Some Bifurcation Results
Including Banach Space Valued Parameters

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Abstract. In this article a contribution to the so-called limit point bifurcation is given. In the paper [4] by Decker and Keller a bifurcation or branching phenomenon which they call multiple limit point bifurcation has been shown for equations $T(\lambda, x) = 0$ with real parameter λ. If one has a solution (λ^o, x^o) of equation $T(\lambda, x) = 0$, then one speaks from a limit point if the Fréchet derivative $T_x(\lambda^o, x^o)$ is singular and $T_\lambda(\lambda^o, x^o)$ is not in the range of $T_x(\lambda^o, x^o)$. Here a method will be given, which generalizes the notion of limit point to that what is called (α, n)-limit point. This makes it possible to handle equations of the form $T(u, x) = 0$ which may have u as a Banach space valued parameter. This equation with singular operator $T_x(u^o, x^o)$ may be "embedded" in a larger system with a linearization, which is non-singular and hence to which an implicit function theorem can apply. An estimation for the number of branching solutions of this new system is given.

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1. Introductory remarks

A constructive method for determining paths or arcs of solutions around a bifurcation element of a general operator equation, say $T(u, x) = 0$ (see (2.1)), operating in a Banach space is studied. Often it is possible to interpret x as a variable characterizing the state of a system and u as a control.

In bifurcation theory one is trying to answer the following question as completely as possible: Let (u^o, x^o) be a solution of the equation $T(u, x) = 0$. Which states $x \neq x^o$ are possible, if the parameter u varies in any neighbourhood of u^o ? Two different cases are considered: Either there is exactly one solution element (u, x) close to (u^o, x^o) for all u belonging to a sufficiently small neighbourhood of u^o or not.

In the last case, the Ljapunov-Schmidt reduction can be applied for solving such problems. Historically (see [12]) this procedure was used to reduce certain infinite-dimensional problems to one of solving finitely many non-linear equations with finitely many real or complex variables.

Today this is a useful tool in analyzing non-linear equations depending upon a parameter. An example is given by the rotation of a viscous fluid between concentric

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cylinders, where critical rotation speeds lead to the formation of so-called Taylor vortices. This is a bifurcation problem for the Navier-Stokes differential equation. Other sources of bifurcation problems are encountered in determining the critical forces for the deformation of rods, plates and shells and in investigating the critical velocities at which traveling waves arise in fluids. Chemical reactions involve bifurcation phenomena, such as those leading to sudden color changes.

Solving these so-called branching equations, which contain all informations about the behaviour of solutions of the original equation is a difficult problem in general. It can involve for example the methods of function theory (Puiseux expansions, Weierstrass preparation theorem), of algebra (elimination theory), of algebraic geometry (Bezout theorem, generalized Morse lemma) and of differential topology (singularities of maps, transversality theory).

In the paper of Decker and Keller [4] the branching of solutions of an equation of the form

$$T(\lambda, x) = 0 \tag{1.1}$$

with $\lambda \in \mathbb{R}$, $x \in B_2$ and $T(\lambda, x) \in B_3$ is studied in Banach spaces. Here (λ^o, x^o) is a solution element of equation (1.1) and $T_x(\lambda^o, x^o)$ is a Fredholm operator of index zero and $\dim N(T_x(\lambda^o, x^o)) = m > 0$, where $N(A)$ denotes the kernel of an operator A.

In [4] a solution element (λ^o, x^o) of equation (1.1) is called *limit point*, if $T_\lambda(\lambda^o, x^o) \notin R(T_x(\lambda^o, x^o))$, where $R(A)$ denotes the kernel of an operator A. In order to prove the existence of solutions of equation $T(\lambda, x) = 0$ in a neighbourhood of (λ^o, x^o) a so-called *limit point bifurcation equation*, which consist of equations of the form

$$A(\mathbf{e})\mathbf{e} + \eta \mathbf{d} = 0 \tag{1.2}$$

$$\mathbf{e}^T \mathbf{e} = 1 \tag{1.3}$$

must be studied. Here

$$A(\mathbf{e}) = (A_{ij}(\mathbf{e})) = \left(\sum_{k=1}^{m} a_{ijk} e_k \right)$$

with

$$a_{ijk} = a_{jik} = \langle \psi_k^*, \ T_{xx}(\lambda^o, x^o)(\varphi_i, \varphi_j) \rangle$$

where $\varphi_i, \varphi_j \in N(T_x(\lambda^o, x^o))$ $(i, j = 1, \ldots, m)$, $e_k \in \mathbf{R}$,

$$\mathbf{d} = (d_1, \ldots, d_m)^T \quad \text{with} \quad d_i = \langle \psi_i^*, T_\lambda(\lambda^o, x^o) \rangle, \quad \psi_i^* \in N(T_x^*(\lambda^o, x^o))$$

and $\eta \in \mathbb{R}$ is fixed.

In the case that $(\mathbf{e}^o, \eta^o)$ is an isolated solution of system (1.2) and (1.3), that is if the Jacobian matrix of the system, evaluated at $(\mathbf{e}^o, \eta^o)$, is non-singular, there is a solution arc of equation (1.1) through (λ^o, x^o). The proof is an application of the blowing-up technique in combination with the implicit function theorem. The equations (1.2) are m equations in $(m + 1)$ unknowns and if one puts $\eta = \zeta^2$, then the equations (1.2) are homogeneous of degree two and thus admit rays of solutions. To avoid this and to fix a definition of a parameter ε one adjoints the normalization equation (1.3). Then the Bezout theorem (see [13/Vol.II: Sec. 89]) tells that the equations (1.2) and (1.3) can

have at most 2^m isolated real roots. Each such isolated root of equations (1.2) and (1.3) generates a locally unique solution arc.

Here, under some other conditions concerning the operator T, a generalization of a method due to Decker and Keller [4] is considered. Also, this method partially includes equations which are mainly investigated numerically by Böhmer and Mei (see [1, 10]). This generalization also gives the possibility of representing all solutions in a neighbourhood of a so-called (α, n)-limit point as a one-parameter family, where the parameter is a new Banach space parameter, called $v \in B_1$.

In Section 2 basic facts concerning the (α, n)-limit point are given. The parameter space B_1 (a Banach space) may have a dimension $\dim B_1 \geq 1$. For the special parameter space $B_1 = I\!R$ and $B_2 = B_3$ it will be shown that the definitions of limit point and (α, n)-limit point are in correspondence.

In Section 3 it will be shown that for a $(0, n)$-limit point (u^o, x^o) of equation (2.1) there exists a solution branch through (u^o, x^o). A representation of the solution branch is given. The proof uses a further equation, which will be added to the equation (2.1) such that the implicit function theorem works.

In Sections 4 and 5 the main results are presented and bounds on the number of solution arcs through the $(0, n)$-limit point (u^o, x^o) of equation (2.1) are given for the cases $n = 1$ and $n = 2$. For $n = 1$ an application of the classical Morse lemma is used and for $n = 2$ a discussion of the number of solutions is carried out by means of the resultant theory.

Section 6 contains an example of a Hammerstein equation, which under certain conditions has only a $(0, 1)$-limit point (u^o, x^o) of equation (2.1).

2. Assumptions and definitions

Let B_i $(i = 1, 2, 3)$ be Banach spaces over $\mathbf{K}$, the set of real or complex numbers. The equation considered is

$$T(u, x) = 0 \tag{2.1}$$

where $T : B_1 \times B_2 \mapsto B_3$. Throughout the paper let $(u^o, x^o) \in B_1 \times B_2$ be a *solution element* of equation (2.1), that means $T(u^o, x^o) = 0$.

If the Fréchet derivative $T_x(u^o, x^o)$ is invertible, that is $T_x^{-1}(u^o, x^o) \in L(B_3, B_2)$ where $L(B_3, B_2)$ means the space of all linear continuous operators from B_3 into B_2, then by the implicit function theorem equation (2.1) can be solved uniquely for x in a neighbourhood of (u^o, x^o) (see Definition 2.1). If on the other hand $\dim N(T_x(u^o, x^o)) \geq 1$, then it is possible that there is a bifurcation at (u^o, x^o) (see Definition 2.2), if u passes u^o.

The following two definitions are adapted to the methods, which are used in this paper.

