Uniqueness Result
for the Generalized Entropy Solutions to the
Cauchy Problem for First-Order
Partial Differential-Functional Equations

Z. Kamont and H. Leszczyński

Abstract. We prove a theorem on differential-functional inequalities in the Carathéodory sense. The proof is based on the fact that we can solve a linear differential equation with first-order partial derivatives. We use also an integral Volterra-type inequality. We obtain a theorem on the uniqueness of generalized entropy solutions to the initial-value problem for non-linear partial differential-functional equations of the first order.

Keywords: *Integral inequalities, method of regularization, Volterra condition, characteristic problem*

AMS subject classification: 35L60, 35R45, 35R10

0. Introduction

Let $b >$, $R_+ = [0, \infty)$, $\tau_0 \in R_+$, $\tau = (\tau_1, \ldots, \tau_n) \in R^n_+$ and denote

$$E_b = [-\tau_0, b] \times R^n, \qquad E_{(b)} = [0, b] \times R^n, \qquad D = [-\tau_0, 0] \times [-\tau, \tau].$$

Further, for arbitrary metric spaces X and Y denote by $C(X, Y)$ the set of all continuous functions defined on X and taking values in Y. For $z : E_b \to R$ and $(x, y) \in E_{(b)}$ we define a function $z_{(x,y)} : D \to R$ by

$$z_{(x,y)}(\xi, \eta) = z(x + \xi, y + \eta) \quad \text{for} \ (\xi, \eta) \in D.$$

At last, let

$$\Omega = E_{(b)} \times R \times C(D, R) \times R^n.$$

Assume that $f : \Omega \to R$ and $\varphi : E_0 \to R$. If $z \in C(E_{(b)}, R)$ is a function of variables $(x, y) = (x, y_1, \ldots, y_n)$ and if there exist derivatives $D_{y_j} z$ $(j = 1, \ldots, n)$ on $E_{(b)}$, then we denote

$$D_y z(x, y) = \Big(D_{y_1} z(x, y), \ldots, D_{y_n} z(x, y) \Big) \qquad \text{for} \ (x, y) \in E_{(b)} .$$

Z. Kamont: Univ. of Gdańsk, Inst. Math., 57 Wit Stwosz Str., 80-952 Gdańsk, Poland
H. Leszczyński: Univ. of Gdańsk, Inst. Math., 57 Wit Stwosz Str., 80-952 Gdańsk, Poland

We consider the differential-functional equation

$$D_x z(x,y) = f\Big(x, y, z(x,y), z_{(x,y)}, D_y z(x,y)\Big) \qquad (0.1)$$

with the initial condition

$$z(x,y) = \varphi(x,y) \qquad \text{for } (x,y) \in E_0 . \qquad (0.2)$$

Let $\mathcal{M}[k,l]$ denote the class of all $k \times l$ real matrices. If $U = [u_{i,j}] \in \mathcal{M}[k,l]$, then $\|U\| = \max_i \{\sum_j |u_{ij}|\}$. Let $\mathcal{N}$ be the set of all natural numbers. For every $k \in \mathcal{N}$ and $\eta = (\eta_1, \ldots, \eta_k) \in R^k$ we define $\|\eta\| = \max_i |\eta_i|$. Throughout the paper we will denote norms of matrices and vectors by the same symbols. At last, if $I = [a_1, a_2]$ is an interval, then $\mathcal{L}(I, R)$ denotes the set of all functions $\Lambda : I \to R$ for which $\int_{a_1}^{a_2} |\Lambda(t)| dt$ exists.

Numerous problems in the theory of differential equations have been solved by means of the differential inequalities methods. Let us mention only the most important problems: an estimation of the domain of the existence of solutions, establishing some uniqueness criteria, the research of stability conditions, finding some estimations of solutions, continuous dependence on initial data and on right-hand-side functions, error estimations for approximate solutions. Some inequalities in a Haar piramid or in an unbounded domain are considered there. For some more reference connected with the subject we send the reader to the papers [1, 3, 6]. The classical theory of differential inequalities with its applications is founded in the monographs [9, 15].

The subject of the uniqueness of solutions to initial or initial-boundary value problems is mostly connected with differential inequalities. Consider the initial value problem

$$\begin{aligned} D_x z(x,y) &= F\Big(x, y, z(x,y), D_y z(x,y)\Big) & &\text{for } (x,y) \in E \\ z(0,y) &= \Phi(y) & &\text{for } y \in [-b, b] \end{aligned} \qquad (0.3)$$

where

$$E = \Big\{ (x,y) \Big| \ x \in [0,a], \ y = (y_1, \ldots, y_n), \ |y_i| \le b_i - M_i x_i \text{ for all } i \Big\},$$

$b = (b_1, \ldots, b_n)$ and $D_y z = (D_{y_1} z, \ldots, D_{y_n} z)$. Assume that $F : E \times R \times R^n \to R$ and that for some function $\sigma : [0,a] \times R_+ \to R_+$ we have the estimate

$$|F(x,y,p,q) - F(x,y,\overline{p},\overline{q})| \le \sigma(x, |p - \overline{p}|) + \sum_{i=1}^{n} M_i |q_i - \overline{q}_i|.$$

If σ is a Perron type comparison function, then there exists at most one classical solution of problem (0.3) that is continuous, possesses first-order partial derivatives in E and total differential in $\partial E \cap ([0,a] \times R^n)$. If we assume that σ is a comparison function of the Kamke type, then a solution of problem (0.3) is unique in the class of functions satisfying the above condition and possessing a continuous partial derivative with respect to x for $x = 0$, see [9, 15].

