On the Oscillatory Behaviour of Solutions
of
Second Order Nonlinear Difference Equations

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Abstract. By using simple discrete inqualities sufficient conditions are provided for the solution $\{y_n\}$ of a difference equation of the form $\triangle(a_n \triangle y_n) + q_{n+1} f(y_{n+1}) = r_n$ $(n \in \mathbb{N}_0; \{a_n\}, \{q_n\}, \{r_n\} \subset \mathbb{R}; f : \mathbb{R} \rightarrow \mathbb{R})$ to be oscillatory or to satisfy $\liminf_{n \to \infty} |y_n| = 0$. Also two other results are established for all solutions of this equation to be oscillatory when $r_n = 0$ for all $n \in \mathbb{N}_0$.

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1. Introduction

The problem of oscillation and non-oscillation of solutions of difference equations has received a great amount of attention in the last few years (see, e.g., [1 - 6, 9, 11 - 15, 17 - 20, 22 - 31, 33 - 48, 51] and the references cited therein). It is interesting to study second order non-linear difference equations because they are discrete analogues of differential equations having physical applications as evidenced, i.e., by [7, 32, 49, 50].

In [3] Szmanda considered the difference equation

$$\triangle(a_n \triangle y_n) + q_n f(y_n) = 0 \qquad (E_1)$$

$(n \in \mathbb{N}_0; \{a_n\}, \{y_n\}, \{q_n\} \subset \mathbb{R}; f : \mathbb{R} \rightarrow \mathbb{R})$ where f is a continuous function such that $uf(u) > 0$ for $u \neq 0$ and the sequence $\{q_n\}$ takes positive as well as negative values for sufficiently large n. He obtained the following

Theorem A: *Let $\{a_n\}$ be a non-decreasing sequence of positive reals and*

$$\lim_{n \to \infty} \frac{1}{a_n} \sum_{s=n_0}^{n-1} s(\mu q_s^+ + q_s^-) = \infty$$

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where $\mu > 0$,

$$q^+ = \max\{q, 0\} \qquad and \qquad q^- = \min\{q, 0\}.$$

Then every bounded solution $\{y_n\}$ of equation (E_1) is either oscillatory or has the property $\liminf_{n \to \infty} |y_n| = 0$.

Hooker and Patula [13] considered the difference equation

$$\Delta^2 y_{n-1} + q_n y_n^\gamma = 0 \qquad (E_2)$$

$(n \in \mathbb{N}_0; \{q_n\} \subset \mathbb{R}; \gamma \in \mathbb{R})$ and proved the following

Theorem B: *Let $\{q_n\} \subset \mathbb{R}$ be a positive sequence such that $q_n > 0$ for $n \geq N$ $(1 < N \in \mathbb{N})$ and let γ be a quotient of positive integers. Then the difference equation (E_2) is oscillatory if and only if*

$$\sum_{n=0}^{\infty} n q_n = \infty \quad in \ case \ \gamma > 1 \qquad and \qquad \sum_{n=0}^{\infty} n^\gamma q_n = \infty \quad in \ case \ 0 < \gamma < 1.$$

Kulenovic and Budincevic [14] considered the difference equation

$$\Delta(a_n \Delta y_n) + q_{n+1} f(y_{n+1}) = 0 \qquad (E_3)$$

$(n \in \mathbb{N}_0; \{a_n\}, \{q_n\} \subset \mathbb{R}; f : \mathbb{R} \to \mathbb{R})$ and they generalized some of the results of Hooker and Patula [13].

In all the above results, the authors obtained conditions for the oscillation of all solutions by assuming some sign condition on the sequence $\{q_n\}$ and only for the oscillation of bounded solutions no sign condition on that sequence is assumed. For details one can refer to the recent monograph by Agarwal [1].