Definition 2.1: Suppose $f : D_1 \subset B_1 \mapsto B_1 \times B_2$ is a map with (open) set $D_1 \neq \emptyset$. The map f is called a *solution arc* or a *continuation* to the solution element (u^o, x^o) if

$$T(R(f)) = 0 \quad \text{with} \quad R(f) = \Big\{ f(v) = (u(v), x(v)) \in B_1 \times B_2 \big| \ v \in D_1 \Big\}$$

and

$$(u(v), x(v)) \to (u^0, x^0) \quad \text{as} \quad v \to v^0 \quad \text{and} \quad v^0 \in D_1$$

(including the map $v \mapsto (v, x(v))$ with $u(v) = v$).

Definition 2.2: A solution element (u^o, x^o) of equation (2.1) is said to be a *bifurcation element* if there are sequences (u_n, x_n), $(u_n, y_n) \subset B_1 \times B_2$ for which the conditions

(i) $T(u_n, x_n) = T(u_n, y_n) = 0$, $x_n \neq y_n$ for all $n \in I\!N$

(ii) $(u_n, x_n) \to (u^o, x^o)$ and $(u_n, y_n) \to (u^o, x^o)$ as $n \to \infty$

are satisfied.

This last definition contains the notion of perpendicular bifurcation. A solution element (u^o, x^o) of equation (2.1) is said to be a *perpendicular bifurcation element* if there is a sequence $(x_n) \subset B_2, x_n \neq x^o$ tending to x^o, such that $T(u^o, x_n) = 0$ for all n (set $u_n = u^o, x_n = x^o, y_n \neq x^o$ for all $n \in I\!N$ with $y_n \to x^o$ in Definition 2.2). In order to prove that a solution element (u^o, x^o) is a bifurcation element it is sufficient to show that (at least) two distinct sequences with properties (i) and (ii) exist.

¿From now on, the following assumptions on T are made to investigate the previous questions concerning the bifurcation behaviour of equation (2.1).

(A1) $T \in C^k[u^o, x^o]$ where $k = 1, 2, \ldots$, $k = \infty$ or $k = \omega$, i.e. T is k-times Fréchet differentiable and the kth derivative is continuous in a neighbourhood of (u^o, x^o), T is Fréchet differentiable of any order or T is analytical, respectively.

Analyticity means that T is represented as an absolutely convergent power series in the variables u and x in a neighbourhood of the point (u^o, x^o).

Definition 2.3: Let B_2 and B_3 be Banach spaces over **K**. An operator $U : B_2 \mapsto B_3$ is called a linear *Fredholm operator* if U is linear and continuous and both the dimensions of the kernel dim $N(U)$ and of the range codim $R(U)$ are finite. The number ind $U = \dim N(U)$ - codim $R(U)$ is called the *index* of U.

For (u^o, x^o) to be a bifurcation element of the equation $T(u, x) = 0$, it is necessary that the partial Fréchet derivative of T (at (u^o, x^o) with respect to x) denoted by $\mathring{T}_x = T_x(u^o, x^o) \in L(B_2, B_3)$ does not possess a continuous inverse defined on B_3. Therefore we make the following assumption.

(A2) $\mathring{T}_x \in L(B_2, B_3)$ is a Fredholm operator with index ind $\mathring{T}_x = 0$.

Let the elements $p_1, \ldots, p_n \in B_2$ be such that $N(\mathring{T}_x) = \text{span}\{p_i| \ i = 1, \ldots, n\}$ and let the elements $q_1^*, \ldots, q_n^* \in B_3^*$ (the dual space to B_3) be such that $N(\mathring{T}_x^*) = \text{span}\{q_j^*| \ j = 1, \ldots, n\}$ with $\mathring{T}_x^* \in L(B_3^*, B_2^*)$. According to the Hahn-Banach theorem, it is possible to complete the systems of basis elements p_i and q_j^* $(i, j = 1, \ldots, n)$ to two biorthonormal systems, i.e. we can choose elements $p_k^* \in B_2^*$ and $q_j \in B_3$ such that

$$\langle p_i, p_k^* \rangle = \delta_{ik} \quad \text{and} \quad \langle q_j, q_l^* \rangle = \delta_{jl}$$

for all $i, j, k, l = 1, \ldots, n$.

In the next section, a procedure called *parametrization method* will be developed. After transition from the equation (2.1) to a system of equations including a new parameter, it is possible to characterize all continuations in a neigbourhood of a special kind of generalized limit points. For this, one needs the following explanation. Set

$$\alpha = \dim \left(N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*) \right) \qquad \text{with } \mathring{T}_u^* \in L(B_3^*, B_1^*) \text{ and } \mathring{T}_x^* \in L(B_3^*, B_2^*)$$

where $\mathring{T}_u = T_u(u^o, x^o)$ denotes again the partial Fréchet derivative at the point (u^o, x^o) with respect to u and $\mathring{T}_u^* \in L(B_3^*, B_1^*)$ the adjoint operator to $\mathring{T}_u \in L(B_1, B_3)$. Because of assumption (A2), it holds $0 \le \alpha \le n$ since $N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*) \subset N(\mathring{T}_x^*)$ and $\dim N(\mathring{T}_x^*) = n$.

Definition 2.4: Let $\mathring{T}_x$ be a Fredholm operator with index zero and $n \ge 1$. Then the solution element (u^o, x^o) of equation (2.1) is called (α, n)-*limit point* if $0 \le \alpha \le n-1$.

The following lemma describes the special case of $(0, n)$-limit point and the case $\alpha = n$.

Lemma 2.5: *Let $T \in C^k[u^o, x^o]$, $k \ge 1$ and let assumption (A2) be fulfilled. Then the following statements are true:*

(1) *The solution element (u^o, x^o) of equation (2.1) is a $(0, n)$-limit point if and only if the elements $\mathring{T}_u^* q_j^*$ $(j = 1, \ldots, n)$ are linearly independent.*

(2) *The condition $\alpha = n$ is fulfilled if and only if $\mathring{T}_u^* q_j^* = 0$ $(j = 1, \ldots, n)$.*

Proof: Statement (1): Let $\alpha = 0$. Suppose the elements $\mathring{T}_u^* q_j^*$ $(j = 1, \ldots, n)$ are linearly dependent. Then the equation $\sum_{j=1}^n c_j \mathring{T}_u^* q_j^* = 0$ is satiesfied with c_j such that $\sum_{j=1}^n |c_j| > 0$. Because $\sum_{j=1}^n c_j \mathring{T}_u^* q_j^* = 0$ it follows that $\mathring{T}_u^* (\sum_{j=1}^n c_j q_j^*) = 0$ and therefore $\sum_{j=1}^n c_j q_j^* \in N(\mathring{T}_u^*)$. The linear independence of the elements q_j^* $(j = 1, \ldots, n)$ yield $0 \ne \sum_{j=1}^n c_j q_j^* \in N(\mathring{T}_x^*)$. Therefore $\sum_{j=1}^n c_j q_j^* \in N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*)$. That means $\alpha \ge 1$ which is inconsistent with $\alpha = 0$.

Let now the elements $\mathring{T}_u^* q_j^*$ $(j = 1, \ldots, n)$ be linearly independent. Suppose $\alpha \ge 1$. Then there is an element $q_o^* \in N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*)$ with $q_o^* \ne 0$ and $q_o^* = \sum_{j=1}^n d_j q_j^*$ where $\sum_{j=1}^n |d_j| > 0$. Because $q_o^* \in N(\mathring{T}_u^*)$ it follows $0 = \mathring{T}_u^* q_o^* = \sum_{j=1}^n \mathring{T}_u^* q_j^*$ and $d_j = 0$ $(j = 1, \ldots, n)$, that is $q_o^* = 0$. This is a contradiction.

Statement (2): Let $\alpha = n$. Then there are n linearly independent elements q_{oj}^* $(j = 1, \ldots, n)$ with span $\{q_{oj}^*| j = 1, \ldots, n\} = N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*)$. From $q_{oj}^* = \sum_{k=1}^n d_{jk} q_k^*$ $(j = 1, \ldots, n)$ with $\det(d_{jk}) \ne 0$ it follows that $q_i^* = \sum_{k=1}^n d_{ki} q_{ok}^*$ $(i = 1, \ldots, n)$ and therefore $\mathring{T}_u^* q_i^* = 0$ $(j = 1, \ldots, n)$.

Let $\mathring{T}_u^* q_j^* = 0$ $(j = 1, \ldots, n)$. Then it follows that $q_j^* \in N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*)$ $(j = 1, \ldots, n)$, that is $\alpha \ge n$. On the other hand there is always $\alpha \le n$ ∎

Remark 2.6: Equivalently a $(0, n)$-limit point is described by $\dim \text{span} \{\mathring{T}_u^* q_j^*| j = 1, \ldots, n\} = n$, $n \ge 1$. It holds $\alpha = n$ if and only if $\mathring{T}_u^* q_j^* = 0$ for $j = 1, \ldots, n$.