The above results can be extended on the differential-functional problems

$$
\begin{aligned}
D_x z(x,y) &= F\Big(x,y,z(\cdot),D_y z(x,y)\Big) && ((x,y) \in E) \\
z(x,y) &= \Phi(x,y) && ((x,y) \in E_0)
\end{aligned}
\tag{0.4}
$$

where $E_0 = [-\tau_0,0] \times [-b,b]$ and $F : E \times C(E,R) \times R^n \to R$. A comparison problem for problem (0.4) is of the form

$$
\begin{aligned}
\eta'(x) &= \tilde{\sigma}(x,\eta(\cdot)) && (x \in [0,a]) \\
\eta(x) &= 0 && (x \in [-\tau_0,0])
\end{aligned}
\tag{0.5}
$$

where $\tilde{\sigma} : [0,a] \times C([-\tau_0,a],R_+) \to R_+$. The essential fact in these considerations is that operators F and $\tilde{\sigma}$ satisfy the Volterra condition, see [1].

If we consider problem (0.3) in an unbounded zone $E = [0,a] \times R^n$, then we can omit the assumption that solutions of the Cauchy problem have the total differential in some points of the domain. In this case we assume that the comparison function is linear with respect to the second variable, which means that F satisfies a Lipschitz condition. The proof of the uniqueness result is based on the fact that we can solve a linear differential inequality with first-order partial derivatives.

An interesting result for the global uniqueness of the Cauchy problem when F satisfies the Hölder condition with respect to the last variable can be found in [2]. In [11] we proved some uniqueness results for differential-functional equations considered on $[0,a] \times R^n$ with the Cauchy data given on $[-\tau_0,0] \times R^n$. The method of differential-functional inequalities is used.

Non-linear equations with partial derivatives have the following property: any classical solution to initial-value problems exists locally with respect to x. This leads to definitions of solutions in the sense "almost everywhere" or *Carathéodory solutions*. The class of weak (Carathéodory) solutions consists of all functions which are continuous, have their derivatives almost everywhere in some domain, and the set of all points where the differential equation is not fulfilled is of Lebesgue measure zero. The main existence and uniqueness results for initial-value problems in the class of generalized solutions can be found in [8, 12] (see also [13, 14]).

Generalized solutions of non-linear equations are also investigated in the case that assumptions for the given functions are extended. A function $z \in C(E_b,R)$ is called a *CC-solution* of equation (0.1) if it is a solution in the Caratheodory sense, $z(\cdot,y)$ is absolutely continuous for $y \in R^n$, the derivative $D_y z$ exists on $E_{(b)}$, and for every $y \in R^n$ equation (0.1) is satisfied almost everywhere on $[0,b]$. CC-solutions to problem (0.1), (0.2) are obtained as solutions of suitable systems of integro-functional equations, see [4], 10], 13]. Theorems on the uniqueness of CC-solutions with non-linear comparison functions are proved by means of integro-functional inequalities of the Volterra type. The papers [4, 10] deal with the existence and uniqueness of CC-solutions. This class of generalized solutions seems to be important if we look for classical solutions because the assumptions that right-hand side of equations is continuous is sufficient to prove that every CC-solution to it is a classical solution. This observation is known in the literature (see [4, 10]).

Solutions of equation (0.1) in the Caratheodory sense are not unique. That is the reason why in the literature two natural subclasses of the set of all solutions in the Carathéodory sense are considered — CC-solutions and generalized entropy solutions, which are the subject of our investigations in the paper. It is easy to construct examples of generalized entropy solutions which are not CC-solutions and vice versa.

Our Theorems 1.1 and 2.1 are some extensions of Kružkov's results on the case of differential-functional equations. Essentially, roots of our proof methods, similarly as in [8], are related to the Holmgren principle: "uniqueness of the solutions to some linear problems follows from existence for the adjoint problem". The proof of Theorem 1.1 (like proofs in [8], 11, 14]) is a non-linear realization of this principle with the Cauchy problem (1.7) as the "adjoint problem".

The existence of generalized entropy solutions is proved in [8] for differential equations and in [7] for equations with deviated variables. We cite the existence theorem for generalized entropy solutions of differential-functional equations from the paper [5]. In this part of the paper we put $n = 1$ and $\tau_0 > 0$. Denote by $C_{0+L}(D, R)$ the class of all functions $w \in C(D, R)$ which satisfy the Lipschitz condition on D. Define

$$C_{0+L}(D, \bar{R}, t) = \left\{ w \in C_{0+L}(D, R) \middle| \ \|w\|_0 + \|w\|_L \le t \right\}$$

where $\| \cdot \|_0$ is the supremum norm in $C(D, R)$ and

$$\|w\|_L = \sup\left\{ |w(t, s) - w(\bar{t}, \bar{s})|(|t - \bar{t}| + |s - \bar{s}|)^{-1}| \ (t, s), (\bar{t}, \bar{s}) \in D \right\} .$$

In a similar way we define $C(D, R, t)$.

Theorem 0.1 [5]: *Suppose that the following conditions are true.*

1. $\varphi \in C(E_0, R)$ *and there exist constants* $\tilde{M}, \tilde{L} \in R_+$ *such that*

$$|\varphi(x, y)| \le \tilde{M} \quad and \quad |\varphi(x, y) - \varphi(x, \bar{y})| \le \tilde{L}|y - \bar{y}| \quad on \ E_0 .$$

2. *There exists* $\tilde{K} \in R_+$ *such that for* $l \in R\backslash\{0\}$ *we have*

$$\left(\varphi(x, y + l) - 2\varphi(x, y) + \varphi(x, y - l) \right) l^{-2} \le \tilde{K} \quad on \ E_0 .$$

3. $f \in C^2(\Omega, R)$ *and there exist a constant* $N \ge \tilde{M}$ *and a non-decreasing function* $V \in C([\tilde{M}, N], R_+)$ *such that for* $t \in R_+$ *we have* $\int_{\tilde{M}}^N V(t)^{-1} dt \ge b$ *and*

$$|f(x, y, p, w, q)| \le V(t) \quad on \ [0, b] \times R \times [-t, t] \times C(D, R, t) \times R.$$

4. *There exist* $N_1 \ge L$, $A > 0$ *and a non-decreasing function* $W \in C([\tilde{L}, N_1], R_+)$ *such that* $\int_{\tilde{L}}^{N_1}[(2t + 1)W(N + 3t)]^{-1}dt \ge b$, *and for* $t \in R_+$, $P = (x, y, p, w, q) \in [0, b] \times R \times [-N, N] \times C_{0+L}(D, R, t) \times R$ *we have*

$$|D_q f(P)| \le A \quad and \quad |D_y f(P)|, |D_p f(P)|, \|D_w f(P)\| \le W(t)$$

where $D_w f$ is the Frechet derivative of f with respect to w.

