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An Embedding Theorem for Functions whose Fourier Transforms are Weighted Square Summable

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Abstract. Our main result is, by elementary means only, an embedding theorem for functions f whose Fourier transforms $\tilde{f}$ are weight square summable, i.e. $\int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda)\, d\lambda < \infty$, where the weight function $w : \mathbb{R}^n \to (0, \infty)$ satisfies the Kolmogorov condition $\int_{\mathbb{R}^n} w(\lambda)^{-1} d\lambda < \infty$. The necessary and sufficient condition on a non-negative Borelian measure on $\mathbb{R}^n$ is given, for the inequality $\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt) \leq A \int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda) d\lambda$ to be hold for every such function f, with a constant $A < \infty$ not depending on f.

Keywords: *Embedding theorems, Fourier analysis, weighted functional spaces*

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1. The space W_ω^2

Let us denote $\mathbb{R}^n$ the n-dimensional real space, $\mathbb{C}^n$ the n-dimensional complex space, $\mathbb{Z}$ the set of all integers and m the standard Lebesgue measure on $\mathbb{R}^n$. If $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, then $|t| = \max_{1 \leq j \leq n} |t_j|$. If $t = (t_1, \ldots, t_n), \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, then $t \cdot \lambda = t_1\lambda_1 + \ldots + t_n\lambda_n$.

In our paper there appears a Borelian function $w : \mathbb{R}^n \to (0, \infty)$. This function is said to be a *weight function*. In what follows we usually assume that the following condition (**K**)

$$K_w := \int_{\mathbb{R}^n} \frac{m(d\lambda)}{w(\lambda)} < \infty \tag{1.1}$$

is satisfied which is called *Kolmogorov condition* because a similar condition was introduced by A. N. Kolmogorov in the theory of stationary random processes with discrete time (see [10: §10] and [11: §5]). In this theory the density of the spectral function of a random process plays the role of such a function w. The constant K_w defined by (1.1) is said to be the *Kolmogorov constant of the weight function* w.

Now we introduce the space W_w^2. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a (generalized, in general) function and $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ its Fourier transform, i.e.

$$f(t) = \int_{\mathbb{R}} \tilde{f}(\lambda) e^{it\cdot\lambda} m(d\lambda) \qquad (t \in \mathbb{R}^n). \tag{1.2}$$

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Definition 1.1: A function f belongs to the *space* W^2_w if its Fourier transform $\tilde{f}$ is a function belonging to the weighted space L^2_w, i.e.

$$\int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda)\, d\lambda < \infty. \tag{1.3}$$

The space W^2_w is the set of all such functions f.

It is clear that the set W^2_w is a complex vector space. By definition, the integral on the left-hand side of (1.3) is the square of the norm of f, i.e.

$$\|f\|_{W^2_w} := \left\{ \int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda)\, m(d\lambda) \right\}^{1/2}. \tag{1.4}$$

This norm is generated by the scalar product

$$\langle f, g\rangle_{W^2_w} := \int_{\mathbb{R}^n} \tilde{f}(\lambda)\overline{\tilde{g}(\lambda)} w(\lambda)\, m(d\lambda). \tag{1.5}$$

The scalar product (1.5) and the norm (1.4) are translation invariant: if $f \in W^2_w$ and $\tau \in \mathbb{R}^n$ and if f_τ is the "translated" function, i.e. $f_\tau(t) = f(t+\tau)$, then $\langle f, g\rangle_{W^2_w} = \langle f_\tau, g_\tau\rangle_{W^2_w}$ for all $\tau \in \mathbb{R}^n$. In particular, $\|f_\tau\| = \|f\|$ for all $\tau \in \mathbb{R}^n$.

The space W^2_w was defined as a space of (generalized) functions which is isometrically isomorphic to the weighted space L^2_w. Since the latter is complete, the space W^2_w is complete as well. Thus, the space W^2_w endowed with the scalar product (1.5) is a Hilbert space. The Schwarz inequality yields

$$\int_{\mathbb{R}^n} |\tilde{f}(\lambda)|\, m(d\lambda) \le K_w^{1/2} \|f\|_{W^2_w} \tag{1.6}$$

where K_w is the Kolmogorov constant of the weight function w.

The following lemma is a direct consequence of the inequality (1.6) and of standard results from the theory of Fourier integrals.

Lemma 1.1: *Let w be a weight function satisfying the Kolmogorov condition (K) (with the Kolmogorov constant K_w). Then each element $f \in W^2_w$ is a function which is continuous, and which is bounded on $\mathbb{R}^n$, i.e.*

$$\max_{t\in\mathbb{R}^n} |f(t)| \le K_w^{1/2} \|f\|_{W^2_w}. \tag{1.7}$$

Moreover, $\lim_{|t|\to\infty} f(t) = 0$.

Spaces W^2_w (under various assumptions for a weight function w) were considered by L. Volevich and B. Paneyakh [20]. In the special case

$$w(\lambda) = (1 + |\lambda|^2)^l \qquad (\lambda \in \mathbb{R}^n) \tag{1.8}$$

the space W^2_w is nothing but the well-known Sobolev-Slobodetskiĭ space. For the weight function (1.8) the condition (**K**) is fulfilled if and only if $l > \frac{n}{2}$.

2. The main embedding theorem

Let w be a weight function which satisfies the Kolmogorov condition (K). Then each function $f \in W_w^2$ is continuous and bounded on $\mathbb{R}^n$ (Lemma 1.1). Let $\mu(dt) \geq 0$ be a non-negative Borelian measure on $\mathbb{R}^n$, and let the total variation $\mathrm{var}_{\mathbb{R}^n}\mu$ be finite. In accordance with the estimate (1.7),

$$\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt) \leq K_w(\mathrm{var}_{R^n}\mu)\|f\|^2_{W^2_{w_r}}. \tag{2.1}$$

However, the estimate (2.1) is too rough. The condition $\mathrm{var}_{\mathbb{R}^n}\mu < \infty$ is not necessary for the integral $\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt)$ to be finite.