Now the question stands on a relation between the two given definitions *limit point* and (α, n)-*limit point*. The following lemma gives the answer.

Lemma 2.7: *Let $B_1 = \mathbb{R}$ and $B_2 = B_3$, and let the assumptions (A1) and (A2) be satisfied. Then the condition $\mathring{T}_u(1) \notin R(\mathring{T}_x)$ is fulfilled if and only if the solution element (u^o, x^o) of equation (2.1) is an (α, n)-limit point.*

Proof: The condition $\mathring{T}_u(1) \notin R(\mathring{T}_x)$ is sufficient, since there exists an index $j_o \in \{1, \ldots, n\}$ with $\langle \mathring{T}_u(1), q_{j_o}^* \rangle \neq 0$. This means that $\mathring{T}_u^* q_{j_o}^* \neq 0$ and therefore $q_{j_o}^* \notin N(\mathring{T}_u^*) \cap N(\mathring{T}_x^*)$, that is $\alpha \leq n - 1$.

Conversely, let (u^o, x^o) be an (α, n)-limit point, that is $\alpha \leq n - 1$. Then there exists again an index $j_o \in \{1, \ldots, n\}$ with $\mathring{T}_u^* q_{j_o}^* \neq 0$ and $\mathring{T}_x^* q_{j_o}^* = 0$. Since $B_1 = \mathbb{R}$ it follows $0 \neq \langle 1, \mathring{T}_u^* q_{j_o}^* \rangle = \langle \mathring{T}_u(1), q_{j_o}^* \rangle$. This means that $\mathring{T}_u(1) \notin R(\mathring{T}_x)$ ∎

For this reason it is clear that the notion of limit point used here generalizes the notion of limit point in [4] to parameter spaces of any dimensions.

3. Construction of a continuation at a (0,n)-limit point

For the following investigation let the assumption

(A3) (u^o, x^o) is a $(0, n)$-limit point of equation (2.1)

be fulfilled.

Now, all examinations of the equation (2.1) are done near a $(0, n)$-limit point. The existence of smooth solution arcs through (u^o, x^o) will be guaranteed by an application of the implicit function theorem. This is a common approach in bifurcation theory, the only problem being that of setting up an appropriate equation to which the implicit function theorem can apply. The choice here is motivated by the definition of the notion (α, n)-limit point and the special choice of the Schmidt operator U (see (3.2) below).

To the linearly independent elements $\mathring{T}_u^* q_j^* \in B_1^*$ $(j = 1, \ldots, n)$ the biorthonormal elements $\hat{u}_i \in B_1$ $(i = 1, \ldots, n)$ can be selected. Any biorthonormal element system can be chosen. Then the relations

$$\langle \hat{u}_i, \mathring{T}_u^* q_j^* \rangle = \delta_{ij} \qquad (i, j = 1, \ldots, n) \tag{3.1}$$

are true again. Since the elements $\mathring{T}_u \hat{u}_i$ are biorthonormal to the elements q_j^* the Schmidt operator U can be set up in the form

$$U = -\mathring{T}_x + \sum_{i=1}^{n} \langle \cdot, p_i^* \rangle \mathring{T}_u \hat{u}_i. \tag{3.2}$$

It can be shown that the linear continuous operator U is continuously invertible (see, e.g., [15: p. 377] or [9: Chapter 6.2]).

Now it is possible to derive a system of equations which describes all continuations of the equation (2.1) to the solution element (u^o, x^o). One considers an operator $S : B_1 \times B_1 \times B_2 \to B_3 \times B_1$ which is defined as

$$S(v, u, x) = \left\{ \begin{array}{c} T(u, x) \\ -v + u - u^o + \sum_{i=1}^{n} \langle x - x^o, p_i^* \rangle \hat{u}_i \end{array} \right\} \tag{3.3}$$

together with the equation

$$S(v, u, x) = 0. \tag{3.4}$$

Here an equation of the form (2.1) with singular operator $\mathring{T}_x$ will be "embedded" in a larger system with S whose linearization $S_{(u,x)}$ is non-singular and hence to which an implicit function theorem can apply. Therefore this gives all solutions of the equation (3.4). The second term in (3.3) is different from one in [4: p. 423/Formula (3.3)]. The connection of the solution elements of equation (2.1) with those of equation (3.4) is as follows.

Lemma 3.1: *Suppose that the map T is of type C^k with $k \geq 1$ and the assumptions (A2) and (A3) are fulfilled. Then the following statements are true:*

(1) *Every solution element (u', x') of equation $T(u, x) = 0$ generates with $v' = u' - u^o + \sum_{i=1}^{n} \langle x' - x^o, p_i^* \rangle \hat{u}_i$ a solution element of equation $S(v, u, x) = 0$ and $(u', x') \to (u^o, x^o)$ implies $(v', u', x') \to (0, u^o, x^o)$.*

(2) *Conversely, every solution element (v', u', x') of equation $S(v, u, x) = 0$ generates a solution element (u', x') of equation $T(u, x) = 0$ and $(v', u', x') \to (0, u^o, x^o)$ implies $(u', x') \to (u^o, x^o)$.*

The proof of the lemma is a straightforward calculation. Properties of the operator S will be summarized in the next lemma.

Lemma 3.2: *The operator S defined in (3.3) has the following properties:*

(1) *Suppose that the map T is of class $C^k[u^o, x^o]$. Then S is in $C^k[0, u^o, x^o]$ with $k \geq 1$.*

(2) *Suppose that the assumptions $(A1), (A2)$ and $(A3)$ are fulfilled. Then the operator $S_{(u,x)}(0, u^o, x^o) \in L(B_1 \times B_2, B_3 \times B_1)$ has a continuous inverse.*

Proof: Statement (1): This is immediately clear because the operator $(v, u, x) \mapsto -v + u - u^o + \sum_{i=1}^{n} \langle x - x^o, p_i^* \rangle \hat{u}_i$ is linear and continuous and therefore of class C^k, $k \geq 1$. The map T is in accordance with assumption (A1) of class C^k. This means $S \in C^k[0, u^o, x^o]$.

Statement (2): The operator $S_{(u,x)}(0, u^o, x^o)$ is injective and maps $B_1 \times B_2$ onto $B_3 \times B_1$. This can be seen as follows. Let the element $(z, w) \in B_3 \times B_1$ be fixed. The system

$$S_{(u,x)}(0, u^o, x^o)(\bar{u}, \bar{x}) = \begin{bmatrix} \mathring{T}_u \bar{u} + \mathring{T}_x \bar{x} \\ \bar{u} + \sum_{i=1}^{n} \langle \bar{x}, p_i^* \rangle \hat{u}_i \end{bmatrix} = \begin{bmatrix} y \\ w \end{bmatrix} \tag{3.5}$$

where $\bar{u}$ and $\bar{x}$ stand for the corresponding increments has the unique solution

$$\bar{u} = w + \sum_{i=1}^{n} \langle y - \mathring{T}_u w, q_i^* \rangle \hat{u}_i \tag{3.6}$$

$$\bar{x} = U^{-1}(\mathring{T}_u w - y) \tag{3.7}$$

where U^{-1} is the inverse Schmidt operator. Therefore $S_{(u,x)}^{-1}(0, u^o, x^o)$ is calculated and by means of the Banach inverse mapping theorem, the operator $S_{(u,x)}^{-1}(0, u^o, x^o)$ is continuous again ∎

Now all preparations for the application of the implicit function theorem are complete. Using Lemma 3.2 all continuations of (3.4) in a neighbourhood of $(0, u^o, x^o)$ can be computed. In the following $K_{B_1}(0, r_1)$ stands for the open ball in B_1 with centre in zero and radius r_1, the meaning of $K_{B_1 \times B_2}((u^o, x^o), r_2)$ is analogous.

