Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 13 (1994), No. 3, 513–535

Asymptotic Expansions for Regularization Methods of Linear Fully Implicit Differential-Algebraic Equations

M. Hanke

Abstract. Differential-algebraic equations with a higher index can be approximated by regularization algorithms. One of such possibilities was introduced by März for linear time varying index 2 systems. In the present paper her approach is generalized to linear time varying index 3 systems. The structure of the regularized solutions and their convergence properties are characterized in terms of asymptotic expansions. In this way it is also possible to characterize the so-called pencil regularization in the index 2 case.

Keywords: *Differential-algebraic equations, regularization methods, asymptotic expansions*

AMS subject classification: Primary 34E05, secondary 34E15

1. Introduction

In recent time, great interest has been spent in the numerical solution of differential-algebraic equations. This notion is adopted to describe dynamical systems subject to constraints and singular systems of differential equations. Such problems arise in a variety of applications, e.g., electrical networks, constraint mechanical systems of rigid bodies, chemical reaction kinetics, and control theory.

Beginning in the early seventies, differential-algebraic equations were solved using appropriately modified numerical methods which were known to work well in case of ordinary differential equations (cf., e.g., [7]), assuming differential-algebraic equations to be more or less implicitly written ordinary differential equations. However, the failure of such methods when being applied to certain types of differential-algebraic equations stimulated an intensive discussion not only of this approach but of the analytic properties of such problems. As a result it was shown that differential-algebraic equations had a much more complex structure than regular ordinary differential equations. Different concepts of an index generalizing the index of a matrix pencil were proposed in order to classify differential-algebraic equations. While problems with index 1 are well-posed in some naturally given topologies, the so-called *higher index problems* (i.e., the index is greater than 1) lead to differentiation problems. Such problems, however, are known to be ill-posed in the sense of Hadamard (cf. [11]). Consequently, boundary value problems for differential-algebraic equations with a higher index lead to ill-posed problems [9, 13].

M. Hanke: Humboldt-Universität zu Berlin, FB Math., PSF 1297, D - 10099 Berlin

ISSN 0232-2064 / $ 2.50

Therefore, it is natural to apply some kind of regularization procedure. Certainly, it is possible to use well-known regularization techniques like the Tikhonov regularization [6] or least-squares collocation [12]. In the present paper we investigate a parametrization which is more closely related to the special structure of differential-algebraic equations. In [19] März proposed a parametrization for linear time varying differential-algebraic equations being tractable with index two. She aimed at the approximation of higher index differential-algebraic equations by transferable ones. In [14] we have already shown that this parametrization leads to a regularization of boundary value problems for such equations in the sense of Tikhonov. As a by-product we have seen that this parametrization is a singular perturbation of some components of the solution. By the way, März' parametrization has a nice interpretation for some differential-algebraic equations describing electrical networks. It is related to the technique of adding small circuit elements to such equations [3]. The relation between different regularization techniques including that of März applied to equations describing constrained multibody systems are considered in [5].

Our aim is now twofold: Firstly, to give a generalization of März' procedure to linear differential-algebraic equations being tractable with index three, and secondly, to give asymptotic expansions of the regularized solutions with respect to the regularization parameter. The latter expansions provide a deeper insight into the convergence properties of the regularized solutions towards the solution of the unperturbed problem and are useful when designing numerical procedures for the approximate solution of the regularized boundary value problems. Having at hand the asymptotic expansions for März' approach it is easy to carry over them to a proposal by Knorrenschild [16], and to generalize earlier results for the so-called pencil regularization [1, 2] in the case of index 2 equations. In the special case of index 2 and 3 differential-algebraic equations in Hessenberg form, similar results for nonlinear systems are proved in [15].