Let Q be the standard unit cube, i.e.

$$Q = \left\{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n \middle| -\frac{1}{2} \leq t_j < +\frac{1}{2} \ (1 \leq j \leq n) \right\}$$

and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ let $Q_\alpha = Q + \alpha$ be the "translated" cube, i.e.

$$Q_\alpha = \left\{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n \middle| -\frac{1}{2} \leq t_j - \alpha_j < +\frac{1}{2} \ (1 \leq j \leq n) \right\}. \tag{2.2}$$

Definition 2.1: A non-negative Borelian measure μ on $\mathbb{R}^n$ satisfies the *Carleson condition* **(C)** if

$$C_\mu := \sup_{\alpha \in \mathbb{R}^n} \mu(Q_\alpha) < \infty. \tag{2.3}$$

The constant C_μ defined by (2.3) is said to be the *Carleson constant of the measure* μ.

Lemma 2.1: *Let w be a weight function which satisfies the Kolmogorov condition (K), and let $\mu \geq 0$ be a Borelian measure on $\mathbb{R}^n$. If $\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt)$ is finite for each function $f \in W_w^2$, then μ satisfies the Carleson condition (C).*

Proof: Let us consider the Hilbert space $L^2(\mu)$ of square summable functions with respect to μ and the embedding operator from W_w^2 into $L^2(\mu)$. Using Lemma 1.1, we obtain that the embedding operator is closed. Hence, the inequality $\int_{R^n} |f(t)|^2 \mu(dt) \leq A\|f\|^2_{W^2_w}$ holds where $A < \infty$ is a constant not depending on $f \in W_w^2$. Let $0 \not\equiv f_0 \in W_w^2$. Without loss of generality, we can assume that $f_0(0) = 1$ (we can "translate" an original function f_0 and multiply it by the appropriate constant). Since the function f_0 is continuous, there exists a "ball" (or a cube) $B = \{t \in \mathbb{R}^n : |t| \leq \rho\}$ (ρ is a positive number) such that $|f_0(t)| \geq 1/2$ for all $t \in B$. The function $f_{-\alpha}$, $f_{-\alpha}(t) = f(t - \alpha)$ satisfies the condition

$$|f_{-\alpha}(t)| \geq \frac{1}{2} \qquad (t \in B_\alpha) \tag{2.4}$$

where $B_\alpha = B + \alpha$ is the "translated" ball B, i.e.

$$B_\alpha = \left\{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n \middle| |t_j - \alpha_j| \leq \rho \ (1 \leq j \leq n) \right\}.$$

Since the W_w^2-norm is translation invariant, $\|f_\alpha\|_{W_w^2} = \|f_0\|_{W_w^2}$. By (2.4), the inequality

$$\int_{\mathbb{R}^n} |f_\alpha(t)|^2 \mu(dt) \leq A\|f_0\|^2_{W_w^2}$$

holds, where the right-hand side does not depend on $\alpha \in \mathbb{R}^n$. Using (2.4), we obtain the inequality

$$\mu(B_\alpha) \leq 2A\|f_0\|^2_{W_w^2} \qquad \text{for all } \alpha \in \mathbb{R}^n. \tag{2.5}$$

This inequality is almost the same as the Carleson condition (C) (or (2.3)). The only difference is that in (2.3) there appears the translated "unit" cube Q_0, and in (2.5) there appears the translated cube B which radius is equal ρ. Standard "covering" reasoning leads to the inequality $\mu(Q_\alpha) \leq C$ for all $\alpha \in \mathbb{R}^n$, where $C \leq \left([\frac{1}{\rho}] + 1\right)^n 2A\|f_0\|^2_{W_l^2}$ (by $[\frac{1}{\rho}]$ we denote the integer part of the number $\frac{1}{\rho}$) ∎

Lemma 2.1 claims that the Carleson condition (C) for the measure $\mu \geq 0$ is *necessary* for the integral $\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt)$ to be finite for every $f \in W_w^2$. It turns out that under some regularity conditions for a weight function w the condition (C) is *sufficient* as well.

Definition 2.2: A weight function $w : \mathbb{R}^n \to (0, \infty)$ satisfies the *regularity condition* **(L)**, if its logarithm $\ln w$ satisfies the Lipschitz condition $L_{\ln w} < \infty$, where

$$L_{\ln w} = \sup_{\substack{\lambda', \lambda'' \in \mathbb{R}^n \\ \lambda' \neq \lambda''}} \frac{|\ln w(\lambda') - \ln w(\lambda'')|}{|\lambda' - \lambda''|}.$$

The constant $L_{\ln w}$ is called the *Lipschitz constant of the function* $\ln w$.

Now we formulate our main result.

The Main Embeding Theorem: *Let w be a weight function which satisfies the Kolmogorov condition (K) (with a Kolmogorov constant K_w). Moreover, suppose that the function $\ln w$ satisfies the Lipschitz condition (with a Lipschitz constant $L_{\ln w}$), and let μ be a non-negative Borelian measure on $\mathbb{R}^n$ which satisfies the Carleson condition (C) (with a Carleson constant C_μ). Then for every function $f \in W_w^2$ the inequality*

$$\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt) \leq E\|f\|^2_{W_w^2} \tag{2.6}$$

holds where $E = A^n C_\mu K_w e^{2L_{\ln w}}$ and $A < \infty$ is an absolute constant.

In principle, our Main Embedding Theorem can be derived from known results, for example from results stated in one of the monographs [14], [15] and [19]. However, these monographs are aimed at a presentation of very general theories and, from their character, can be classified into so-called 'hard analysis methods'. Moreover, the mentioned monographs are not self-contained. More precisely, some facts on maximal functions or from potential theory which are essential for these extremely general theories are not proved there. Our Main Embedding Theorem will be obtained by using rather elementary means. We want to give a very simple and self-contained proof. This is the main goal of this paper.

3. Auxiliary results

Our proof of the Main Embedding Theorem is based on two results which may be interesting themselves.

