector space, span A denotes its linear hull. If I is a set, $\omega(I)$ denotes as usual the vector space

$\mathbb{C}^I$ endowed with the product topology. A subset $\{v_i : i \in I\}$ of the algebraic dual of a vector space E has the *interpolation property* if, for every family $\{c_i : i \in I\}$ of $\mathbb{C}$, there is $x \in E$ such that $\langle x, v_i \rangle = c_i$ for every $i \in I$.

For future reference, let us mention the following result of M. Eidelheit, a generalization of which to the case when E is a B-complete space can be found as Theorem 1 of [4].

Theorem 1.1 (Interpolation): *A subset $\{v_i : i \in I\}$ of the topological dual E' of a Fréchet space E has the interpolation property if and only if its elements are linearly independent and such that, for every equicontinuous subset B of E', the vector space* $\operatorname{span}\{v_i : i \in I\} \cap \operatorname{span} B$ *has finite dimension.*

b) About the asymptotic expansions. We begin with the following

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Definition: A holomorphic function f on a non-void domain Ω of $\mathbb{C}$ *has an asymptotic expansion at* $u \in \partial\Omega$ if the limits

$$a_0 = \lim_{z\in\Omega,\, z\to u} f(z)$$

and, for every $n \in \mathbb{N}$,

$$a_n = \lim_{z\in\Omega,\, z\to u} \frac{f(z) - \sum_{j=0}^{n-1} a_j(z-u)^j}{(z-u)^n}$$

exist and are finite. In such a case,

a) we say that the series $\sum_{n=0}^{\infty} a_n(z-u)^n$ *is the asymptotic expansion of* f *at* u

b) we write $f(z) \approx \sum_{n=0}^{\infty} a_n(z-u)^n$ at u

c) we use the notations

$$f^{[n]}(u) = a_n \qquad \text{for all } n \in \mathbb{N}_0$$

$$f^{[0]}(z,u) = f(z) \qquad \text{for all } z \in \Omega$$

$$f^{[n]}(z,u) = \frac{f^{[n-1]}(z,u) - f^{[n-1]}(u)}{z-u} \qquad \text{for all } z \in \Omega, n \in \mathbb{N}.$$

(So, in fact, we have

$$f^{[n]}(z,u) = \frac{f(z) - \sum_{j=0}^{n-1} a_j(z-u)^j}{(z-u)^n} \qquad \text{for all } z \in \Omega, n \in \mathbb{N}$$

as well as

$$\lim_{z\in\Omega,\, z\to u} f^{[n]}(z,u) = f^{[n]}(u) \qquad \text{for all } n \in \mathbb{N}_0.)$$

c) Notations. Unless explicitly stated, throughout this paper, we use the following notations:

a) Ω is a non-void domain of $\mathbb{C}$.

b) $\{K_m : m \in \mathbb{N}\}$ is a compact cover of Ω such that $(K_1)^\circ \neq \emptyset$ and $K_m \subset (K_{m+1})^\circ$ for every $m \in \mathbb{N}$.

c) $\mathcal{H}(\Omega)$ is the Fréchet-Montel space of holomorphic functions on Ω, endowed with the compact-open topology (i.e., for instance, with the countable system of norms $\{\|\cdot\|_{K_m} : m \in \mathbb{N}\}$).

d) D is a non-void subset of $\partial\Omega$.

e) $A(\Omega; D)$ is the set of the elements of $\mathcal{H}(\Omega)$ which have an asymptotic expansion at every point of D. Of course, it is a vector subspace of $\mathcal{H}(\Omega)$. We endow it canonically with the topology induced by $\mathcal{H}(\Omega)$. Let us insist on this fact: from now on, $A(\Omega; D)$ is a topological vector subspace of $\mathcal{H}(\Omega)$.

f) The notation T refers to the linear map

$$T : A(\Omega; D) \to \omega(D \times \mathbb{N}_0), \qquad f \mapsto \left(f^{[n]}(u)\right)_{(u,n)\in D\times\mathbb{N}_0}.$$

g) For every $u \in D$ and $n \in \mathbb{N}_0$, the notation $\eta_{u,n}$ refers to the linear functional

$$\eta_{u,n} : A(\Omega; D) \to \mathbb{C}, \qquad f \mapsto f^{[n]}(u).$$

As the set $\mathcal{P}(\Omega)$ of the restrictions to Ω of the polynomials is a vector subspace of $A(\Omega; D)$, the next result proves that the linear functionals $\eta_{u,n}$ for $u \in D$ and $n \in \mathbb{N}_0$ are linearly independent on any vector subspace of $A(\Omega; D)$ containing $\mathcal{P}(\Omega)$.

Lemma 1.2: *For every subset D of $\partial\Omega$, $\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\}$ is a set of linearly independent linear functionals on $\mathcal{P}(\Omega)$.*

Proof: If it is not the case, there are a finite subset D' of D, an integer $N \in \mathbb{N}_0$ and elements $c_{u,n}$ of $\mathbb{C}$ for $u \in D'$ and $n \in \{0, \dots, N\}$ such that $\langle P, \sum_{u \in D'} \sum_{n=0}^{N} c_{u,n}\eta_{u,n}\rangle$ is 0 for every $P \in \mathcal{P}(\Omega)$, although the coefficients are not all equal to 0. So we may suppose the existence of $u_0 \in D'$ and $n_0 \in \{0, \dots, N\}$ such that $c_{u_0,n_0} \neq 0$ and $c_{u_0,n} = 0$ for every $n \in \{n_0 + 1, \dots, N\}$. Then the consideration of the polynomial

$$P(z) = (z - u_0)^{n_0} \prod_{u \in D' \setminus \{u_0\}} (z - u)^{N+1}$$

leads immediately to a contradiction ∎

d) About the regularly asymptotic sets. We begin with the following

Definition. The set D is *regularly asymptotic for* Ω if, for every family

$$\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$$

of $\mathbb{C}$, there is a function $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z - u)^n$ at u for every $u \in D$. So D is regularly asymptotic for Ω if and only if the linear map T is surjective which happens if and only if the set $\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\}$ has the interpolation property on $A(\Omega; D)$.

In such a case, it is clear that no element of D can be an isolated point of $\partial\Omega$. The next result gives another restriction on such sets D.

Proposition 1.3: *If D is regularly asymptotic for Ω, then it has no accumulation point. Hence it is countable.*

Proof: If it is not the case, there is a sequence $(u_m)_{m \in \mathbb{N}}$ of distinct points of D converging to some $u_0 \in D$ with $u_m \neq u_0$ for every $m \in \mathbb{N}$. As D is regularly asymptotic for Ω, there is then $f \in A(\Omega; D)$ such that

$$f(z) \approx \sum_{n=0}^{\infty} c_{m,n}(z - u_m)^n \qquad \text{at } u_m \text{ for all } m \in \mathbb{N}_0$$

with

$$c_{m,n} = \begin{cases} 1 & \text{if } m \in \mathbb{N} \\ 0 & \text{if } m = 0 \end{cases} \qquad \text{for all } n \in \mathbb{N}_0.$$