2. The regularization approach

Consider implicit ordinary differential equations

$$A(t)x'(t) + B(t)x(t) = q(t) \qquad (t \in [0,1]) \tag{2.1}$$

subject to the boundary conditions

$$D_0x(0) + D_1x(1) = \gamma. \tag{2.2}$$

Here, $A(t)$ and $B(t)$ are $m \times m$ matrices depending continuously on $t \in [0,1]$. In the notation we will follow [10]. Equation (2.1) is called a *normal differential-algebraic equation* if $A(t)$ is singular for all t, and the null space $\mathcal{N}(A(t))$ is smooth, i.e., there exists a continuously differentiable matrix function $Q = Q(t)$ such that $Q(t)$ projects $\mathbb{R}^m$ onto $\mathcal{N}(A(t))$ for every $t \in [0,1]$. Remark that, for a normal differential-algebraic equation, rank $(A(t))$ is constant on the interval $[0,1]$. Let $\|\cdot\|$ denote the norm in $C[0,1]^m$ whereas $|\cdot|$ be the Euclidean norm in $\mathbb{R}^r$. In the following we omit the argument t if no confusion can arise. The null space of the leading coefficient matrix

$A(t)$ determines what kind of functions we should accept as solutions of equation (2.1). Letting $P = I - Q$, equation (2.1) is obviously equivalent to

$$A(Px)' + (B - AP')x = q. \tag{2.3}$$

This system represents the fact that only some components of x (namely Px) are determined by differential relations while others are simply given by algebraic equations. Regarding equation (2.3) it is natural to adopt the following definition: A function $x \in C[0,1]^m$ is called *a solution* of equation (2.1) if $Px \in C^1[0,1]^m$ and it fulfils equation (2.3).

For equation (2.3), we introduce the following notations:

$$\begin{aligned}
A_0 &:= A & B_0 &:= B - AP' \\
A_1 &:= A_0 + B_0 Q & B_1 &:= B_0 P \\
A_2 &:= A_1 + B_1 Q_1 & \tilde{B}_1 &:= (B_0 - A_1(PP_1)')P \\
\tilde{A}_2 &:= A_1 + \tilde{B}_1 Q_1 & \tilde{B}_2 &:= \tilde{B}_1 P_1 \\
\tilde{A}_3 &:= \tilde{A}_2 + \tilde{B}_2 Q_2. & &
\end{aligned}$$

Here $Q_1(t)$ denotes a projection onto $\mathcal{N}(A_1(t))$, and $Q_2(t)$ is a projection onto $\mathcal{N}(\tilde{A}_2(t))$, $P_i = I - Q_i$. Working with $\tilde{B}_1$ we suppose PP_1 to be continuously differentiable. The following definition was introduced by März and extensively discussed in a number of papers (see, e.g., [9, 10, 20]).

Definition: Equation (2.1) is called

(i) *transferable* if $A_1(t)$ is non-singular for every $t \in [0,1]$.

(ii) *tractable with index 2* if $A_1(t)$ is singular but $A_2(t)$ is non-singular for every $t \in [0,1]$.

(iii) *tractable with index 3* if $A_1(t)$, $\tilde{A}_2(t)$ are singular but $\tilde{A}_3(t)$ is non-singular for every $t \in [0,1]$.

Remarks: (i) The definition is independent of the special choice of the projector functions Q, Q_1 and Q_2. **(ii)** The notions of transferability, tractability with index k ($k = 2,3$) slightly generalize the notions of a global index 1 and k, respectively, introduced in [8]. **(iii)** If PP_1 is continuously differentiable, $A_2(t)$ is singular if and only if $\tilde{A}_2(t)$ is so. **(iv)** If equation (2.1) is tractable with index 2, Q_1 can be chosen such that $Q_1 Q = 0$. Analogously, if equation (2.1) is tractable with index 3 and some smoothness conditions are fulfilled, Q_1 and Q_2 can be chosen such that $Q_2 Q_1 = 0$, $Q_2 Q = 0$ and $Q_1 Q = 0$.

The proofs of statements (i), (ii), (iv) can be found in [9, 10], statement (iii) is a consequence of [9: Theorem A.13].

Example: Very important special cases of differential-algebraic equations are semi-explicit systems. The following three types arise frequently in applications:

$$\begin{aligned}
u' &= B_{11}u + B_{12}v + q_1 \\
0 &= B_{21}u + B_{22}v + q_2
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned} u' &= B_{11}u + B_{12}v + q_1 \\ 0 &= B_{21}u + q_2 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} u' &= B_{11}u + B_{12}v + B_{13}w + q_1 \\ v' &= B_{21}u + B_{22}v + q_2 \\ 0 &= B_{32}v + q_3. \end{aligned} \tag{2.6}$$

These systems are said to be in *Hessenberg form* [4]. Assume that

(i) B_{22} is non-singular in system (2.4)
(ii) $B_{21}B_{12}$ is non-singular in system (2.5)
(iii) $B_{32}B_{21}B_{13}$ is non-singular in system (2.6).