Theorem 3.3: *If the assumptions $(A1)-(A3)$ are fulfilled, then in a neighbourhood of $(0, u^o, x^o)$ the equation (3.4) is uniquely solvable with respect to u and x as functions of v. More precisely: There are positive numbers r_1 and r_2 and uniquely determined maps $\varphi : K_{B_1}(0, r_1) \to B_1$ and $\psi : K_{B_1}(0, r_1) \to B_2$ with the following properties:*

i) $\varphi, \psi \in C^k[0], \; k \geq 1$.

ii) $\varphi(0) = u^o$ *and* $\psi(0) = x^o$.

iii) $T(\varphi(v), \psi(v)) = 0$ *and* $-v + \varphi(v) - u^o + \sum_{i=1}^{n} \langle \psi(v) - x^o, p_i^* \rangle \hat{u}_i = 0$ *for* $v \in K_{B_1}(0, r_1)$.

iv) $(\varphi(v), \psi(v)) \in K_{B_1 \times B_2}((u^o, x^o), r_2)$ *for* $v \in K_{B_1}(0, r_1)$.

Proof: The point $(0, u^o, x^o)$ is a solution element of equation (3.4). In conjunction with Lemma 3.2 all conditions for the application of the implicit function theorem (see, e.g., [15: p.150]) to equation (3.4) are verified. The results are the statements i) - iv) ∎

Consequently, all solution elements of equation (2.1) can be constructed as solutions of system (3.4).

Theorem 3.4: *Let the assumptions $(A1)-(A3)$ be fulfilled. Then for the operators φ and ψ from Theorem 3.3 the representations*

$$\varphi(v) = u^o + v - \sum_{i=1}^{n} \langle v, \mathring{T}_u^* q_i^* \rangle \hat{u}_i - \sum_{i=1}^{n} \left\langle \frac{1}{2} W(v, v) + U(R(v)), q_i^* \right\rangle \hat{u}_i \qquad (3.9)$$

$$\psi(v) = x^o + U^{-1} \mathring{T}_u v + U^{-1} \left(\frac{1}{2} W(v, v) + U(R(v)) \right) \qquad (3.10)$$

with

$$\begin{aligned}
W(v, v) = \mathring{T}_{uu} & \left(\left(v - \sum_{i=1}^{n} \langle v, \mathring{T}_u^* q_i^* \rangle \hat{u}_i \right)^2 \right) \\
& + 2\mathring{T}_{ux} \left(v - \sum_{i=1}^{n} \langle v, \mathring{T}_u^* q_i^* \rangle \hat{u}_i, U^{-1} \mathring{T}_u v \right) \\
& + \mathring{T}_{xx} ((U^{-1} \mathring{T}_u v)^2)
\end{aligned} \qquad (3.11)$$

and

$$R : \; K_{B_1}(0, r_3) \to B_2, \qquad \|R(v)\| = o(\|v\|^2) \quad \text{for} \quad v \to 0$$

are valid.

Proof: The representations (3.9) and (3.10) follow from the Taylor formula for the maps φ and ψ in connection with the statement of Theorem 3.3/iii). The derivatives $\varphi_v(0), \psi_v(0), \varphi_{vv}(0)$ and $\psi_{vv}(0)$ are calculated from (3.8) by differentiation to v. For details see [9: Chapter 6.2] ∎

The Fredholm property of $\overset{\circ}{T}_x$ transfers to the operator $\varphi_v(0) \in L(B_1, B_1)$ in the sense that $\varphi_v(0)(\cdot) = I - \sum_{i=1}^{n} \langle \cdot, \overset{\circ}{T}^*_u q^*_i \rangle \hat{u}_i$ is a Fredholm operator with index zero.

Corollary 3.5: *The operator $\varphi_v(0) \in L(B_1, B_1)$ is a Fredholm operator with index zero, $N(\varphi_v(0)) = span\{\hat{u}_i\}_{i=1}^{n}$ and $N(\varphi^*_v(0)) = span\{\overset{\circ}{T}^*_u q^*_j\}_{j=1}^{n}$.*

Remark 3.6: To eliminate the parameter v, the equation $u = \varphi(v)$ must be solvable for v since one looks for a representation of the solution x as a function of u.

Remark 3.7: If one considers the equation

$$\bar{\varphi}(u, v) = 0 \tag{3.12}$$

with

$$\bar{\varphi} : \ B_1 \times B_1 \to B_1, \qquad \bar{\varphi}(u, v) = u - \varphi(v),$$

then the implicit function theorem does not work for this equation since $\varphi_v(0)$ ($n \geq 1$) is a Fredholm operator with index zero. For this reason, the equation (3.12) is to be called a *bifurcation equation*. Unfortunately, in opposite to the Ljapunov-Schmidt reduction (3.12) is also an infinite-dimensional equation. But in many practical cases, where the parameter space B_1 is n-dimensional, the branching equation consists of n non-linear C^k-equations in n variables over $\mathbf{K}$. This justifies the expression bifurcation equation also in case of infinite dimension.

In the following, the solution behaviour of the equation (3.4) as well as the equation (2.1) are investigated in a neighbourhood of the limit point (u^o, x^o).

4. Bifurcation statements at a (0,1)-limit point

In this section, let $\alpha = 0$ and $n = 1$. A bifurcation of solution at a $(0,1)$-limit point under the assumptions (A1) - (A3) with respect to the line $v = t\hat{u}$ ($t \in \mathbb{R}$) will be shown by application of the Morse lemma. The abbreviations $p = p_1, p^* = p^*_1, q = q_1, \ldots$ are chosen.

For the special selection of the parameter $v = t\hat{u}$, the equation (3.9) reads as

$$u = \varphi(t\hat{u}) = u^o - \left[\frac{1}{2} \langle \overset{\circ}{T}_{xx}(p, p), q^* \rangle t^2 + \langle U(R(t\hat{u})), q^* \rangle \right] \hat{u}. \tag{4.1}$$

With

$$g(t) = \frac{1}{2} \langle \overset{\circ}{T}_{xx}(p, p), q^* \rangle t^2 + \langle U R(t\hat{u}), q^* \rangle \tag{4.2}$$

equation (4.1) changes to

$$u = \varphi(t\hat{u}) = u^o - g(t)\hat{u}. \tag{4.3}$$

Under the assumption

(A4) $\langle \overset{\circ}{T}_{xx}(p, p), q^* \rangle \neq 0$

the Morse lemma is applicable and therefore solution bifurcation at (u^o, x^o) can be proved.

Lemma 4.1 (see [6: p. 63]): *Let* $G : K_R(0,d) \to I\!R, \quad x \mapsto G(x) \quad (d > 0)$ *be a map with the properties* $G \in C^2[0]$, $G'(0) = 0$ *and* $G''(0) \neq 0$. *Then there exists a local* C^1-*diffeomorphism*

$$\Phi : \ K_R(0,d_1) \to K_R(0,d_1'), \qquad y \mapsto \Phi(y) \ (d_1, d_1' > 0), \quad \Phi'(0) \neq 0$$

such that

$$G(\Phi(y)) - G(0) = \frac{1}{2} G''(0) y^2$$

for every $y \in K_R(0,d_1)$.

Applying this lemma to the function g in (4.2), one gets with the assumptions (A1) - (A4) that $g \in C^2[0]$, $g(0) = 0, g'(0) = 0$ and $g''(0) = \langle \mathring{T}_{xx}(p,p), q^* \rangle \neq 0$. Therefore, Lemma 4.1 is applicable: There exists a local C^1-diffeomorphism $\Phi : \ r \mapsto \Phi(r) = t$, and (4.2) can be transformed to

$$g(\Phi(r)) = \frac{1}{2} \langle \mathring{T}_{xx}(p,p), q^* \rangle r^2. \tag{4.4}$$

With this preliminary examination, the following bifurcation result can be proved.