5. *For $\overline{w} \in D(D, R_+$ we have $D_w f(P)(\overline{w}) \geq 0$ on*

$$\tilde{\Omega} = [0, b] \times R \times [-N, N] \times C_{0+L}(D, R, N + 3N_1) \times [-N_1, N_1].$$

6. *The second order derivatives of f are bounded on $\tilde{\Omega}$ and $D_{qq}^2 f(P) \leq 0$ on $\tilde{\Omega}$.*

Then there is a function $u \in C(E_b, R)$ which is a generalized entropy solution to problem (0.1), (0.2).

The proof of the theorem is based on the difference method. Assumptions of existence theorems are by far more restrictive than assumptions of uniqueness theorems, which is typical for this class of problems.

Now, we give an example of the Cauchy problem for which there are at least two Caratheodory solutions. For $X \subset [0, 1] \times R$ we denote by χ_x the characteristic function of X. Put

$$X_1 = \Big\{ (x, y) \Big| \; x \in [0, 1], \; 2 \leq y < x + 2 \Big\}$$

$$X_2 = \Big\{ (x, y) \Big| \; x \in [0, 1], \; -x + 2 \leq y < 2 \Big\}$$

$$X_3 = \Big\{ (x, y) \Big| \; x \in [0, 1], \; x - 2 \leq y < -x + 2 \Big\}$$

$$X_4 = \Big\{ (x, y) \Big| \; x \in [0, 1], \; -2 \leq y < x - 2 \Big\}$$

$$X_5 = \Big\{ (x, y) \Big| \; x \in [0, 1], \; -x - 2 \leq y < -2 \Big\}.$$

We define $g : [0, 1] \times R \to R$ by

$$g(x, y) = -\frac{1}{2}(x - y + 2)^2 \chi_{x_1}(x, y) + \left[\frac{1}{2}(y - 2 + x)^2 - x^2\right] \chi_{x_2}(x, y)$$

$$- x^2 \chi_{x_3}(x, y) + \left[\frac{1}{2}(x - y - 2)^2 - x^2\right] \chi_{x_4}(x, y)$$

$$- \frac{1}{2}(y + 2 + x)^2 \chi_{x_5}(x, y).$$

Then $g \in C([0, 1] \times R, R)$. Consider the differential-functional equation

$$D_x z(x, y) = \left[\int_{-2}^{2} z(x, y + s) ds - g(x, y)\right] \sin z(x, y) - \big(D_y z(x, y)\big)^2 \qquad (0.6)$$

on $[0, 1] \times R$ with the initial condition

$$z(0, y) = 0 \qquad \text{for } y \in R. \qquad (0.7)$$

Then functions $u(x, y) = 0$ for $(x, y) \in [0, 1] \times R$ and $v : [0, 1] \times R \to R$ given by

$$v(x, y) = \begin{cases} 0 & \text{for } x \in [0, 1], \; y < -x \text{ or } y > x \\ y - x & \text{for } x \in [0, 1], \; 0 \leq y \leq x \\ -y - x & \text{for } x \in [0, 1], \; -x \leq y < 0 \end{cases}$$

are Carathéodory solutions to problem (0.6), (0.7).

1. Differential-functional inequalities in Carathéodory sense

We shall prove a theorem on differential-functional inequalities. As a consequence of this theorem we get a uniqueness criterion for generalized entropy solutions to the Cauchy problem.

If $G : [a, \bar{a}] \times R^n \to R$ ($a, \bar{a} \in R_+$ and $a \leq \bar{a}$), then let supp G denote the support of function G. Given any normed spaces X and Y let $C^1(X, Y)$ stand for the set of all continuously differentiable functions defined on X and taking values in Y. For $a, \varepsilon \in R_+$ such that $a, a + \varepsilon \in [0, b]$ we shall denote

$$\mathcal{P}^{1,0}_{a,\varepsilon} = \left\{ G \in C^1\left([a, a + \varepsilon] \times R^n, R^+\right) \,\middle|\, \begin{array}{l} G(a + \varepsilon, y) = 0 \text{ for } y \in R^n \\ \text{supp } G \text{ is a compact set} \end{array} \right\}$$

and

$$\mathcal{P}_{a,\varepsilon} = \left\{ \phi \in C([a, a + \varepsilon] \times R^n, R_+) \,\middle|\, \text{supp } \phi \subset E_b \setminus E_0 \text{ is a compact set} \right\} .$$

Let $K_2(0, M_1) = \{y \in R^n : \|y\|_2 \leq M_1\}$, where $M_1 \in R_+$. The symbol $\| \cdot \|_2$ denotes here the Euclidean norm. If $M_0, M_1 \in R_+$ then let

$$\Omega(M_0, M_1) = E_{(b)} \times [-M_0, M_0] \times C(D, [-M_0, M_0]) \times K_2(0, M_1).$$

If X is a non-empty compact normed space and $T : C(X, R) \to R$ is a continuous linear functional, then

$$\|T\|_* = \sup\left\{ \|T\phi\| \,\middle|\, \phi \in C(X, R), \ \|\phi\| \leq 1 \right\} .$$

Assumption H_1. Suppose the following.

