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Spline Approximation Methods Cutting Off Singularities

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Abstract. The topic of this paper is some types of singular behavior of spline approximation methods for one-dimensional singular integral operators which are caused by discontinuities in the coefficients or by non-smooth geometries of the underlying curves. These singularities can be cutted off by a modification of the approximation method which is closely related to the finite or infinite section method for discrete Toeplitz operators. Using Banach algebra techniques, one can derive stability criteria for a large variety of modified methods (including Galerkin, collocation, and qualocation

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1. Introduction

As a model situation, we consider the Galerkin method for solving the singular integral equation

$$Au := \Big((a_+\chi_+ + a_-\chi_-)I + (b_+\chi_+ + b_-\chi_-)S_{\mathbb{R}}\Big)u = f \tag{1}$$

where $a_\pm$ and $b_\pm$ are complex numbers, $\chi_\pm$ is the characteristic function of the positive and negative semi-axis, respectively, and $S_{\mathbb{R}}$ is the operator of singular integration,

$$(S_{\mathbb{R}}u)(t) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{u(s)}{s-t}\,ds \qquad (t \in \mathbb{R}).$$

For the *Galerkin method* for solving equation (1) we replace (1) by a sequence of approximation equations

$$L_n A u_n = L_n f \qquad (u_n \in S_n) \tag{2}$$

where S_n is a spline space and L_n is the associated Galerkin projection. The spline spaces considered here are supposed to be of a very natural structure, namely, we start with a *mother spline* φ, that is, with a bounded, measurable, and compactly supported function φ satisfying the conditions

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) \equiv 1 \qquad \text{for } x \in \mathbb{R} \tag{3}$$

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and

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \varphi(t+k)\overline{\varphi(t)}\, dt \cdot z^k \neq 0 \qquad \text{for } z \in \boldsymbol{T}. \tag{4}$$

(observe that the sums in (3) and (4) are actually finite by the boundedness of supp φ). Then we set $\varphi_{kn}(t) := \varphi(nt-k)$ and define the spline space S_n as the smallest closed subspace of $L^p = L^p(\mathbb{R})$ containing all functions φ_{kn} with $k \in \mathbb{Z}$. For example one can take $\varphi = \chi_{[0,1]}$, the characteristic function of the interval $[0,1]$, or $\varphi = \chi_{[0,1]} * \dots * \chi_{[0,1]}$, the d-fold convolution of $\chi_{[0,1]}$ by itself. Then (3) and (4) are satisfied, and S_n is just the space of all L^p-functions which are polynomials of degree d over each interval $[k, k+1]$, and which are $(d-1)$-times continuously differentiable on $\mathbb{R}$.

A basic observation which is usually attributed to de Boor is that the spaces $S_n \subseteq L^p$ and $l^p = l^p(\mathbb{Z})$ are isomorphic : If the function $\sum x_k\varphi_{kn}$ is in S_n, then the coefficient sequence (x_k) is in l^p and conversely, and, moreover, the mappings

$$E_n : l^p \to S_n,\ (x_k) \mapsto \sum x_k\varphi_{kn} \qquad \text{and} \qquad E_{-n} : S_n \to l^p,\ \sum x_k\varphi_{kn} \mapsto (x_k)$$

are continuous and $\sup_n \|E_n\|\,\|E_{-n}\| < \infty$.

The *Galerkin projection* L_n is the operator mapping L^p onto S_n such that $(L_n f, \varphi_{kn}) = (f, \varphi_{kn})$ for all $f \in L^p$ and $k \in \mathbb{Z}$. Condition (4) ensures the existence of L_n, whereas (3) involves the strong convergence of L_n to the identity operator I as $n \to \infty$. The natural question for (2) is whether this method *applies* to (1), i.e., whether (2) is uniquely solvable for all $n \geq n_0$ and for all right sides $f \in L^p$, and whether the sequence $(u_n)_{n\geq n_0}$ converges to a solution u of equation (1). Since L_n converges strongly to I, the applicability of (2) is equivalent to the *stability* of the sequence $(L_nA|_{S_n})$ (a sequence (A_n) of operators is *stable* if A_n is invertible for all sufficiently large n, and if $\sup_n \|A_n^{-1}\| < \infty$).

Let us first consider stability of $(L_nA|_{S_n})$ in the special case of constant coefficients, that is, $a_+ = a_- =: a$ and $b_+ = b_- =: b$. Clearly, this sequence is stable if and only if the sequence $(E_{-n}L_nAE_n)$ of operators on l^p is stable. This sequence shows two peculiarities: the operators $E_nL_nAE_n$ are independent of n (hence, this sequence is stable if and only if the operator $E_{-1}L_1AE_1$ is invertible), and the representation of the operator $E_{-n}L_nAE_n$ with respect to the standard basis of l^p yields a Toeplitz matrix,

$$E_{-n}L_nAE_n = aI + bT^o(g) \tag{5}$$

where g is a certain piecewise continuous function on the unit circle $\boldsymbol{T}$.

If φ is sufficiently smooth, then g can be represented as

$$g(e^{2\pi iy}) = \frac{\sum_{j\in\mathbb{Z}} \operatorname{sgn}(y+1/2)|(\mathcal{F}\varphi)(y+j)|^2}{\sum_{j\in\mathbb{Z}} |(\mathcal{F}\varphi)(y+j)|^2}$$

where $\mathcal{F}$ denotes Fourier transform (see [4]). The operator $T^0(g)$ is defined as follows: Let g_n denote the nth Fourier coefficient of g. Then $T^0(g)$ acts on finitely supported sequences $(x_k) \in l^p$ via $T^0(g)(x_k) = (y_k)$, where $y_k = \sum_{l\in\mathbb{Z}} g_{k-l}x_l$. Since $T^0(g)$ is invertible if and only if g is invertible (thought of a function in $L^\infty(\boldsymbol{T})$), we have the following:

The approximation equations for the Galerkin method (2) for the operator A with constant coefficients are of Toeplitz structure, and this method is stable if and only if the function $a + bg$ is invertible.

Let us now return to the general case of equation (1). The operators $E_{-n}L_nAE_n$ are again independent of n, but now

$$E_{-n}L_nAE_n = (a_+P + a_-Q) + (b_+P + b_-Q)T^0(g) + K \tag{6}$$

where $P: (x_k) \mapsto (\ldots, 0, 0, x_0, x_1, x_2, \ldots), Q = I - P$, and K is a certain compact perturbation which originates from basis splines φ_{kn} having the discontinuity point 0 in the interior of their supports. The appearance of a non-zero perturbation K involves serious complications: It is by no means easy to determine K explicitely. Even if K would be explicitely known, there seems to be no general way to decide invertibility of the operators (6). And, thirdly, K destroies the nice structure of the approximation equations (observe that (6) is a block Toeplitz operator if $K = 0$).

Analogous effects can be observed in other situations: Galerkin methods for Mellin convolution operators over $\mathbb{R}^+$, collocation and qualocation methods for these operators, Galerkin methods for singular integral operators with piecewise continuous coefficients over smooth closed curves and for singular integral operators with continuous coefficients over curves with corners and endpoints, etc. To have an example for the latter case consider the singular integral operator $aI + bS_\square$ with constant coefficients over the unit square $\square$. Provide $\square$ with equidistant partitions, and let $S_n^\square$ and $L_n^\square$ stand for the corresponding spline space and Galerkin projection, respectively.