Let $\chi : I\!R \to \boldsymbol{C}$ be a function which admits an estimate

$$|\chi(\xi)| \le C_\chi \frac{1}{1+\xi^2} \qquad (\xi \in I\!R) \tag{3.1}$$

where $C_\chi < \infty$ is a constant, and let

$$\chi_n(t) = \chi(t_1)\chi(t_2) \cdots \chi(t_n) \qquad (t = (t_1, ..., t_n) \in I\!R^n). \tag{3.2}$$

We define the operator $\mathbf{X}_\alpha$, $\alpha = (\alpha_1, \dots, \alpha_n) \in I\!R^n$, in the following way. If $f : I\!R^n \to \boldsymbol{C}$ is a function from $L^2(m)$, *i.e.* $\int_{I\!R} |f(t)|^2 m(dt) < \infty$, then

$$(\mathbf{X}_\alpha f)(t) = \int_{I\!R} e^{i\alpha\cdot(t-\tau)} \chi_n(t-\tau) f(\tau) m(d\tau) \qquad (t \in I\!R^n). \tag{3.3}$$

Lemma 3.1: *Let $\mu(dt) \ge 0$ be a Borelian measure on $I\!R^n$ for which the Carleson condition (C) is satisfied with a Carleson constant C_μ (see (2.3)). Then the operator $\mathbf{X}_\alpha$ acts continuously from the space $L^2(m)$ into the space $L^2(\mu)$, and its norm does not exceed $(A_2^n C_\chi^{2n} C_\mu)^{1/2}$ where A_2 is an absolute constant:*

$$\int_{I\!R^n} |(\mathbf{X}_\alpha f)(t)|^2 \mu(dt) \le (A_2 C_\chi^2)^n C_\mu \int_{I\!R^n} |f(t)|^2 m(dt) \qquad (f \in L^2(I\!R^n)). \tag{3.4}$$

In particular, the right-hand side of (3.4) does not depend on α.

We derive Lemma 1 from the Schur inequality. Its proof will be given later, after this inequality.

Lemma 3.2: *Let $\mu(dt) \ge 0$ be a Borelian measure on $I\!R^n$ for which the Carleson condition (C) is satisfied with a Carleson constant C_μ (see (2.3)). Then*

$$\sup_{\tau \in I\!R^n} \int_{I\!R^n} |\chi_n(t-\tau)|\, \mu(dt) \le A_1^n C_\chi^n C_\mu \tag{3.5}$$

where A_1 is an absolute constant:

$$A_1 = 2 \sup_{\xi \in I\!R} \sum_{k \in Z\!\!Z} \frac{1}{(k-\xi)^2+1}. \tag{3.6}$$

Proof: As before, the cube Q_α is defined by (2.2). Because cubes Q_p, where $p = (p_1, \dots, p_n)$ runs over $Z\!\!Z^n$, form a tiling of the space $I\!R^n$, we can split the integral into a sum:

$$\int_{I\!R^n} |\chi_n(t-\tau)|\, \mu(dt) = \sum_{p \in Z\!\!Z^n} \int_{Q_p} |\chi_n(t-\tau)|, \mu(dt). \tag{3.7}$$

Further,

$$\int_{Q_p} |\chi_n(t-\tau)\,\mu(dt) \leq C_\mu \sup_{t\in Q_p} |\chi_n(t-\tau)|. \tag{3.8}$$

From (3.1), (3.2) it follows the estimate

$$\sup_{t\in Q_p} |\chi_n(t-\tau)| \leq (2C_\chi)^n \prod_{1\leq j\leq n} \frac{1}{1+(p_j-\tau_j)^2} \tag{3.9}$$

where $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$. From (3.7) - (3.9) we obtain

$$\int_{\mathbb{R}^n} |\chi_n(t-\tau)|\,\mu(dt) \leq (2C_\chi)^n \sum_{p\in\mathbb{Z}^n} \left(\prod_{1\leq j\leq n} \frac{1}{1+(p_j-\tau_j)^2} \right).$$

Thus, the estimate (3.4) holds with A_1 from (3.6) ∎

We derive Lemma 3.1 from the so-called

Schur Inequality: *Let $\mu(dt) \geq 0$ and $\nu(dt) \geq 0$ be Borelian measures on $\mathbb{R}^n$, and let $K(t,\tau)$ $((t,\tau) \in \mathbb{R}^n \times \mathbb{R}^n)$ be a kernel, i.e. a Borelian function. Assume that $M_1 < \infty$ and $M_\infty < \infty$ where*

$$M_1 := \sup_{\tau\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,\tau)|\,\mu(dt) \qquad \textit{and} \qquad M_\infty := \sup_{t\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,\tau)|\,\nu(d\tau).$$

Then the expression

$$(\mathbf{K}f)(t) = \int_{\mathbb{R}^n} K(t,\tau) f(\tau)\,\nu(d\tau) \tag{3.10}$$

determines a bounded operator $\mathbf{K}$ from the space $L^2(\nu)$ into the space $L^2(\mu)$, and

$$\|\mathbf{K}\|_{L^2(\nu)\to L^2(\mu)} \leq \sqrt{M_1 M_\infty}. \tag{3.11}$$

Proof: Rather than estimate the norm of the operator $\mathbf{K}$, we estimate the bilinear form

$$(\mathbf{K}f, g) = \iint_{\mathbb{R}^n\times R^n} K(t,\tau) f(\tau) g(t)\,\mu(dt)\nu(d\tau).$$

Obviously,

$$|(\mathbf{K}f, g)| \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} |K(t,\tau)|\,|f(\tau)|\,|g(t)|\,\mu(dt)\nu(d\tau).$$

Applying the Schwarz inequality, we obtain

$$|(\mathbf{K}f, g)|^2 \leq \left(\iint_{\mathbb{R}^n\times R^n} |K(t,\tau)|\,|f(\tau)|^2 \mu(dt)\nu(d\tau) \right)$$
$$\times \left(\iint_{\mathbb{R}^n\times R^n} |K(t,\tau)|\,g(t)|^2\,\mu(dt)\nu(d\tau) \right).$$

Clearly,

$$\iint_{\mathbb{R}^n \times R^n} |K(t,\tau)|\,|f(\tau)|^2\,\mu(dt)\nu(d\tau) \le \left(\sup_{\tau \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,\tau)|\,\mu(dt)\right) \int_{\mathbb{R}^n} |f(\tau)|^2 \nu(d\tau)$$

and

$$\iint_{\mathbb{R}^n \times R^n} |K(t,\tau)|\,|g(t)|^2 \mu(dt)\nu(d\tau) \le \left(\sup_{t \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,\tau)|\,\nu(d\tau)\right) \int_{\mathbb{R}^n} |g(t)|^2 \mu(dt).$$

