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Steiner Symmetrization and Periodic Solutions of Boundary Value Problems

F. Brock

Abstract. Let $f^* = f^*(x,y)$ denote the monotone decreasing rearrangement of a function $f = f(x,y)$ with respect to y. If $-\Delta u = f$, $-\Delta v = f^*$ in the domain $\Omega = (0,1) \times (0,1)$ and $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on the boundary $\partial\Omega$ of Ω, then $\operatorname{osc} u \le \operatorname{osc} v$, where the quantity $\operatorname{osc} w$ for a function w is defined as the difference $\sup w - \inf w$. Similar results are proved for periodic solutions of some boundary value problems in cylindrical domains.

Keywords: *Symmetrization, periodic solutions, boundary value problems*

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The motivation for this paper is the following problem stated by Kawohl (see [1: p.61]):

Let $\Omega = (0,1) \times (0,1)$ and $f = f(x_1,x_2)$ a sufficiently smooth function with $\iint_\Omega f(x_1,x_2)\,dx_1dx_2 = 0$. Further, let f^* denote the monotone decreasing rearrangement of f with respect to x_2 (for definition see [1: p.11]) and let $u,v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be two solutions of the boundary value problem

$$\begin{aligned} \Omega: &\quad -\Delta u = f,\ -\Delta v = f^* \\ \partial\Omega: &\quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad (\nu - \text{the outer normal.} \end{aligned} \tag{1}$$

Is it then true that

$$\operatorname{osc}_\Omega u \le \operatorname{osc}_\Omega v \tag{2}$$

where $\operatorname{osc} w$ for a function w is defined as the difference $\sup_\Omega w - \inf_\Omega w$? We shall give a positive answer to this question and derive similar results for some boundary value problems in cylindrical domains with periodic solutions. The proofs are based on symmetry properties of the Greens function and a Hardy-Littlewood-type inequality for a certain integral representation of the solution.

First we introduce some notations. Let be $n \ge 2$ some integer. We use the following partition of points $x, \xi \in I\!R^n$:

$$x = (x', y) \qquad \text{and} \qquad \xi = (\xi', \eta)$$

F. Brock: Univ. Leipzig, Institut für Mathematik, Augustusplatz 10, D - 04109 Leipzig

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with

$$x' = (x_1, \dots, x_{n-1}), \qquad y = x_n, \qquad \xi' = (\xi_1, \dots, \xi_{n-1}), \qquad \eta = \xi_n.$$

By L^k we denote the k-dimensional L-measure, $1 \le k \le n$. If F is any L-measurable function on a measurable set $M \subseteq I\!R^n$, which is bounded below by a constant C and for which the level sets $\{F > t\}\,(t > C)$ have a finite L-measure, then let F^* denote the Steiner symmetrization of F with respect to the variable y. The function F^* is defined on the Steiner symmetrization M^* of M with respect to y (for a definition of F^* and M^* see [1: p.11]). We mention here that for almost any x' and $t > C$ the level sets $\{F^*(x', \cdot) > t\}$ are open intervals lying symmetrically to the hyperplane $\{y = 0\}$ and have the same L-measure as $\{F(x', \cdot) > t\}$. The function $F^*(x', y)$ is symmetric in y (i.e., $F^*(x', y) = F^*(x', -y)$) and monotone decreasing in y for $y > 0$. Further we shall also use the notation $F_* = -(-F)^*$.

The following inequality for an integral of a product of two functions and their symmetrizations is due to Hardy and Littlewood (see, e.g., [1: p.23]) and will play the central role in our proof.

Lemma 1: *Let F, G be two L-measurable functions on a measurable set $M \subseteq I\!R^n$. Then*

$$\int_{M^*} F_*(\xi) G^*(\xi)\, d\xi \le \int_M F(\xi) G(\xi)\, d\xi \le \int_{M^*} F^*(\xi) G^*(\xi)\, d\xi \tag{3}$$

if the integrals converge.