Theorem 4.2: *Let the assumptions* $(A1) - (A4)$ *be satisfied. Then* (u^o, x^o) *is a bifurcation element of equation (2.1), which in no case is a perpendicular bifurcation element.*

Proof: According to Definition 2.2, there must be found sequences $(u_m, x_m^{(i)})$ $(i = 1,2)$ with the stated properties. Let $(s_m) \subset I\!R, s_m \neq 0, s_m \to 0$ for $m \to \infty$ be a sequence which lies in the range of g, for all $m \in I\!N$. Then one gets the equation

$$s_m = g(\Phi(r_m)) = \frac{1}{2} \langle \mathring{T}_{xx}(p,p), q^* \rangle r_m^2$$

which has the solutions

$$r_m^{(i)} = \pm \sqrt{2 s_m \langle \mathring{T}_{xx}(p,p), q^* \rangle^{-1}}$$

for all $m \in I\!N$. Thus one has $t_m^{(i)} = \Phi(r_m^{(i)})$ $(i = 1,2)$ with $t_m^{(1)} \neq t_m^{(2)}$ for all $m \in I\!N$. The sequences $(t_m^{(i)})$ give rise to sequences $(x_m^{(i)})$:

$$x_m^{(i)} = \psi(t_m^{(i)}\hat{u}) = x^o + t_m^{(i)} p + \frac{1}{2} U^{-1} W(t_m^{(i)}\hat{u}, t_m^{(i)}\hat{u}) + R(t_m^{(i)}\hat{u}) \qquad (i = 1,2). \tag{4.5}$$

Because of this construction these sequences have the properties $T(u_m, x_m^{(i)}) = 0$ and $(u_m, x_m^{(i)}) \to (u^o, x^o)$ as $m \to \infty$ $(i = 1,2)$. It remains to show that $x_m^{(1)} \neq x_m^{(2)}$ for all $m \geq m_o$, or equivalently, if t' and t'' are sufficiently small with $t' \neq t''$, then it follows $\psi(t'\hat{u}) \neq \psi(t''\hat{u})$. Assume the opposite, i.e. $\psi(t'\hat{u}) = \psi(t''\hat{u})$ for $t' \neq t''$. Then one gets

$$(t' - t'')p = \psi_1(t''\hat{u}) - \psi_1(t'\hat{u}) \tag{4.6}$$

with

$$\psi_1(t\hat{u}) = \frac{1}{2} U^{-1} W(t\hat{u}, t\hat{u}) + R(t\hat{u}) = \frac{1}{2} U^{-1} \mathring{T}_{xx}(p,p) t^2 + R(t\hat{u}).$$

The application of the mean value theorem of differential calculus (see [15: p. 76]) at $h(t) = \psi_1(t\hat{u})$ yields

$$\|h(t'') - h(t')\| \le |t'' - t'| \sup_{\varrho \in (0,1)} \|h'(t' + \varrho(t'' - t'))\|. \tag{4.7}$$

Passing to the norm in (4.6) and applying (4.7) one gets the strong positiveness of the supremum

$$0 < \|p\| \le \sup_{\varrho \in (0,1)} \|h'(t' + \varrho(t'' - t'))\|.$$

This is a contradiction since the derivative h' is continuous in a neighbourhood of zero with $h'(0) = 0$. An analogous argument shows that (u^o, x^o) is not a perpendicular bifurcation element ∎

Remark 4.3: In proving that (u^o, x^o) is a bifurcation element for equation (2.1) the assumption (A4) is mainly used. Theorem 4.2 also follows from [3: Theorem 6.2.1/p. 218]. There a direct application of the Ljapunov-Schmidt bifurcation equation is made while here the discussion of the number of solution of equation (2.1) is based on the Morse lemma.

Remark 4.4: An other opportunity for the investigation of equation (2.1) is given through [15: Theorem 8A or Theorem 8B]. But a comparison is not possible, since these theorems use a non-vanishing mixed second derivative term which does not occur in assumptions (A3) and (A4).

The Morse lemma has another advantage: one can handle the case $\langle T_{xx}(p,p), q^* \rangle = 0$, that is the case when assumption (A4) is violated. For that purpose one can make use of an extension of the Morse lemma (see [6] again).

Lemma 4.5: *Let* $f : \mathbb{R} \to \mathbb{R}$ *have the properties*

1) $f \in C^k[0]$

2) $f(0) = f'(0) = \ldots = f^{(l-1)}(0) = 0$ *and* $f^{(l)}(0) \ne 0$ *for* $2 \le l < k$.

Then there exists a local C^1*-diffeomorphism* $\Psi : (-\delta_1, \delta_1) \to (-\delta_2, \delta_2)$ $(\delta_1, \delta_2 > 0)$ *with* $\Psi(0) = 0$ *and* $\Psi'(0) \ne 0$ *such that*

$$f(\Psi(r)) = \frac{1}{l!} f^{(l)}(0) r^l$$

for all $r \in (-\delta_1, \delta_1)$.

This Lemma is applicable to the bifurcation function $\varphi(u, v) = u - \varphi(v)$. For the selected direction $v = t\hat{u}$ $(0 \ne t \in \mathbb{R})$ one can prove the following result.

Theorem 4.6: *Let assumptions* $(A1) - (A3)$ *be fulfilled and let* φ *have the representation*

$$u = \varphi(t\hat{u}) = u^o + \bar{g}(t)\hat{u}$$

with

$$\bar{g}(t) = a_l t^l + \bar{R}_l(t) \quad (0 \ne a_l \in \mathbb{R}) \qquad and \qquad \bar{R}_l(t) = o(|t|^l) \quad (l \ge 2).$$

Then concerning the direction $\hat{u}$ the following statements are true:

i) *For every even number $l \geq 2$, (u^o, x^o) is a bifurcation element of the equation* (2.1).

ii) *For every odd number $l \geq 3$ there is exactly one continuation to the solution element (u^o, x^o) of equation* (2.1) *(in that direction $\hat{u}$).*

Proof: All assumptions for application of Lemma 4.3 are fulfilled. With the diffeomorphism Ψ for $\bar{g}$ one gets $\bar{g}(\Psi(r)) = a_l r^l$ for r in a neighbourhood of zero. Now the equation

$$a_l r^l = s \qquad (s \neq 0) \tag{4.8}$$

is investigated again. The following two cases are possible:

a) For every even l, $l \geq 2$ and sign $s = $ sign a_l equation (4.8) has exactly two different real solutions, which yield two different continuations near the solution element (u^o, x^o) (conclusions analogous to Theorem 4.2).

b) For odd l, $l \geq 3$ equation (4.8) possesses precisely one continuation to the solution element (u^o, x^o) of equation (2.1) ∎

Remark 4.7: In the case b) one can only say that bifurcation does not occur in the direction $\hat{u}$. Here, further investigations in other directions are necessary and deserve separate considerations.

5. Bifurcation results at a (0,2)-limit point

In this section, under several assumptions bifurcation at a $(0, 2)$-limit point is proved. The investigations will be carried out using a theorem by Buchner, Marsden and Schecter (see [2]).

Let X and Y be Banach spaces and $B : X^l \to Y$ $(l \geq 1)$ an l-linear continuous operator. Let the map $Q : X \to Y$ be defined as $Q(x) = \frac{1}{l!} B(x^l)$. The operator Q is Fréchet differentiable at every point x^o and its derivative is given by

$$Q_x(x^o)\bar{x} = \frac{1}{(l-1)!} B((x^o)^{l-1}, \bar{x})$$

for all $\bar{x} \in X$.

Definition 5.1: The map Q is called

1) *regular at x^o if $Q_x(x^o) \in L(X, Y)$ is surjective.*

2) *regular on the set $Q^{-1}(0)$ if Q is regular for all $x \in Q^{-1}(0) \setminus \{0\}$ (remark that $Q^{-1}(0) \setminus \{0\} = \emptyset$ is admissible).*

Theorem 5.2 (see [2: p. 409]): *Let $l, k \in \mathbb{N}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ with $2 \leq l < k$ and $g \in C^k[0]$, $g(0) = 0$, $Dg(0) = 0, \ldots$, $D^{l-1}g(0) = 0$. Then the following statements are true:*

i) *If Q, $Q(\bar{x}) = D^l g(0)\bar{x}^l$ is regular on $Q^{-1}(0)$, then there exist neighbourhoods $U_1, U_2 \subset \mathbb{R}^n$ containing zero and a C^1-diffeomorphism Ψ with the properties $\Psi(Q^{-1}(0) \cap U_1) = g^{-1}(0) \cap U_2$ and $\Psi(0) = 0$, $D\Psi(0) = I$.*

ii) *If* Q, $Q(\bar{x}) = D^l g(0)\bar{x}^l$ *is regular for all* $\bar{x}$, $\bar{x} \neq 0$, *then according to statement* i) *the diffeomorphism* Ψ *can be selected such that* $g(\Psi(x)) = Q(x)$ *for all* $x \in U_1$.

Remark 5.3: The proof of Theorem 5.2 is based on typical arguments of algebraic geometry, the Zariski topology on algebraic varieties, and the main theorem of elimination theory, which can be found in the monograph of Mumford [11].

Remark 5.4: Conclusion i) of Theorem 5.2 implies that g and Q have homeomorphic zero sets. The diffeomorphism Ψ is close to the identity map near zero. From conclusion ii) of Theorem 5.2 notice that if $Q(v) = 0$, where $v \neq 0$, then the line $l(t) = tv$ in $Q^{-1}(0)$ is mapped by Ψ into the C^1-curve $\Psi(l(t))$ in the zero set of g, which is also tangent to v at $t = 0$. One thus speaks of each $v \in Q^{-1}(0)$, $v \neq 0$, as a *direction* of bifurcation. For $m = 1$, $l = 2$ and Q regular on its zero set, Theorem 5.2 follows from the Morse lemma.