Combining these inequalities, we obtain $|(\mathbf{K}f,g)|^2 \le M_1 M_\infty \|f\|^2_{L^2(\nu)} \|g\|^2_{L^2(\mu)}$ which is equivalent to (3.11) ∎

Remark 3.1: The above-given proof of the inequality (3.4), (3.10) follows the reasoning due to I. Schur [16: §2 /Statement 1]). I. Schur considered an operator $\mathbf{A}$ in the space $\mathbb{R}^n$ (or $\mathbb{C}^n$), whose matrix representation in the natural basis of this space is $\mathbf{A} = \|a_{jk}\|_{1 \le j,k \le n}$. I. Schur proved the inequality

$$\|\mathbf{A}\|_{l^2 \to l^2} \le \sqrt{M_1 M_\infty}$$

where

$$M_1 = \sup_{1 \le k \le n} \sum_{1 \le j \le n} |a_{jk}| \quad \text{and} \quad M_\infty = \sup_{1 \le j \le n} \sum_{1 \le k \le \infty} |a_{jk}|.$$

Here $\|\mathbf{A}\|_{l^p \to l^p}$ is the norm of the operator $\mathbf{A}$ acting in the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) equipped with the standard l^p-norm. It is clear that $M_1 = \|\mathbf{A}\|_{l^1 \to l^1}$ and $M_\infty = \|\mathbf{A}\|_{l^\infty \to l^\infty}$. Thus above inequality has the form $\|\mathbf{A}\|_{l^2 \to l^2} \le \|\mathbf{A}\|^{1/2}_{l^1 \to l^1} \|\mathbf{A}\|^{1/2}_{l^\infty \to l^\infty}$. The last inequality is a special case of the so-called *convexity inequality* of M. Riesz and G. O. Thorin (see [9: Theorem 295 and Theorem 408]). Inequalities of such types are considered in the theory of interpolation of linear operators.

Proof of Lemma 3.1: We derive Lemma 3.1 from the Schur inequality. Now $K(t,\tau) = e^{i\alpha \cdot (t-\tau)} \chi_n(t-\tau), \nu(d\tau) = m(d\tau), \mathbf{K} = \mathbf{X}_\alpha$. The inequality (3.5) means that $M_1 \le (A_1 C_\chi)^n C_\mu$. In view of (3.1) and (3.2), $\int_{\mathbb{R}^n} |\chi_n(t-\tau)|\, m(d\tau) \le (C_\chi \pi)^n$, i.e., $M_\infty \le (C_\chi \pi)^n$. Thus, Lemma 3.1 is a special case of the Schur inequality which corresponds to this choice of k, μ and ν, and we can take $A_2 = \pi A_1$ ∎

Remark 3.2: Let us explain why we use the term "Carleson measure" for a measure μ satisfying the condition (2.3). We have restricted ourself to the case $n = 1$. Let μ be a Borelian measure on the real axis $\mathbb{R}$ satisfying the Carleson condition (C) (i.e. (2.3)). We consider the real axis as a subset of the half-plane $H = \{z = x + iy : y > -1\}$ and the measure μ as a Borelian measure on the *half-plane* H. Thus, the Borelian measure μ is supported in the subset $\mathbb{R}$ of the half-plane H. The condition (**C**) implies that the measure μ is a *Carleson measure* in the half-plane H (concerning the definition of a Carleson measure and the Carleson Theorem we refer, for example, to the books of P. Koosis [12: Chapter VIII/Section E] and of J. Garnett [8: Chapter I/Section 5]). Traditionally one considers Carleson measures either on the unit disc or on the upper half-plane. However, we have to consider Carleson measures in the half-plane $\{z : \operatorname{Im} z > -1\}$. An estimate of the form (3.4) can be derived from Carleson's theorem. However, in fact we need not the Carleson theorem in its *general form*. The measure which we have to consider is very special: it is supported on a straight line parallel to the boundary of the half-plane. For such measures we have done the proof of Carleson's theorem in a very simple way.

The Main Lemma: *Let $\mu(dt)$ be a Borelian measure on $\mathbb{R}^n$ satisfying the Carleson condition (C) with the Carleson constant C_μ. Let Q_α ($\alpha \in \mathbb{R}^n$) be a unit cube of the space $\mathbb{R}^n$ and let $\tilde{\varphi}$ be a square summable (with respect to Lebesgue measure) function supported on the cube Q_α, i.e.*

$$\tilde{\varphi}(\lambda) = 0 \qquad (\lambda \notin Q_\alpha). \tag{3.12}$$

Let φ be the Fourier transform of the function $\tilde{\varphi}$, i.e.

$$\varphi(t) = \int_{Q_\alpha} \tilde{\varphi}(\lambda) e^{it\cdot\lambda} m(d\lambda) \qquad (t \in \mathbb{R}^n). \tag{3.13}$$

Then the inequality

$$\int_{\mathbb{R}^n} |\varphi(t)|^2 \mu(dt) \le (A_3)^n C_\mu \int_{Q_\alpha} |\tilde{\varphi}(\lambda)|^2 m(d\lambda) \tag{3.14}$$

holds where $A_3 < \infty$ is an absolute constant.