The spline Galerkin method

$$L_n^\square(aI + bS_\square)u_n = L_n^\square f \qquad (u_n \in S_n^\square) \tag{7}$$

can be studied by localizing over $\square$. For example, at a corner of $\square$, (7) behaves as the equation

$$L_n^\angle(aI + bS_\angle)u_n = L_n^\angle f \qquad (u_n \in S_n^\angle)$$

where $\angle$ is the infinite angle $\mathbb{R}^+ \cup i\mathbb{R}^+$, and $S_n^\angle$ and $L_n^\angle$ are spline spaces and Galerkin projections related with $\angle$. If $\angle$ is mapped one-to-one onto the real axis $\mathbb{R}$ by identifying $i\mathbb{R}^+$ with $\mathbb{R}_-$, the space $S_n^\angle$ goes over into S_n, the projection $L_n^\angle$ into L_n, and the operator $aI + bS_\angle$ into

$$aI + b\Big(\chi_+S_{\mathbb{R}}\chi_+ + \chi_-S_{\mathbb{R}}\chi_- + \chi_-JN_{\frac{\pi}{2}}\chi_+ - \chi_+N_{\frac{3}{2}\pi}J\chi_-\Big)I$$

where $(Jf)(t) = f(-t)$, and N_β refers to the Hankel operator

$$(N_\beta f)(t) = \frac{1}{\pi i}\int_0^\infty \frac{f(s)}{s - e^{i\beta}t}\,ds \qquad (t \in \mathbb{R}^+).$$

The naturally appearing functions $\chi_\pm$ again involve compact perturbations.

2. Doubly-indiced approximation sequences

Probably, it was Chandler and Graham [2] who first proposed a way to overcome these difficulties. When studying integral equations of the second kind they realized that the behavior of the approximation system becomes much better if the spline space S_n is replaced by the modified spline space

$$S_{n,i} := \text{clos}_{L^p} \text{span}\,\{\varphi_{kn}\}_{k\in\mathbb{Z}} \qquad (|k| \geq i).$$

This space results from the original one by omitting a finite number of basis splines in a neighbourhood of the "singular point" 0. Prößdorf and Rathsfeld [5] successfully applied this approach to collocation and quadrature methods for Mellin operators, and they pointed put its close relationship to the finite section method for operators in the Toeplitz algebra. Elschner [3] combined modified spline spaces with Mellin techniques to study stability and convergence of Galerkin and collocation methods with piecewise polynomial splines for Mellin convolution equations and Wiener-Hopf equations.

Instead of modifying the spline space S_n we prefer to work in the original space but with modified approximation sequences. Define operators

$$Q_i : l^p \to l^p, \qquad (x_k) \mapsto (\ldots, x_{-i-2}, x_{-i-1}, 0, \ldots, 0, x_i, x_{i+1}, \ldots)$$

and consider instead of (2) the approximation system

$$E_n Q_i E_{-n} L_n A E_n Q_i E_{-n} u_{ni} = E_n Q_i E_{-n} L_n f \qquad (u_{ni} \in S_n). \tag{8}$$

The operators $E_n Q_i E_{-n} L_n$ are projections into S_n again. To force their strong convergence to the identity operator as $(n,i) \to \infty$ we have to restrict the natural index set $\mathbb{Z}^+ \times \mathbb{Z}^+$ by choosing a suitable subset $T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $i/n \to 0$ as $n \to \infty$ and $(n,i) \in T$ (otherwise we could fix n and let i go to infinity to obtain $E_n Q_i E_{-n} L_n \to 0$). For example one can take

$$T = \left\{(n,i) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \,\middle|\, i \leq n^{1-\delta} \quad \text{with some fixed} \quad \delta > 0\right\}.$$

Generally, an approximation method $(A_{ni})_{(n,i)\in T}$ is called *applicable* to the equation $Au = f$ if there is an $(n_0, i_0) \in T$ such that the equations

$$A_{ni} u_{ni} = E_n Q_i E_{-n} L_n f \qquad (u_{ni} \in S_n) \tag{9}$$

possess unique solutions u_{ni} for all $(n,i) \geq (n_0, i_0)$ (that means $n \geq n_0$ and $i \geq i_0$), if the sequence $(u_{ni})_{(n,i)\in T}$ is bounded, and if the sequence $(u_{t(n)})_{n\in\mathbb{Z}^+}$ converges in the norm of L^p to a solution u of the equation $Au = f$ for each monotonically increasing sequence $t : \mathbb{Z}^+ \to T$. The sequence $(A_{ni})_{(n,i)\in T}$ is said to be *stable* if there exists an (n_0, i_0) such that A_{ni} is invertible whenever $(n,i) \geq (n_0, i_0)$ and if $\sup_{(n,i)\geq(n_0,i_0)} \|A_{ni}^{-1}\| < \infty$. Let us finally agree to call a bounded sequence $(A_{ni})_{(n,i)\in T}$ *strongly convergent* to the operator A if the (common) sequence $(A_{t(n)})_{n\in\mathbb{Z}^+}$ converges strongly to A for each monotonically increasing sequence $t : \mathbb{Z}^+ \to T$.

If A is the model operator (1), then the sequence $(E_nQ_iE_{-n}L_nAE_nQ_iE_{-n})$ is stable if and only if the sequence

$$(Q_iE_{-n}L_nAE_nQ_i) = \Big(Q_i((a_+P + a_-Q) + (b_+P + b_-Q)T^o(g) + K)Q_i\Big) \tag{10}$$

is stable. The stability of this sequence is subject to a general theorem concerning finite and infinite sections of operators in a Toeplitz algebra (see [1: Section 7.68], [7] and [9] and Theorem 3 below; for references on the history of the topic see [1]). What results is that (10) is stable if and only if

1. the operator $(a_+P + a_-Q) + (b_+P + b_-Q)T^0(g)$ is Fredholm
2. the operators $QT^0(a_- + b_-g)Q + P$ and $PT^0(a_+ + b_+g)P + Q$ are invertible
3. the point 0 lies outside a certain curved triangle (which is a common straight triangle in case $p = 2$) with vertices $1, (a_- + b_-)/(a_- - b_-)$ and $(a_+ + b_+)/(a_+ - b_+)$.

This result indeed solves our problems: all conditions are verifiable, and the structure of the approximation system becomes sufficiently nice since $\|Q_iKQ_i\| \to 0$ as $i \to \infty$ and, thus, the unpleasent operator K can be neglected.

3. Banach algebras of approximation sequences

Stability problems can be reformulated into invertibility problems in Banach algebras in the following way: Let $\mathcal{F}$ stand for the set of all bounded sequences (A_n) of operators $A_n : S_n \to S_n$ such that $\|(A_n)\| := \sup_n \|A_nL_n\| < \infty$. Provided with elementwise operations, $\mathcal{F}$ becomes a Banach algebra, and the subset $\mathcal{K}$ of $\mathcal{F}$ containing all sequences (K_n) with $\|K_n\| \to 0$ forms a closed two-sided ideal in $\mathcal{F}$. It is elementary to show that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{K}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{K}$.

For doubly-indiced sequences one analogously defines an algebra $\mathcal{F}^T$ consisting of sequences $(A_{ni})_{(n,i)\in T}$, and the ideal $\mathcal{K}$ has to be replaced by the ideal $\mathcal{K}^T$ of all sequences (K_{ni}) such that, given $\varepsilon > 0$, there is an $(n_0, i_0) \in T$ with $\|K_{ni}L_n\| < \varepsilon$ whenever $(n,i) \geq (n_0, i_0)$. The latter will be abbreviated henceforth to $\|K_{ni}L_n\| \to 0$ as $(n,i) \to \infty$.

In [4], a subalgebra of $\mathcal{F}$ was introduced which contains a large variety of approximation sequences for singular integral operators (but without cutting-off factors), and a stability criterion for these sequences had been derived. Similarly, we are going to define a subalgebra of $\mathcal{F}^T$ which is, on the one hand, large enough to contain a bulk of approximation sequences (including sequences with cutting-off factors) and, on the other hand, small enough to be effectively treatable.