Here we consider the second order difference equation

$$\Delta(a_n \Delta y_n) + q_{n+1} f(y_{n+1}) = r_n \qquad (E)$$

$(n \in \mathbb{N}_0; \{a_n\}, \{q_n\} \subset \mathbb{R}; F : \mathbb{R} \to \mathbb{R})$ where the sequence $\{q_n\}$ is allowed to change signs and we give sufficient conditions for any solution $\{y_n\}$ of equation (E) to be either oscillatory or to satisfy the condition $\liminf_{n \to \infty} |y_n| = 0$. Two other results give sufficient conditions for all solutions of equation (E) to be oscillatory in the case when $r_n = 0$ for all n. The results presented here differ in several aspects from those of other authors due firstly to the fact that the sequence $\{q_n\}$ can change signs and secondly to the fact that our results will cover also unbounded solutions. Examples illustrating some of our results are also inserted. The results obtained here are motivated from those of [8] and [16]. For general background on difference equations see [1, 10, 21].

2. Some basic lemmas

Consider the difference equation

$$\Delta(a_n \Delta y_n) + q_{n+1} f(y_{n+1}) = r_n \tag{1}$$

$(n \in I\!N_0; \{a_n\}, \{q_n\}, \{r_n\} \subset I\!R; f : I\!R \to I\!R)$ where $a_n > 0$ for all $n \in I\!N_0$. By a *solution* of equation (1) we mean a non-trivial sequence $\{y_n\} \subset I\!R$ satisfying (1) for all $n \in I\!N_0$. A solution $\{y_n\}$ of equation (1) is said to be *oscillatory* if it is neither essentially positive nor essentially negative and it is said to be *non-oscillatory* otherwise.

The following conditions will be utilized as they are needed:

$$\sum_{n=0}^{\infty} 1/a_n = \infty \tag{2}$$

$$u f(u) > 0 \qquad \text{for all } u \neq 0 \tag{3}$$

$$f(u) - f(v) = g(u,v)(u-v) \quad \text{for } u, v \neq 0 \ (g \text{ a non-negative function}) \tag{4}$$

$$\sum_{n=0}^{\infty} |r_n| < \infty. \tag{5}$$

We let

$$Z_{n_0}^{\alpha} = \left\{ n_0, n_0 + 1, \ldots, \alpha \right\}$$

where $\alpha, n_0 \in I\!N_0$ are such that $n_0 < \alpha$ or $\alpha = \infty$. In the last case $Z_{n_0}^{\alpha}$ is denoted by Z_{n_0}. For convenience we assume empty sums to be zero and empty products to be one. Also, to simplify notation we let $w_n = a_n \Delta y_n$ for any non-oscillatory solution $\{y_n\}$ of equation (1).

In this section we present two lemmas which are interesting in their own right and which will be used in the proofs of our main results given in Section 3.

Lemma 1: *Let the function*

$$K = K(n,s,y) : Z_{n_0} \times Z_{n_0} \times I\!R^+ \longrightarrow I\!R$$

be such that, for fixed n and s, the function $K(n,s,\cdot)$ is non-decreasing. Let $\{P_n\} \subset I\!R$ be a given sequence and $\{y_n\}, \{x_n\}$ be sequences defined on $n \in Z_{n_0}$ and satisfying, for all $n \in Z_{n_0}$,

$$y_n \geq P_n + \sum_{s=n_0}^{n-1} K(n,s,y_s) \qquad \text{and} \qquad x_n = P_n + \sum_{s=n_0}^{n-1} K(n,s,x_s). \tag{6}$$

Then $x_n \leq y_n$ for all $n \in Z_{n_0}$.

Proof: When $n = n_0$, the result is obvious. Suppose there exists an integer $t \in Z_{n_0}$ such that

$$x_{t+1} > y_{t+1} \qquad \text{and} \qquad x_s < y_s \quad \text{for all } s \leq t.$$

Then

$$x_{t+1} - y_{t+1} \leq \sum_{s=n_0}^{n-1} \Big(K(t+1,s,x_s) - K(t+1,s,y_s) \Big) \leq 0$$

which is a contradiction ∎

Remark: The importance of the lemma is that once the discrete inequality in (6) is known, then a lower bound $\{x_n\}$ can be found by replacing the inequality by an equation and solving the latter.