This leads directly to the following contradiction:

$$\lim_{z \in \Omega,\, z \to u_m} f(z) = f^{[0]}(u_m) = 1 \qquad \text{for all } m \in \mathbb{N}$$

and

$$\lim_{z \in \Omega,\, z \to u_0} f(z) = f^{[0]}(u_0) = 0.$$

Thus the statement is proven ∎

Remark. However these restrictions on D are not sufficient to ensure that D is regularly asymptotic for Ω. If we set $\Omega = \mathbb{C} \setminus (\{0\} \cup \{1/m : m \in \mathbb{N}\})$, it is clear that $D = \{0\}$ has no isolated point of $\partial\Omega$ and that D has no accumulation point. However, for every $f \in A(\Omega; D)$, there is a neighbourhood U of 0 such that f is holomorphic and bounded on $U \setminus (\{0\} \cup \{1/m : m \in \mathbb{N}\})$. Hence f must have a holomorphic extension to $\mathbb{C} \setminus \{0, 1, \ldots, 1/M\}$ for some $M \in \mathbb{N}$ therefore to $\mathbb{C} \setminus \{1, \ldots, 1/M\}$.

e) Aim of this paper. In [1], T. Carleman has proved the following:

a) Every finite subset D of the boundary of a bounded, convex and open subset Ω of $\mathbb{C}$ is regularly asymptotic for Ω; for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$ of $\mathbb{C}$, there are infinitely many functions $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ at u for every $u \in D$ (cf. [1: p. 31]).

b) $\{0\}$ is regularly asymptotic for the open subset

$$\Omega = \{z \in \mathbb{C} : \|z\| < R\} \setminus \{(x, 0) : x \leq 0\}$$

of $\mathbb{C}$ (cf. [1: p. 37]).

We are going to generalize these results. We will first introduce some definitions and a basic result when D is a singleton, then consider the case when D is finite, next discuss the case when D is countable and finally state the generalizations we have obtained.

2. If D is a singleton

Let u be a point of $\partial\Omega$. For every $r \in \mathbb{N}$, we set

$$A_r(\Omega; \{u\}) = \left\{ f \in A(\Omega; \{u\}) : \sup_{z \in \Omega, |z-u| \leq 1/r} |f(z)| < \infty \right\}.$$

Clearly $A_r(\Omega; \{u\})$ is a vector subspace of $A(\Omega; \{u\})$ and

$$\bigcup_{r=1}^{\infty} A_r(\Omega; \{u\}) = A(\Omega; \{u\}).$$

Moreover an easy recursion on $j \in \mathbb{N}$ establishes that the boundedness of $f \in A(\Omega; \{u\})$ on $\{z \in \Omega : |z-u| \leq 1/r\}$ implies for every $j \in \mathbb{N}$ the boundedness of $f^{[j]}(\cdot, u)$ on the same set. Therefore, for every $n \in \mathbb{N}$,

$$p_{u,r,n} : A_r(\Omega; \{u\}) \to [0, +\infty), \quad f \mapsto \|f\|_{K_n} + \left\| \sum_{j=0}^{n} |f^{[j]}(\cdot, u)| \right\|_{\{z \in \Omega : |z-u| \leq 1/r\}}$$

is a semi-norm on $A_r(\Omega; \{u\})$; it is even a norm since Ω is a domain and since $(K_n)^\circ \neq \emptyset$ for every $n \in \mathbb{N}$. So

$$P_{u,r} = \{p_{u,r,n} : n \in \mathbb{N}\}$$

is a countable system of semi-norms on $A_r(\Omega; \{u\})$, endowing this space with a metrizable locally convex topology which is finer than the one induced by $\mathcal{H}(\Omega)$. In fact more can be said: we will prove in the next section that $(A_r(\Omega; \{u\}), P_{u,r})$ is a Fréchet space.

The sets

$$V_{u,r,n} = \left\{ f \in A_r(\Omega; \{u\}) : p_{u,r,n}(f) \leq \frac{1}{n} \right\} \qquad (n \in I\!N)$$

constitute a fundamental basis of the neighbourhoods of the origin in $(A_r(\Omega; \{u\}), P_{u,r})$. So, for every sequence $\mu = (m_n)_{n \in I\!N}$ of $I\!N$,

$$B_{u,r,\mu} = \bigcap_{n=1}^{\infty} m_n V_{u,r,n}$$

is an absolutely convex and bounded subset of that space but more can be said here too.

Proposition 2.1: *For every $u \in \partial\Omega$, $r \in I\!N$ and $\mu \in I\!N^{I\!N}$, the set $B_{u,r,\mu}$ is an absolutely convex and compact subset of $\mathcal{H}(\Omega)$.*

Proof: As $B_{u,r,\mu}$ is an absolutely convex and bounded subset of the Fréchet-Montel space $\mathcal{H}(\Omega)$, it is enough to prove that $B_{u,r,\mu}$ is sequentially closed. Let $(f_k)_{k \in I\!N}$ be a sequence of $B_{u,r,\mu}$ converging in $\mathcal{H}(\Omega)$ to f. First we establish that f has an asymptotic expansion at u with

$$f^{[j]}(u) = \lim_{k \to \infty} f_k^{[j]}(u)$$

for every $j \in I\!N_0$. For every $k \in I\!N$, as we have $f_k \in B_{u,r,\mu}$, we get at once $|f_k^{[0]}(u)| \leq m_1$. So from every subsequence of $(f_k^{[0]}(u))_{k \in I\!N}$, we can extract a subsequence converging to some $a_0 \in \mathbb{C}$. As we also have

$$\left|\tilde{f}_k^{[1]}(z,u)\right| = \left|\frac{f_k^{[0]}(z,u) - f_k^{[0]}(u)}{z-u}\right| \leq m_1$$

for every $k \in I\!N$ and $z \in \Omega$ such that $|z-u| \leq 1/r$, we get

$$\left|\frac{f(z) - a_0}{z-u}\right| \leq m_1$$

for every $z \in \Omega$ such that $|z-u| \leq 1/r$, hence $\lim_{z \in \Omega,\, z \to u} f(z) = a_0$. The conclusion is then direct by use of a recursion.

It is then an easy matter to check that f belongs to $B_{u,r,\mu}$. Hence the conclusion ∎

3. If D is finite

If $D = \{u_1, \ldots, u_J\}$ is finite, we use the following notations:

a) For every $r \in \mathbb{N}$,

$$A_r(\Omega; D) = \bigcap_{u \in D} A_r(\Omega; \{u\})$$
$$= \left\{ f \in A(\Omega; D) : \sup_{z \in \Omega,\, |z-u| \leq 1/r} |f(z)| < \infty \quad \text{for all } u \in D \right\}.$$

So $A_r(\Omega; D)$ is a vector subspace of $A(\Omega; D)$ and

$$\bigcup_{r=1}^{\infty} A_r(\Omega; D) = A(\Omega; D).$$

b) For every $r, n \in \mathbb{N}$,

$$p_{D,r,n} : A_r(\Omega; D) \to [0, +\infty), \qquad f \mapsto \sup\{p_{u,r,n}(f) : u \in D\}$$

is a norm on $A_r(\Omega; D)$ and

$$P_{D,r} = \{p_{D,r,n} : n \in \mathbb{N}\}$$

is a countable system of norms on $A_r(\Omega; D)$ endowing it with a finer locally convex topology than the one induced by $\mathcal{H}(\Omega)$. From now on, if D is finite and if $r > 0$, the notation $A_r(\Omega; D)$ will refer to the locally convex space $(A_r(\Omega; D), P_{D,r})$. So

$$V_{D,r,n} = \left\{ f \in A_r(\Omega; D) : p_{D,r,n}(f) \leq \frac{1}{n} \right\} \qquad (n \in \mathbb{N})$$

is a fundamental sequence of neighbourhoods of the origin in $A_r(\Omega; D)$.

c) For every sequence $\rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$,

$$B_{D,\rho} = \bigcap_{u \in D} B_{u,r_1,\rho_1}$$

where ρ_1 is the sequence $(r_{n+1})_{n \in \mathbb{N}}$. Of course, $B_{D,\rho}$ is an absolutely convex and compact subset of $A(\Omega; D)$; moreover it is a bounded subset of the space $A_{r_1}(\Omega; D)$.