Assumptions: Let Ω' be a bounded domain in $I\!R^{n-1}$ with Lipschitz boundary, and assume that Σ' is some relatively open portion of $\partial\Omega'$. Choosing some number $p < 0$, we set

$$\begin{array}{lll}
\Omega = \Omega' \times I\!R, & \Omega_0 = \Omega' \times (0, p), & \Omega_1 = \Omega' \times (-p, +p) \\
\Sigma = \Sigma' \times I\!R, & \Sigma_1 = \Sigma' \times (-p, +p) & \\
S_- = \Omega' \times \{-p\}, & S_+ = \Omega' \times \{+p\} & \\
\Gamma = \partial\Omega \setminus \Sigma, & \Gamma_1 = (\partial\Omega' \setminus \Sigma') \times (-p, +p). &
\end{array}$$

Let $a^{ij}, b^i \in L^\infty(\overline{\Omega'})$ $(i, j = 1, \dots, n)$, $c \in L^\infty(\Omega')$, $\sigma \in L^\infty(\Sigma')$ and $f \in L^2(\Omega_1)$. We assume that

$$\lambda|\xi|^2 \le a^{ij}(x')\xi_i\xi_j \le \Lambda|\xi|^2 \qquad (x' \in \Omega', 0 < \lambda < \Lambda < +\infty) \tag{4}$$

$$c \ge 0, \quad \sigma \ge 0 \tag{5}$$

$$a^{in} = a^{nj} = 0, \quad b-^n = 0 \qquad (i, j = 1, \dots, n-1). \tag{6}$$

where as in the following we take into account Einstein's rule and for simplicity let all summations go from 1 to n. If we have at the same time

$$L^{n-1}(\Gamma_1) = 0, \qquad c = 0 \qquad \text{and} \qquad \sigma = 0, \tag{7}$$

then we demand in addition

$$\int_{\Omega_1} f(x)\, dx = 0. \tag{8}$$

We denote with f^* the Steiner symmetrization of f with respect to y in the domain Ω_1. Note that, if f is symmetric in the variable y, then the function f^* is at the same time the monotone decreasing rearrangement of f with respect to y in the domain Ω_0.

We continue f and f^* onto the domain Ω by

$$f(x', y + 2pm) = f(x) \qquad \text{and} \qquad f^*(x', y + 2pm) = f^*(x) \tag{9}$$

for all $x \in \Omega_1$ and $m \in \mathbb{Z}$.

Now with the linear homogeneous operators

$$\begin{aligned} L &\equiv -D_i(a^{ij}D_j) + b^iD_i + c \qquad && \text{in} \quad \Omega \\ B &\equiv D_\nu + \sigma && \text{on} \quad \Sigma \end{aligned}$$

(where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal and $D_\nu = \nu_i a^{ij} D_j$ denotes the conormal derivative), we look for functions u, v which are $2p$-periodic in y and satisfy

$$Lu = f \qquad \text{and} \qquad Lv = f^* \quad \text{in } \Omega \tag{10}$$

and the mixed Robin-Dirichlet boundary conditions

$$Bu = Bv = 0 \quad \text{on } \Sigma \qquad \text{and} \qquad u = v = 0 \quad \text{on } \Gamma \tag{11}$$

in a weak sense. Note here that the condition (11) also covers the case of the pure Neumann boundary conditions $D_\nu u = D_\nu v = 0$ on $\partial\Omega$, namely if (7) and (8) are satisfied.

The facts that the operators L and B contain derivatives in y only of even order (condition (6)) and that the coefficients of L and B are independent of y, we need to apply comparison arguments in connection with periodicity and symmetry properties of the solutions.

We give now a precise formulation of the above problems (10), (11). If G is any bounded subdomain of Ω, we set

$$\begin{aligned} V(G) &= \left\{ v \in C^1(G) \cap C(\Omega_1 \cup \Gamma_1) : v = 0 \text{ near } \Gamma \cup (\partial G \cap \Omega) \right\} \\ H(G) &= \text{ closure of } V(G) \text{ in } W^{1,2}(G). \end{aligned}$$

By Π we denote the set of all measurable functions on Ω, which are $2p$-periodic in the variable y. We consider the following boundary value problems:

(P) Find $u \in \Pi$ such that for any bounded subdomain G of Ω we have $u \in H(G)$ and

$$\int_G \left(a^{ij} D_j u D_i h + b^i h D_i u + cuh \right) dx + \int_{\Sigma \cap \partial G} \sigma u h \, ds = \int_G f h \, dx$$

for any $h \in V(G)$.