The following lemma is a discrete analogue of [8: Lemma 2].

Lemma 2: *Suppose that conditions (3) and (4) hold. Let $\{y_n\}$ be a positive (negative) solution of difference equation (1) for $n \in Z^\alpha_{n_{<0}}$, for some positive integer N_1 such that $n_0 \leq N_1 < \alpha < \infty$. Let there exist $N \in Z^\alpha_{N_1}$ and a positive constant m such that*

$$-\frac{w_{N_1}}{f(y_{N_1})} + \sum_{s=N_1}^{n-1}\left(q_{s+1} - \frac{r_s}{f(y_{s+1})}\right) + \sum_{s=N_1}^{N-1}\frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s)f(y_{s+1})} \geq m \qquad (7)$$

for all $n \in Z^\alpha_N$. Then

$$w_n \leq -mf(y_N) \qquad or \qquad w_n \geq -mf(y_N)$$

for all $n \in Z^\alpha_N$.

Proof: Since

$$\frac{\triangle w_n}{f(y_{n+1})} + q_{n+1} = \frac{r_n}{f(y_{n+1})},$$

we have

$$-\frac{w_n}{f(y_n)} = -\frac{w_{N_1}}{f(y_{N_1})} + \sum_{s=N_1}^{n-1}\left(q_{s+1} - \frac{r_s}{f(y_{s+1})}\right) + \sum_{s=N_1}^{n-1}\frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s)f(y_{s+1})}$$

for all $n \in Z^\alpha_{N_1}$. Thus from (7) we see that

$$-\frac{w_n}{f(y_n)} \geq m + \sum_{s=N}^{n-1}\frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s)f(y_{s+1})} \qquad (8)$$

for all $n \in Z^\alpha_N$. Since the sum in (8) is non-negative, we have $a_n \triangle y_n \leq 0$ for all $n \in Z^\alpha_N$. If $\{y_n\}$ is positive, let $u_n = -a_n \triangle y_n$. Then (8) becomes

$$u_n \geq mf(y_n) + \sum_{s=N}^{n-1}\frac{f(y_n)g(y_s, y_{s+1})(-\triangle y_s)}{f(y_s)f(y_{s+1})}u_s.$$

Define

$$K(n, s, x) = \frac{f(y_n)g(y_s, y_{s+1})(-\triangle y_s)}{f(y_s)f(y_{s+1})}x \qquad \left(n, s \in Z^\alpha_N; x \in \mathbb{R}^+\right).$$

Notice that, for each fixed n and s, the function $K(n, s, \cdot)$ is non-decreasing. Hence Lemma 1 applies with $p_n = mf(y_n)$, to obtain

$$v_n = mf(y_n) + \sum_{s=N}^{n-1}K(n, s, v_s)$$

provided $v_s \in \mathbb{R}^+$ for each $s \in Z^\alpha_N$. Multiplying the last equation by $1/f(y_n)$ and then applying the operator $\triangle$ we obtain $\frac{\triangle v_n}{f(y_n)} = 0$ so that $v_n = v_N = mf(y_N)$ for all $n \in Z^\alpha_N$. Thus by Lemma 1, $a_n \triangle y_n \leq -mf(y_N)$ for all $n \in Z^\alpha_N$.

The proof for the case when $\{y_n\}$ is negative follows from a similar argument by taking $u_n = a_n \triangle y_n$ and $p_n = -mf(y_n)$ ∎

3. Oscillatory and asymptotic behaviour

The first result is concerned with situations when the solutions of equation (1) are bounded away from zero and it generalizes Theorem 2 in [44].