Proposition 3.1: *If D is finite, then, for every $r \in \mathbb{N}$, $A_r(\Omega; D)$ is a Fréchet space.*

Proof: Let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence. As it clearly is a Cauchy sequence in $\mathcal{H}(\Omega)$, it converges in $\mathcal{H}(\Omega)$ to some f. Let us prove now that f belongs to $A_r(\Omega; D)$. Let us fix $u \in D$. Let us also fix $j \in \mathbb{N}_0$. On one hand, for every $k \in \mathbb{N}$, the function $f_k^{[j]}(\cdot,u)$ on Ω has a finite limit $f_k^{[j]}(u)$ at u. On the other hand, the sequence $(f_k^{[j]}(\cdot,u))_{k\in\mathbb{N}}$ is uniformly Cauchy on $\{z\in\Omega : |z-u|\le 1/r\}$. Therefore the two limits

$$\lim_{k\to\infty}\lim_{z\in\Omega,\, z\to u} f_k^{[j]}(z,u) = \lim_{k\to\infty} f_k^{[j]}(u)$$

and

$$\lim_{z\in\Omega,\, z\to u}\lim_{k\to\infty} f_k^{[j]}(z,u)$$

exist, are finite and are equal. A direct recursion on $j\in\mathbb{N}_0$ establishes then that f has an asymptotic expansion at u.

It is finally a standard matter to prove that the sequence $(f_k)_{k\in\mathbb{N}}$ converges in $A_r(\Omega;D)$ to f ∎

The previous result is also a consequence of the following considerations which will prove to be very fruitful (we refer the reader to [3] for the definition and the properties of the quasi-LB spaces and representations).

Proposition 3.2: *If D is finite, then the family $\{B_{D,\rho} : \rho\in\mathbb{N}^{\mathbb{N}}\}$ is a quasi-*LB *representation of the space $A(\Omega;D)$, made of absolutely convex and compact sets.*

Proof: By Proposition 2.1, for every $\rho\in\mathbb{N}^{\mathbb{N}}$, $B_{D,\rho}$ is an absolutely convex and compact subset of $\mathcal{H}(\Omega)$ hence of $A(\Omega;D)$ since it is a subset of $A(\Omega;D)$. It is also clear that, for every $\rho,\sigma\in\mathbb{N}^{\mathbb{N}}$ such that $\rho\le\sigma$ (i.e. $r_n\le s_n$ for every $n\in\mathbb{N}$ if $\sigma=(s_n)_{n\in\mathbb{N}}$), we have $B_{D,\rho}\subset B_{D,\sigma}$.

To conclude, it is then enough to check that $\cup_{\rho\in\mathbb{N}^{\mathbb{N}}} B_{D,\rho}$ is equal to $A(\Omega;D)$. This is a direct matter: every $f\in A(\Omega;D)$ has an asymptotic expansion at every element of D so it is a bounded function on $\cup_{u\in D}\{z\in\Omega : |z-u|\le 1/r\}$ for some $r\in\mathbb{N}$. Moreover, for every $n\in\mathbb{N}_0$, there is $s_n\in\mathbb{N}$ such that $f\in s_nV_{D,r,n}$. Therefore f belongs to $B_{D,\rho}$ with $r_1=r$ and $r_{n+1}=s_n$ for every $n\in\mathbb{N}$ ∎

Now, as in [3], for every $\rho=(r_n)_{n\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$, we may introduce successively:

a) For every $n\in\mathbb{N}$, the set

$$B_{D,r_1,\ldots,r_n} = \bigcup\{B_{D,\sigma} : \sigma=(s_n)_{n\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}; s_1=r_1,\ldots,s_n=r_n\}.$$

In fact, it is a direct matter to check that in our case

$$B_{D,r_1} = A_{r_1}(\Omega;D)$$

and, for every $n=2,3,\ldots$,

$$B_{D,r_1,\ldots,r_n} = (r_2V_{D,r_1,1})\cap\ldots\cap(r_nV_{D,r_1,n-1}).$$

b) For every $n\in\mathbb{N}$, the linear hull $F_{D,r_1,\ldots,r_n}$ of $B_{D,r_1,\ldots,r_n}$. Of course, in our case, we get at once $F_{D,r_1,\ldots,r_n}=A_{r_1}(\Omega;D)$ for every $n\in\mathbb{N}$.

c) The vector space $F_{D,\rho}=\cap_{n=1}^{\infty}F_{D,r_1,\ldots,r_n}$, i.e. $F_{D,\rho}=A_{r_1}(\Omega;D)$ in our case.

At this point, [3] provides another way to prove that, for every $r\in\mathbb{N}$, $A_r(\Omega;D)$ is a Fréchet space.

Definition: If D is finite, we have

$$A(\Omega; D) = \bigcup_{r=1}^{\infty} A_r(\Omega; D)$$

and, for every $r, s \in \mathbb{N}$ such that $r < s$, the canonical injection from $A_r(\Omega; D)$ into $A_s(\Omega; D)$ is continuous. Therefore we may endow $A(\Omega; D)$ with an (LF)-topology τ by considering the inductive limit of the sequence $(A_r(\Omega; D))_{r\in\mathbb{N}}$ of Fréchet spaces. Of course, τ is finer than the topology of $A(\Omega; D)$.

Notations: If $D = \{u_1, \ldots, u_J\}$ is a finite subset of $\partial\Omega$, then, in this section and unless specifically stated, we use the following notations:

a) For every $j \in \{1, \ldots, J\}$, the notation T_j refers to the linear map

$$T_j : A(\Omega; D) \to \omega(\mathbb{N}_0), \qquad f \mapsto \left(f^{[n]}(u_j)\right)_{n\in\mathbb{N}_0}$$

which is continuous if we endow $A(\Omega; D)$ with the topology τ and L_j is the vector subspace $\operatorname{span}\{\eta_{u_j,n} : n \in \mathbb{N}_0\}$ of the topological dual $(A(\Omega; D), \tau)'$.

b) $L = \operatorname{span}(\cup_{j=1}^{J} L_j)$.

Now we are looking for necessary and sufficient conditions under which D is regularly asymptotic for Ω.

Definition: If r belongs to $\mathbb{N}$, the finite subset D of $\partial\Omega$ is *r-regularly asymptotic for Ω* if the restriction of the map T to $A_r(\Omega; D)$ is also surjective onto $\omega(D \times \mathbb{N}_0)$.

Proposition 3.3: *The finite subset D of $\partial\Omega$ is regularly asymptotic for Ω if and only if there is $r \in \mathbb{N}$ such that D is r-regularly asymptotic for Ω.*

Proof: The proof of Proposition 9 in [4] applies also to this case ∎

Corollary 3.4: *If D is finite and is regularly asymptotic for Ω, then the kernel of T has $2^{\aleph_0}$ as algebraic dimension.*

Proof: By the previous proposition, there is $r \in \mathbb{N}$ such that the restriction S of T to $A_r(\Omega; D)$ is a continuous and surjective linear map. To conclude, we just have to prove that the Fréchet subspace $\ker S$ of $A_r(\Omega; D)$ is not finite-dimensional. If it were finite-dimensional, it would have a topological complement M and the restriction of T to M would appear as an isomorphism in between M and
$\omega(D \times \mathbb{N})$ although M has a continuous norm ∎

The next result comes from [5]. We repeat it here, with proof, for the sake of completeness since the reference [5] is not readily accessible.