(P*) Find $v \in \Pi$ such that for any bounded subdomain G of Ω we have $v \in H(G)$ and

$$\int_G \left(a^{ij} D_j v D_i h + b^i h D_i v + cvh \right) dx + \int_{\Sigma \cap \partial G} \sigma v h \, ds = \int_G f^* h \, dx$$

for any $h \in V(G)$.

Existence, uniqueness and symmetry properties of the solutions of the problems (P)and (P*) are given in the following

Lemma 2: *Let Ω, f and f^* be as in the Assumptions. Then there exist solutions of the problems (P) and (P*). The functions u,v are unique up to a constant, if conditions (7) and (8) are satisfied, and are unique otherwise. Further, if f is symmetric in y, so is also u and $D_n u(x) = 0$ on $S_- \cup S_+$. If f is antisymmetric (odd) in y, so is also u and $u = 0$ on $S_- \cup S_+$, if we demand $\int_{\Omega_1} u(x)\,dx = 0$ in case that (7) and (8) are satisfied. Finally we have $v = v^*$ in Ω_1.*

Proof: The idea is to transform the problems (P) and (P*) into known boundary value problems in the domain Ω_1 by means of a suitable decomposition of their solutions. To this we introduce the spaces

$$\begin{aligned} V_1 &= \left\{ v \in C^1(\Omega_1) \cap C(\Omega_1 \cup \Gamma_1) : v = 0 \text{ near } \Gamma_1 \right\} \\ V_2 &= \left\{ v \in V_1 : v = 0 \text{ near } S_- \cup S_+ \right\} \\ H_1 &= \text{closure of } V_1 \text{ in } W^{1,2}(\Omega_1) \\ H_2 &= \text{closure of } V_2 \text{ in } W^{1,2}(\Omega_1) \end{aligned}$$

the bilinear form

$$\mathcal{L}(u,h) = \int_{\Omega_1} \left(a^{ij} D_j u D_i h + b^i h D_i u + cuh \right) dx + \int_{\Sigma_1} \sigma u h\, ds \quad (u, h \in W^{1,2}(\Omega_1))$$

and decompose the function f into its symmetric and antisymmetric part:

$$\left.\begin{aligned} f_1(x) &= \frac{1}{2}\Big(f(x',y) + f(x',-y) \Big) \\ f_2(x) &= \frac{1}{2}\Big(f(x',y) - f(x',-y) \Big) \end{aligned}\right. \qquad (x \in \Omega_1). \tag{12}$$

We consider the following boundary value problems:

($\mathbf{P}_k$) Find $u_k \in H_k$ such that

$$\mathcal{L}(u_k, h) = \int_{\Omega_1} f_k h\, dx \quad \text{for any } h \in V_k \qquad (k = 1,2).$$

Note that the functions u_k $(k = 1,2)$ satisfy in a weak sense the equations

$$Lu_k = f_k \ \text{ in } \Omega_1, \quad Bu_k = 0 \ \text{ on } \Sigma_1, \quad u_k = 0 \ \text{ on } \Gamma_1 \qquad (k = 1,2) \tag{13}$$

$$\frac{\partial u_1}{\partial \nu} = 0, \quad u_2 = 0 \qquad \text{on } S_- \cup S_+. \tag{14}$$

From [2: p.215 ff.] and (4) - (8) we can conclude that the solutions of the problems (P_1) and (P_2) exist. The function u_2 is unique. Further, the function u_1 is unique up to a constant, if (7) and (8) are satisfied, and is unique otherwise. Finally in view of (6), the symmetry of f_1 and the antisymmetry of f_2 it follows through comparison considerations that u_1 is symmetric (even) and u_2 is antisymmetric (odd) in y.