Theorem 1: *Suppose that conditions (2) - (5) hold and that*

$$\sum_{s=n_0}^{\infty} q_{s+1} \quad converges \tag{9}$$

and

$$g(u, v) \geq \mu > 0 \quad for \ all \ \ u, v \neq 0. \tag{10}$$

Let $\{y_n\}$ be a solution of the difference equation (1) such that $\liminf_{n \to \infty} |y_n| > 0$. Then

$$\sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} \quad converges \tag{11}$$

for all sufficiently large n,

$$\frac{w_n}{f(y_n)} \to 0 \quad as \ \ n \to \infty \tag{12}$$

and

$$\frac{w_n}{f(y_n)} = \sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} + \sum_{s=n}^{\infty} \left(q_{s+1} - \frac{r_s}{f(y_{s+1})} \right) \tag{13}$$

for sufficiently large n.

Proof: There exist $m_1, m_2 > 0$ and an integer $n_1 > n_0$ such that $|y_n| \geq m_1$ and $|f(y_n)| \geq m_2$ for $n \in Z_{n_1}$. This, together with (5) implies that

$$\left| \sum_{s=n_1}^{n} \frac{r_s}{f(y_{s+1})} \right| \leq \sum_{s=n_1}^{n} \left| \frac{r_s}{f(y_{s+1})} \right| \leq m_3 \tag{14}$$

for some $m_3 > 0$ and all $n \in Z_{n_1}$. Now, suppose that (11) does not hold. Then, in view of (9) there exist $m > 0$ and an integer $n_2 \geq n_1$ such that (7) hold for all $n \in Z_{n_2}$.

If $\{y_n\}$ is positive for all $n \in Z_{n_1}$, it follows from Lemma 2 and its proof that

$$\triangle y_n < 0 \quad and \quad a_n \triangle y_n \leq -m f(y_{n_2}) \quad for \ \ n \in Z_{n_2}.$$

After summing we have

$$y_n \leq y_{n_2} - m f(y_{n_2}) \sum_{s=n_2}^{n-1} \frac{1}{a_s}$$

which in view of (2) contradicts the fact that $y_n > 0$ for $n \in Z_{n_1}$.

A similar argument handles the case when $\{y_n\}$ is negative for all $n \in Z_{n_1}$. Since

$$\Delta\left(\frac{w_n}{f(y_n)}\right) + \frac{w_n^2 g(y_n, y_{n+1})}{a_n f(y_n) f(y_{n+1})} = \frac{r_n}{f(y_{n+1})} - q_{n+1}$$

we have

$$\frac{w_k}{f(y_k)} + \sum_{s=n}^{k-1} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} = \frac{w_n}{f(y_n)} + \sum_{s=n}^{k-1}\left(\frac{r_s}{f(y_{s+1})} - q_{s+1}\right). \tag{15}$$

From (9), (11), (14) and (15) we see that $\beta := \lim_{n\to\infty} w_k/f(y_k)$ exists so that from (15) we have

$$\frac{w_n}{f(y_n)} = \beta + \sum_{s=n}^{\infty}\left(q_{s+1} - \frac{r_s}{f(y_{s+1})}\right) + \sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} \tag{16}$$

for $n \in Z_{n_1}$. To show that (12) and (13) hold, we have to show that $\beta = 0$. Suppose first that $y_n > 0$ for all $n \in Z_{n_1}$. If $\beta < 0$, then because of (10), (11) and (15) there exists an integer $N_1 \geq n_1$ such that

$$\left|\sum_{s=N_1}^{n-1} q_{s+1}\right| \leq -\frac{\beta}{6} \quad \text{and} \quad \left|\sum_{s=N_1}^{n-1} \frac{r_s}{f(y_{s+1})}\right| \leq \frac{\beta}{6}$$

and

$$\left|\sum_{s=N_1}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})}\right| \leq -\frac{\beta}{6} \quad \text{for all } n \in Z_{N_1}.$$