For the readers convenience, we start with recalling and specifying to the one-dimensional context some definitions and results of [4]. Given an integer n we let V_n denote the discrete shift operator $V_n : l^p \to l^p, (x_k) \mapsto (x_{k-n})$, and for each real number x we let $\{x\}$ stand for the smallest integer which is not less than x. Further we fix a real number r. Now define $\mathcal{A}_r$ as the smallest closed subalgebra of $\mathcal{F}$ which encloses all sequences of the form $(E_nT^0(a)E_{-n})$ where a is an arbitrary piecewise continuous function which possesses a finite total variation and which is continuous on $\boldsymbol{T} \setminus \{1\}$, and all sequences

$(E_n V_{\{tn+r\}} P V_{-\{tn+r\}} E_{-n})_{n \in \mathbb{Z}^+}$ with t running through the reals. One can show for example that $\mathcal{A}_0$ contains the Galerkin approximation sequences with trial space $S_n = S_n^{\varphi}$ and test space S_n^{ψ} both for singular integral operators $aI + bS_{\mathbb{R}}$ (where a and b are piecewise continuous on the one-point compactification $\dot{\mathbb{R}}$ of $\mathbb{R}$ by ∞ and continuous on $\dot{\mathbb{R}} \setminus \mathbb{Z}$) as for Mellin convolutions, whereas $\mathcal{A}_{-\varepsilon}$ contains ε-collocation sequences for these operators (even with general piecewise continuous coefficients), and the algebra $\mathcal{A}_0$ is also suitable to investigate certain qualocation and quadrature methods.

In the following proposition we introduce certain homomorphisms W and W_s associating with each sequence (A_n) a linear bounded operator. For brevity, we simply denote these operators by $W(A_n)$ and $W_s(A_n)$ in place of the – more correct – notations $W((A_n))$ and $W_s((A_n))$, respectively. Let further $L(X)$ stand for the algebra of all bounded linear operators and $K(X)$ for the ideal of all compact linear operators on the Banach space X.

Proposition 1: *Let $(A_n) \in \mathcal{A}_r$. Then the following is true:*

(a) There is an operator $W(A_n) \in L(L^p)$ such that the sequence (A_n) converges strongly to $W(A_n)$ as $n \to \infty$, and the mapping $W : \mathcal{A}_r \to L(L^p)$ is a continuous algebra homomorphism.

(b) For each $s \in \mathbb{R}$, there is an operator $W_s(A_n) \in L(l^p)$ such that the sequence $(V_{-\{sn+r\}} E_{-n} A_n E_n V_{\{sn+r\}})$ converges strongly to $W_s(A_n)$ as $n \to \infty$, and the mapping $W_s : \mathcal{A}_r \to L(l^p)$ is a continuous algebra homomorphism.

(c) The cosets $E_{-n} A_n E_n + K(l^p)$ are independent of n, and if $W_\infty(A_n)$ denotes one of them, then the mapping $W_\infty : \mathcal{A}_r \to L(l^p)/K(l^p)$ is a continuous algebra homomorphism.

Theorem 1 (see [4]): *A sequence $(A_n) \in \mathcal{A}_r$ is stable if and only if the operators $W(A_n)$ and $W_s(A_n)$ $(s \in \mathbb{R})$ as well as the coset $W_\infty(A_n)$ are invertible.*

Our result concerning cutting-off methods will be of the same form: we are going to construct a subalgebra $\mathcal{A}_r^T$ of $\mathcal{F}^T$ and homomorphisms W_T and W_s^T with $s \in \dot{\mathbb{R}}$ such that the analogue of Theorem 1 holds. Given $r \in \mathbb{R}$ we let $\mathcal{A}_r^T$ stand for the smallest closed subalgebra of $\mathcal{F}^T$ which contains

(i) all (constant with respect to i) sequences $(A_n)_{(n,i) \in T}$ with $(A_n)_{n \in \mathbb{Z}^+} \in \mathcal{A}_r$ (that is, $\mathcal{A}_r$ can be viewed as a subalgebra of $\mathcal{A}_r^T$ when identifying the sequences $(A_n)_{n \in \mathbb{Z}^+}$ and $(A_n)_{(n,i) \in T}$).

(ii) all sequences $(E_n V_{\{yn+r\}} Q_i V_{-\{yn+r\}} E_{-n})_{(n,i) \in T}$ with $y \in \mathbb{R}$.

The following proposition gives some hint how to introduce the desired homomorphisms.

Proposition 2: *Let $(A_{ni}) \in \mathcal{A}_r^T$. Then the following is true:*

(a) The strong limits $W((A_{ni})_{n \in \mathbb{Z}^+})$ exist for each fixed i, and they are independent of i.

(b) The strong limits $W_s((A_{ni})_{n \in \mathbb{Z}^+})$ exist for each fixed i and $s \in \mathbb{R}$.

(c) The cosets $E_{-n} A_{ni} E_n + K(l^p)$ are independent of $(n, i) \in T$.

Proof: All occurring mappings are continuous algebra homomorphisms. So it suffices to verify the assertions for the sequences in (i) and (ii) in place of (A_{ni}). If

$(A_{ni}) = (A_n)_{(n,i)\in T}$ with $(A_n)_{n\in\mathbb{Z}^+} \in \mathcal{A}_r$, then Proposition 1 yields immediately all what we want, and for sequences of the form (ii) the assertion is easily verified ∎

The preceding proposition shows that it is correct to define the homomorphism $W_T : \mathcal{A}_r^T \to L(L^p)$ by

$$W_T((A_{ni})_{(n,i)\in T}) := W((A_{ni})_{n\in\mathbb{Z}^+}) \qquad \text{for some } i \tag{11}$$

and the homomorphism $W_\infty^T : \mathcal{A}_r^T \to L(l^p)/K(l^p)$ by

$$W_\infty^T((A_{ni})_{(n,i)\in T}) := E_{-n}A_{ni}E_n + K(l^p) \qquad \text{for some } (n,i). \tag{12}$$

The definition of W_s^T for $s \in \mathbb{R}$ is less obvious: we let $\mathcal{F}(l^p)$ refer to the algebra of all bounded sequences (A_n) with $A_n \in L(l^p)$ provided with elementwise operations and the supremum norm. The subset $\mathcal{G}(l^p)$ of $\mathcal{F}(l^p)$ consisting of all sequences (K_n) of compact operators K_n with $\|K_n\| \to 0$ forms a closed two-sided ideal in $\mathcal{F}(l^p)$. Now define

$$W_s^T((A_{ni})_{(n,i)\in T}) := (W_s((A_{ni})_{n\in\mathbb{Z}^+}))_{i\in\mathbb{Z}^+} + \mathcal{G}(l^p). \tag{13}$$

One can straightforwardly check that W_s^T is a continuous algebra homomorphism mapping $\mathcal{A}_r^T$ into $\mathcal{F}(l^p)/\mathcal{G}(l^p)$.

Here is the main result on stability of sequences with cutting-off factors.

Theorem 2: *A sequence $(A_{ni}) \in \mathcal{A}_r^T$ is stable if and only if*

(a) the operator $W_T(A_{ni})$ is invertible

(b) the cosets $W_s^T(A_{ni})$ are invertible for all $s \in \mathbb{R}$

(c) the coset $W_\infty^T(A_{ni})$ is invertible.

Condition (b) is equivalent to the stability of the sequence $(W_s((A_{ni})_{n\in\mathbb{Z}^+}))_{i\in\mathbb{Z}^+}$. A criterion for this stability will be quoted in Theorem 3 below.