Theorem 3.5: *The finite subset D of $\partial\Omega$ is regularly asymptotic for Ω if and only if, for every $u \in D$ and $(c_n)_{n\in\mathbb{N}_0} \in \omega(\mathbb{N}_0)$, there is $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_n(z-u)^n$ at u.*

Proof: The condition is trivially necessary. The condition is also sufficient.Indeed, if D is a singleton, the result is trivial. So let us consider the case $D = \{u_1, \ldots, u_J\}$ with an integer $J \geq 2$.

For every $j \in \{1,\dots,J\}$, T_j is clearly a surjective, linear and continuous map. So the Fréchet space $\omega(\mathbb{N}_0)$ is equal to $\cup_{r=1}^{\infty} T_j A_r(\Omega; D)$ hence there is $r_j \in \mathbb{N}$ such that $T_j A_{r_j}(\Omega; D)$ is a second category vector subspace of $\omega(\mathbb{N}_0)$. As a surjective, linear and continuous map from the Fréchet space $A_{r_j}(\Omega; D)$ onto its image, the restriction of T_j is a topological homomorphism. Therefore $T_j A_{r_j}(\Omega; D)$ is a Fréchet space hence is equal to $\omega(\mathbb{N}_0)$.

To conclude, we are going to prove that, for $r = \sup\{r_1,\dots,r_J\}$, the set

$$\{\eta_{u,n} : u \in D,\ n \in \mathbb{N}_0\}$$

has the interpolation property on $A_r(\Omega; D)$. By Theorem 1.1, as these

continuous linear functionals on $A_r(\Omega; D)$ are linearly independent, we just need to prove that, for every $s \in \mathbb{N}$, the dimension of the vector space $L \cap \operatorname{span} V^{\circ}_{D,r,s}$ is finite.

Let us fix $s \in \mathbb{N}$ and, in order to simplify the notations, let us set $U = V_{D,r,s}$. For every $j \in \{1,\dots,J\}$, the restriction of T_j to $A_r(\Omega; D)$ is surjective hence the dimension of $L_j \cap \operatorname{span} U^{\circ}$ is finite: there is $N(j) \in \mathbb{N}$ such that

$$\left(\beta_0,\dots,\beta_N \in \mathbb{C} : \beta_N \neq 0; \sum_{n=0}^{N} \beta_n \eta_{u_j,n} \in \operatorname{span} U^{\circ}\right) \Longrightarrow N \leq N(j).$$

To conclude, it is then sufficient to prove that

$$\zeta = \sum_{j=1}^{J} \sum_{n=0}^{N} \alpha_{j,n} \eta_{u_j,n} \in \operatorname{span} U^{\circ}$$

with $N > \sup\{s, N(1),\dots,N(J)\}$ implies $\alpha_{j,N} = 0$ for every $j = 1,\dots,J$.

This we do by contradiction: let us suppose the existence of such a functional ζ with $N > \sup\{s, N(1),\dots,N(J)\}$ and $\alpha_{k,N} \neq 0$ for some $k \in \{1,\dots,J\}$, belonging to qU° for some $q \in \mathbb{N}$.

We need some preparation. Let us denote by $P(z)$ the polynomial

$$\prod_{1 \leq j \leq J,\, j \neq k} (z - u_j)^{N+1}.$$

It can also be written as

$$P(z) = c_0 + c_1(z - u_k) + \ldots + c_{(N+1)(J-1)}(z - u_k)^{(N+1)(J-1)}$$

with $c_0 \neq 0$. Then we choose a closed disk B in $\mathbb{C}$, centered at the origin and containing

$$K_s \bigcup \left(\bigcup_{j=1}^{J} \left\{ z \in \Omega : |z - u_j| \leq \frac{1}{r} \right\} \right).$$

Finally we set

$$C_1 = \sup\left\{ \left\| \frac{P(\cdot)}{(\cdot - u_j)^h} \right\|_B : h \in \{0,\dots,s\},\ j \in \{1,\dots,J\} \setminus \{k\} \right\}$$

$$C_2 = \sup\left\{ \left\| c_h + \cdots + c_{(N+1)(J-1)}(z-u_k)^{(N+1)(J-1)-h} \right\|_B : h = 0,\ldots,s \right\}$$

and introduce the polynomial

$$Q(z) = \frac{P(z)}{2(C_1+C_2)(s+1)^2}.$$

For every $g \in U$, the function Qg clearly belongs to $A_r(\Omega; D)$. In fact, it belongs to U:

a) For every $j \in \{1,\ldots,J\} \setminus \{k\}$ and $h \in \{0,\ldots,s\}$, we have

$$(Qg)^{[h]}(z,u_j) = \frac{Q(z)}{(z-u_j)^h}\, g(z)$$

hence

$$\left|(Qg)^{[h]}(z,u_j)\right| \le \frac{C_1}{2(C_1+C_2)(s+1)^2}\cdot\frac{1}{s} \le \frac{1}{2s(s+1)^2}$$

for every $z \in \Omega$ such that $|z-u_j| \le 1/r$.

b) For $h \in \{0,\ldots,s\}$, we also have

$$(Qg)^{[h]}(z,u_k) = \frac{\sum_{l=0}^{h-1} c_l g^{[h-l]}(z,u_k) + \left(c_h + \cdots + c_{(N+1)(J-1)}(z-u_k)^{(N+1)(J-1)-h}\right) g^{[0]}(z,u_k)}{2(C_1+C_2)(s+1)^2}$$

hence

$$\left|(Qg)^{[h]}(z,u_k)\right| \le \frac{(s+1)C_2}{2(C_1+C_2)(s+1)^2}\cdot\frac{1}{s} \le \frac{1}{2s(s+1)}$$

for every $z \in \Omega$ such that $|z-u_k| \le 1/r$.

c) For every $z \in K_s$, it is clear that

$$|(Qg)(z)| \le \frac{C_1}{2(C_1+C_2)(s+1)^2}\cdot\frac{1}{s} \le \frac{1}{2s(s+1)^2}.$$

At this stage, we consider the continuous linear functional

$$\eta = \sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k)\eta_{u_k,j}.$$

The coefficient of $\eta_{u_k,N}$ is $\alpha_{k,N}Q^{[0]}(u_k)$ and differs from 0. Moreover as $N > N(k)$, η does not belong to $\operatorname{span} U^\circ$. Therefore there is $f \in U$ such that $|\langle f,\eta\rangle| > q$. As Qf belongs to U, we finally arrive at the following contradiction:

We have

$$|\langle Qf,\zeta\rangle| \le q$$

because $\zeta \in qU^\circ$, as well as

$$\begin{aligned}
|\langle Qf, \zeta\rangle| &= \left|\sum_{j=1}^{J}\sum_{n=0}^{N} \alpha_{j,n}\langle Qf, \eta_{u_j,n}\rangle\right| \\
&= \left|\sum_{n=0}^{N} \alpha_{k,n}\langle Qf, \eta_{u_k,n}\rangle\right| \\
&= \left|\sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k) f^{[j]}(u_k)\right| \\
&= \left|\left\langle f, \sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k)\eta_{u_k,j}\right\rangle\right| \\
&= |\langle f, \eta\rangle| \\
&> q.
\end{aligned}$$

Thus our statement is proved ∎

At this stage, we can extend Theorem 4 of [4] in the following way.