From (16) we see that (7) holds on Z_{N_1} with $N = N_1$. But then, as argued above, Lemma 2 and (2) contradict the assumption that $y_n > 0$ for all $n \in Z_{n_1}$. If $\beta > 0$, it follows from (9), (11), (14) and (16) that $w_n/f(y_n) \to \beta$ as $n \to \infty$, so there exists an integer $N_2 > n_1$ such that $w_n/f(y_n) \geq \beta/2$ for all $n \in Z_{N_2}$. We use (4) and (10) to obtain

$$\frac{w_n g(y_n, y_{n+1})}{a_n f(y_{n+1})} \geq \frac{\mu\beta}{2a_n + \mu\beta}.$$

Thus

$$\sum_{s=n_1}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} > \lim_{n\to\infty} \sum_{s=n_1}^{n} \frac{\mu\beta^2}{2(2a_s + \mu\beta)} = \infty$$

by condition (2), which contradicts (11). This completes the proof that $\beta = 0$ for the case $y_n > 0$ for all $n \in Z_{n_1}$. The proof of $\beta = 0$ for the case that $y_n < 0$ for all $n \in Z_{n_1}$ is similar and will be omitted ∎

Consider the difference equation

$$\Delta^2 y_n + \frac{(-1)^n}{(n+1)^5} y_{n+1}^3 = \frac{(-1)^n}{(n+1)^2} \quad (n \in I\!N). \tag{E_4}$$

It is easy to verify that all conditions of Theorem 1 are satisfyed for this equation. Hence every non-oscillatory solution of this equation satisfies (11) - (13). One such solution is $\{y_n\} = \{n\}$ satisfying the condition $\liminf_{n \to \infty} |y_n| > 0$.

Before stating our next theorem, we observe that if conditions (5) and (9) hold, then

$$h_0(n) = \sum_{s=n}^{\infty} (q_{s+1} - \sigma|r_s|) \qquad (n \in Z_{n_0})$$

is well defined for every positive constant σ (i.e. the above series converges) and $h_0(n) \geq 0$ for all sufficiently large n. As long as the above series converges we can define

$$h_1(n) = \sum_{s=n}^{\infty} \frac{(h_0(s))^2}{a_s + Lh_0(s)} \qquad \text{and} \qquad h_{m+1}(n) = \sum_{s=n}^{\infty} \frac{(h_0(s) + Lh_m(s))^2}{a_s + L(h_0(s) + Lh_m(s))}$$

for $m \in I\!N$, where L is a positive constant. Now in the next two theorems we need the condition

(H) For every constant $L > 0$, there exists a positive integer M such that h_m exists for $m = 1, 2, \ldots, M - 1$ and h_M does not exist.

Theorem 2: *Suppose that conditions (2) - (5), (9) and (H) hold and, for any $\sigma_1 > 0$, there exists $\sigma_2 > 0$ such that*

$$g(u, v) \geq \sigma_2 \qquad \text{for all} \quad |u|, |v| \geq \sigma_1. \tag{17}$$

Then any solution $\{y_n\}$ of the difference equation (1) is either oscillatory or satisfies the condition $\liminf_{n \to \infty} |y_n| = 0$.

Proof: Assume that the conclusion of the theorem is false. Then there is a non-oscillatory solution $\{y_n\}$ of equation (1) such that $\liminf_{n \to \infty} |y_n| > 0$. It then follows from (4) that $|f(y_n)| \geq d$ for some $d > 0$ and all $n \in Z_{n_1}$. From (13) and (14) we have

$$\frac{w_n}{f(y_n)} \geq h_o(n) + \sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} \tag{18}$$

for all $n \in Z_{n_1}$ and from (12) we have

$$\sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} < \infty$$

for all $n \in Z_{n_1}$. Thus from (17) we have $w_n/f(y_n) \geq h_0(n) \geq 0$ and using (17) we then have

$$\frac{w_n^2 g(y_n, y_{n+1})}{a_n f(y_n) f(y_{n+1})} \geq \frac{L(h_0(n))^2}{a_n + Lh_0(n)} \tag{20}$$

for $n \in Z_{n_1}$ and some $L > 0$. If $M = 1$, then (19) and (20) imply that the series