Of course, Theorem 1 can be rediscovered from Theorem 2 since, for $(A_{ni}) = (A_n)_{(n,i)\in T}$ with $(A_n)_{n\in\mathbb{Z}^+} \in \mathcal{A}_r$,

$$W_T(A_{ni}) = W(A_n), \quad W_s^T(A_{ni}) = (W_s(A_n))_{i\in\mathbb{Z}^+} + \mathcal{G}(l^p), \quad W_\infty^T(A_{ni}) = W_\infty(A_n),$$

and since a constant sequence is stable if and only if its generating operator is invertible.

Proof of Theorem 2: The proof proceeds analogously as those of Theorem 1 in [4]. We only mark the essential steps.

Step 1. The set $\mathcal{G}^T := \mathcal{A}_r^T \cap \mathcal{K}^T$ is a closed two-sided ideal of $\mathcal{A}_r^T$, and if $(G_{ni}) \in \mathcal{G}^T$, then

$$W_T(G_{ni}) = 0, \qquad W_s^T(G_{ni}) = 0 \quad (s \in \mathbb{R}), \qquad W_\infty^T(G_{ni}) = 0. \tag{14}$$

The identities (14) are immediate consequences of the definition of the norm convergence of doubly-indiced sequences.

Step 2. The collection $\mathcal{J}^T$ of all sequences of the form $(L_nK|_{S_n})_{(n,i)\in T} + (G_{ni})$ with $K \in K(L^p)$ and $(G_{ni}) \in \mathcal{G}^T$ belongs to $\mathcal{A}_r^T$ and forms a closed two-sided ideal of this algebra. Indeed, for the inclusion $\mathcal{J}^T \subseteq \mathcal{A}_r^T$ it would clearly suffice to show that $(L_n K|_{S_n})$ is in $\mathcal{A}_r$. But now we cannot refer to [4] since these sequences were included into $\mathcal{A}_r$ in the multidimensional context by definition. In the one-dimensional setting considered here one can actually prove the desired inclusion in the following way:

- Show that $(L_n f I|_{S_n}) \in \mathcal{A}_r$ whenever f is continuous on $\dot{\mathbb{R}}$. This can be done by approximating f by a piecewise constant function which, on its hand, is constituted by shifted characteristic functions χ_+. Using the inclusion $(E_n P E_{-n}) \in \mathcal{A}_r$ one easily derives that the sequence $(L_n \chi_+ I|_{S_n})$ is in $\mathcal{A}_r$.
- Show that $(L_n S_{\mathbb{R}}|_{S_n}) \in \mathcal{A}_r$. For this, it remains to show that the function g introduced in Section 1 is piecewise continuous on $\boldsymbol{T}$ and continuous on $\boldsymbol{T}\setminus\{1\}$, but this is well known (see [6: 10.4 and 10.8]).
- Show that $(L_n(S_{\mathbb{R}} f I - f S_{\mathbb{R}})|_{S_n}) \in \mathcal{A}_r$. This is a simple consequence of the commutator property
$$\|L_n f I - f L_n\| \to 0 \qquad \text{as } n \to \infty \tag{15}$$
(see [4: Theorem 3/(a)]). Indeed, (15) entails that
$$(L_n f I|_{S_n})(L_n S_{\mathbb{R}}|_{S_n}) - (L_n S_{\mathbb{R}}|_{S_n})(L_n f I|_{S_n}) - (L_n(f S_{\mathbb{R}} - S_{\mathbb{R}} f I)|_{S_n}) \in \mathcal{K}.$$
- The operators $S_{\mathbb{R}} f I - f S_{\mathbb{R}}$ are compact, and each compact operator on L^p can be approximated by sums of products of operators of this form.

The proof that $\mathcal{J}^T$ is even an ideal follows easily: if $(A_{ni}) \in \mathcal{A}_r^T$, then

$$(A_{ni})(L_n K|_{S_n} + G_{ni}) = \Big(L_n W(A_{ni}) K|_{S_n} + A_{ni} G_{ni} + (A_{ni} L_n - L_n W(A_{ni})) K|_{S_n}\Big).$$

The first term is in $\mathcal{J}^T$ since $W(A_{ni})K$ is compact, and the other two terms are even in $\mathcal{G}^T$. Thus, $\mathcal{J}^T$ is a left ideal, and analogously one verifies the right ideal property.

Step 3. Denote the canonical homomorphism from $\mathcal{A}_r^T$ onto the quotient algebra $\mathcal{A}_r^T/\mathcal{J}^T$ by Φ. The analogue of Proposition 15/(b) in [4] is the following: A coset $(A_{ni}) + \mathcal{G}^T \in \mathcal{A}_r^T/\mathcal{G}^T$ is invertible in $\mathcal{A}_r^T/\mathcal{G}^T$ if and only if the operator $W(A_{ni})$ is invertible in $L(L^p)$ and if the coset $\Phi(A_{ni})$ is invertible in $\mathcal{A}_r^T/\mathcal{J}^T$ (observe that, by (14), the operator $W(A_{ni})$ actually depends on the coset $(A_{ni}) + \mathcal{G}^T$ only). The proof goes as in [1: 7.9 - 7.11].

Step 4. If $f \in C(\dot{\mathbb{R}})$, then the coset $\Phi((L_n f I|_{S_n})_{(n,i)\in T})$ belongs to the center of $\mathcal{A}_r^T/\mathcal{J}^T$. For a proof it remains to show that

$$(L_n f I|_{S_n})(A_{ni}) - (A_{ni})(L_n f I|_{S_n}) \in \mathcal{J}^T$$

for all sequences of the form (i) and (ii). Concerning the first ones, see Proposition 17 in [4], and for (ii) one has to take into account that the operators A_{ni} are of diagonal form and that the operators $E_{-n} L_n f E_n$ are also of diagonal form up to certain perturbations tending to zero in the operator norm. The simple proof of the latter assertion is based on the commutator property (Theorem 3/(a) of [4]) and can be implicitely found in the proof of Proposition 17 of [4].

Step 5. Taking into account the inclusion

$$(L_n fI|_{S_n})(L_n gI|_{S_n}) - (L_n fgI|_{S_n}) \in \mathcal{G},$$

which is involved by the commutator property (15) again, it is not hard to derive that the set

$$\mathcal{B} := \left\{ \Phi(L_n fI|_{S_n}) \quad \text{with } f \in C(\dot{\mathbb{R}}) \right\}$$

is a commutative Banach algebra belonging to the center of $\mathcal{A}_r^T/\mathcal{J}^T$, that the maximal ideal space of $\mathcal{B}$ is homeomorphic to $\dot{\mathbb{R}}$, and that the maximal ideal associated with $s \in \dot{\mathbb{R}}$ is

$$\left\{ \Phi(L_n fI|_{S_n}) \quad \text{with } f \in C(\dot{\mathbb{R}}) \text{ and } f(s) = 0 \right\}. \tag{16}$$

Let $\mathcal{J}_s^T$ denote the smallest closed two-sided ideal of $\mathcal{A}_r^T/\mathcal{J}^T$ containing the maximal ideal (16), write $\mathcal{A}_{r,s}^T$ for the quotient algebra $(\mathcal{A}_r^T/\mathcal{J}^T)/\mathcal{J}_s^T$, and let Φ_s refer to the canonical homomorphism from $\mathcal{A}_r^T$ onto $\mathcal{A}_{r,s}^T$. Then Allan's local principle (compare [4] or [1: 1.34]) gives the following:

The coset $\Phi(A_{ni})$ is invertible in $\mathcal{A}_r^T/\mathcal{J}^T$ if and only if all cosets $\Phi_s(A_{ni})$ with $s \in \dot{\mathbb{R}}$ are invertible in $\mathcal{A}_{r,s}^T$.