1. $f : \Omega \to R$ and for all $P = (x, y, p, w, q) \in \Omega$ there exist the derivatives

$$D_y f(P) , \quad D_q f(P) , \quad D_w f(P) , \quad D_p f(P)$$
$$D_{yq} f(P) , \quad D_{qq} f(P) , \quad D_{qw} f(P) , \quad D_{qp} f(P).$$

2. There exist constants $M_0, \ M_1 > 0$ such that the functions

$$D_y f, \quad D_q f, \quad D_w f, \quad D_p f, \quad D_{yq} f, \quad D_{qq} f, \quad D_{qw} f$$

are continuous on $\Omega(M_0, M_1)$.

3. There exist constants $c_1, c_2 \in R_+$ such that

$$\|D_y f(P)\|, \ \|D_q f(P)\|, \ \|D_w f(P)\|_*, \ D_p f(P) \leq c_1$$
$$\|D_{qp} f(P)\|, \ \|D_{yq} f(P)\|, \ \|D_{qw} f(P)\|_* \leq c_2$$

for every $P = (x, y, p, w, q) \in \Omega(M_0, M_1)$.

4. For every $(x, y, p, w, q) \in \Omega(M_0, M_1)$ and $\tilde{w} \in C(D, [-M_0, M_0])$ with $\tilde{w} \geq w$ the inequality

$$0 \leq f(x, y, p, \tilde{w}, q) - f(x, y, p, w, q)$$

is true.

5. There exist non-negative constants c_3 and c_4 with $c_3 > c_4$ such that, for all $(x, y, p, w, q) \in \Omega(M_0, M_1)$, we have the estimates

$$c_3 \|\eta\|_2^2 \geq \sum_{j,l=1}^{n} \left(-D_{q_j q_l} f_i(x, y, p, w, q) \right) \eta_j \eta_l \geq c_4 \|\eta\|_2^2$$

for arbitrary $\eta = (\eta_1, \ldots, \eta_n) \in R^n$.

Let $\lambda : [0, b] \to R_+$ be a bounded function on $[\delta, b]$ for every $\delta \in (0, b]$. A function z is said to be of *class* $C(E_b, R; M_0, M_1, \lambda)$ if the following conditions are satisfied:

(i) $z \in C(E_b, R)$ and $\|z(x, y)\| \leq M_0$ for $(x, y) \in E_b$.

(ii) $D_x z, D_y z$ exist a. e. on $E_{(b)}$ and $D_x z \in L^2_{loc}(E_{(b)}, R), D_y z \in L^2_{loc}(E_{(b)}, R^n)$.

(iii) $\|D_y z(x, y)\|_2 \leq M_1$ for almost all $(x, y) \in E_{(b)}$.

(iv) $z(x, y + \eta) - 2z(x, y) + z(x, y - \eta) \leq \lambda(x) \|\eta\|_2^2$ for every $\eta \in R^n, (x, y) \in E_{(b)}$.

Now we are in positions to formulate the following theorem on differential-functional inequalities.

Theorem 1.1. *Suppose that the following conditions are true.*
1. *Assumption H_1 is satisfied and $u, v \in C(E_b, R)$.*

2. *There are constants $M_0, M_1 \in R_+$ and a function $\lambda : [0, b] \to R_+$, bounded on $[\delta, b]$ for every $\delta \in (0, b]$, such that $u, v \in C(E_b, R; M_0, M_1, \lambda)$.*

3. *The differential-functional inequalities*

$$D_x u(x, y) \leq f\Big(x, y, u(x, y), u_{(x,y)}, D_y u(x, y)\Big) \tag{1.1}$$

$$D_x v(x, y) \geq f\Big(x, y, v(x, y), v_{(x,y)}, D_y v(x, y)\Big) \tag{1.2}$$

are satisfied almost everywhere on $E_{(b)}$ and

$$u(x, y) \leq v(x, y) \quad \text{for all } (x, y) \in E_0. \tag{1.3}$$

Then for every $(x, y) \in E_b$ we have the inequality

$$u(x, y) \leq v(x, y) \tag{1.4}$$

provided either $D_w f(x, y, p, w, q) = 0$ for $(x, y, p, w, q) \in \Omega(M_0, M_1)$ or the function $\lambda : [0, b] \to R_+$ is integrable on the interval $[0, b]$.

Proof. Let $\{u^{(\nu)}\}_{\nu \geq 1}$ and $\{v^{(\nu)}\}_{\nu \geq 1}$ be two fixed sequences of functions satisfying the following conditions:

(i) For every ν the functions $u^{(\nu)}, v^{(\nu)} : E_b \to R$ are of class C^2.

(ii) For arbitrary compact subset $X \subset E_b$ we have the limits

$$\lim_{\nu \to \infty} u^{(\nu)}|_X = u|_X \quad \text{and} \quad \lim_{\nu \to \infty} v^{(\nu)}|_X = v|_X \quad \text{(uniformly)}$$

$$\lim_{\nu \to \infty} \|D_y u^{(\nu)}|_X - D_y v|_X\|_{L^2(X,R)} = 0 = \lim_{\nu \to \infty} \|D_y v^{(\nu)}|_X - D_y v|_X\|_{L^2(X,R)}.$$

(iii) For every $(x,y) \in E_{(b)}$ and $\eta \in R^n$ we have the estimates

$$D_{\eta\eta} u^{(\nu)}(x,y) \leq \lambda(x) \quad \text{and} \quad D_{\eta\eta} v^{(\nu)}(x,y) \leq \lambda(x)$$

where $D_{\eta\eta}z$ denotes the second order derivative of the function z in the direction of the vector η (note that sequences $\{u^{(\nu)}\}$ and $\{v^{(\nu)}\}$ can be constructed by the regularization method, see [8]). Denote $\omega = v-u$ and $\omega^{(\nu)} = v^{(\nu)} - u^{(\nu)}$ for $\nu \geq 1$. Then from inequality (1.3) it follows that $\omega(x,y) \geq 0$ for $(x,y) \in E_0$. Suppose that condition (1.4) is satisfied on E_a for some $a \in [0,b)$. We shall show that it is satisfied on $E_{\bar{a}}$ for some $\bar{a} \in (a,b]$.