Proposition 3.6: *The finite subset $D = \{u_1, \ldots, u_J\}$ of $\partial\Omega$ is regularly asymptotic for Ω if and only if the following condition (*) is satisfied:*

(*) *There is $r \in \mathbb{N}$ such that, for every compact subset K of Ω and $j_0 \in \{1, \ldots, J\}$, there is an integer $p \in \mathbb{N}$ such that, for every $h > 0$, there is $f \in A_r(\Omega; D)$ verifying*

$$|f(z)| \le 1 \qquad \text{for all } z \in K \bigcup \left(\bigcup_{j=1}^{J} \left\{ u \in \Omega : |u - u_j| \le \frac{1}{r} \right\} \right)$$

and

$$|f^{[p]}(u_{j_0})| > h.$$

Proof: The condition is necessary. Indeed, by Proposition 3.3, there is an integer $r \in \mathbb{N}$ such that

$$S : A_r(\Omega; D) \to \omega(D \times \mathbb{N}_0), \qquad f \mapsto \left(f^{[n]}(u)\right)_{(u,n)\in D\times\mathbb{N}_0}$$

is a continuous, surjective and linear map. Let us fix a compact subset K of Ω and an integer j_0 in $\{1, \ldots, J\}$. There is then an integer $s \in \mathbb{N}$ such that $K \subset K_s$. As, for $n \in \mathbb{N}_0$, the continuous linear functionals $\eta_{u_{j_0},n}$ on $A_r(\Omega; D)$ are linearly independent, Theorem 1.1 tells us that the dimension of the vector space

$$\operatorname{span}\left\{\eta_{u_{j_0},n} : n \in \mathbb{N}_0\right\} \cap \operatorname{span} V_{D,r,s}^\circ$$

is finite. Therefore there is an integer $p \in \mathbb{N}$ such that $\eta_{u_{j_0},p}$ does not belong to $\operatorname{span} V_{D,r,s}^\circ$. Hence, for every $h > 0$, there is $f \in V_{D,r,s}$ such that

$$|f^{[p]}(u_{j_0})| = |\langle f, \eta_{u_{j_0},p}\rangle| > h.$$

Hence the conclusion.

The condition is sufficient. Indeed, by Theorem 3.5, it is enough to prove that, for every $j \in \{1, \dots, J\}$, the map

$$S_j : A_r(\Omega; D) \to \omega(\mathbb{N}_0), \qquad f \mapsto \left(f^{[n]}(u_j)\right)_{n \in \mathbb{N}_0}$$

is surjective. Let us fix $j \in \{1, \dots, J\}$. As, for $n \in \mathbb{N}_0$, the continuous linear functionals $\eta_n = \eta_{u_j,n}$ are linearly independent, by Theorem 1.1, we just need to prove that, for every $s \in \mathbb{N}$, the vector space

$$\operatorname{span}\{\eta_n : n \in \mathbb{N}_0\} \cap \operatorname{span} V^{\circ}_{D,r,s}$$

has finite dimension. Let us fix $s \in \mathbb{N}$ and denote by p_j the least integer satisfying condition (*) for $K = K_s$ and $j_0 = j$. There is then $k > 0$ such that, for every $g \in A_r(\Omega; D)$ verifying $|g(z)| \leq 1$ for every element z of the set

$$\mathcal{K} = K_s \bigcup \left(\bigcup_{u \in D} \left\{ t \in \Omega : |t - u| \leq \frac{1}{r} \right\} \right),$$

one has

$$|g^{[n]}(u_j)| \leq k \qquad \text{for all } n \in \{0, \dots, p_j - 1\}.$$

At this point, to conclude, it is sufficient to prove that a functional of the type

$$\delta = \sum_{n=0}^{N} \alpha_n \eta_n \qquad \text{with } N > p_j + s,\ \alpha_n \in \mathbb{C} \text{ and } \alpha_N \neq 0$$

never belongs to $\operatorname{span} V^{\circ}_{D,r,s}$. This we do by contradiction. Let us suppose that such a functional δ belongs to $hV^{\circ}_{D,r,s}$ for some $h > 0$. We need some preparation. We choose an integer d greater than the diameter of $\mathcal{K}$ and set successively

$$\alpha = \sup\{|\alpha_0|, \dots, |\alpha_N|\}$$
$$P(z) = (z - u_j)^{N - p_j} \prod_{1 \leq k \leq J,\, k \neq j} (z - u_k)^N$$
$$L = \sup\left\{|P^{[n]}(u_j)| : n = 0, \dots, N\right\}.$$

Now we choose $f \in A_r(\Omega; D)$ such that $|f(z)| \leq 1$ for every $z \in \mathcal{K}$ and

$$|f^{[p_j]}(u_j)| > \frac{s(s+2)hd^{NJ} + 2L\alpha N^2 k}{|\alpha_N P^{[N-p_j]}(u_j)|}.$$

Finally we set $g = Pf$. Of course g belongs to $A_r(\Omega; D)$; more precisely from

$$\begin{aligned} p_{D,r,s}(g) &= \sup_{u \in D} \left(\|Pf\|_{K_s} + \sup\left\{ \sum_{l=0}^{s} \left| \frac{P(z)f(z)}{(z-u)^l} \right| : z \in \Omega, |z - u| \leq \frac{1}{r} \right\} \right) \\ &\leq d^{NJ} + (s+1)d^{NJ} \\ &= (s+2)d^{NJ} \end{aligned}$$

we get that g belongs to $s(s+2)d^{NJ}V_{D,r,s}$ and hence g satisfies

$$|\langle g,\delta\rangle| \le hs(s+2)d^{NJ}.$$

But we also have

$$|f^{[N-t]}(u_j)| \le k \qquad \text{for every } t \in \{N-p_j+1,\dots,N\}$$

and this leads to the following contradiction:

$$\begin{aligned}
|\langle g,\delta\rangle| &= \left|\sum_{n=0}^{N} \alpha_n g^{[n]}(u_j)\right| \\
&= \left|\sum_{n=0}^{N} \alpha_n (Pf)^{[n]}(u_j)\right| \\
&= \left|\sum_{n=0}^{N} \alpha_n \sum_{t=0}^{n} P^{[t]}(u_j) f^{[n-t]}(u_j)\right| \\
&= \left|\sum_{n=N-p_j}^{N} \alpha_n \sum_{t=N-p_j}^{n} P^{[t]}(u_j) f^{[n-t]}(u_j)\right| \\
&\ge |\alpha_N| \left|P^{[N-p_j]}(u_j) f^{[p_j]}(u_j)\right| - |\alpha_N| \sum_{t=N-p_j+1}^{N} \left|P^{[t]}(u_j)\right| \left|f^{[N-t]}(u_j)\right| \\
&\quad - \sum_{n=N-p_j}^{N-1} |\alpha_n| \sum_{t=N-p_j}^{n} \left|P^{[t]}(u_j)\right| \left|f^{[n-t]}(u_j)\right| \\
&\ge s(s+2)hd^{NJ} + 2L\alpha N^2 k - \alpha N L k - N^2 \alpha L k \\
&> s(s+2)hd^{NJ}.
\end{aligned}$$

Thus the assertion is proved ∎

Now by use of the previous result and of ideas of [4], we are going to establish a first generalization of Carleman's result.

Notations: For $u \in \mathbb{C}$ and $A \subset \mathbb{C}$, let us set

$$\delta_1(u,A) = \inf\{|u-z| : z \in A\} \quad \text{and} \quad \delta_2(u,A) = \sup\{|u-z| : z \in A\}.$$

Definition: The boundary $\partial\Omega$ is *quasi-connected at* $u \in \partial\Omega$ if, for every $\varepsilon, \delta > 0$ such that $0 < \varepsilon < \delta$, there is a connected subset A of $\partial\Omega$ such that

$$\delta_2(u,A) < \delta \qquad \text{and} \qquad \delta_1(u,A) < \varepsilon\delta_2(u,A).$$

Let us mention that in Section 5, we will prove that $\partial\Omega$ *is quasi-connected at* $u \in \partial\Omega$ *if the connected component of* u *in* $\partial\Omega$ *contains more than one point. So if* Ω *is simply connected,* $\partial\Omega$ *is quasi-connected at every point of* $\partial\Omega$.