Then systems (2.4), (2.5), (2.6) are transferable, tractable with index 2, and tractable with index 3, respectively.

With the above notation, equation (2.3) can be equivalently written as

$$A_1\{P(Px)' + Qx\} + B_0Px = q. \tag{2.7}$$

If we assume that equation (2.1) is transferable, equation (2.7) can be multiplied by PA_1^{-1} and QA_1^{-1}, respectively. Because of $P(Px)' = (Px)' - P'Px$ and $PQ = 0$ we obtain the equivalent system

$$\begin{aligned} (Px)' - P'Px + PA_1^{-1}BPx &= PA_1^{-1}q \\ Qx + QA_1^{-1}BPx &= QA_1^{-1}q. \end{aligned}$$

Now a solution expression results immediately. We have

$$x = Px + Qx = (I - QA_1^{-1}B)z + QA_1^{-1}q \tag{2.8}$$

where u solves the regular linear explicit differential equation

$$z' - P'z + PA_1^{-1}Bz = PA_1^{-1}q. \tag{2.9}$$

Equation (2.9) has the remarkable property that, if $Q(t_0)z(t_0) = 0$ for some $t_0 \in [0,1]$, then $Q(t)z(t) = 0$ for all $t \in [0,1]$. As a consequence of equation (2.8) boundary conditions (2.2) are only allowed for the Px-components of the solution. Resuming we obtain the following

Theorem 2.1 [9: Theorem 1.2.25, Corollary 1.2.26]: *Let equation (2.1) be transferable. Assume $D_0 = D_0P(0)$ and $D_1 = D_1P(1)$ to hold in (2.2) and rank $\begin{pmatrix} D_0 & D_1 \\ Q(0) & 0 \end{pmatrix} = m$. If the homogeneous boundary value problem*

$$\begin{aligned} z' + (PA^{-1}B_0 - P')z &= 0 \\ D_0z(0) + D_1z(1) &= 0 \\ Q(0)z(0) &= 0 \end{aligned}$$

has the trivial solution only, then, for every $q \in C[0,1]^m$ and $\gamma \in M := \mathcal{R}(D_0, D_1)$, the problem (2.1), (2.2) has exactly one solution x_0. Moreover, the estimate

$$\|(Px_0)'\| + \|x_0\| \leq C(\|q\| + |\gamma|) \tag{2.10}$$

holds with a constant C independent of q and γ.

Estimation (2.10) shows that the solution of the linear boundary value problem depends continuously on the data if equation (2.1) is transferable and the boundary conditions (2.2) are stated appropriately. If equation (2.1) is not transferable, this is no longer true.

Let us now turn to linear differential-algebraic equations being tractable with index 2. If A_2 is non-singular, $Q_{1,s} := Q_1 A_2^{-1} B_1$ is a projection onto $\mathcal{N}(A_1)$, too. We suppose $Q_1 \equiv Q_{1,s}$ and PP_1 to be continuously differentiable in the following. Obviously, $Q_{1,s}Q = 0$. Note that $Q_1 Q = 0$ implies PQ_1 and PP_1 to be projections. Equation (2.7) yields now

$$A_2\left[P_1\{P(Px)' + Qx\} + Q_1 x\right] + B_0 PP_1 x = q. \tag{2.11}$$

Multiplying by $PP_1A_2^{-1}$, $PQ_1A_2^{-1}$ and $QP_1A_2^{-1}$, respectively, we obtain the equivalent system

$$\begin{aligned} PP_1P(Px)' + PP_1Qx + PP_1A_2^{-1}B_0PP_1x &= PP_1A_2^{-1}q \\ PQ_1x + PQ_1A_2^{-1}B_0PP_1x &= PQ_1A_2^{-1}q \\ QP_1P(Px)' + QP_1Qx + QP_1A_2^{-1}B_0PP_1x &= QP_1A_2^{-1}q. \end{aligned}$$

Since

$$PP_1P = PP_1, \qquad PP_1Q = 0, \qquad QP_1Q = Q, \qquad Q_1A_2^{-1}B_0PP_1 = Q_1$$

this system can be simplified.