The following preparations are necessary in order to derive the announced bifurcation result by means of Theorem 5.2/ii). For the special direction $v = t_1\hat{u}_1 + t_2\hat{u}_2$ ($t = (t_1, t_2) \in \mathbb{R}^2, (t_1, t_2) \neq (0,0)$) the bifurcation equation $\bar{\varphi}(u, v) = u - \varphi(v) \doteq 0$ (take (3.9) into consideration) is

$$u = \varphi(t_1\hat{u}_1 + t_2\hat{u}_2) = u^o - \sum_{i=1}^{2} \left\langle \frac{1}{2}W(\cdot,\cdot) + UR(\cdot), q_i^* \right\rangle \hat{u}_i.$$

Here $W(\cdot,\cdot)$ stands for $W(t_1\hat{u}_1 + t_2\hat{u}_2, t_1\hat{u}_1 + t_2\hat{u}_2)$, the meaning of $R(\cdot)$ is analogous. With the abbreviations

$$g^i(t) = \frac{1}{2}\langle W(\cdot,\cdot), q_i^* \rangle + \langle UR(\cdot), q_i^* \rangle \qquad (i = 1, 2) \tag{5.1}$$

the latter equation can be written as

$$u = u^o - \sum_{i=1}^{2} g^i(t)\hat{u}_i. \tag{5.2}$$

This is the bifurcation equation. The terms $\langle W(\cdot,\cdot), q_i^* \rangle$ can be calculated from (3.11). With the further abbreviations

$$a_{jk}^{(i)} = \langle \overset{*}{T}_{xx}(p_j, p_k), q_i^* \rangle, \qquad a_{jk}^{(i)} = a_{kj}^{(i)}$$
$$R^{(i)}(t) = \langle UR(\cdot), q_i^* \rangle \qquad (i, j, k = 1, 2)$$

now one can write the $g^i(t)$ as

$$g^i(t) = \frac{1}{2}\left(a_{11}^{(i)} t_1^2 + 2a_{12}^{(i)} t_1 t_2 + a_{22}^{(i)} t_2^2\right) + R^{(i)}(t) \qquad (i = 1, 2). \tag{5.3}$$

To prove that (u^o, x^o) is a bifurcation element of equation (2.1), one must show that the system of equations

$$g^{(i)}(t) = w_i \qquad (i = 1, 2) \tag{5.4}$$

has several solutions.

Using the map

$$g : \; I\!R^2 \to I\!R^2, \qquad g(t_1, t_2) = \begin{bmatrix} g^{(1)}(t_1, t_2) \\ g^{(2)}(t_1, t_2) \end{bmatrix}$$

Theorem 5.2/ii) can be applied to g. In accordance with Theorem 3.3 it follows that with $T \in C^k[u^o, x^o]$ one also gets $\varphi \in C^k[0]$ and $\psi \in C^k[0]$, $k \geq 3$. Furthermore, g has the properties $g(0, 0) = 0$ and $Dg(0, 0) = 0$. The function $Q(t_1, t_2) = D^2 g(0, 0)(t_1, t_2)^2$ has the form

$$Q(t_1, t_2) = \begin{bmatrix} Q^{(1)}(t_1, t_2) \\ Q^{(2)}(t_1, t_2) \end{bmatrix} \qquad \text{with} \quad Q^{(i)} : \; I\!R^2 \to I\!R$$

and

$$Q^{(i)}(t_1, t_2) := a_{11}^{(i)} \, t_1^2 + 2a_{12}^{(i)} \, t_1 t_2 + a_{22}^{(i)} \, t_2^2 \qquad (i = 1, 2). \tag{5.5}$$

Lemma 5.5: *The Q-form (5.5) has the following properties:*
(1) *The derivative of Q is*

$$Q_t(t_1^o, t_2^o) = \begin{bmatrix} 2a_{11}^{(1)}t_1^o + 2a_{12}^{(1)}t_2^o & 2a_{12}^{(1)}t_1^o + 2a_{22}^{(1)}t_2^o \\ 2a_{11}^{(2)}t_1^o + 2a_{12}^{(2)}t_2^o & 2a_{12}^{(2)}t_1^o + 2a_{22}^{(2)}t_2^o \end{bmatrix}. \tag{5.6}$$

(2) *Q is regular for all $(t_1, t_2) \neq (0, 0)$ if the coefficients fulfil the conditions $A > 0$ and $B > 0$ or $A < 0$ and $B > 0$ where*

$$\begin{aligned} A &= \begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{11}^{(2)} & a_{12}^{(2)} \end{vmatrix} \\ B &= \begin{vmatrix} a_{11}^{(1)} a_{12}^{(2)} - a_{12}^{(1)} a_{11}^{(2)} & \frac{1}{2}(a_{11}^{(1)} a_{22}^{(2)} - a_{22}^{(1)} a_{11}^{(2)}) \\ \frac{1}{2}(a_{11}^{(1)} a_{22}^{(2)} - a_{22}^{(1)} a_{11}^{(2)}) & a_{12}^{(1)} a_{22}^{(2)} - a_{22}^{(1)} a_{12}^{(2)} \end{vmatrix}. \end{aligned} \tag{5.7}$$

Proof: Statement (1): Equation (5.6) follows by differentiation of Q with respect to (t_1, t_2). Statement (2): It must be shown that $Q_t(t^o) \in L(I\!R^2, I\!R^2)$ is surjective for each $t^o \in I\!R^2, t^o \neq 0$. Therefore one has to investigate the linear equation $Q_t(t^o)x = y$ $(x, y \in I\!R^2)$. This equation has a solution for each $y \in I\!R^2, y \neq 0$, if the conditions mentioned in assertion (2) are fulfilled ∎

Now all preparations about the bifurcation equation (5.2) are met and one can apply Theorem 5.2 and gets the following

Theorem 5.6: *Let the assumptions $(A1)$ (with $k \geq 3$), $(A2)$ and $(A3)$ be fulfilled and suppose that $A > 0$ and $B > 0$ or $A < 0$ and $B > 0$. Then there exist neighbourhoods U_1 and U_2 in $I\!R^2$ containing zero and a C^1-diffeomorphism $\Psi : \; U_1 \to U_2$, $s \mapsto \Psi(s) = t$ such that $g(\Psi(s)) = Q(s)$ holds for all $s = (s_1, s_2) \in U_1$.*

Proof: The assumptions $(A1)$ - $(A3)$ ensure that the map g (see (5.3) has properties $g \in C^k[0]$ $(k \geq 3)$, $g(0, 0) = 0$, $Dg(0, 0) = 0$ and $D^2 g(0, 0)(t^2) = Q(t)$. The assumptions $A > 0$ and $B > 0$ or $A < 0$ and $B > 0$, respectively, guarantee the regularity of the Q-form for all $t \neq 0$ (see Lemma 5.5). Theorem 5.2/ii) can be applied and the proof is finished ∎

Theorem 5.6 yields the nice conclusion that the search for solutions of the equation (5.4) is now reduced to the system

$$Q^{(i)}(s_1, s_2) = w_i \qquad (i = 1, 2). \tag{5.8}$$

In the following, this system will be handled in the form

$$s_1^2 + \frac{2a_{12}^{(i)}}{a_{11}^{(i)}} s_1 s_2 + \frac{a_{22}^{(i)}}{a_{11}^{(i)}} s_2^2 - \frac{w_i}{a_{11}^{(i)}} = 0 \qquad (i = 1, 2) \tag{5.9}$$

assuming $a_{11}^{(i)} \neq 0$, i.e. $\langle \mathring{T}_{xx}(p_1, p_1), q_i^* \rangle \neq 0$.