Step 6. The mapping $\Phi_s(A_{ni}) \mapsto W_s^T(A_{ni})$ is correctly defined, and the image of $\mathcal{A}_r^T/\mathcal{J}^T$ under this mapping is inverse closed in $\mathcal{F}(l^p)/\mathcal{G}(l^p)$. Indeed, we have already remarked that $W_s^T(G_{ni}) = 0$ whenever $(G_{ni}) \in \mathcal{G}$, and from [4: Proposition 18] we conclude that $W_s^T((L_n K|_{S_n})_{(n,i)\in T}) = 0$ for all compact K. Thus, $W_s^T(A_{ni})$ depends on $\Phi(A_{ni})$ only. Furthermore, [4: Proposition 18] entails that

$$W_s^T((L_n fI\,|_{S_n})) = \begin{cases} (f(s)I|_{S_n}) + \mathcal{G}(l^p) & \text{if } s \neq \infty \\ f(\infty) + K(l^p) & \text{if } s = \infty \end{cases}$$

whence easily follows that $W_s^T(A_{ni})$ depends on $\Phi_s(A_{ni})$ only. This shows the correctness of the definitions, and for characterizing the images of the mappings W_s^T, we let $\mathcal{T}^p$ denote the smallest closed subalgebra of $L(l^p)$ which contains the operators P and $T^0(a)$ with a running again through the piecewise continuous functions with finite total variation on $\boldsymbol{T}$ which are continuous on $\boldsymbol{T} \setminus \{1\}$, and we let $\mathcal{F}(\mathcal{T}^p)$ refer to the smallest closed subalgebra of $\mathcal{F}(l^p)$ containing all constant sequences (A) with $A \in \mathcal{T}^p$ as well as the sequence (Q_n). It is well known that the algebra $\mathcal{T}^p$ encloses the ideal $K(l^p)$ and that $\mathcal{F}(\mathcal{T}^p)$ contains $\mathcal{G}(l^p)$, and that the algebras $\mathcal{T}^p/K(l^p)$ and $\mathcal{F}(\mathcal{T}^p)/\mathcal{G}(l^p)$ are inverse closed in $L(l^p)/K(l^p)$ and $\mathcal{F}(l^p)/\mathcal{G}(l^p)$, respectively. Further it is immediate that

$$W_s^T((E_n T^0(a) E_{-n})_{(n,i)\in T}) = (T^0(a))_{i\in\mathbb{Z}^+} + \mathcal{G}(l^p)$$
$$W_s^T((E_n V_{\{sn+r\}} P V_{-\{sn+r\}} E_{-n})_{(n,i)\in T}) = (P)_{i\in\mathbb{Z}^+} + \mathcal{G}(l^p)$$
$$W_s^T((E_n V_{\{sn+r\}} Q_i V_{-\{sn+r\}} E_{-n})_{(n,i)\in T}) = (Q_i)_{i\in\mathbb{Z}^+} + \mathcal{G}(l^p)$$

whence follows that the image of $\mathcal{A}_r^T$ under W_s^T is just $\mathcal{F}(\mathcal{T}^p)/\mathcal{G}(l^p)$ in case $s \neq \infty$ and, similarly, the image of $\mathcal{A}_r^T$ under W_∞^T is $\mathcal{T}^p/K(l^p)$. This gives our claim.

Step 7. The mappings

$$\hat{W}_s^T : \mathcal{F}(\mathcal{T}^p)/\mathcal{G}(l^p) \to \mathcal{A}_{r,s}^T, \quad (A_i) + \mathcal{G}(l^p) \mapsto \Phi_s(E_n V_{\{sn+r\}} A_i V_{-\{sn+r\}} E_{-n}) \tag{17}$$

with $s \neq \infty$ and

$$\hat{W}_\infty^T : \mathcal{T}^p / K(l^p) \to \mathcal{A}_{r,\infty}^T, \quad A + K(l^p) \mapsto \Phi_\infty(E_n A E_{-n}) \tag{18}$$

are correctly defined, and

$$\Phi_s(A_{ni}) = \hat{W}_s^T(W_s^T(A_{ni})) \qquad \text{for all } (A_{ni}) \in \mathcal{A}_r^T. \tag{19}$$

Let us verify the correctness of (17) in case $s = 0$ for example. What we have to show is that $(E_n V_{\{r\}} G_i V_{-\{r\}} E_{-n})$ is in $\mathcal{G}^T$ whenever $(G_i) \in \mathcal{G}(l^p)$. For, we can suppose without loss that

$$G_i = \begin{cases} K_{kl} & \text{if } i = i_0 \\ 0 & \text{if } i \neq i_0 \end{cases}$$

where K_{kl} is the rank one operator having a 1 at the klth place of its matrix representation whereas the other entries are zero (this is justified since each compact operator on l^p can be uniformly approximated by finite sums of operators of the form K_{kl}). Let us further agree to abbreviate the sequence $(E_n V_{\{r\}} B V_{-\{r\}} E_{-n})$ by $[B]$ for the moment. Now we get the desired assertion as follows: The sequences $[Q_i]$ and $[A]$ with $A \in \mathcal{T}^p$ belong to $\mathcal{A}_r^T$ by definition; hence the sequence

$$\begin{aligned} &\Big([V_{k-i_0}][P][Q_i][V_{i_0-k}] - [V_{k+1-i_0}][P][Q_i][V_{i_0-k-1}]\Big)[K_{kl}] \\ &\quad = \Big[\Big(V_{k-i_0} P Q_i V_{i_0-k} - V_{k+1-i_0} P Q_i V_{i_0-k-1}\Big) K_{kl}\Big] \end{aligned}$$

belongs to $\mathcal{A}_r^T$, too. But, as one easily checks,

$$\Big(V_{k-i_0} P Q_i V_{i_0-k} - V_{k+1-i_0} P Q_i V_{i_0-k-1}\Big) K_{kl} = G_i$$

which gives the correctness of (17) for $s = 0$. The proof for $s \neq 0$ is similar. For the correctness of (18), it remains to verify that $\Phi_s(E_n K E_{-n}) = 0$ whenever $K \in K(l^p)$. We again suppose without loss that $K = K_{kl}$. If f is continuous on $\dot{\mathbb{R}}$ and $f(\infty) = 1$, then it is evident that $f E_n K_{kl} E_{-n} = 0$ whenever the support of f is sufficiently small. This in combination with the commutator relation gives the desired correctness.

Finally, the mappings $\hat{W}_s^T$ with $s \in \dot{\mathbb{R}}$ are continuous algebra homomorphisms, and so we verify (19) only for the generating sequences of the form (i) or (ii). For (i) we can refer to [4: Prop. 20], and for $A_{ni} = E_n V_{\{yn+r\}} Q_i V_{-\{yn+r\}} E_{-n}$ the verification is straightforward. Indeed, let for example $s = 0$. Then

$$W_0^T(A_{ni}) = \begin{cases} (Q_i) + \mathcal{G}(l^p) & \text{if } y = 0 \\ (I) + \mathcal{G}(l^p) & \text{if } y \neq 0 \end{cases}.$$

Thus, in case $y = 0$, (19) follows at once, and in case $y \neq 0$, (19) reduces to

$$\Phi_0(E_n V_{\{yn+r\}} Q_i V_{-\{yn+r\}} E_{-n}) = \Phi_0(I|_{S_n}). \tag{20}$$

This is evidently true if the generating mother spline φ is the function $\chi_{[0,1]}$. Indeed, in this situation one has

$$E_n V_{\{yn+r\}} Q_i V_{-\{yn+r\}} E_{-n} = L_n \chi_{\mathbb{R}\setminus M} I|_{S_n}$$

with $M = [(-i + \{yn + r\})/n, (i - 1 + \{yn + r\})/n]$, and for $(n, i) \in T$ and n large enough, the functions $\chi_{\mathbb{R}\setminus M}$ and 1 coincide locally in a neighborhood of 0. The case of general φ can be traced back to this case again by invoking the commutator relation (compare [4]).