$$h_1(n) = \sum_{s=n}^{\infty} \frac{(h_0(s))^2}{a_s + Lh_0(s)}$$

converges which contradicts the non-existence of $h_M(n) = h_1(n)$. If $M = 2$, then from (18) and (20) we have

$$\frac{w_n}{f(y_n)} \geq h_0(n) + L \sum_{s=n}^{\infty} \frac{(h_0(s))^2}{a_s + Lh_0(s)} = h_0(n) + Lh_1(n)$$

from which it follows that

$$\frac{w_n^2 g(y_n, y_{n+1})}{a_n f(y_n) f(y_{n+1})} \geq \frac{L(h_0(s) + Lh_1(n))^2}{a_n + L(h_0(n) + Lh_1(n))}.$$

Thus in view of (19), a summation of the last inequality would give a contradiction to the non-existence of $h_M(n) = h_2(n)$. A similar argument provides a contradiction for any integer $M > 2$ ∎

As an example of an equation satisfying the hypotheses of Theorem 2, consider

$$\Delta^2 y_n + \left(\frac{(1 + 1/n)^{1/2} - 1}{(n+1)^{1/2}} \right) y_{n+1}^3 = \frac{2}{n(n+1)(n+2)} + \frac{(1 + 1/n)^{1/2} - 1}{(n+1)^{7/2}} \qquad (E_5)$$

($n \in I\!N$) which has the non-oscillatory solution $\{y_n\} = \{1/n\}$. Here $\sum_{s=n}^{\infty} q_{s+1} = n^{-1/2}$. Now

$$|r_n| \leq \frac{2}{n(n+1)(n+2)} + \frac{(1 + 1/n)^{3/2} - 1}{(n + 1/2)^{3/2}}$$

so $\sum_{s=n}^{\infty} |r_s| \leq 2/n^{3/2}$ and hence $h_0(n) \geq 0$ for sufficiently large n. Since

$$\sum_{s=n}^{\infty} \frac{(h_0(s))^2}{a_s + Lh_0(s)} \geq \sum_{s=n}^{\infty} \frac{(1/s^{1/2} - 2p/s^{3/2})^2}{1 + L/s^{1/2}} = \sum_{s=n}^{\infty} \frac{s^{1/2}(1/s^{1/2} - 2p/s^{3/2})^2}{s^{1/2} + L} = \infty$$

we have $M = 1$.

Our next two theorems are oscillation results for the case $r_n = 0$ ($n \in I\!N$). Observe that in this case the difference equation (1) becomes

$$\Delta(a_n \Delta y_n) + q_{n+1} f(y_{n+1}) = 0 \qquad (21)$$

and $h_0(n) = \sum_{s=n}^{\infty} q_{s+1}$.

Theorem 3: *Suppose that conditions (2) - (4), (9), (10) and (H) hold. Then all solutions of the difference equation (21) are oscillatory.*

Proof: Let $\{y_n\}$ be a non-oscillatory solution of equation (21). Then there exists an integer $n_1 \geq n_0$ such that $|y_n| > 0$ for all $n \in Z_{n_1}$. Since (10) implies that $f(u)$ is strictly increasing for $u \neq 0$, we have $|f(y_n)| > 0$ for all $n \in Z_{n_1}$. It is easy to see that Lemma 2 is valid for equation (21) with condition (7) replaced by

$$-\frac{w_{N_1}}{f(y_{N_1})} + \sum_{s=N_1}^{n-1} q_{s+1} + \sum_{s=N_1}^{N-1} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} \geq m.$$

Proceeding as in the proof of Theorem 1, we again obtain (11) since (13) obviously holds. Using (15) with $r_n = 0$ for all n and continuing as in the proof of Theorem 1, we again obtain (12) and (13) with $r_n = 0$ for all sufficiently large n. The remaining part of the proof is similar to that of Theorem 2 and hence will be omitted ∎

Remark: Notice that obtaining (11) - (13) in the proof of Theorem 3 extends Theorem 2 of [43].