We continue the functions f_k and u_k onto the infinite cylinder Ω analogously as f by (9) and conclude from the boundary conditions on $S_- \cup S_+$ that the functions u_k are in $W^{1,2}(G)$ and satisfy $Lu_k = f_k$ weakly in any bounded subdomain G of Ω, i.e. they are solutions of the problem (P) with the function f replaced by f_k $(k = 1, 2)$. Now it is easy to see that the function $u = u_1 + u_2$ is a solution of problem (P).

Reversely we can decompose any solution of problem (P) into its symmetric and antisymmetric part analogously as f in Ω_1 by formula (12), and then identify these parts as solutions of the above problems (P_1) and (P_2). This yields the uniqueness properties of u.

If f is symmetric in y, then f_2 and u_2 vanish. If f is antisymmetric in y, then f_1 vanishes, which means that u_1 vanishes in the case that (7) is not satisfied and is constant if (7) and (8) are valid. This yields the symmetry properties from the assertion.

Obviously we can replace the function f by f^* in the above considerations. Since also $v = v_1$, it follows from comparison considerations that $v = v^*$ in Ω_1 ∎

Remark 1: The non-uniqueness of the solutions u and v in the case of pure Neumann boundary conditions on $\partial\Omega$ (i.e. (7) and (8) are satisfied) can be easily removed, if we demand

$$\int_{\Omega_1} u(x)\,dx = \int_{\Omega_1} v(x)\,dx = 0. \tag{15}$$

Now let $g(x;\xi)$ denote the Greens function associated with the the boundary value problem (P_2), i.e. with the operator L in Ω_1 and the mixed Robin-Dirichlet boundary conditions on $\Gamma_1 \cup \Sigma_1$ and Neumann boundary conditions on $S_- \cup S_+$ (see (13),(14)). More precisely we demand that for any $x \in \Omega_1$ and $h \in V_1$

$$\mathcal{L}(h, g(x;\cdot)) = h(x) - \begin{cases} (L^n(\Omega_1))^{-1} \int_{\Omega_1} h(\xi)\,d\xi & \text{if (7) is satisfied} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Note that g is unique up to a constant, if (7) is satisfied, and is unique otherwise. We normalize in the first case: $\int_{\Omega_1} g(x;\xi)\,d\xi = 0,\ x \in \Omega_1$.

Now exploiting the periodicity of the solutions, we derive a special representation formula for u and v.

Lemma 3: *Let u, v be the solutions of the problems (P) and (P^*), respectively, which are normalized by (15) in the case that (7) and (8) are satisfied. Then*

$$u(x) = \int_{\Omega_1} g(x', 0; \xi) f(\xi', y + \eta)\,d\xi \tag{17}$$

$$v(x) = \int_{\Omega_1} g(x', 0; \xi) f^*(\xi', y + \eta)\,d\xi \tag{18}$$

for all $x \in \Omega_1$.

Proof: We fix some $y_0 \in (-p, +p)$ and introduce the functions

$$\overline{u}(x', y) = u(x', y + y_0) \qquad \text{and} \qquad \overline{f}(x', y) = f(x', y + y_0) \qquad (x \in \Omega_1.$$

Clearly, $\overline{u}$ is a solution of the problem (P) with the function f replaced by $\overline{f}$. Again we decompose $\overline{u}$ into its symmetric and antisymmetric parts $\overline{u_1}$ and $\overline{u_2}$, respectively, analogously as the function f in (12). Then we conclude that

$$\overline{u_1}(x',y) = \int_{\Omega_1} g(x',y;\xi)\overline{f_1}(\xi)\,d\xi \qquad (x \in \Omega_1). \tag{19}$$

Note, that if (7) and (8) are satisfied, formula (19) yields a solution which is normalized by $\int_{\Omega_1} \overline{u_1}(x)\,dx = 0$. Further, from comparison considerations we conclude that $g(x',0;\cdot)$ is symmetric (even) in the variable η, i.e.

$$g(x',0;\xi',\eta) = g(x',0;\xi',-\eta) \qquad (\xi \in \Omega_1,\ x' \in \Omega').$$