Theorem 3.7: *If $\partial\Omega$ is quasi-connected at every point of the finite subset D of $\partial\Omega$, then D is regularly asymptotic for Ω.*

Proof: As $D = \{u_1, \ldots, u_J\}$ is finite, there is $r \in \mathbb{N}$ such that the disks

$$\left\{z \in \mathbb{C} : |z - u_j| \leq \frac{1}{r}\right\}$$

are pairwise disjoint for $j \in \{1, \ldots, J\}$. Let us fix $j \in \{1, \ldots, J\}$ and a compact subset K of Ω. We are going to prove that Proposition 3.6 applies with $p = 1$. Of course there is $C > 0$ such that $|z - \alpha| \cdot |z - \beta| < C^2$ for every

$$z \in \mathcal{K} = K \bigcup \left(\bigcup_{u \in D} \left\{t \in \mathbb{C} : |t - u| \leq \frac{1}{r}\right\}\right)$$

and every $\alpha, \beta \in \mathbb{C}$ such that $|\alpha - u_j| \leq 1/r$ and $|\beta - u_j| \leq 1/r$. Given $h > 0$, by hypothesis, there is a connected subset A of $\partial\Omega$ such that

$$\delta_2(u_j, A) < \frac{1}{r} \qquad \text{and} \qquad 0 < \delta_1(u_j, A) < \inf\left\{\frac{1}{2r}, \frac{1}{16C^2h^2}\right\} \delta_2(u_j, A).$$

Therefore there are points α and β in A such that

$$|u_j - \alpha| < \inf\left\{\frac{1}{2r}, \frac{1}{16C^2h^2}\right\} |u_j - \beta|.$$

Then there is a determination g of $\sqrt{(\cdot - \alpha)(\cdot - \beta)}$ which is holomorphic on $\Omega \cup V$ for some open neighbourhood V of D. So $f = g/C$

a) belongs to $A_r(\Omega; D)$

b) verifies $|f(z)| < 1$ for every $z \in \mathcal{K}$

c) verifies

$$\begin{aligned}
|f^{[1]}(u_j)| &= \frac{1}{C} \lim_{z \in \Omega,\, z \to u_j} \left| \frac{g(z) - g(u_j)}{z - u_j} \right| \\
&= \frac{|2u_j - (\alpha + \beta)|}{2C|u_j - \alpha|^{1/2}|u_j - \beta|^{1/2}} \\
&\geq \frac{1 - \left|\frac{u_j - \alpha}{u_j - \beta}\right|}{2C\left|\frac{|u_j - \alpha|}{u_j - \beta}\right|^{1/2}} \\
&> \frac{1 - 1/2}{2C \cdot \frac{1}{4Ch}} \\
&= h,
\end{aligned}$$

hence the conclusion by Proposition 3.6 ∎

Let us also mention the following result.

Proposition 3.8: *Let D be a finite subset of $\partial\Omega$. Let us suppose moreover the existence of a subset $\{d_{u,n} : u \in D, n \in \mathbb{N}_0\}$ of $(0,+\infty)$ such that, for every subset $\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$ of $\mathbb{C}$ such that $|c_{u,n}| \leq d_{u,n}$ for every $u \in D$ and $n \in \mathbb{N}_0$, there is $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ at every $u \in D$.*

Then there is $\rho \in \mathbb{N}^{\mathbb{N}}$ such that, for every subset $\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$ as above, there is $g \in B_{D,r_1,\rho_1}$ such that $g(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ at every $u \in D$.

Proof: The set

$$C = \left\{(c_{u,n})_{(u,n)\in D\times\mathbb{N}_0} : |c_{u,n}| \leq d_{u,n} \quad \text{for all} \quad (u,n) \in D \times \mathbb{N}_0\right\}$$

is clearly an absolutely convex and compact subset of $\omega(D \times \mathbb{N}_0)$. By the Proposition 12/(b) of [3], there are then $\rho \in \mathbb{N}^{\mathbb{N}}$ and a subset M of B_{D,r_1,ρ_1} such that $TM = C$. Hence the conclusion ∎

4. If D is infinite

Let us recall that every subset D of $\partial\Omega$ which is regularly asymptotic for Ω is countable.

Notations: Let $D = \{u_j : j \in \mathbb{N}\}$. Then:

1) For every $\rho = (r_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, we set

$$\begin{aligned} A_\rho(\Omega; D) &= \bigcap_{j=1}^{\infty} A_{r_j}(\Omega; \{u_j\}) \\ &= \left\{ f \in A(\Omega; D) : \sup_{z\in\Omega,\, |z-u_j|\leq 1/r_j} |f(z)| < \infty \text{ for all } j \in \mathbb{N} \right\}. \end{aligned}$$

So $A_\rho(\Omega; D)$ is a vector subspace of $A(\Omega; D)$ and

$$\bigcup_{\rho\in\mathbb{N}^{\mathbb{N}}} A_\rho(\Omega; D) = A(\Omega; D).$$

2) For every $\rho \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$p_{D,\rho,n} : A_\rho(\Omega; D) \to [0,+\infty), \qquad f \mapsto \sup\{p_{u_j,r_j,n} : j \in \{1,\ldots,n\}\}$$

is a norm on $A_\rho(\Omega; D)$ and

$$P_{D,\rho} = \{p_{D,\rho,n} : n \in \mathbb{N}\}$$

a countable system of norms on $A_\rho(\Omega; D)$ endowing it with a finer locally convex topology than the one induced by $\mathcal{H}(\Omega)$. From now on, if $D = \{u_j : j \in \mathbb{N}\}$ and if $\rho \in \mathbb{N}^{\mathbb{N}}$, the notation $A_\rho(\Omega; D)$ will refer to the locally convex space $(A_\rho(\Omega; D), P_{D,\rho})$. So

$$V_{D,\rho,n} = \left\{ f \in A_\rho(\Omega; D) : p_{D,\rho,n}(f) \leq \frac{1}{n} \right\} \qquad (n \in \mathbb{N})$$

is a fundamental sequence of neighbourhoods of the origin in $(A_\rho(\Omega; D), P_{D,\rho})$.

3) We fix once for all an infinite partition

$$\{\{n_{j,k} : k \in \mathbb{N}\} : j \in \mathbb{N}\}$$

of $\{2n : n \in \mathbb{N}\}$. Then, for every $\rho \in \mathbb{N}^{\mathbb{N}}$,

$$B_{D,\rho} = \bigcap_{j=1}^{\infty} B_{u_j, r_{2j-1}, \rho(j)}$$

where $\rho_{(j)}$ is the sequence $(r_{n_{j,k}})_{k \in \mathbb{N}}$. Of course, $B_{D,\rho}$ is an absolutely convex and compact subset of $A(\Omega; D)$; moreover it is a bounded subset of the space $(A_{\rho'}(\Omega; D), P_{D,\rho'})$ where ρ' is the sequence $(r_{2n-1})_{n \in \mathbb{N}}$.

Let us remark that these notations depend heavily on the enumeration of the points of D but any enumeration will do.