Now one can apply the resultant theory to the system (5.9). First of all it follows the definition and a result about the resultant. Consider the system of equations

$$\begin{aligned} f_1(s) &:= b_{10}s^n + b_{11}s^{n-1} + \ldots + b_{1n} = 0 \\ f_2(s) &:= b_{20}s^m + b_{21}s^{m-1} + \ldots + b_{2m} = 0 \end{aligned} \tag{5.10}$$

where $b_{10} \neq 0$ and $b_{20} \neq 0$ and n and m are natural numbers. By the *resultant* $R(f_1, f_2)$ of f_1 and f_2 one means the determinant

$$R(f_1, f_2) = \begin{vmatrix} b_{10} & b_{11} & \cdots & & b_{1n} & & & \\ & b_{10} & b_{11} & \cdots & & b_{1n} & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_{10} & b_{11} & \cdots & & b_{1n} \\ b_{20} & b_{21} & \cdots & & b_{2m} & & & \\ & b_{20} & b_{21} & \cdots & & b_{2m} & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_{20} & b_{21} & \cdots & & b_{2m} \end{vmatrix}$$

where the upper part of the matrix consists of m rows and the under one of n rows. The remaining elements are all zero. Now f_1 and f_2 have a common divisor of positive degree if and only if $R(f_1, f_2) = 0$. This result is shown, for example, by Krasnoselski [7: Theorem 21.1] or by Wainberg and Trenogin [14: Theorem 4.3].

For (5.9) one can calculate the resultant and one gets (the variable s_1 being fixed)

$$R(s_2) := R(Q^{(1)} - w_1, Q^{(2)} - w_2) = -\frac{4B}{(a_{11}^{(1)} a_{11}^{(2)})^2} \left[s_2^4 - \frac{E}{B} s_2^2 - \frac{D^2}{4B} \right] \tag{5.11}$$

with

$$C = a_{11}^{(1)} a_{22}^{(2)} - a_{22}^{(1)} a_{11}^{(2)}, \quad D = a_{11}^{(1)} w_2 - a_{11}^{(2)} w_1, \quad E = A(a_{12}^{(1)} w_2 - a_{12}^{(2)} w_1) - \frac{C}{2} D.$$

Lemma 5.7: *Under the assumptions of Theorem 5.6 the resultant* (5.11) *has the following real zeroes:*

(1) *In case $D = 0$ the zeroes of the resultant are*

$$s_2^{(1,2)} = \pm \frac{|A|}{\sqrt{B}} \sqrt{-\frac{w_2}{a_{11}^{(2)}}}, \quad s_2^{(3,4)} = 0 \;\; for \;\; \frac{w_2}{a_{11}^{(2)}} < 0, \quad s_2^{(1,2)} = 0 \;\; for \;\; \frac{w_2}{a_{11}^{(2)}} > 0.$$

(2) *In case $D \neq 0$ the zeros of the resultant are*

$$s_2^{(1,2)} = \pm \sqrt{\frac{1}{2B} E + \sqrt{E^2 + BD^2}}. \tag{5.12}$$

Proof: Solve the equation $R(s_2) = 0$ (see (5.11)) for the cases $D = 0$ and $D \neq 0$. Take into account that $D = 0$ implies $E = -A^2(w_2/a_{11}^{(2)})$ and for $D \neq 0$ only two solutions of (5.12) are real ∎

Remark 5.8: The real numbers w_1 and w_2 play the role of small parameters. Therefore, D can be zero for $w_2 = (a_{11}^{(2)}/a_{11}^{(1)})w_1$ only. The identical vanishing of D would mean $a_{11}^{(1)} = a_{11}^{(2)} = 0$. This is a contradiction to $A \neq 0$.

Remark 5.9: In the case that one or both of the coefficients equal to zero one must work with (5.8) again.

Remark 5.10: Every zero of the resultant produces a common divisor of degree one or two in (5.8). The degree determines the number of solutions of the system (5.9).

Our aim is to prove bifurcation for the equation (2.1). So, properly, it is not necessary to find out all solutions of (5.9). But the following lemma guarantees the existence of exactly two different real zeros of the system (5.9).

Lemma 5.11: *Let the assumptions of Theorem 5.6 be fulfilled and $a_{11}^{(i)} \neq 0$ ($i = 1, 2$). For $D = 0$ and $w_2/a_{11}^{(2)} > 0$ there are exactly two solutions of the system (5.9) of the form $\left(-\sqrt{w_2/a_{11}^{(2)}}, 0\right)$ and $\left(+\sqrt{w_2/a_{11}^{(2)}}, 0\right)$.*

Proof: If $w_2/a_{11}^{(2)} > 0$ in system (5.9), then the zeros $s_2^{(1,2)}$ of the resultant (5.11) generate the common square divisor $s_1^2 - (w_2/a_{11}^{(2)}) = 0$ (since $D = 0$ means $w_1/a_{11}^{(1)} = w_2/a_{11}^{(2)}$) ∎

Remark 5.12: One can select w_1 and w_2 such that $D = 0$ and $w_1/a_{11}^{(1)} = w_2/a_{11}^{(2)} > 0$ and that w_1 and w_2 lie in the range of $Q^{(1)}$ and $Q^{(2)}$, respectively.

Remark 5.13: As a conclusion from Lemma 5.11 in connection with Theorem 5.6 it follows that two different zeros of the system (5.9) produce, by means of C^1-diffeomorphism Ψ, exactly two different real solutions of the system (5.4).

With these results the following theorem is easily proved.

Theorem 5.14: *Let assumtions (A1) ($k \geq 3$), (A2) and (A3) be fulfilled and assume $A \neq 0$ and $B > 0$ (see(5.7)). Then there exist (at least) two different continuations $(u_m, x_m^{(i)})_{m \in I\!N}$ ($i = 1, 2$) for the equation $T(u, x) = 0$ to the solution element (u^o, x^o), i.e. (u^o, x^o) is a bifurcation element.*

Proof: With some small modifications the proof can be carried out in the same way as for Theorem 4.2 ∎

Remark 5.15: If one of the two square forms $Q^{(i)}(t_1, t_2)$ $(i = 1, 2)$ is positive definite or negative definite, then the assumptions $A > 0$ and $B > 0$ of Theorem 5.6 are not fulfilled and Theorem 5.14 is not applicable. If $Q^{(1)}$ is positive definite, one can transform Q into

$$Q(s) = \begin{bmatrix} s_1^2 + s_2^2 \\ \lambda_1 s_1^2 + \lambda_2 s_2^2 \end{bmatrix}$$

with numbers λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ being the zeros of the equation

$$\det\left(\langle \mathring{T}_{xx}(p_i, p_j), q_2^* - \lambda q_1^* \rangle\right)_{i,j=1,2} = 0.$$

The map Q would be regular, if

$$\det \begin{pmatrix} 2s_1 & 2s_2 \\ 2\lambda_1 s_1 & 2\lambda_2 s_2 \end{pmatrix} = 4 s_1 s_2 (\lambda_2 - \lambda_1) \neq 0 \qquad \text{for all } (s_1, s_2) \in I\!\!R^2 \setminus \{0\}.$$

But this is not the case.

Remark 5.16: For the case that $Q^{(1)}$ is positive definite, Lorenz has shown in [8], with the aid of Leray-Schauder mapping degree, that there exist at least four distinct real continuations of the equation (2.1) in the solution element (u^o, x^o) with respect to the same direction $v = t_1 \hat{u}_1 + t_2 \hat{u}_2$. The requirement $A \neq 0$ and $B > 0$ could restrict the bifurcation possibilities at the solution element (u^o, x^o) (this is a conjecture!).

6. An example

Now the results of the previous section will be applied to the Hammerstein equation

$$T(u, x) = x - g(u)KFx = 0 \qquad ((u, x) \in B_1 \times B_2). \tag{6.1}$$

Here

$$g : B_1 \to \mathbf{K}, \qquad K \in L(B, B_2), \qquad F : B_2 \to B$$

and B, B_1, B_2 are Banach spaces again. The following assumptions about the operators g and F are met:

(A6.1) $g \in C^k[u^o]$ and $F \in C^k[x^o]$ $(k \geq 1)$.

(A6.2) $\mathring{T}_x := I - g(u^o)KF_x(x^o) \in L(B_2, B_2)$ is a Fredholm operator with index zero and $\dim N(\mathring{T}_x) = n \geq 1$.