As an immediate consequence of (19) we conclude that $\Phi_s(A_{ni})$ is invertible in $\mathcal{A}^T_{r,s}$ if and only if $W^T_s(A_{ni})$ is invertible, and summarizing Steps 3, 5 and 7 we obtain the following:

The coset $(A_{ni}) + \mathcal{G}^T$ is invertible in $\mathcal{A}^T_r/\mathcal{G}^T$ if and only if $W_T(A_{ni})$ and $W^T_s(A_{ni})$ are invertible for all $s \in \mathbb{R}$.

Step 8. We claim that the stability of $(A_{ni}) \in \mathcal{A}^T_r$ involves the invertibility of $W_T(A_{ni})$ and $W^T_s(A_{ni})$ for all $s \in \mathbb{R}$. Suppose

$$\sup_{(n,i)\geq(n_0,i_0)} \|A^{-1}_{ni}\| < \infty. \tag{21}$$

Then the sequences $(A_{ni})_{n\in\mathbb{Z}^+}$ are stable for all $i \geq i_0$, and these sequences belong to $\mathcal{A}_r$ (recall that $Q_i = I$ + *compact*). From [4] we conclude that $W_s((A_{ni})_{n\in\mathbb{Z}^+})$ and $W((A_{ni})_{n\in\mathbb{Z}^+})$ are invertible and, thus, Proposition 2 yields our claim for W_T and W_∞. Concerning W^T_s we emphasize that (21) gives moreover

$$\sup_{i\geq i_0} \|W_s((A_{ni})_{n\in\mathbb{Z}^+})^{-1}\| < \infty$$

whence the stability of $(W_s((A_{ni})_{n\in\mathbb{Z}^+}))_{i\in\mathbb{Z}^+}$ follows.

Now we can finish the proof of Theorem 2: If (A_{ni}) is stable, then $W_T(A_{ni})$ and $W^T_s(A_{ni})$ are invertible. Since the images of W^T_s are inverse closed, this implies invertibility of $W^T_s(A_{ni})$ in $\mathcal{F}(\mathcal{T}^p)/\mathcal{G}(l^p)$ and $\mathcal{T}^p/K(l^p)$ if $s \neq \infty$ and $s = \infty$, respectively. Then, by (19), $\Phi_s(A_{ni})$ is invertible and, as a consequence of Allan's local principle, $\Phi(A_{ni})$ is invertible in $\mathcal{A}^T_r/\mathcal{J}^T$. Now, by Step 3, the coset $(A_{ni}) + \mathcal{G}^T$ is invertible in $\mathcal{A}^T_r/\mathcal{G}^T$, but this of course implies invertibility of $(A_{ni}) + \mathcal{K}^T$ in $\mathcal{F}^T/\mathcal{K}^T$ which, finally, involves stability of (A_{ni}) again ∎

Remark. Theorems 1 and 2 can be generalized to larger algebras, say $\mathcal{A}_{\Omega,r}$ and $\mathcal{A}^T_{\Omega,r}$, which contain $\mathcal{A}_r$ and $\mathcal{A}^T_r$ as their subalgebras. Thereby, Ω stands for any fixed closed subset of the unit circle, and the algebra $\mathcal{A}_{\Omega,r}$ originates from $\mathcal{A}_r$ by adding all sequences of the form $(E_n T^0(a) E_{-n})_{n\in\mathbb{Z}^+}$ where a is piecewise continuous on $\boldsymbol{T}$, continuous on $\boldsymbol{T}\setminus\Omega$, and has finite total variation. The algebra $\mathcal{A}^T_{\Omega,r}$ is defined analogously. Let $t \in \boldsymbol{T}$ and write Y_t for the operator mapping l^p onto itself via $(x_k) \mapsto (x_k\, t^{-k})$. One can show that, for all $(A_n) \in \mathcal{A}_{\Omega,r}$ and $t \in \Omega$, there is an operator $W^t(A_n)$ on L^p such that

$$E_n Y_t^{-1} E_{-n} A_n E_n Y_t E_{-n} L_n \rightarrow W^t(A_n)$$

strongly as $n \to \infty$, and the mapping $W^t : \mathcal{A}_{\Omega,r} \to L(L^p)$ is a continuous algebra homomorphism. Clearly, $W^1(A_n) = W(A_n)$ for all $(A_n) \in \mathcal{A}_r$. Let now $(A_{ni}) \in \mathcal{A}^T_{\Omega,r}$. Then the strong limits $W^t((A_{ni})_{n \in \mathbb{Z}^+})$ exist for each i, and they are independent of i. We denote one of them by $W^t_T(A_{ni})$. The generalization of Theorem 2 reads as follows:

A sequence $(A_{ni}) \in \mathcal{A}^T_{\Omega,r}$ is stable if and only if $W^t_T(A_{ni})$ and $W^T_s(A_{ni})$ are invertible for all $t \in \Omega$ and $s \in \dot{\mathbb{R}}$.

One example for an approximation sequence which belongs to $\mathcal{A}_{\{1,-1\},0}$ (but not to $\mathcal{A}_0 = \mathcal{A}_{\{1\},0}$) is the modified quadrature method considered in [5: Equation 0(10)] (which is actually equivalent to a trigonometric collocation method).

4. Finite sections of Toeplitz operators

In this section we are going to derive a stability criterion for sequences in $\mathcal{F}(\mathcal{T}^p)$ (resp. an invertibility criterion in $\mathcal{F}(\mathcal{T}^p)/\mathcal{G}(l^p)$) which is needed to examine condition (b) in Theorem 2. The basic observation is the following one.

Proposition 3: *If $(A_n) \in \mathcal{F}(\mathcal{T}^p)$, then $(E_n A_n E_{-n}) \in \mathcal{A}_0$.*

Proof: It is obviously sufficient to show that the sequences $(E_n A E_{-n})$ with $A \in \mathcal{T}^p$ and $(E_n Q_n E_{-n})$ belong to $\mathcal{A}_0$. For the first one, this is immediate from the definition; for the second one we find

$$(E_n Q_n E_{-n}) = (E_n V_{-n} P V_n E_{-n}) - (E_n V_n P V_{-n} E_{-n})$$

which is also in $\mathcal{A}_0$ ∎

Since the sequences (A_n) and $(E_n A_n E_{-n})$ are simultaneously stable or not, Theorem 1 yields in this situation: The sequence (A_n) is stable if and only if $W(E_n A_n E_{-n})$ and $W_s(E_n A_n E_{-n})$ are invertible for all $s \in \dot{\mathbb{R}}$. A detailed analysis shows that most of these conditions are redundant.