Proof: Let us consider some function $\widetilde{\chi} : \mathbb{R} \to \mathbb{C}$ such that

$$\widetilde{\chi}(\eta) = 1 \qquad \left(-\frac{1}{2} \le \eta \le \frac{1}{2}\right) \tag{3.15}$$

and its Fourier transform

$$\chi(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \widetilde{\chi}(\eta) e^{i\xi\eta} d\eta \qquad (\xi \in \mathbb{R}) \tag{3.16}$$

where $\widetilde{\chi}$ admits an estimate of the form (3.1). For concretness, we choose

$$\widetilde{\chi}(\eta) = \begin{cases} 1 & \text{if } -\frac{1}{2} \le \eta \le \frac{1}{2} \\ 2 - 2|\eta| & \text{if } \frac{1}{2} \le \eta \le 1 \\ 0 & \text{if } |\eta| \ge 1 \end{cases}. \tag{3.17}$$

The Fourier transform χ of $\widetilde{\chi}$ can be calculated explicitely:

$$\chi(\xi) = \frac{2}{\pi} \frac{\cos\frac{\xi}{2} - \cos\xi}{\xi^2} \qquad (\xi \in \mathbb{R}).$$

From here we derive the estimate (3.1) with

$$C_\chi = \frac{8}{\pi}. \tag{3.18}$$

Let us introduce the function χ_n by (3.2). In accordance with (3.16)

$$\chi_n(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widetilde{\chi}_n(\lambda) e^{it\lambda} m(d\lambda) \qquad (t \in \mathbb{R}^n)$$

where

$$\widetilde{\chi}_n(\lambda) = \widetilde{\chi}(\lambda_1)\widetilde{\chi}(\lambda_2)\cdots\widetilde{\chi}(\lambda_n) \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n)$$

and $\widetilde{\chi}$ is the same that in (3.17). From (3.15) it follows that

$$\widetilde{\chi}_n(\lambda) = 1 \qquad (\lambda \in Q_\alpha). \tag{3.19}$$

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, then

$$e^{i\alpha\cdot t}\chi_n(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widetilde{\chi}_n(\lambda - \alpha) e^{it\lambda} m(d\lambda) \qquad (t \in \mathbb{R}^n) \tag{3.20}$$

where, in view of (3.19), $\widetilde{\chi}_n(\lambda - \alpha) = 1$ for all $\lambda \in Q_\alpha$. Since the function $\tilde{\varphi}$ is supported in Q_α (see (3.12)),

$$\tilde{\varphi}(\lambda) = \tilde{\varphi}(\lambda)\widetilde{\chi}_n(\lambda - \alpha) \qquad (\lambda \in \mathbb{R}^n). \tag{3.21}$$

In accordance with the convolution theorem for Fourier integrals from (3.21), (3.20) and (3.13) it follows

$$\varphi(t) = \int_{\mathbb{R}^n} e^{i\alpha(t-\tau)}\chi_n(t-\tau)\varphi(t)\, m(d\tau) \qquad (t \in \mathbb{R}^n).$$

This representation provides us the possibility to estimate the integral $\int_{\mathbb{R}^n} |\varphi(t)|^2 \mu(dt)$ by the integral $\int_{\mathbb{R}^n} |\varphi(t)|^2 m(dt)$ from above. This representation has the form $\varphi = \mathbf{X}_\alpha \varphi$, where $\mathbf{X}_\alpha$ was introduced in (3.3). By Lemma 3.1, the estimate

$$\int_{\mathbb{R}^n} |\varphi(t)|^2 \mu(dt) \le (A_2 C_\chi^2)^n C_\mu \int_{\mathbb{R}^n} |\varphi(t)|^2 m(dt) \tag{3.22}$$

holds with C_χ from (3.18). Combining (3.22) with the Parseval identity

$$\int_{\mathbb{R}^n} |\varphi(t)|^2 m(dt) = (2\pi)^n \int_{Q_\alpha} |\tilde{\varphi}(\lambda)|^2 m(d\lambda)$$

we obtain (3.14) with $A_3 = 2\pi A_2 C_\chi^2$ ∎

This Main Lemma is not new. In more general setting it can be found in [18] or in [19: Sections 1.3.3 and 1.3.4]. However, in [18, 19] there is considered the case of L^p spaces with $p \in (0, \infty)$. To cover uniformly this general case (especially, the case $0 < p < 1$), some estimates of the so-called Hardy-Littlewood maximal function are used in [18, 19], which are not proved there. Our proof is obtained by using rather elementary methods.

4. Proof of the main embedding theorem

Let $p \in \mathbb{Z}^n$ and let Q_p be a cube Q_α (see (2.2)) with $\alpha = p$. The system $\{Q_p\}_{p \in \mathbb{Z}}$ forms a tiling of the space $\mathbb{R}^n$. Let w be a weight function satisfying the Kolmogorov condition (K) and let $f \in W^2_w$. Then the Fourier transform $\tilde{f}$ is summable (see (1.6)). The integral (1.2) can be splitted into the sum

$$f(t) = \sum_{p \in \mathbb{Z}^n} f_p(t) \qquad \text{where} \quad f_p(t) = \int_{Q_p} \tilde{f}(\lambda) e^{it\cdot\lambda} m(d\lambda) \qquad (t \in \mathbb{R}^n).$$

From the triangle inequality for the space $L^2(\mu)$ and from standard theorems of Lebesgue integration theory we obtain

$$\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt) \le \left(\sum_{p \in \mathbb{Z}^n} \left(\int_{\mathbb{R}^n} |f_p(t)|^2 \mu(dt) \right)^{1/2} \right)^2. \tag{4.1}$$

Applying the Main Lemma from Section 3, we obtain

$$\int_{\mathbb{R}^n} |f_p(t)|^2 \mu(dt) \le A_3^n C_\mu \int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda). \tag{4.2}$$

Inserting (4.2) into (4.1), we derive the inequality

$$\int_{\mathbb{R}^n} |f(t)|^2 \mu(dt) \le A_3^n C_\mu \left(\sum_{p \in \mathbb{Z}} \left(\int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2} \right)^2. \tag{4.3}$$

Let now $\{w_p\}_{p \in \mathbb{Z}^n}$ be an arbitrary family of positive numbers. Applying the Schwarz inequality to the sum in the right-hand side of (4.3):

$$\sum_{p \in \mathbb{Z}} \left(\int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2} = \sum_{p \in \mathbb{Z}} \frac{1}{\sqrt{w_p}} \left(w_p \int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2}$$

we obtain

$$\sum_{p \in \mathbb{Z}} \left(\int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2} \le \left(\sum_{p \in \mathbb{Z}} \frac{1}{w_p} \right)^{1/2} \left(\sum_{p \in \mathbb{Z}} w_p \int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2}. \tag{4.4}$$