The question is now under which conditions on g and F the solution element (u^o, x^o) may be a $(0, n)$-limit point of equation (6.1). The answer will be given by the following

Lemma 6.1: *Let the assumptions* (A6.1) *and* (A6.2) *be fulfilled and let* $n \geq 1$, *as well as* $\langle x^o, q_1^* \rangle \neq 0$ *and* $g_u(u^o) \neq 0$. *Then for* $n = 1$ *the solution element* (u^o, x^o) *of equation* (6.1) *is a* $(0,1)$-*limit point while for* $n > 1$ *it is not.*

Proof: First of all, the operators $\overset{\circ}{T}_u$ and $\overset{\circ}{T}_u^*$ are calculated. The first one is

$$\overset{\circ}{T}_u(\cdot) = -\langle \cdot, g_u(u^o) \rangle K F x^o = -\frac{1}{g(u^o)}\langle \cdot, g_u(u^o) \rangle x^o$$

if $g(u^o) \neq 0$ (in the case $g(u^o) = 0$ the operator $\overset{\circ}{T}_x$ is the identity and therefore $N(\overset{\circ}{T}_x) = \{0\}$). From

$$\langle \overset{\circ}{T}_u \bar{u}, x^* \rangle = -\frac{1}{g(u^o)}\langle \bar{u}, g_u(u^o) \rangle \langle x^o, x^* \rangle = -\frac{1}{g(u^o)}\langle \bar{u}, \langle x^o, x^* \rangle g_u(u^o) \rangle$$

$$= \left\langle \bar{u}, -\frac{1}{g(u^o)}\langle x^o, x^* \rangle g_u(u^o) \right\rangle = \langle \bar{u}, \overset{\circ}{T}_u^* x^* \rangle$$

it follows

$$\overset{\circ}{T}_u^*(\cdot) = -\frac{1}{g(u^o)}\langle x^o, \cdot \rangle g_u(u^o).$$

This calculation shows that

$$\overset{\circ}{T}_u^* q_1^* = -\frac{1}{g(u^o)}\langle x^o, q_1^* \rangle g_u(u^o) \neq 0,$$

that is (u^o, x^o) is a $(0,1)$-limit point if $n = 1$.

If $n > 1$, then the elements $\overset{\circ}{T}_u^* q_i^*$ $(i = 1, \ldots, n)$ had to be linearly independent in order to qualify (u^o, x^o) as a limit point. But since

$$\sum_{i=1}^n \lambda_i \overset{\circ}{T}_u^* q_i^* = -\frac{g_u(u^o)}{g(u^o)} \sum_{i=1}^n \lambda_i \langle x^o, q_i^* \rangle$$

there is always an n-tupel $(\lambda_1, \cdots, \lambda_n)$ with $\sum_{i=1}^n |\lambda_i| > 0$ such that the sum is zero. Thus the elements $\overset{\circ}{T}_u^* q_i^*$ $(i = 1, \ldots, n)$ are linearly dependent ∎

To get a bifurcation result for a $(0,1)$-limit point (u^o, x^o) of equation (6.1) and to apply Theorem 4.2 an additional condition on the second derivative on F is assumed.

Theorem 6.2: *Let the assumptions* (A6.1) *(with* $k \geq 2$*) and* (A6.2) *(with* $n = 1$*) be fulfilled and suppose* $\langle x^o, q_1^* \rangle \neq 0$, $g_u(u^o) \neq 0$ *and* $\langle F_{xx}(x^o)(p_1, p_1), K^* q_1^* \rangle \neq 0$. *Then* (u^o, x^o) *is a bifurcation element of equation* (6.1) *but not a perpendicular bifurcation element.*

Proof: The above conditions garantee assumptions (A1) - (A3), assumption (A4) is fulfilled because of

$$\overset{\circ}{T}_{xx}(\cdot, \cdot) = -g(u^o) K F_{xx}(x^o)(\cdot, \cdot)$$

and

$$\langle \overset{\circ}{T}_{xx}(p_1, p_1), q_1^* \rangle = \langle -g(u^o) K F_{xx}(x^o)(p_1, p_1), q_1^* \rangle$$

$$= -g(u^o)\langle F_{xx}(x^o)(p_1, p_1), K^* q_1^* \rangle$$

$$\neq 0.$$

Therefore Theorem 4.2 applies and the assertion is proved ∎

.   If $\langle F_{xx}(x^o)(p_1,p_1), K^*q_1^* \rangle = 0$, then higher order derivations of T have to be used in order to gain statements about bifurcation at a solution element (u^o, x^o) and Theorem 4.6 must be applied again.

7. Concluding remarks

In comparison with the Ljapunov-Schmidt reduction, the parametrization method presented here works with the additionally assumption of linear independence of the elements $\overset{\circ}{T}_u^* q_j^*$ $(j = 1,\dots,n)$. This makes it possible to represent all solutions in a neighbourhood of a $(0,n)$-limit point (u^o, x^o) of the equation $T(u,x) = 0$.

With the aid of the operator S only continuations of solution elements (u^o, x^o) which are $(0,n)$-limit points of the equation $T(u,x) = 0$ can be obtained.

It is an open question whether the treatment of (α,n)-limit points with $\alpha \geq 1$ and $n \geq 1$ is possible by means of some other operator S.

It is also unclear which bifurcation solutions can appear for the cases $\alpha = 0$ and $n \geq 3$. Indeed the amount of calculations will increase rapidly.

In contrast to the Ljapunov-Schmidt reduction the bifurcation equation $\bar{\varphi}(u,v) = u - \varphi(v) = 0$ used above is an infinite-dimensional equation, if the parameter space is infinite-dimensional. So, an infinite-dimensional equation had to be solved for v as a function of u, which is in general a difficult task. But in many practical cases the parameter space is finite-dimensional and we have a finite-dimensional bifurcation equation analogously to the Ljapunov-Schmidt reduction.

Another important investigation method uses the finite-symmetry-group invariance, which frequently occurs, e.g., in the theory of crystals or in the theory of elasticity (buckling models). The discussion of the branching equations typically can be simplified substantially if the problem admits a symmetry group. Then one can apply purely group-theoretical considerations to show that many coefficients of the branching equations are zero. Among others this is done in the book by Golubitsky and Schaeffer [5] which is also a good introduction to this subject.

All our investigation here are of local nature. Global aspects, mapping degree theory and symmetry group investigations are not applied to the equation (2.1).

References

[1] Böhmer, K. and Z. Mei: *On a numerical Lyapunov-Schmidt method.* Lect. Appl. Math. 26 (1990), 79 - 98.

[2] Buchner, M., Marsden, J. and S. Schecter: *Applications of the blowing-up construction and algebraic geometry to bifurcation problems.* J. Diff. Equ. 48 (1983), 404 - 433.

[3] Chow, S.-N. and J. K. Hale: *Methods of Bifurcation Theory.* New York: Springer - Verlag 1982.

[4] Decker, D. W. and H. B. Keller: *Multiple limit point bifurcation.* J. Math. Anal. & Appl. 75 (1980), 417 - 430.

[5] Golubitsky, M. and M. Schaeffer: *Singularities and Bifurcation Theory.* New York: Springer - Verlag 1984.

[6] Jongen, H., Jonker, P. and F. Twilt: *Nonlinear Optimization in $I\!R^n$*. Part I: *Morse Theory, Chebyshev Approximation*. Frankfurt: Peter Lang Verlag 1983.

[7] Krasnoselski, M. A. et al.: *Näherungsverfahren zur Lösung von Operatorgleichungen*. Berlin: Akademie - Verlag 1973.

[8] Lorentz, R.: *Grenzpunkte für nichtlineare Gleichungen in Banachräumen und Normalitätskriterien für Optimalsteuerprobleme*. Dissertation. Ilmenau (Germany): Techn. Hochschule 1986.

[9] Marx, B.: *Bifurkationsresultate für eine Klasse nichtlinearer Gleichungen in Banachräumen*. Dissertation. Ilmenau (Germany): Techn. Hochschule 1987.

[10] Mei, Z.: *A numerical approximation for the simple bifurcation problems*. Numer. Funct. Anal. and Optimiz. 10 (1989), 383 - 400.

[11] Mumford, D.: *Algebraic Geometrie*. Part I: *Complex Projective Varieties*. New York: Springer - Verlag 1976.

[12] Schmidt, E.: *Zur Theorie der linearen und nichtlinearen Integralgleichungen*. Part III: *Über die Auflösung der nichtlinearen Integralgleichung und die Verzweigung ihrer Lösungen*. Math. Ann. 65 (1908), 370 - 399.

[13] Van der Waerden, B.: *Algebra*. Vols. I and II. Berlin: Springer - Verlag 1955.

[14] Wainberg, M. M. and W. A.Trenogin: *Theorie der Lösungsverzweigung bei nichtlinearen Gleichungen*. Berlin: Akademie - Verlag 1973.

[15] Zeidler, E.: *Nonlinear Functional Analysis and its Applications*. Part I: *Fixed Point Theorems*. New York: Springer - Verlag 1986.