Let $\varepsilon > 0$ such that $a + \varepsilon \in [0,b]$. Suppose $G \in \mathcal{P}_{a,\varepsilon}^{1,0}$. Then from inequalities (1.1), (1.2) we get the inequality

$$
\begin{aligned}
\int_{R^n} \int_a^{a+\varepsilon} & G(x,y) \Big\{ D_x \omega^{(\nu)}(x,y) \\
& - \Big[f\Big(x,y,v^{(\nu)}(x,y),v^{(\nu)}_{(x,y)},D_y v^{(\nu)}(x,y)\Big) \\
& - f\Big(x,y,u^{(\nu)}(x,y),u^{(\nu)}_{(x,y)},D_y u^{(\nu)}(x,y)\Big) \Big] \Big\} \, dxdy \\
& \geq \int_{R^n} \int_a^{a+\varepsilon} G(x,y) \Gamma^{(\nu)}(x,y) \, dxdy
\end{aligned}
\tag{1.5}
$$

for $\nu \geq 1$, where the function $\Gamma^{(\nu)}$ is defined by

$$
\begin{aligned}
\Gamma^{(\nu)}(x,y) = \; & D_x v^{(\nu)}(x,y) - f\Big(x,y,v^{(\nu)}(x,y),v^{(\nu)}_{(x,y)},D_y v^{(\nu)}(x,y)\Big) \\
& - \Big(D_x v(x,y) - f\big(x,y,v(x,y),v_{(x,y)},D_y v(x,y)\big)\Big) \\
& - \Big(D_x u^{(\nu)}(x,y) - f\big(x,y,u^{(\nu)}(x,y),u^{(\nu)}_{(x,y)},D_y u^{(\nu)}(x,y)\big)\Big) \\
& + \Big(D_x u(x,y) - f\big(x,y,u(x,y),u_{(x,y)},D_y u(x,y)\big)\Big).
\end{aligned}
$$

It follows from assumptions (i) - (iii) and Assumption H_1 that $\lim_{\nu \to \infty} \|\Gamma^{(\nu)}|_X\|_{L^2(X,R)} = 0$ for arbitrary compact subset $X \subset E_{(b)}$. Define

$$
\begin{aligned}
& P^{(\nu)}(x,y,\theta) \\
& = \Big(x,y,u^{(\nu)}(x,y) + \theta\omega^{(\nu)}(x,y),(u^{(\nu)} + \theta\omega^{(\nu)})_{(x,y)},D_y(u^{(\nu)} + \theta\omega^{(\nu)})(x,y)\Big).
\end{aligned}
$$

Applying the Hadamard mean-value theorem and integrating by parts in (1.5) we obtain the inequality

$$
\int\limits_{R^n} \int\limits_{a}^{a+\epsilon} \left\{ -\omega^{(\nu)}(x,y) \left\{ D_x G(x,y) + G(x,y) \int\limits_0^1 D_p f\left(P^{(\nu)}(x,y,\theta) \right) d\theta \right. \right.
$$

$$
\left. - \sum_{j=1}^n \frac{\partial}{\partial y_j} \left\{ G(x,y) \int\limits_0^1 D_{q_j} f\left(P^{(\nu)}(x,y,\theta) \right) d\theta \right\} \right\}
$$

$$
\left. - G(x,y) \int\limits_0^1 D_w f\left(P^{(\nu)}(x,y,\theta) \right) d\theta \left(\omega^{(\nu)} \right)_{(x,y)} \right\} dx\,dy
$$

$$
\geq \int\limits_{R^n} \int\limits_{a}^{a+\epsilon} G(x,y) \Gamma(\nu)(x,y)\,dx\,dy + \int\limits_{R^n} G(a,y)\omega(a,y)\,dy
$$

(1.6)

for $\nu \geq 1$. Take arbitrary $\phi \in \mathcal{P}_{a,\epsilon}$ and consider the Cauchy problem

$$
D_x z(x,y) + z(x,y) \int\limits_0^1 D_p f\left(P^{(\nu)}(x,y,\theta) \right) d\theta
$$

$$
- \sum_{j=1}^n \frac{\partial}{\partial y_j} \left\{ z(x,y) \int\limits_0^1 D_{q_j} f\left(P^{(\nu)}(x,y,\theta) \right) d\theta \right\} = -\phi(x,y)
$$

$$
z(a+\epsilon, y) = 0.
$$

(1.7)

Let

$$
g^{(\nu)}(\cdot\,; x,y) : [a, a+\epsilon] \to R^n \qquad \text{for} \quad (x,y) \in E_{(a+\epsilon)} \setminus E_{(a)}
$$

be solutions of the characteristic initial-value problem corresponding to problem (1.7):

$$
\eta'(t) = -\int\limits_0^1 D_q f\left(P^{(\nu)}(t,\eta(t),\theta) \right) d\theta \qquad \text{and} \qquad \eta(x) = y.
$$

(1.8)

For $(x,y) \in E_{(a+\epsilon)} \setminus E_{(a)}$ we denote

$$
H^{(\nu)}(x,y) = \int\limits_0^1 D_p f\left(P^{(\nu)}(x,y,\theta) \right) d\theta
$$

$$
- \sum_{j=1}^n \int\limits_0^1 A_{jj}\left(P^{(\nu)}(x,y,\theta); u^{(\nu)} + \theta\omega^{(\nu)} \right) d\theta
$$