Lemma 2.2: *Let equation (2.1) be tractable with index 2. Assume $D_0 = D_0PP_1(0)$ and $D_1 = D_1PP_1(1)$ to hold in (2.2). If x is a solution of problem (2.1), (2.2), then*

$$\begin{aligned} & z' + (PP_1A_2^{-1}B_1 - (PP_1)')z - (PP_1)'y = PP_1A_2^{-1}q \\ & y = PQ_1A_2^{-1}q \\ & v = QP_1A_2^{-1} - QP_1A_2^{-1}B_0z - QP_1P(z' + y') \\ & D_0z(0) + D_1z(1) = \gamma, \quad (I - PP_1(0))z(0) = 0 \end{aligned} \tag{2.12}$$

where $z = PP_1x$, $y = PQ_1x$ and $v = Qx$. Conversely, if z, y and v are solutions of problem (2.12), then $PP_1z = z$, $PQ_1y = y$ and $Qv = v$, and $x = z + y + v$ is a solution of problem (2.1), (2.2).

Corollary: *Let equation (2.1) be tractable with index 2. Equation (2.1) is solvable if and only if $q \in C[0,1]^m$ and $PQ_1A_2^{-1}q$ is continuously differentiable.*

As an immediate consequence we obtain the following theorem.

Theorem 2.3: *Let the assumptions of Lemma 2.2 be fulfilled. Moreover, assume rank* $\begin{pmatrix} D_0 & D_1 \\ I-PP_1(0) & 0 \end{pmatrix} = m$. *If the homogeneous boundary value problem*

$$\begin{aligned} z' + (PP_1A_2^{-1}B_1 - (PP_1)')z &= 0 \\ D_0z(0) + D_1z(1) &= \gamma \\ (I - PP_1(0)z(0) &= 0 \end{aligned}$$

has only the trivial solution, then problem (2.1), (2.2) has exactly one solution for every $q \in C[0,1]^m$ *with* $PQ_1A_2^{-1}q \in C^1[0,1]^m$ *and every* $\gamma \in M := \mathcal{R}(D_0, D_1)$. *Moreover, the estimates*

$$\begin{aligned} \|Px_0\| &\le C(\|q\| + |\gamma|) \\ \|Qx_0\| &\le C(\|q'\| + \|q\| + |\gamma|) \end{aligned}$$

are true with a constant C *independent of* q *and* γ.

Consequently, it is not possible to obtain a continuous dependence of the solution on the data since it is not natural to measure the data in some norm containing their derivatives. Hence, the boundary value problem (2.1), (2.2) is ill-posed. From this point of view it is not surprising that usual numerical methods that proved their value for ordinary differential equations do not work. For this reason we look for regularization algorithms.

Let now equation (2.1) be tractable with index 3. Again, $Q_{2,s} := Q_2A_3^{-1}\tilde{B}_2$ is a projection onto $\mathcal{N}(\tilde{A}_2)$. We choose $Q_2 \equiv Q_{2,s}$ and assume Q_2 to be continuously differentiable, and $Q_1Q = 0$. This gives $Q_2Q_1 = 0$ and $Q_2Q = 0$ additionally. Obviously, PQ_1, PP_1Q_2 and PP_1P_2 are projections.

Lemma 2.4: *Let (2.1) be tractable with index 3. Assume* $D_0 = D_0PP_1P_2(0)$ *and* $D_1 = D_1PP_1P_2(1)$ *to hold in (2.2). If* x *is a solution of problem (2.1), (2.2), then*

$$\begin{aligned} & z' + (PP_1P_2\tilde{A}_3^{-1}\tilde{B}_1 - (PP_1P_2)')z - (PP_1P_2)'y = PP_1P_2A_3^{-1}q \\ & y = PP_1Q_2\tilde{A}_3^{-1}q \\ & v = PQ_1P_2\tilde{A}_3^{-1}q + PQ_1Q_2(y' + z') - PQ_1P_2\tilde{A}_3^{-1}\tilde{B}_1z \\ & w = QP_1P_2A_3^{-1}q + QQ_1(v' + y' + z') - QP_1P_2(y' + z') \\ & D_0z(0) + D_1z(1) = \gamma, \quad (I - PP_1P_2(0))z(0) = 0 \end{aligned} \tag{2.13}$$

where

$$z = PP_1P_2x, \qquad y = PP_1Q_2x, \qquad v = PQ_1x, \qquad w = Qx.$$

Conversely, if z, y, v *and* w *are solutions of problem (2.13), then*

$$PP_1P_2z = z, \qquad PP_1Q_2y = y, \qquad PQ_1v = v, \qquad Qw = w$$

and $x = z + y + v + w$ *is a solution of problem (2.1), (2.2).*

A complete proof of Lemma 2.4 can be found in [21]. The essential steps are sketched in the appendix.