Now we specify the sequence $\{w_p\}_{p \in \mathbb{Z}}$:

$$w_p = \int_{Q_p} w(\lambda)\, m(d\lambda). \tag{4.5}$$

Let us suppose that the weight function w satisfies the regularity condition (L), and let $L_{\ln w}$ be the Lipschitz constant of the function $\ln w$. Then the inequalities

$$\frac{1}{w_p} \le e^{L_{\ln w}} \int_{Q_p} \frac{m(d\lambda)}{w(\lambda)}$$

and

$$w_p \int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \le e^{L_{\ln w}} \int_{Q_p} |\tilde{f}(\lambda)|^2 w(\lambda) m(d\lambda)$$

are fulfilled. Inserting these inequalities into the sums in the right-hand side of (4.5), we obtain the inequality

$$\begin{aligned} &\sum_{p\in\mathbb{Z}} \left(\int_{Q_p} |\tilde{f}(\lambda)|^2 m(d\lambda) \right)^{1/2} \\ &\qquad \le e^{L_{\ln w}} \left(\int_{\mathbb{R}^n} \frac{m(d\lambda)}{w(\lambda)} \right)^{1/2} \left(\int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda)\, m(d\lambda) \right)^{1/2} \end{aligned} \tag{4.6}$$

Combining (4.4) with (4.6), we derive the inequality

$$\int_{\mathbb{R}^n} |f(\lambda)|^2 \mu(dt) \le A_3^n C_\mu K_w e^{2L_{\ln w}} \int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 w(\lambda)\, m(d\lambda)$$

where A_3 is the constant from (3.14) ∎

5. A regularity condition for a weight function is essential

In this section we present an example which shows that only the Kolmogorov condition (K) for the weight function w (without any regularity condition like the Lipschitz condition) does **not** imply an inequality of the form (2.6). Moreover, we give an example of a weight function $w : \mathbb{R} \to (0, \infty)$ satisfying the Kolmogorov condition (K) and a function f belonging to the space W_w^2 but not belonging to the space $L^2(m)$:

$$\int_{\mathbb{R}} |f(t)|^2 m(dt) = \infty \tag{5.1}$$

$$\int_{\mathbb{R}} |\tilde{f}(t)|^2 w(\lambda)\, m(d\lambda) < \infty. \tag{5.2}$$

Of course, the Lebesgue measure m on $\mathbb{R}$ satisfies the Carleson condition (C) with the Carleson constant $C_m = 1$. In view of the Parseval identity, (5.1) is equivalent to

$$\int_{\mathbb{R}} |\tilde{f}(t)|^2 m(d\lambda) = \infty. \tag{5.3}$$

Thus, we have to construct such functions $w : \mathbb{R} \to (0,\infty)$ and $\tilde{f} : \mathbb{R} \to \mathbb{C}$, for which the conditions (5.2), (5.3) and

$$\int_{\mathbb{R}} \frac{m(d\lambda)}{w(\lambda)} < \infty \tag{5.4}$$

hold. Let $\{\delta_k\}_{k\in\mathbb{N}_0}$ and $\{w_k\}_{k\in\mathbb{N}_0}$ be sequences consisting of strictly positive numbers. We choose these sequences later. We require a technical condition $\delta_k < \frac{1}{2}$ $(k \in \mathbb{N}_0)$ which ensures that the intervals $[k-\delta_k, k+\delta_k]$ do not overlap. Pick

$$w(\lambda) = \begin{cases} k^2 & \text{for } \lambda \in [k-\frac{1}{2}, k+\frac{1}{2}] \setminus [k-\lambda_{|k|}, k+\lambda_{|k|}] \\ w_{|k|} & \text{for } \lambda \in [k-\lambda_k, k+\lambda_k] \end{cases}.$$

Put

$$\tilde{f}(\lambda) = \begin{cases} a_{|k|} & \text{for } \lambda \in [k-\lambda_{|k|}, k+\lambda_{|k|}] \\ 0 & \text{for } \lambda \in \mathbb{R} \setminus (\cup_{k\in\mathbb{Z}}[k-\lambda_{|k|}, k+\lambda_{|k|}]) \end{cases}$$

for some sequence $\{a_k\}_{k\in\mathbb{N}_0}$ of positive numbers which will be concretized later. The condition (5.4) is equivalent to the condition

$$\sum_{0\le k<\infty} \frac{\delta_k}{w_k} < \infty\,. \tag{5.5}$$

The conditions (5.2) and (5.3) are equivalent to the conditions

$$\sum_{0\le k<\infty} |a_k|^2 w_k \delta_k < \infty \tag{5.6}$$

and

$$\sum_{0\le k<\infty} |a_k|^2 \delta_k = \infty, \tag{5.7}$$

respectively. Thus it remains to be seen whether the conditions (5.5) - (5.7) are compatible. We present such sequences $\{a_k\}, \{\delta_k\}$ and $\{w_k\}$ explicitly. Choose $\delta > 1$, then choose α: $2-\delta < \alpha \le 1$. Choose now γ satisfying the conditions $1-\alpha < \gamma < \delta - 1$. Pick now

$$\delta_k = (1+k)^{-\delta}, \qquad w_k = (1+k)^{-\gamma}, \qquad a_k = (1+k)^{\frac{\delta-\alpha}{2}}.$$

For such a_k, δ_k, w_k the conditions (5.5) - (5.7) are fulfilled.

Remark that in the above-constructed example the weight function w behaves extremely non-regularly. On the set $S = \cup_{k\in\mathbb{Z}}(k-\delta_{|k|}, k+\delta_{|k|})$ the function $w = w(\lambda)$ tends to zero as $|\lambda| \to \infty$. The condition $\int_S \frac{m(d\lambda)}{w(\lambda)} < \infty$ is satisfied because the set S is the union of intervals whose lengths tend to zero rapidly. The condition $\int_{\mathbb{R}\setminus S} \frac{m(d\lambda)}{w(\lambda)} < \infty$ is fulfilled because of the growth of the function w on the set $\mathbb{R} \setminus S$.