(1.9)

where the matrix $A(P; z) = [A_{jl}(P; z)]_{j,l=1}^{n}$ is defined by

$$
\begin{aligned}
A_{jl}(P; z) = {} & D_{q_j\, y_l} f(P) + D_{q_j\, p_k} f(P) D_{y_l} z(x, y) \\
& + D_{q_j\, w_k} f(P)(D_{y_l} z)_{(x,y)} + \sum_{\mu=1}^{n} D_{q_\mu q_j} f(P) D_{y_\mu y_l} z(x, y)
\end{aligned}
\tag{1.10}
$$

for $z \in C(E_b, R; M_0, M_1, \lambda)$ and $P = (x, y, p, w, q) \in \Omega(M_0, M_1)$. Consider inequality (1.6) with $G^{(\nu)}$ satisfying problem (1.7). Then we have the inequality

$$
\begin{aligned}
\int_{R^n} \int_a^{a+\varepsilon} & \left\{ \omega^{(\nu)}(x, y)\phi(x, y) \right. \\
& \left. - G^{(\nu)}(x, y) \int_0^1 D_w f\left(P^{(\nu)}(x, y, \theta)\right) d\theta \left(\omega^{(\nu)}\right)_{(x,y)} \right\} dx\, dy \\
\geq {} & \int_{R^n} \int_a^{a+\varepsilon} G^{(\nu)}(x, y)\Gamma^{(\nu)}(x, y)\, dx\, dy + \int_{R^n} G(a, y)\omega(a, y)\, dy
\end{aligned}
\tag{1.11}
$$

for $\nu \geq 1$, where

$$
G^{(\nu)}(x, y) = \int_x^{a+\varepsilon} \phi\left(t, g^{(\nu)}(t; x, y)\right) \exp\left(\int_x^t H^{(\nu)}\left(s, g^{(\nu)}(s; x, y)\right) ds\right) dt.
\tag{1.12}
$$

Formula (1.12) defines a transformation of the set $\mathcal{P}_{a,\varepsilon}$ onto $\mathcal{P}_{a,\varepsilon}^{1,0}$. For $(x, y) \in E_{a+\varepsilon} \backslash E_a$ the matrix $D_y g^{(\nu)}(\cdot\, ; x, y) = [D_{y_l} g_j^{(\nu)}(\cdot\, ; x, y)]_{j,l=1}^{n}$ satisfies the equation

$$
\left[\frac{d}{dt}\zeta_{jl}\right]_{j,l=1}^{n} = -A\left(P^{(\nu)}(t, g^{(\nu)}(t; x, y), \theta); u^{(\nu)} + \theta\omega^{(\nu)}\right) [\zeta_{jl}]_{j,l=1}^{n}
\tag{1.13}
$$

with the initial condition

$$
\zeta_{jl}(x) = \delta_{jl} \qquad (j, l = 1, \ldots, n)
\tag{1.14}
$$

where δ_{jl} is the Kronecker symbol. Denote

$$
J^{(\nu)}(t; x, y) = \det\left[D_y g^{(\nu)}(t; x, y)\right] \quad \text{for } (x, y) \in E_{a+\varepsilon}\backslash E_a,\ t \in [a, a+\varepsilon].
$$

Observe that $J^{(\nu)}(t; x, y) > 0$, moreover we have the equality

$$
J^{(\nu)}(t; x, y) = \exp\left(\int_t^x \sum_{j=1}^{n} A_{jj}\left(P^{(\nu)}(s, g^{(\nu)}(s; x, y), \theta); u^{(\nu)} + \theta\omega^{(\nu)}\right) ds\right).
\tag{1.15}
$$

Now we use the change of variables

$$E_{a+e}\backslash E_a \ni (x,y) \longrightarrow (x,\eta) = \left(x, g^{(\nu)}(t;x,y)\right) \in E_{a+e}\backslash E_a$$

and obtain the equality

$$\int\limits_{R^n}\int\limits_a^t \phi\left(t, g^{(\nu)}(t;x,y)\right) \exp\left(\int\limits_x^t H^{(\nu)}\left(s,g^{(\nu)}(s;x,y)\right) ds\right)$$

$$\times \int\limits_0^1 D_w f\left(P^{(\nu)}(x,y,\theta)\right) d\theta \left(\omega^{(\nu)}\right)_{(x,y)} dx\, dy$$

$$= \int\limits_{R^n}\int\limits_a^t \phi(t,\eta) J^{(\nu)}\left(t;x,g^{(\nu)}(x;t,\eta)\right) \qquad (1.16)$$

$$\times \exp\left(\int\limits_x^t H^{(\nu)}\left(s,g^{(\nu)}(s;x,g^{(\nu)}(x;t,\eta))\right) ds\right)$$

$$\times \int\limits_0^1 D_w f\left(P^{(\nu)}(x,g^{(\nu)}(x;t,\eta),\theta)\right) d\theta \left(\omega^{(\nu)}\right)_{(x,g^{(\nu)}(x;t,\eta))} dx\, d\eta.$$

From (1.6), (1.7), (1.11), (1.16) it follows that for every $\phi \in \mathcal{P}_{a,e}$ we have the inequality

$$\int\limits_{R^n}\int\limits_a^{a+e} \phi(x,y)\left\{\omega^{(\nu)}(x,y) - \int\limits_0^x J^{(\nu)}\left(x;t,g^{(\nu)}(t;x,y)\right)\right.$$

$$\times \exp\left(\int\limits_t^x H^{(\nu)}\left(s,g^{(\nu)}(s;t,g^{(\nu)}(t;x,y))\right) ds\right)$$

$$\qquad (1.17)$$

$$\left.\times \int\limits_0^1 D_w f\left(P^{(\nu)}(t,g^{(\nu)}(t;x,y),\theta)\right) \left(\omega^{(\nu)}\right)_{(t,g^{(\nu)}(t;x,y))} d\theta\, dt\right\} dx\, dy$$

$$\geq \int\limits_{R^n}\int\limits_a^{a+e} G(x,y)\Gamma^{(\nu)}(x,y)\, dx\, dy + \int\limits_{R^n} G(a,y)\omega(a,y)\, dy$$

for $\nu \geq 1$.