Corollary: *Let equation (2.1) be tractable with index 3. Equation (2.1) is solvable if and only if* $q \in C[0,1]^m$ *and* $PP_1Q_2\tilde{A}_3^{-1}q$, v *are continuously differentiable, where* $v = v(q)$ *is given by (2.13).*

Again, we have differentiation problems. The representation (2.13) shows that some components of q must be twice differentiable for equation (2.1) in order to be solvable. The ill-posedness is more severe than in the index 2 case. Using Lemma 2.4 it is now straightforward to formulate an existence and uniqueness theorem.

Theorem 2.5: *Let the assumptions of Lemma 2.4 be fulfilled. Moreover, assume rank* $\begin{pmatrix} D_0 & D_1 \\ I-PP_1P_2(0) & 0 \end{pmatrix} = m$. *If the homogeneous boundary value problem*

$$\begin{aligned} z' + (PP_1P_2\tilde{A}_3^{-1}\tilde{B}_1 - (PP_1P_2)')z &= 0 \\ D_0z(0) + D_1z(1) &= 0 \\ (I - PP_1P_2(0))z(0) &= 0 \end{aligned}$$

has only the trivial solution, then problem (2.1), (2.2) has exactly one solution for every $q \in C[0,1]^m$ *such that equation (2.1) is solvable, and every* $\gamma \in M := \mathcal{R}(D_0, D_1)$. *Moreover, the estimates*

$$\begin{aligned} \|Px_0\| &\le C(\|q'\| + \|q\| + |\gamma|) \\ \|Qx_0\| &\le C(\|q''\| + \|q'\| + \|q\| + |\gamma|) \end{aligned}$$

are true with a constant C *independent of* q *and* γ.

Note that the estimates are not sharp since only certain projections of q' and q'' are really involved.

From an analytical point of view, higher index systems (i.e. systems with an index greater than 1) are ill-posed problems in the sense of Hadamard in our topologies. This observation is the basis for regularization methods.Let us remark that regularization methods can also be considered as another way of index reduction [5].

März [19] proposed a parametrization of the linear index 2 problem (2.1), namely

$$(A + \varepsilon B_1)(Px)' + B_0x = q. \tag{2.14}$$

She aimed at obtaining a transferable differential-algebraic equation. Indeed, if equation (2.1) is tractable with index 2, equation (2.10) is transferable for sufficiently small $\varepsilon > 0$. In a Hilbert space setting, it was shown in [14] that (2.10) is a regularization in the sense of Tikhonov for equation (2.1). Unfortunately, if equation (2.1) is tractable with index 3, equation (2.10) is no transferable differential-algebraic equation in general.

Example: The following equation is taylored around a well-known example introduced by Petzold [23] in connection with the application of BDF methods for higher index differential-algebraic equations. The present variation is due to März [18]. Let $m = 3$. Choose

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha t & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha & 0 \\ 0 & \alpha t & 1 \end{pmatrix} \qquad (\alpha \in \mathbb{R}). \tag{2.15}$$

With these matrices, equation (2.1) is tractable with index 3 for every $\alpha \in I\!R$. The relevant matrices can be computed to be

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha t & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1+\alpha & 0 \\ 0 & \alpha t & 1 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha t & 0 \end{pmatrix} \qquad \tilde{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha t & 1 \end{pmatrix}$$

$$\tilde{A}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\alpha t & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad PP_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha t & 1 \end{pmatrix}$$

$$Q_{2,s} = \begin{pmatrix} 0 & \alpha t & 1 \\ 0 & -\alpha t & -1 \\ 0 & \alpha t(1+\alpha t) & 1+\alpha t \end{pmatrix} \qquad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha t & 1 \end{pmatrix}.$$

$$\tilde{A}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\alpha t & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Obviously, $\tilde{A}_3$ is non-singular for every $\alpha \in I\!R$ while A_1 and $\tilde{A}_2$ are not. In order to gain more insight note that

$$\det(\lambda A + B) \equiv \alpha + 1$$

such that the matrix pencil (A, B) is non-singular if and only if $\alpha \neq -1$. Constructing (2.14) we obtain

$$A_\varepsilon := A + \varepsilon B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha t + \varepsilon(1+\alpha) & 1 \\ 0 & \varepsilon \alpha t & \varepsilon \end{pmatrix}. \tag{2.16}$$