Proposition 4.1: *If $D = \{u_j : j \in \mathbb{N}\}$, then, for every $\rho \in \mathbb{N}^{\mathbb{N}}$, $A_\rho(\Omega; D)$ is a Fréchet space.*

Proof: One can go on with a direct proof as in Proposition 3.1 or use the technique of the (LB)-spaces as we do hereafter ∎

Proposition 4.2: *If $D = \{u_j : j \in \mathbb{N}\}$, the family $\{B_{D,\rho} : \rho \in \mathbb{N}^{\mathbb{N}}\}$ is a quasi-*LB *representation of the space $A(\Omega; D)$ made of absolutely convex and compact sets.*

Now as in [3], once again, for every $\rho \in \mathbb{N}^{\mathbb{N}}$, we may successively introduce the sets

$$B_{D,r_1,\ldots,r_n} = \bigcup \{B_{D,\sigma} : \sigma \in \mathbb{N}^{\mathbb{N}}, s_1 = r_1, \ldots, s_n = r_n\}$$

$$F_{D,r_1,\ldots,r_n} = \operatorname{span} B_{D,r_1,\ldots,r_n}$$

for every $n \in \mathbb{N}$ and

$$F_{D,\rho} = \bigcap_{n=1}^{\infty} F_{D,r_1,\ldots,r_n}.$$

One can describe directly these sets in our case. In particular, one gets $F_{D,\rho} = A_{\rho'}(\Omega; D)$. This provides another way to establish that, for every $\rho \in \mathbb{N}^{\mathbb{N}}$, $A_\rho(\Omega; D)$ is a Fréchet space.

Definition: If $D = \{u_j : j \in \mathbb{N}\}$, we have

$$A(\Omega; D) = \bigcup_{\rho \in \mathbb{N}^{\mathbb{N}}} A_\rho(\Omega; D)$$

and, for every $\rho, \sigma \in \mathbb{N}^{\mathbb{N}}$ such that $\rho \le \sigma$, the canonical injection from $A_\rho(\Omega; D)$ into $A_\sigma(\Omega; D)$ is continuous. Therefore we may endow $A(\Omega; D)$ with the locally convex topology τ of the inductive limit of these Fréchet spaces. Of course τ is finer than the topology of $A(\Omega; D)$ but we must insist on the fact that $(A(\Omega; D), \tau)$ is *not* an (LF)-space.

However some results similar to those obtained in the case when D is finite, can be established.

Definition: If $\rho = (r_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, the subset $D = \{u_j : j \in \mathbb{N}\}$ of $\partial\Omega$ is *ρ-regularly asymptotic for Ω* if the restriction of the map T to $A_\rho(\Omega; D)$ is surjective.

Theorem 4.3: *Let $D = \{u_j : j \in \mathbb{N}\}$ be a subset of $\partial\Omega$ having no accumulation point. The following conditions are equivalent:*

(a) *D is regularly asymptotic for Ω.*

(b) *For every $j \in \mathbb{N}$ and every sequence $(c_n)_{n\in\mathbb{N}}$ of complex numbers, there is $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_n(z-u_j)^n$ at u_j.*

(c) *There is $\rho \in \mathbb{N}^{\mathbb{N}}$ such that D is ρ-regularly asymptotic for Ω.*

Proof: (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are trivial.

(b) $\Rightarrow$ (c): Let us first introduce a sequence $\rho \in \mathbb{N}^{\mathbb{N}}$. As D has no accumulation point, there is a sequence $\tau = (t_j)_{j\in\mathbb{N}}$ such that the disks $\{z \in \mathbb{C} : |z - u_j| \le 1/t_j\}$ are pairwise disjoint. We set $r_1 = t_1$ and obtain the other elements as follows, by induction. If $r_1, \ldots, r_m$ are determined, we denote by d_m the distance of u_{m+1} to

$$H_m = K_m \bigcup \left(\bigcup_{j=1}^{m} \left\{ z \in \mathbb{C} : |z - u_j| \le \frac{1}{t_j} \right\} \right).$$

As $\{u_{m+1}\}$ is a regularly asymptotic set for Ω, u_{m+1} is not an isolated point in $\partial\Omega$; therefore there is $v_m \in \partial\Omega$ such that

$$0 < |v_m - u_{m+1}| < \inf\left\{ \frac{d_m}{4}, \frac{1}{t_{m+1}} \right\} \qquad \text{and} \qquad \mathrm{d}(v_m, H_m) \ge \frac{d_m}{2}.$$

Then we choose r_{m+1} in $\mathbb{N}$ such that $r_{m+1} \ge t_{m+1}$ and $1/r_{m+1} < |v_m - u_{m+1}|$.

As $\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\}$ is a subset of the dual space $(A_\rho(\Omega; D), P_{D,\rho})'$, the elements of which are linearly independent, by Theorem 1.1, all we need to prove is that, for every $s \in \mathbb{N}$, the vector space

$$\mathrm{span}\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\} \cap \mathrm{span}\, V^\circ_{D,\rho,s}$$

has finite dimension. Let us fix $s \in \mathbb{N}$ and, to simplify the notations, let us set $q = p_{D,\rho,s}$ and $U = V_{D,\rho,s}$. For every $j \in \mathbb{N}$,

we know already that

$$\mathrm{span}\{\eta_{u_j,n} : n \in \mathbb{N}_0\} \cap \mathrm{span}\, U^\circ$$

has finite dimension so there is an integer $N(j) \in \mathbb{N}$ such that

$$\left(\alpha_1, \ldots, \alpha_N \in \mathbb{C} : \alpha_N \ne 0;\ \sum_{n=0}^{N} \alpha_k \eta_{u_j,n} \in \mathrm{span}\, U^\circ \right) \Longrightarrow N \le N(j).$$

To conclude, it is then sufficient to prove that, if

$$\zeta = \sum_{j=1}^{m+1} \sum_{n=0}^{N} \alpha_{j,n} \eta_{u_j,n} \qquad \text{with } m \ge s$$

belongs to span U°, then a) and b) hereunder hold.

a) We have $\alpha_{m+1,n} = 0$ for every $n = 0, \ldots, N$. Indeed, if it is not the case, let $q \in \mathbb{N}$ be such that $\zeta \in qU^\circ$ and let l be the greatest integer such that $\alpha_{m+1,l} \neq 0$. We then introduce the polynomial

$$P(z) = (z-u_1)^{m+N} \cdots (z-u_m)^{m+N}(z-u_{m+1})^l$$

and an integer d larger than the diameter of H_m, choose an integer t such that

$$|u_{m+1}-u_1|^{m+N} \cdots |u_{m+1}-u_m|^{m+N} 2^t > qs(s+2)d^{(m+N)(m+1)}$$

and finally set

$$g(z) = P(z) \cdot \left(\frac{d_m}{2(z-v_m)}\right)^t \qquad \text{for all } z \in \Omega.$$

It is clear that g belongs to $A_\rho(\Omega; D)$. Moreover from

$$\left|\frac{d_m}{2(z-v_m)}\right|^t \leq 1 \qquad \text{for all } z \in H_m$$

we easily get

$$\begin{aligned} q(g) &\leq \sup_{1 \leq j \leq s} p_{u_j, r_j, s}(g) \\ &\leq d^{(m+N)(m+1)} + (s+1)d^{(m+N)(m+1)} \\ &= (s+2)d^{(m+N)(m+1)} \end{aligned}$$

hence g belongs to $s(s+2)d^{(m+N)(m+1)}U$ therefore

$$|\langle g, \zeta\rangle| \leq qs(s+2)d^{(m+N)(m+1)}.$$

But we also have

$$\left|\frac{d_m}{2(u_{m+1}-v_m)}\right|^t > 2^t$$

hence

$$\begin{aligned} |\langle g, \zeta\rangle| &= \left|\sum_{j=1}^{m+1}\sum_{n=0}^{N} \alpha_{j,n} g^{[n]}(u_j)\right| \\ &= \left|g^{[l]}(u_{m+1})\right| \\ &= |u_{m+1}-u_1|^{m+N} \cdots |u_{m+1}-u_m|^{m+N} \cdot \left|\frac{d_m}{2(u_{m+1}-v_m)}\right|^t \\ &> |u_{m+1}-u_1|^{m+N} \cdots |u_{m+1}-u_m|^{m+N} \cdot 2^t \\ &> qs(s+2)d^{(m+N)(m+1)}. \end{aligned}$$

Hence a contradiction.

b) We have $\alpha_{j,N} = 0$ if $N > \sup\{s, N(1), \ldots, N(s)\}$, by a proof very similar to the one of Theorem 3.5.