Theorem 3: *Let $(A_n) \in \mathcal{F}(\mathcal{T}^p)$. The sequence (A_n) is stable if and only if $W(E_n A_n E_{-n})$ and $W_s(E_n A_n E_{-n})$ are invertible for $s \in \{-1, 0, 1, \infty\}$.*

Proof: One has

$$W_s(E_n P E_{-n}) = \begin{cases} O & \text{if } s \in (-\infty, 0) \\ P & \text{if } s = 0 \\ I & \text{if } s \in (0, \infty) \\ P + K(l^p) & \text{if } s = \infty \end{cases}$$

$$W_s(E_n Q_n E_{-n}) = \begin{cases} I & \text{if } s \in (-\infty, -1) \cup (1, \infty) \\ Q & \text{if } s = -1 \\ O & \text{if } s \in (-1, 1) \\ P & \text{if } s = 1 \\ I + K(l^p) & \text{if } s = \infty \end{cases}$$

$$W_s(E_n T^0(a) E_{-n}) = \begin{cases} T^0(a) & \text{if } s \in (-\infty, \infty) \\ T^o(a) + K(l^p) & \text{if } s = \infty \end{cases}$$

(compare [4] and recall that $Q = I - P$). Taking these identities into account one easily gets $W_s(E_n A_n E_{-n}) = W_{s'}(E_n A_n E_{-n})$ whenever s and s' belong to the same of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$. Further, the upper-semicontinuity of the mapping $s \mapsto \|\Phi_s(E_n A_n E_{-n})\|$ (see, e.g., [1: Theorem 1.34/(b)]) involves that, if $\Phi_s(E_n A_n E_{-n})$ is invertible for some s, then $\Phi_{s'}(E_n A_n E_{-n})$ is invertible for all s' in a certain neighbourhood of s. Finally, we have already seen that $\Phi_s(E_n A_n E_{-n})$ is invertible if and only if $W_s(E_n A_n E_{-n})$ is invertible. Combining these things we conclude that invertibility of $W_s(E_n A_n E_{-n})$ for $s \in \{-1, 0, 1, \infty\}$ implies invertibility of $W_s(E_n A_n E_{-n})$ for all $s \in \dot{\mathbb{R}}$ ∎

Theorem 3 can be generalized to sequences involving operators $T^0(a)$ with a being piecewise continuous on Ω and continuous on $\mathbb{T} \setminus \Omega$. One only has to include into its formulation the invertibility of all operators $W^t(E_n A_n E_{-n})$ with $t \in \Omega$. Furthermore, Proposition 3 remains valid with $\mathcal{A}_0$ replaced by $\mathcal{A}_r$ with arbitrary r.

5. Examples

We are going to illustrate the preceding theorems by a few examples.

Example 1. Let the functions a and b be piecewise continuous on $\dot{\mathbb{R}}$ and continuous on $\mathbb{R} \setminus \{0, 1\}$, and let K be a compact operator. Then the Galerkin approximation sequence

$$(A_n) := (L_n(aI + bS_{\mathbb{R}} + K)|_{S_n}) \tag{22}$$

for the singular integral operator $A = aI + bS_{\mathbb{R}} + K$ belongs to the algebra $\mathcal{A}_0$ (see [4]), and Theorem 1 yields the following

Proposition 4: *The sequence (A_n) given by (22) is stable if and only if*
(a) the operator A is invertible on L^p
(b) the operators $a(s) + b(s)T^0(g)$ are invertible on l^p for $s \in \mathbb{R} \setminus \{0, 1\}$
(c) the operators

$$(a(s+0)P + a(s-0)Q) + (b(s+0)P + b(s-0)Q)T^0(g) + K_s$$

(with certain compact operators K_s) are invertible on l^p for $s \in \{0, 1\}$
(d) the operator

$$(a(+\infty)P + a(-\infty)Q) + (b(+\infty)P + b(-\infty)Q)T^0(g)$$

is Fredholm on l^p.

In this form, the proposition holds even for matrix-valued coefficients. Conditions (b) and (d) are effective: the operator in (b) is invertible if and only if the function $a(s) + b(s)g$ is invertible, and the Fredholmness of the operator in (d) is subject to the Gohberg–Krupnik symbol calculus for singular integral operators. On the other hand there seems to be no evident way to verify condition (c). This suggests to modify (22) by introducing cutting-off factors at 0 and 1. The resulting approximation sequence

$$(A_{ni}) = \left(E_n Q_i V_n Q_i V_{-n} E_{-n} A_n E_n Q_i V_n Q_i V_{-n} E_{-n}|_{S_n}\right) \tag{23}$$

belongs to $\mathcal{A}_0^T$ since

$$(E_n Q_i V_n Q_i V_{-n} E_{-n}) = (E_n Q_i E_{-n})(E_n V_{\{n\}} Q_i V_{-\{n\}} E_{-n})$$

and Theorem 2 states that (23) is stable if and only if conditions (a), (b) and (d) hold and if the condition

(c)' *the sequences*

$$\Big(Q_i\Big((a(s+0)P + a(s-0)Q) + (b(s+0)P + b(s-0)Q)T^0(g) + K_s\Big)Q_i\Big)_{i\in\mathbb{Z}^+}$$

are stable for $s = 0$ *and* $s = 1$

holds. This stability can be examined by means of Theorem 3, and summarizing these results we finally get the following

Proposition 5: *The sequence (23) is stable if and only if conditions (a), (b) and (d) of Proposition 4 are satisfied and if, for s=0 and s=1,*

(c)" the operators

$$(a(s+0)P + a(s-0)Q) + (b(s+0)P + b(s-0)Q)T^0(g)$$

are Fredholm on l^p, *the Toeplitz operators*

$$PT^0(a(s-0) + b(s-0)\tilde{g})P + Q \quad \text{and} \quad PT^0(a(s+0) + b(s+0)g)P + Q$$

(with $\tilde{g}(z) = g(1/z)$*) are invertible on* l^p, *and the singular integral operators*

$$\chi_{[-1,1]}I + \chi_{\mathbb{R}\setminus[-1,1]}\Big((a(s+0)\chi_{\mathbb{R}^+} + a(s-0)\chi_{\mathbb{R}^-})I$$
$$+ (b(s+0)\chi_{\mathbb{R}^+} + b(s-0)\chi_{\mathbb{R}^-})S_{\mathbb{R}}\Big)\chi_{\mathbb{R}\setminus[-1,1]}I$$

are invertible on L^p.

Now condition (c)" is also verifiable (at least in scalar case): the Fredholmness of the first operator is a matter of symbol calculus again and, concerning the other operators, recall that scalar Toeplitz operators and singular integral operators are invertible if (and only if) they are Fredholm and if their index is zero (Coburn's theorem). So it remains to employ the well known index formulas for these operators.

Example 2. Here we consider the same approximation method as in Example 1 for the operator

$$A = \chi_{[0,1]}(aI + bS_{\mathbb{R}^+} + M^0(c))\chi_{[0,1]}I + \chi_{\mathbb{R}\setminus[0,1]}I$$

where a and b are complex numbers, $M^0(c)$ is the Mellin convolution operator

$$(M^0(c)u)(t) = \int_0^\infty k\left(\frac{t}{s}\right)u(s)\frac{ds}{s} \qquad (t \in \mathbb{R}^+)$$

and c is just the Mellin transform of the kernel function k. Under the additional assumption $\varphi = \chi_{[0,1]}$ it is shown in [5] and [8] that

$$E_{-n}L_nM^0(c)E_n = G(c) + K$$

where $G(c)$ has matrix representation $(k(\frac{l+1}{m+1})\frac{1}{m+1})_{l,m=1}^{\infty}$ and K is compact. Moreover, $G(c)$ belongs to the Toeplitz algebra $\mathcal{T}^p$, and the Gohberg–Krupnik symbol of $G(c)$ is given on $\boldsymbol{T} \times \bar{\mathbb{R}}$ (with $\bar{\mathbb{R}}$ referring to the compactification of $\mathbb{R}$ by the two points $+\infty$ and $-\infty$) by

$$(G(c))(t,z) = \begin{cases} c(z) & \text{if } t = 1 \\ 0 & \text{if } t \neq 1. \end{cases}$$

In particular, this involves that $(L_nM^0(c)|_{S_n}) \in \mathcal{A}_0$. These results remain valid for arbitrary generating functions φ.