Now,

$$\det(\lambda A_\varepsilon + B) = (1 + \varepsilon\lambda)^2(1+\alpha).$$

Therefore, this matrix pencil is non-singular if and only if $\alpha \neq -1$. In this case, the infinite eigenvalue is simple such that (2.14) is transferable. This could equally well be proved using the definition. Replacing now B_1 by $\tilde{B}_1$ in (2.16) leads to a transferable differential-algebraic equation for every $\alpha \in I\!R$. Moreover, one computes easily

$$\det(\lambda(A + \varepsilon\tilde{B}_1) + B) = (1+\varepsilon\lambda)(1+\alpha+\varepsilon\lambda)$$

such that the matrix pencil $(A + \varepsilon\tilde{B}_1, B)$ is non-singular. If $\alpha = -1$, equation (2.14) is tractable with index 2 on every interval not containing 0.

Therefore, if equation (2.1) is tractable with index 3, we consider the parametrization

$$(A + \varepsilon\tilde{B}_1)(Px)' + B_0 x = q. \tag{2.17}$$

We will show that equation (2.17) is transferable provided that $\varepsilon > 0$ is sufficiently small. Unfortunately, for a general index 3 problem, it is very hard to realize (2.17) in practice because of the additional term $A_1(PP)'P$ compared with (2.14). In the general case, (2.17) is only of theoretical interest. However, (2.17) may become useful for systems with a special structure and/or where an analytic preprocessing is possible.

Remark: The parametrization (2.14) applied to (2.5) gives

$$\begin{aligned} (I + \varepsilon B_{11})u' &= B_{11}u + B_{12}v + q_1 \\ \varepsilon B_{21}u' &= B_{21}u + q_2. \end{aligned} \tag{2.18}$$

A similar parametrization was considered by Knorrenschild [16], namely,

$$\begin{aligned} u' &= B_{11}u + B_{12}v + q_1 \\ \varepsilon B_{21}u' &= B_{21}u + q_2. \end{aligned} \tag{2.19}$$

Obviously, both parametrizations are closely related and have the same asymptotic properties. The approach (2.19) can be generalized to equation (2.1) if (2.14) is replaced by

$$(A + \varepsilon(I - R)B_1)(Px)' + B_0x = q \tag{2.20}$$

where $R(t)$ denotes a projection of $\mathbb{R}^m$ onto the range of $A(t)$.

Example: Both parametrizations (2.14) and (2.20) have a nice interpretation for some differential-algebraic equations describing electrical networks. Consider the electrical circuit of Figure 1.

The circuit equations are

$$\begin{aligned} i_1' &= -L^{-1}v_0 \\ 0 &= i_1 - I_0. \end{aligned}$$

Taking into account the inner resistance of the current source, a better model would be the circuit given in Figure 2 with a large R. Now, the equations read

$$\begin{aligned} i_1' &= -L^{-1}v_0 \\ 0 &= i_1 + LR^{-1}i_1' - I_0. \end{aligned}$$

Letting $R = \varepsilon^{-1}L$ we just obtain (2.20). The parametrization (2.14) arises if, additionally, the inductivity L is perturbed by a factor $1 + \varepsilon$.

Remark: Another possibility for introducing a small parameter, the so-called *pencil regularization,* was given independently by Boyarincev [1: pp. 103 and 166] and Campbell [2: p. 118]. For this, equation (2.1) is replaced by

$$(A+\varepsilon B)x' + Bx = q. \tag{2.21}$$

This parametrization aims at ordinary differential equations. For transferable differential-algebraic equations or tractable constant coefficient differential-algebraic equations, the matrix pencil (A, B) is non-singular [9: pp. 14 and 35] such that this goal is reached. Unfortunately, this is no longer true for time varying coefficients. Consider, e.g., the differential-algebraic equation (2.1) with the coefficients (cf. [23])

$$A(t) = \begin{pmatrix} 0 & 0 \\ 1 & -t \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix} \qquad (t \in [0,1]).$$

This differential-algebraic equation is tractable with index 2. Obviously, for all ε, $A+\varepsilon B$ is singular. Even more, the parametrized equation is tractable with index 2 having, consequently, the same ill-posedness as the original problem. As will be shown later, for a time independent null space $\mathcal{N}(A(t))$, the pencil regularization can be successfully applied.