6. The compactness of the embedding operator

Let w be a weight function satisfying the Kolmogorov condition (K) and the regularity condition (L). Let $\mu \geq 0$ be a Borelian measure on $\mathbb{R}^n$. We considered an embedding operator from the space W_w^2 into the space $L^2(\mu)$. We have established that the Carleson condition (C) for the measure μ is necessary and sufficient for this embedding operator to be continuous. Now we are interested in the compactness of this embedding operator.

To investigate the compactness of the embedding operator we need some refinement of the inequality (1.6) and of Lemma 1.1. Let E be an arbitrary Borelian subset of $\mathbb{R}^n$. The Schwarz inequality yields

$$\int_E |\tilde{f}(\lambda)|\, m(d\lambda) \leq \left(\int_E \frac{m(d\lambda)}{w(\lambda)}\right)^{1/2} \left(\int_E |\tilde{f}(\lambda)|^2 w(\lambda)\, m(d\lambda)\right).$$

Roughing the last inequality, we obtain the inequality

$$\int_E |\tilde{f}(\lambda)|\, m(d\lambda) \leq \left(\int_E \frac{m(d\lambda)}{w(\lambda)}\right)^{1/2} \|f\|_{W_w^2}. \tag{6.1}$$

Let $B(r) = \{\lambda \in \mathbb{R}^n : |\lambda| \leq r\}$. If for a weight function w the Kolmogorov condition (K) is fulfilled, then, of course, $\lim_{r\to\infty} \int_{\mathbb{R}^n \setminus B(r)} \frac{m(d\lambda)}{w(\lambda)} = 0$. From this and from the estimate (6.1) we derive

Lemma 6.1: *Let w be a weight function satisfying the Kolmogorov condition (K). Then the limit relation*

$$\lim_{r\to\infty} \int_{\mathbb{R}^n \setminus B(r)} |\tilde{f}(r)|\, m(d\lambda) = 0$$

is uniform with respect to f taken from the unit ball of the space W_w^2.

From (1.5), Lemma 6.1 and standard results concerning Fourier integrals we obtain

Lemma 6.2: *Assume that the weight function w satisfies the Kolmogorov condition (K). Then the family of functions $\{f \in W_w^2 : \|f\|_{W_w^2} \leq 1\}$ is uniformly continuous on $\mathbb{R}^n$, i.e. for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that, for all $t', t'' \in \mathbb{R}^n$ with $|t' - t''| < \delta$, $|f(t') - f(t'')| < \varepsilon$ follows for all f belonging to this family.*

Definition 6.1: A Borelian measure $\mu \geq 0$ on $\mathbb{R}^n$ satisfies the *condition* $(\mathbf{C_0})$ if $\lim_{|\alpha|\to\infty} \mu(Q_\alpha) = 0$.

Of course, if a measure μ satisfies the condition (C_0), then all the more the measure μ satisfies the Carleson condition (C).

Theorem (On the compactness of the embedding operator): *Let w by a weight function on $\mathbb{R}^n$ satisfying the Kolmogorov condition (K) and the regularity condition (L), and let $\mu(dt) \geq 0$ be a Borelian measure on $\mathbb{R}^n$. The embedding operator from W_w^2 into $L^2(\mu)$ is compact if and only if the measure μ satisfies the condition (C_0).*

Proof: *Necessity.* Let $f_0 \not\equiv 0$ be an element of the space W_w^2. If the condition (C_0) is not fulfilled for the measure μ, then there exists a sequence $\{\tau_n\}$ of elements $\tau \in \mathbb{R}^n$ such that

$$\lim_{n\to\infty} |\tau_n| = \infty \qquad \text{and} \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^n} |f_0(t + \tau_n)|^2 \mu(dt) > 0. \tag{6.2}$$

However, the sequence $\{f_0(t+\tau_n)\}$ tends to zero weakly in the space W^2_w. This follows from the identity

$$\langle f(\cdot+\tau), g(\cdot)\rangle_{W^2_w} = \int_{\mathbb{R}^n} \tilde{f}_0(\lambda)\overline{\tilde{g}(\lambda)}w(\lambda)e^{i\tau\lambda}m(d\lambda)$$

and from the Riemann-Lebesgue theorem (the function $\tilde{f}_0(\lambda)\overline{\tilde{g}(\lambda)}w(\lambda)$ is summable). The weak convergence of the sequence $\{f(\cdot+\tau_n\}$ to zero in the space W^2_w and the compactness of the embedding operator imply the strong convergence of this sequence to zero in $L^2(\mu)$. This strong convergence to zero contradicts (6.2).

Sufficiency. Let $\varepsilon > 0$ be a prescribed number. Let the condition (C_0) be fulfilled for the measure μ. Then this measure can be decomposed into the sum

$$\mu = \mu_1 + \mu_2 \tag{6.3}$$

of two measures μ_1 and μ_2, where the measure μ_1 has a compact support and the measure μ_2 satisfies the condition Carleson (C) with Carleson constant $C_{\mu_2} \le \varepsilon$. Moreover, such a decomposition (6.3) can be done by means of a decomposition of the space $\mathbb{R}^n$: there exists a compact subset S of $\mathbb{R}^n$ such that the measure μ_1 is supported on S, $\mu_1(\mathbb{R}^n \setminus S) = 0$, and the measure μ_2 is supported outside S, $\mu_2(S) = 0$. Thus, $L^2(\mu) = L^2(\mu_1) \oplus L^2(\mu_2)$. Hence, the embedding operator E_μ from W^2_w into L^2_μ is equal to the orthogonal sum $E_\mu = E_{\mu_1} \oplus E_{\mu_2}$ where E_{μ_j} $(j=1,2)$ are embedding operators from W^2_w into $L^2(\mu_j)$. In view of Lemma 1.4, the operator E_{μ_1} is compact. By the Main Embedding Theorem, $\|E_{\mu_2}\| \le A\varepsilon$ where A is an absolute constant ∎

In a special case (w has the form (1.8) with an integral $l > \frac{n}{2}$ and μ is absolutely continuous) the theorem was proved in [4: Theorem 2]. By more restrictive assumptions it was proved earlier in [2: Theorem 2.2].

7. Some related results

The following theorem is closely related to our Main Embedding Theorem.