The proof is now complete ∎

Corollary 4.4: *If $D = \{u_j : j \in \mathbb{N}\}$ is regularly asymptotic for Ω, then the kernel of T has $2^{\aleph_0}$ as algebraic dimension.*

Proof: The proof of Corollary 3.4 applies here too: one has just to replace $r \in \mathbb{N}$ by some appropriate $\rho \in \mathbb{N}^{\mathbb{N}}$ and the space $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$ ∎

Proposition 4.5: *Let $D = \{u_j : j \in \mathbb{N}\}$ be a subset of $\partial\Omega$ having no accumulation point. This implies the existence of a sequence $\tau = (t_j)_{j \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that the disks $\{z \in \mathbb{C} : |z - u_j| \leq 1/t_j\}$ are pairwise disjoint. Then the following conditions on a sequence $\rho \in \mathbb{N}^{\mathbb{N}}$ verifying $\rho \geq \tau$ are equivalent:*

(a) *D is ρ-regularly asymptotic for Ω.*

(b) *For every compact subset K of Ω and every $j_0 \in \mathbb{N}$, there is $p \in \mathbb{N}$ such that, for every $h > 0$, there is $f \in A_\rho(\Omega; D)$ verifying*

$$|f(z)| \leq 1 \qquad \text{for all } z \in K \bigcup \left(\bigcup_{j=1}^{J} \left\{ u \in \Omega : |u - u_j| \leq \frac{1}{r_j} \right\} \right),$$

and

$$|f^{[p]}(u_{j_0})| > h.$$

Proof: The proof of (a) ⇒ (b) is essentially the same as the one of the necessity of the condition in Theorem 3.6: one just has to replace $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$.

Slight modifications to the proof of the sufficiency of the condition in Theorem 3.6 give (b) ⇒ (a). One just needs to replace $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$, to fix j in $\mathbb{N}$, to impose moreover the condition $s > j$ on s, to replace $V_{D,r,s}$ by $V_{D,\rho,s}$, to set

$$\mathcal{K} = K_s \bigcup \left(\bigcup_{j=1}^{s} \left\{ z \in \Omega : |z - u_j| \leq \frac{1}{r_j} \right\} \right)$$

and

$$P(z) = (z - u_j)^{N - p_j} \prod_{1 \leq k \leq s,\, k \neq j} (z - u_k)^N,$$

and to replace $p_{D,r,s}$ by $p_{D,\rho,s}$ ∎

Theorem 4.6: *Let $D = \{u_j : j \in \mathbb{N}\}$ be a subset of $\partial\Omega$ having no accumulation point. This implies the existence of a sequence $\rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that the disks $\{z \in \mathbb{C} : |z - u_j| \leq 1/r_j\}$ for $j \in \mathbb{N}$ are pairwise disjoint.*

If $\partial\Omega$ is quasi-connected at every point of D, then D is ρ-regularly asymptotic, hence regularly asymptotic, for Ω.

Proof: The proof is very similar to the one of Theorem 3.7. One fixes j in $\mathbb{N}$, sets

$$\mathcal{K} = K \bigcup \left(\bigcup_{k=1}^{j} \left\{ t \in \mathbb{C} : |t - u_j| \leq \frac{1}{r_j} \right\} \right)$$

and chooses A with the condition $\delta_2(u_j, A) < 1/r_j$ instead of $\delta_2(u_j, A) < 1/r$. The conclusion then follows from the preceding proposition ∎

5. Generalizations of Carleman's results

We begin with the following

Theorem 5.1: *If D is a non-empty subset of $\partial\Omega$ having no accumulation point and if $\partial\Omega$ is quasi-connected at every point of D, then D is regularly asymptotic for Ω.*

Moreover, for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}\}$ of complex numbers, the set of the elements f of $A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ for every $u \in D$ is a linear variety of dimension $2^{\aleph_0}$.

Proof: Having no accumulation point, D must be countable. So if D is finite, this is Theorem 3.7 and the Corollary 3.4, and if D is infinite but countable, it is a trivial consequence of Theorem 4.6 and of Corollary 4.4 ∎

The following result gives an easy way to verify that $\partial\Omega$ is quasi-connected at some point.

Proposition 5.2: *If the connected component C_u of $u \in \partial\Omega$ has more than one point, then $\partial\Omega$ is quasi-connected at u.*

Proof: We are going to use twice the following property (cf. [6: (10.1)]): *If A is a closed, connected subset and G a bounded, open subset of $\mathbb{C}$ such that $A \neq G \cap A \neq \emptyset$, then every connected component of $\overline{G} \cap A$ has some point in the boundary of G.*

Given $0 < \varepsilon < \delta < 1$, as C_u contains more than one point, there is $r_1 \in (0, \delta/2)$ such that $C_1 = C_u \cap \{z \in \mathbb{C} : |z-u| \leq r_1\} \neq C_u$. Let C be the connected component of C_1 containing u. Of course, C_u is a closed, connected subset and $b = \{z \in \mathbb{C} : |z-u| < r_1\}$ is a bounded, open subset of $\mathbb{C}$ such that $C_u \neq b \cap C_u \neq \emptyset$. Therefore there is $z_1 \in C$ such that $|z_1 - u| = r_1$. Now we chose $r_2, r_3 > 0$ such that $r_2 < \varepsilon r_1$ and $r_1 < r_3$, and set $G = \{z \in \mathbb{C} : r_2 < |z-u| < r_3\}$. So we have $u \notin G$ and $z_1 \in G$, hence $C \neq G \cap C \neq \emptyset$ and the connected component P of $C \cap \overline{G}$ containing z_1 contains a point z_2 of the boundary of G. As $|z_2 - u| \neq r_3$, we must have $|z_2 - u| = r_2$. Therefore P is a connected subset of $\partial\Omega$ such that $\delta_1(u, P) = |z_2 - u| = r_2$ and $\delta_2(u, P) = |z_1 - u| = r_1$ hence $\delta_2(u, P) = r_1 < \delta$ and $0 < \delta_1(u, P) = r_2 < \varepsilon r_1 = \varepsilon \delta_2(u, P)$ ∎

Combining the previous two results, we get the following statement which constitutes the practical form of our result. Let us mention that it generalizes Proposition 10 of [4]:

If the connected component of $u \in \partial\Omega$ has more than one point, then $\{u\}$ is regularly asymptotic for Ω

as well as Corollary 1 of [4]:

If Ω is simply connected, then every point of $\partial\Omega$ is regularly asymptotic for Ω.

Theorem 5.3: *If D is a non-empty subset of $\partial\Omega$ having no accumulation point and if the connected component of every point of D in $\partial\Omega$ has more than one point, then D is regularly asymptotic for Ω.*

Moreover, for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}\}$ *of complex numbers, the set of the elements* f *of* $A(\Omega; D)$ *such that* $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n$ *at every* $u \in D$ *is a linear variety of dimension* $2^{\aleph_0}$.

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