Theorem 3 implies that all solutions of the difference equation

$$\Delta^2 y_n + \left(\frac{(1+1/n)^{1/2} - 1}{(n+1)^{1/2}}\right)(y_{n+1} + y_{n+1}^3) = 0 \qquad (n \in I\!\!N) \qquad (E_6)$$

are oscillatory.

In the following theorem $h_0(n) = A(n) = \sum_{s=n}^{\infty} q_{s+1}$ need not be non-negative.

Theorem 4: *Suppose that conditions (2) - (4), (9) and (11) hold. Further assume*

$$(a_n + \mu A(n)) > 0 \qquad (n \in Z_{n_0}). \tag{22}$$

If the series

$$\sum_{n=n_0}^{\infty} \frac{A_+^2(n)}{a_n + \mu A(n)} \qquad where \quad A_+(n) = \max\{A(n), 0\}$$

diverges, then all solutions of the difference equation (21) are oscillatory.

Proof: Suppose that the difference equation (21) has a non-oscillatory solution $\{y_n\}$. Then there exists an integer $n_1 \geq n_0$ such that $|y_n| > 0$ for all $n \in Z_{n_1}$. Since (10) implies that $f(u)$ is strictly increasing for $u \neq 0$, we have $|f(y_n)| > 0$ for all $n \in Z_{n_1}$. And hence, by Theorem 3, we have

$$\frac{w_n}{f(y_n)} \geq A(n) \qquad (n \in Z_{n_1}) \tag{24}$$

and condition (11) is fulfilled. Use (4), (10) and (24) to get

$$\frac{w_n g(y_n, y_{n+1})}{a_n f(y_{n+1})} \geq \frac{A(n)}{a_n + \mu A(n)} \qquad (n \in Z_{n_1}). \tag{25}$$

Now from (24) and (25), we obtain

$$\sum_{s=n}^{\infty} \frac{w_s^2 g(y_s, y_{s+1})}{a_s f(y_s) f(y_{s+1})} \geq \sum_{s=n}^{\infty} \frac{(A_+(s))^2}{a_s + \mu A(s)} \qquad (n \in Z_{n_1})$$

which contradicts (19) ∎

Remark: When $a_n = 1$ for all $n \in I\!\!N$, Theorem 4 reduces to Theorem 3 of [43]. Also Theorem 4 is a discrete analogue of Theorem 3 given in [16] but the conditions and the proof are different from the continuous case.

The following theorem is a discrete analogue of a result of Waltman [49] but the proof is different from the continuous case.

Theorem 5: *In addition to conditions (2) - (4), assume that the sequence $\{q_n\}$ in the difference equation (1) satisfies the condition*

$$\sum_{s=n_0}^{\infty} q_n = \infty. \tag{26}$$

Then every solution of the difference equation (21) is oscillatory.

Proof: Let to the contrary $\{y_n\}$ be a non-oscillatory solution of equation (21) which we may (and we do) assume to be positive on Z_{n_0}. In view of (26), condition (7) is satisfied on Z_{n_1} for some sufficiently large n_1. Thus from Lemma 2 $w_n \leq -m f(y_{n_1})$ for all $n \in Z_{n_1}$ which after summing yields $y_n \leq y_{n_1} - m f(y_{n_1}) \sum_{s=n_1}^{n-1} \frac{1}{a_s}$. Thus in view of condition (2), $y_n \to -\infty$ as $n \to \infty$ which contradicts the fact that $y_n > 0$ for all $n \in Z_{n_1}$. The proof for the case when the solution $\{y_n\}$ is negative is similar and hence will be omitted ∎

The difference equation

$$\triangle(n\triangle y_n) + 2^{2n+1}(qn + 3)y_{n+1}^3 = 0 \qquad (n \in I\!N) \tag{E_6}$$

satisfies all conditions of Theorem 5 and hence every solution of equation (E_6) is oscillatory. One such solution is $\{y_n\} = \{(-1)^n/2^n\}$.

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