Since $\overline{f_2}(\xi',\eta)$ is odd in the variable η, we get from this

$$\int_{\Omega_1} g(x',0;\xi)\overline{f_2}(\xi)\,d\xi = 0. \tag{20}$$

Now adding (19) with $y = 0$ and (20), and taking into account that $\overline{u_2}(x',0) = 0$, we conclude that

$$\overline{u}(x',0) = \int_{\Omega_1} g(x',0;\xi)\overline{f}(\xi)\,d\xi \qquad (x' \in \Omega'). \tag{21}$$

If we set $y_0 = y$ in (21), then (17) follows in view of the definition of $\overline{u}$ and $\overline{f}$. Analogously we can derive formula (18) ∎

Now we are in a position to prove the main result of the paper.

Theorem: *Let Ω, f, f^* be as in the Assumptions, and let u, v be solutions of the problems (P) and (P*), respectively. Then if (7) is not satisfied we have*

$$v(x',p) \le u(x) \le v(x',0) \qquad (x \in \Omega_1) \tag{22}$$

$$\inf\{v(x) : x \in \Omega\} \le \inf\{u(x) : x \in \Omega\}$$

$$\sup\{u(x) : x \in \Omega\} \le \sup\{v(x) : x \in \Omega\} \tag{23}$$

$$\|u\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} \tag{24}$$

$$osc_\Omega\, u \le osc_\Omega\, v. \tag{25}$$

If (7) and (8) are satisfied, then (25) follows, and the relations (22) - (24) remain valid for solutions u, v which are normalized by (15).

Remark 2: If we choose $n = 2, L = -\Delta, p = 1, \Omega' = (0,1), f$ symmetric in y and assume that (7) and (8) are satisfied, then it follows from Lemma 2 that u, v satisfy (1), and we get the assertion (2) from (25).

Proof of the Theorem: Let u, v be normalized by (15) in the case that (7) and (8) are satisfied. We fix some $x \in \Omega_1$ and set

$$f(\xi',y+\eta) = F(\xi) \qquad \text{and} \qquad g(x',0;\xi',\eta) = G(\xi).$$

From (16) we see that G can be approximated by solutions of the adjoint boundary value problems of (P_1):

$$(\overline{\mathbf{P_1}}) \quad \mathcal{L}(h, w_m) = \int_{\Omega_1} f_m h\, dx \qquad (h \in V_1)$$

with smooth functions $f_m = f_m^*$ $(m = 1, 2, \ldots)$ such that

$$w_m \longrightarrow G \qquad \text{a.e. in } \Omega_1 \text{ as } m \to \infty. \tag{26}$$

As in the proof of Lemma 2 we can conclude by comparison considerations that $w_m = w_m^*$ $(m = 1, 2, \ldots)$. By (26) it follows then that also $G = G^*$. Further we have

$$F^*(\xi) = f^*(\xi) \qquad \text{and} \qquad F_*(\xi) = f^*(\xi', \pm p + \eta).$$

Then applying (3) with $M = M^* = \Omega_0$ we obtain (22) by Lemma 3. The relations (23) - (25) are immediate consequences. ∎

Remark 3: If we assume that $a^{ij}, b^i \in C^{0,1}(\Omega')$, then we can conclude (compare with [2: Theorem 8.8]) that $u, v \in W^{2,2}(M)$ for any bounded subdomain $M \subset\subset \Omega$, and the equations (10) are valid throughout Ω ∎

Remark 4: Let be $0 < \alpha < 1$ and $\partial\Omega$ of class C^α, $a^{ij}, b^i \in C^{1,\alpha}(\overline{\Omega'})$ $(i, j = 1, \ldots, n), c, f \in C^\alpha(\overline{\Omega'})$, $\sigma \in C^{1,\alpha}(\overline{\Sigma'})$. Then we can conclude that also $f^* \in C^\alpha(\overline{\Omega})$ and that the solution u, v of the problems (P) and (P*), respectively, are in $C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\Omega \cup \Sigma) \cap C^\alpha(\overline{\Omega})$ (see [2: Theorem 6.31]).

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