Theorem: *Let $\tilde{f}(\lambda) \in L^2[-\sigma,\sigma]$, where $\sigma < \infty$, and let*

$$f(t) = \int_{[-\sigma,\sigma]} \tilde{f}(\lambda)e^{it\lambda}d\lambda \qquad (t \in \mathbb{R}).$$

Let $\{t_k\}$ be a sequence of real points which is separating: $|t_k - t_l| \ge \delta > 0$ $(t_k \ne t_l)$. Then

$$\sum_k |f(t_k)|^2 \le A\frac{\sigma}{\delta}\int_{\mathbb{R}} |f(t)|^2 dt$$

where $A < \infty$ is an absolute constant.

This theorem has been proved firstly by M. Plancherel and G. Polya (see [13], especially Sections 27 - 31 of this paper). Their proof is based on some considerations dealing with subharmonic functions. This proof is reproduced, for example, in [1: pp.

97 - 103], and in [21: Chapter 2, Pt. 2, Section 3] (especially Theorem 17 there). Of course, the theorem is an immediate consequence of our Main Embedding Theorem. The last inequality is related with the notion of *a frame in a Hilbert space* (see [21]). Some inequalities of the form

$$C_1 \int_{\mathbb{R}^n} |f(t)|^p \mu(dt) \le \int_{\mathbb{R}^n} |f(t)|^p m(dt) \le C_2 \int_{\mathbb{R}^n} |f(t)|^p \mu(dt)$$

where $0 < p < \infty$, and the Fourier transform $\tilde{f}$ has a compact support, have been considered by H. Triebel (see [19] and [20: Section 1.3.4]).

A special case was considered by L. S. Frank [7]. There is contained the inequality

$$\sum_{k \in \mathbb{Z}^n} |f(kh)|^2 \le A(l,n) \frac{1}{h} \int_{\mathbb{R}^n} |\tilde{f}(\lambda)|^2 (1+\lambda^2)^l m(d\lambda) \tag{7.1}$$

where $h > 0$ is an arbitrary fixed number and $2l > n$. The inequality (7.1) has the form (2.6). The weight function w has the form (1.8). Condition $l > \frac{n}{2}$ implies the Kolmogorov condition (K) for the weight function. A measure μ has the form $\mu = \sum_{k \in \mathbb{Z}} \delta(t - kh)$ where δ is the δ-measure concentrated at the zero point. However, the proof given in [7] uses a special structure of the measure μ (it is concentrated in the lattice which is a subgroup of $\mathbb{R}^n$). This proof can not be extended for more general measures.

The inequality

$$\|u\|_{L^2(\mu)} \le C \|u\|_{W_l^2} \qquad \text{for all } u \in W_l^2$$

with a constant C not depending on u has been considered by V. G. Maz'ya (see books [14: §§2.3 and 8.3] and [15: §1.1]; the norm $\|\cdot\|_{W_l^2}$ corresponds to the form (1.8) of w). V. G. Maz'ya did not assume that the condition $l > \frac{n}{2}$ is satisfied. If this condition is not satisfied, a function u from the space W_l^2 must not be continuous on $\mathbb{R}^n$. However, even in the case $l \le \frac{n}{2}$ the integral $\int_{\mathbb{R}^n} |u(t)|^2 \mu(dt)$ is finite for some measures μ. The assumption which ensures the finiteness of this integral has the form

$$\sup_e \frac{\mu(e)}{\operatorname{cap}(e, W_l^2)} < \infty \tag{7.2}$$

where e ($e \in \mathbb{R}^n$) runs over all compacts of positive capacity $\operatorname{cap}(e, W_l^2)$. V.G. Maz'ya has considered a more general case: spaces W_l^p with $p \in [1, \infty]$ and even Orlicz spaces. In the case $l > \frac{n}{2}$ the condition (7.2) is reduced to the Carleson condition (C) on the measure μ. V.G. Maz'ya considered capacities corresponding to w of the form (1.8). It is possible to develope the capacity theory for the spaces W_w^2 with a more or less arbitrary weight function w. We will not do this here.

Let us suppose now that the measure μ is absolutely continuous. We represent the measure μ in the form $\mu(dt) = |b(t)|^2 m(dt)$ and the weight function w in the form $w(t) = |a(t)|^{-2}$ where a and b are Borelian functions on $\mathbb{R}^n$. The Carleson condition (C) and Kolmogorov condition (K) can be written in the forms

$$\sup_{\tau \in \mathbb{R}^n} \int_{Q_0} |b(t+\tau)|^2 m(dt) < \infty \tag{7.3}$$

and

$$\int_{\mathbb{R}^n} |a(t)|^2 m(dt) < \infty, \tag{7.4}$$

respectively. The inequality (2.6) can be rewritten in the following way:

$$\int_{\mathbb{R}^n} |b(t)f(t)|^2 m(dt) \le E \int_{\mathbb{R}^n} |\tilde{f}(\lambda)a^{-1}(\lambda)|^2 m(d\lambda).$$

Let us introduce the function $\tilde{\varphi}(\lambda) = \tilde{f}(\lambda)a^{-1}(\lambda)$ $(\lambda \in \mathbb{R}^n)$. We assume that $\varphi \in L^2(m)$. Then $\tilde{f}(\lambda) = a(\lambda)\varphi(\lambda)$ and $f(t) = \int_{\mathbb{R}^n} e^{it\cdot\lambda}a(\lambda)\varphi(\lambda)\, m(d\lambda)$. Let us define the operator $b(X)a(D)$. By definition,

$$(b(X)a(D)\varphi)(t) = b(t) \int_{\mathbb{R}^n} e^{it\cdot\lambda}a(\lambda)\varphi(\lambda)\, m(d\lambda) \qquad (t \in \mathbb{R}^n).$$

Our Main Embedding Theorem claims that under the conditions (7.3) and (7.4) (and under the Lipschitz condition on the function $\ln|a(\cdot)|$) the operator $b(X)a(D)$ acts continuously in $L^2(m)$. Under another assumptions on the functions a and b the operator $b(X)a(D)$ was considered in [17: Section 4], [3], [6: §3] and in [5: Proposition 4.2 and Formula (2.7)].

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