For $P(\nu)(x,y,\theta)$ there are orthonormal vectors $\bar{\eta}^{(j)} = \eta^{(\nu,j)}(x,y,\theta) \in R^n$ ($j = 1,\dots,n$) and eigenvalues $\lambda^{(\nu,j)}(x,y,\theta) \in R_+$ such that $c_3 \geq \lambda^{(\nu,j)}(x,y,\theta) \geq c_4 \geq 0$ and

$$-\sum_{j,l=1}^n D_{q_j q_l} f\left(P^{(\nu)}(x,y,\theta)\right) D_{y_j y_l} z^{(\nu)}(x,y)$$

$$\qquad (1.18)$$

$$= \sum_{j=1}^n \lambda^{(\nu,j)}(x,y,\theta) D_{\bar{\eta}^{(j)} \bar{\eta}^{(j)}} z^{(\nu)}(x,y).$$

Thus from Assumption H_1 and (1.18) combined with (1.9) it follows that the function $H^{(\nu)}$ is upper-bounded by a measurable function independent of $\nu \geq 1$. The same property for the function $J^{(\nu)}$ can be stated.

Let $\delta \in (0, \varepsilon)$ be an arbitrary number. Since for $\phi \in \mathcal{P}_{a,\varepsilon}$ with $\operatorname{supp} \phi \in E_{a+\varepsilon} \backslash E_{a+\delta}$ the right-hand side of inequality (1.17) tends to 0 as $\nu \to \infty$, we have the inequality

$$
\int_{R^n} \int_a^{a+\varepsilon} \phi(x,y) \left\{ \omega(x,y) - \liminf_{\nu \to \infty} \int_0^x J^{(\nu)} \left(x; t, g^{(\nu)}(t; x, y) \right) \right.
$$

$$
\times \exp \left(\int_t^x H^{(\nu)} \left(s, g^{(\nu)}(s; t, g^{(\nu)}(t; x, y)) \right) ds \right) \tag{1.19}
$$

$$
\left. \times \int_0^1 D_w f \left(P^{(\nu)}(t, g^{(\nu)}(t; x, y), \theta) \right) \left(\omega^{(\nu)} \right)_{(t, g^{(\nu)}(t; x, y))} d\theta dt \right\} dx dy \geq 0.
$$

If $D_w f = 0$, then we get what was to be proved (compare [8]). If $D_w f \neq 0$, then from inequality (1.19) we get the Volterra integral inequality

$$
\omega(x,y) \geq \liminf_{\nu \to \infty} \int_0^x J^{(\nu)} \left(x; t, g^{(\nu)}(t; x, y) \right)
$$

$$
\times \exp \left(\int_t^x H^{(\nu)} \left(s, g^{(\nu)}(s; t, g^{(\nu)}(t; x, y)) \right) ds \right) \tag{1.20}
$$

$$
\times \int_0^1 D_w f \left(P^{(\nu)}(t, g^{(\nu)}(t; x, y), \theta) \right) \left(\omega^{(\nu)} \right)_{(t, g^{(\nu)}(t; x, y))} d\theta dt
$$

for $(x, y) \in E_{a+\varepsilon} \backslash E_{a+\delta}$, where $\delta \in (0, \varepsilon)$. Denote by ω^- and $(\omega^{(\nu)})^-$ for $\nu \geq 1$ the functions that are given by equality

$$
\omega^-(x,y) = \min\{0, \omega(x,y)\} \qquad \text{and} \qquad (\omega^{(\nu)})^-(x,y) = \min\{0, \omega^{(\nu)}(x,y)\} \tag{1.21}
$$

for $(x, y) \in E_b$. From inequality (1.20) it is easy to see that

$$
\omega^-(x,y) \geq \liminf_{\nu \to \infty} \int_0^x J^{(\nu)} \left(x; t, g^{(\nu)}(t; x, y) \right)
$$

$$
\times \exp \left(\int_t^x H^{(\nu)} \left(s, g^{(\nu)}(s; t, g^{(\nu)}(t; x, y)) \right) ds \right) \tag{1.22}
$$

$$
\times \int_0^1 D_w f \left(P^{(\nu)}(t, g^{(\nu)}(t; x, y), \theta) \right) \left(\left(\omega^{(\nu)} \right)^- \right)_{(t, g^{(\nu)}(t; x, y))} d\theta dt
$$

for $(x, y) \in E_{a+\epsilon} \backslash E_{a+\delta}$, where $\delta \in (0, \epsilon)$. In view of conditions (1.9), (1.10), (1.15), (1.18) and Assumption H_1 we have the inequality

$$
\begin{aligned}
J^{(\nu)} &\left(x; t, g^{(\nu)}(t; x, y) \right) \exp \left(\int\limits_t^x H^{(\nu)} \left(s, g^{(\nu)}(s; t, g^{(\nu)}(t; x, y)) \right) ds \right) \\
&\leq \exp \left(\int\limits_t^x \left(c_1 + 2nc_2(1 + 2M_1) + c_3\lambda(s) \right) ds \right).
\end{aligned}
\tag{1.23}
$$

Let a function $W : [a, a + \epsilon] \to R_-$ be given by

$$
W(x) = \inf_{(t,y) \in E_x} \omega^-(t, y) \qquad \text{for } x \in [a, a + \epsilon].
\tag{1.24}
$$

Next, from (1.18), (1.22) - (1.23) and Assumption H_1 we obtain the inequality

$$
W(x) \geq \int\limits_0^x \exp \left(\int\limits_t^x \left(c_1 + 2nc_2(1 + 2M_1) + c_3\lambda(s) \right) ds \right) c_1 W(t) \, dt
\tag{1.25}
$$

for $x \in [a, a + \epsilon]$. From definition (1.21) we get the inequalities $\omega(x, y) \geq W(x) \geq 0$ for $(x, y) \in E_{a+\epsilon}$. As $a \in [0, b)$ was taken to be arbitrary, we have obtained inequality (1.4), which finishes the proof of Theorem 1.1 ∎

Remark 1. Let $L^p(X, R)\,(p \geq 1)$ denote the Banach space of real functions defined on a metric space X that are integrable with power p. In the above given theorem one can replace the constants $M_0, M_1, c_1, c_2, c_3 \in R_+$ by measurable functions $\tilde{M}_0, \tilde{M}_1, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 : [0, b] \to R_+$. It is enough to assume then that there are constants $P_0, P_1, P_2, P_3 \in R_+$ with $\frac{1}{P_0} + \frac{1}{P_1} \leq 1$ and $\frac{1}{P_2} + \frac{1}{P_3} \leq 1$ such that $\tilde{M}_1 \in L^{P_0}([0, b], R_+), \tilde{c}_1 \in L^1([0, b], R_+), \tilde{c}_2 \in L^{P_1}([0, b], R_+), \tilde{c}_3 \in L^{P_3}([0, b], R_+)$ and $\lambda \in L^{P_2}([0, b], R_+)$.