The common Galerkin method $(L_nA|_{S_n})$ again possesses two singular points, namely 0 and 1, and so we modify this method by introducing cutting-off factors as follows:

$$(A_{ni}) := \Big(E_nQ_iV_nQ_iV_{-n}E_{-n}L_nAE_nQ_iV_nQ_iV_{-n}E_{-n}\Big). \tag{24}$$

Proposition 6: *The sequence (24) is stable if and only if*

(a) the operator A is invertible on L^p

(b) the operator $PT^0(a+bg)P + G(c) + Q$ is Fredholm on l^p, the Toeplitz operator $PT^0(a+bg)P + Q$ is invertible on l^p, and the singular integral operator

$$\chi_{[1,\infty)}(aI + bS_{\mathbb{R}^+} + M^0(c))\chi_{[1,\infty)}I + \chi_{(-\infty,1]}$$

is invertible on L^p

(c) the Toeplitz operator $PT^0(a+b\tilde{g})P + Q$ is invertible on l^p, and the singular integral operator

$$a\chi_{(-\infty,-1]}I + b\chi_{(-\infty,-1]}S_{(-\infty,-1]} + \chi_{[-1,\infty)}I$$

is invertible on L^p.

Herein, conditions (a), (b) and (c) correspond to the invertibility of the operators $W_T(A_{ni}), W_0^T(A_{ni})$ and $W_1^T(A_{ni})$, respectively. The other homomorphisms figuring in Theorem 2 are not relevant here since

$$W_s^T(A_{ni}) = \begin{cases} I & \text{if } s \in \mathbb{R}\setminus[0,1] \\ PT^0(a+bg)P + Q & \text{if } s \in (0,1) \\ I + K(l^p) & \text{if } s = \infty \end{cases}$$

and so their invertibility is either evident or a consequence of condition (b) in this proposition.

The case $b = 0$, i.e. the pure Mellin operator, is of particular interest. In this setting, the conditions in Proposition 6 reduce to the following ones:

(a)' *the Mellin operator $A = \chi_{[0,1]}(aI + M^0(c))\chi_{[0,1]}I + \chi_{\mathbb{R}\setminus[0,1]}I$ is invertible on L^p*

(b)' *the operator $aP + G(c) + Q$ is Fredholm on l^p, and the operator*

$$\chi_{[1,\infty)}(aI + M^0(c))\chi_{[1,\infty)}I + \chi_{(-\infty,1]}I$$

is invertible on L^p

whereas condition (c) is already implied by (a). Examining conditions (a)' and (b)' via symbol calculus we finally get

Proposition 7: *Let $A = \chi_{[0,1]}(aI + M^0(c))\chi_{[0,1]}I + \chi_{\mathbb{R}\setminus[0,1]}I$. Then sequence (24) is stable if and only if*

(a) the function $a + c$ does not vanish on $\dot{\mathbb{R}}$

(b) the winding number of this function with respect to the origin is equal to 0.

It is rather interesting to observe that the conditions in this proposition are completely independent of the choice of the generating mother spline φ (whereas the function g in Proposition 6 depends heavily on φ). Thus, if one possible spline Galerkin method with cutting-off applies to the Mellin operator in condition (a) of Proposition 7, then all of these methods apply!

Example 3. As a last application we consider the operator

$$(Au)(t) = u(t) + \frac{1}{\pi i}\int_0^1 \frac{u(s)ds}{s+t} + \frac{1}{\pi i}\int_0^1 \frac{u(s)ds}{2-s-t} \qquad (t \in [0,1]) \tag{25}$$

with two fixed singularities at 0 and 1 and, more generally, operators of the form

$$A = a_1 I + a_2 S_{\mathbb{R}} b_2 I + a_3 M^0(c_3) b_3 I + a_4 U_1 J M^0(c_4) J U_{-1} b_4 I \tag{26}$$

where $U_{\pm 1}$ are the shift operators $(U_{\pm 1}f)(t) = f(t \pm 1)$ and J is the flip operator $(Jf)(t) = f(-t)$. Further we suppose that $a_1(t) = 1$ for $t \in \mathbb{R} \setminus [0,1]$ and that $a_j(t) = b_j(t) = 0$ whenever $t \in \mathbb{R} \setminus [0,1]$ and $j > 1$. The operator (25) actually results from (26) by choosing $a_1 \equiv 1, a_2 \equiv 0, a_3 = a_4 = b_3 = b_4 = \chi_{[0,1]}$, and $c_3 = c_4 = n$ with $n(z) = 1/\sinh \pi(z + i/p)$.

For the approximative solution of the equation $Au = f$ we again consider a Galerkin method which cuts off the singular points 0 and 1:

$$(A_{ni}) = \Big(E_n Q_i V_n Q_i V_{-n} E_{-n} L_n A E_n Q_i V_n Q_i V_{-n} E_{-n}\Big). \tag{27}$$

This sequence is in $\mathcal{A}_0^T$. To verify this one has to prove that

$$E_{-n} L_n J M^0(c_4) J E_n - \tilde{J} G(c_4) \tilde{J} \in K(l^p)$$

where $\tilde{J}$ is the discrete flip operator sending (x_k) into (x_{-k-1}), and one has to take into account that the algebra T^p is invariant under the mapping $C \mapsto \tilde{J} C \tilde{J}$.

For the next proposition, we set for brevity

$$A(s) = a_1(s) + a_2(s)b_2(s)g \qquad \text{and} \qquad \hat{A}(s) = a_1(s) + a_2(s)b_2(s)\tilde{g}$$

$$B = a_3(0+0)b_3(0+0)c_3$$
$$C = a_4(1-0)b_4(1-0)c_4.$$

Proposition 8: *The method (27) for the operator (26) is stable if and only if*

(a) the operator A is invertible on L^p

(b) the operator $PT^0(A(0+0))P+G(B)+Q$ *is Fredholm on* l^p*, the Toeplitz operator* $PT^0(A(0+0))P+Q$ *is invertible on* l^p*, and the operator*

$$\chi_{[1,\infty)}\Big(a_1(0+0)I + a_2(0+0)b_2(0+0)S_{\mathbb{R}^+} + M^0(B)\Big)\chi_{[1,\infty)}I + \chi_{(-\infty,1]}I$$

is invertible on L^p

(c) the operators $T^0(A(s))$ *are invertible for all* $s \in [0,1]$ *(or, what is the same, the functions* $A(s)$ *are invertible)*

(d) the operator $PT^0(\hat{A}(1-0))P+G(C)+Q$ *is Fredholm on* l^p*, the Toeplitz operator* $PT^0(\hat{A}(1-0))P+Q$ *is invertible on* l^p*, and the operator*

$$\chi_{[1,\infty)}\Big(a_1(1-0)I - a_2(1-0)b_2(1-0)S_{\mathbb{R}^+} + M^o(C)\Big)\chi_{[1,\infty)}I + \chi_{(-\infty,1]}I$$

is invertible on L^p.

Conditions (a) – (d) correspond to the invertibility of $W_T(A_{ni}), W_0^T(A_{ni}), W_s^T(A_{ni})$ with $s \in (0,1)$, and $W_1^T(A_{ni})$, respectively. The invertibility of $W_s^T(A_{ni})$ for $s \in \mathbb{R}\backslash[0,1]$ is obvious since

$$W_s^T(A_{ni}) = \begin{cases} I & \text{if } s \in \mathbb{R}\setminus[0,1] \\ I + K(l^p) & \text{if } s = \infty \end{cases}.$$

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