2. Uniqueness result

The following theorem on the uniqueness of generalized entropy solutions to problem (0.1), (0.2) is a simple consequence of Theorem 1.1.

Theorem 2.1. *Suppose that the following assumptions are true.*
1. Assumption H_1 is satisfied and $u, v \in C(E_b, R)$.

2. There are constants $M_0, M_1 \in R_+$ and a function $\lambda : [0, b] \to R_+$, bounded on $[\delta, b]$ for every $\delta \in (0, b]$, such that $u, v \in C(E_b, R; M_0, M_1, \lambda)$.

3. The functions u, v are solutions to the Cauchy problem (0.1),(0.2), where $\varphi \in C(E_0, R)$.

Then for every $(x, y) \in E_b$ we have the equality

$$
u(x, y) = v(x, y)
\tag{2.1}
$$

provided either $D_w f(x, y, p, w, q) = 0$ *for* $(x, y, p, w, q) \in \Omega(M_0, M_1)$ *or the function* $\lambda : [0, b] \to R_+$ *is integrable on the interval* $[0, b]$.

Proof. The functions u, v satisfy assumptions of Theorem 1.1, thus from (2.1) we have $u \le v$ for $(x, y) \in E_b$. The inverse inequality is obtained in the same way.

Remark 2. If we omit condition (iv) in the definition of class $C(E_b, R; M_0, M_1, \lambda)$, then Theorem 2.1 is not true, because there may exist two different solutions of problem (0.1), (0.2). An adequate example of f without a functional variable is given in [8].

Remark 3. Our uniqueness result and a theorem on differential-functional inequalities remain valid if we replace problem (0.1),(0,2) by the following Cauchy problem for the weakly-coupled system of non-linear equations:

$$D_x z_i(x, y) = F_i(x, y, z(x, y), z_{(x,y)}, D_y z_i(x, y)), \quad i = 1, \dots, m,$$

$$z(x, y) = \Phi(x, y) \text{ on } E_0 .$$

We assume in this case that the function

$$F(x, y, p, w, q) = (F_1(x, y, p, w, q), \dots, F_m(x, y, p, w, q))$$

satisfies the quasi-monotonicity condition with respect to p. While proving a theorem on differential-functional inequalities we get a system of Volterra integral inequalities, which implies an appropriate inequality for functions.

References

[1] Augustynowicz, A. and Z. Kamont: *On Kamke's functions in uniqueness theorems for first order partial differential-functional equations.* J. Nonlin. Anal., T.M.A. 14 (1990)10, 837 - 850.

[2] Besala, P.: *On solutions of first order partial differential equations defined in an unbounded zone.* Bull. Acad. Polon. Sci. 12 (1964)2, 95 - 99.

[3] Brandi, P. and R. Ceppitelli: *Existence, uniqueness and continuous dependence for a first order non linear partial differential equation in a hereditary structure.* Ann. Polon. Math. 47 (1986), 121 - 136.

[4] Cinquini Cibrario, M.: *Sopra una classe di sistemi di equazioni non lineari a derivate parziali in piu variabili indipendenti.* Ann. Mat. Pura Appl. (IV) 140 (1985), 223 - 253.

[5] T. Człapiński: *On the existence of generalized solutions of nonlinear first order partial differential-functional equations in two independent variables.* Czech. Math J. 41 (116) (1991), 490 - 506.

[6] Kamont, Z.: *On the Cauchy problem for systems of first order of partial differential equations.* Serdica 7 (1979), 327 - 339.

[7] Kamont, Z. and S. Zacharek: *On the existence of weak solutions of nonlinear first order partial differential equations in two independent variables.* Boll. Un. Mat. Ital. (6) 5-B (1986), 851 - 870.

[8] Kružkov, S. N.: *Generalized solutions of nonlinear equations of first order with several independent variables* (in Russian). Mat. Sb. 70 (1966), 394 - 415. Engl. transl.: Math. USSR-Sb. 1 (1967), 93 - 116.

[9] Lakshmikantham, V. and S. Leela: *Differential and Integral Inequalities.* New York - London: Acad. Press 1969.

[10] Leszczyński, H.: *On the existence and uniqueness of weak solutions of the Cauchy problem for weakly coupled systems of non-linear partial differential-functional equations of the first order.* Boll. Un. Mat. Ital. (7) 7-B (1993), 323 - 354.

[11] Leszczyński, H.: *A contribution to the uniqueness theory for first-order partial differential-functional systems.* Demonstr. Math. 26 (1993), 561 - 579.

[12] Lions, P. L.: *Generalized Solutions of Hamiltonian-Jacobi Equations.* Boston - London - Melbourne: Pitman Adv. Publ. Program 1982.

[13] Myshkis, A. D. and A. S. Šlopak: *A mixed problem for systems of differential-functional equations with partial derivatives and with operators of the Volterra type* (in Russian). Mat. Sb. 41 (1957), 239 - 256.

[14] Oleinik, O. A.: *Discontinuous solutions of nonlinear differential equations* (in Russian). Usp. Mat. Nauk 12 (1957), 3 - 73; AMS Transl. Ser. (2) 26 (1957), 95 - 172.

[15] Szarski, J.: *Differential Inequalities.* Warsaw: Polish Sci. Publ. 1967.