For the parametrizations (2.14) and (2.17) we will give asymptotic expansions in powers of ε. We obtain singular perturbation problems in differential-algebraic equations. Moreover, we will provide the necessary form of the additional boundary conditions in order to obtain convergence of the solutions of the parametrized equations towards the solution of the unperturbed equations. Similar results are shown to be valid for the pencil regularization as an immediate consequence provided that Q is constant. Our main tool is the following well-known theorem (cf. [24: p.88] and [22: pp. 46 ff]).

Theorem 2.6: *Consider the boundary value problem*

$$\begin{aligned} z' &= f(z,y,\cdot,\varepsilon) \\ \varepsilon y' &= g(z,y,\cdot,\varepsilon) \\ h(z(0),z(1)) &= 0, \quad y(0) = y_\varepsilon^0 \end{aligned} \qquad (t \in [0,1]) \tag{2.22}$$

for $0 < \varepsilon \le \varepsilon_*$. *Suppose the following:*

(i) The functions f, g, h are sufficiently smooth.

(ii) The functions f, g, y_ε^0 have asymptotic expansions with respect to ε.

(iii) The boundary value problem

$$\begin{aligned} z' &= f(z,y,\cdot,0) \\ 0 &= g(z,y,\cdot,0) \\ h(z(0),z(1)) &= 0 \end{aligned} \qquad (t \in [0,1])$$

has a solution (z_0, y_0).

(iv) $W(t) := g_y(z_0(t), y_0(t), t, 0)$ *has only eigenvalues* $\lambda_i(t)$ *such that* $Re\lambda_i(t) \le -\sigma < 0$ *for all* $t \in [0,1]$.

(v) The equation of the first variation

$$\begin{aligned} z' &= f_z(z_0, y_0, \cdot, 0)z + f_y(z_0, y_0, \cdot, 0)y \\ 0 &= g_z(z_0, y_0, \cdot, 0)z + g_y(z_0, y_0, \cdot, 0)y \end{aligned} \qquad (t \in [0,1])$$
$$h_{z(0)}(z_0(0), z_0(1))z(0) + h_{z(1)}(z_0(0), z_0(1))z(1) = 0$$

has only the trivial solution.

(vi) $y_0^0 := \lim_{\varepsilon\to 0} y_\varepsilon^0$ *belongs to the domain of attraction of the attraction point* $y_0(0)$ *of the equation* $\tilde{y}' = g(z_0(0), \tilde{y}, 0)$.

Then, for sufficiently small $\varepsilon_* > 0$, *problem (2.22) has exactly one solution* $(z_\varepsilon, y_\varepsilon)$ *in a neighbourhood of* (z_0, y_0). *There, the asymptotic expansions*

$$z_\varepsilon(t) \simeq \sum_{j=0}^{N} (z_j(t) + \bar{z}_j(\tau))\varepsilon^j \qquad \text{and} \qquad y_\varepsilon(t) \simeq \sum_{j=0}^{N} (y_j(t) + \bar{y}_j(\tau))\varepsilon^j$$

hold true, where $\tau = t/\varepsilon$, $|\bar{z}_j(\tau)|$, $|\bar{y}_j(\tau)| \le C\exp(-\alpha\tau)$ *and* $\bar{z}_0(\tau) \equiv 0$. *Moreover,* $\bar{y}_0(\tau) \equiv 0$ *if and only if* $y_0^0 = y(0)$.

In the following we will always assume that all functions involved are sufficiently smooth.

3. The index 2 problem

In this section we are concerned with linear equations being tractable with index 2 as well as with their parametrizations. Consider the problem

$$A(t)x' + B(t) = q \qquad (t \in [0,1]) \tag{3.1}$$

$$D_0x(0) + D_1x(1) = \gamma \tag{3.2}$$

and the associated parametrization

$$(A + \varepsilon B_1)(Px)' + B_0x = q. \tag{3.3}$$

We need the following assumption (cf. Theorem 2.3).

Assumption (LP2):
(i) Equation (3.1) is tractable with index 2, $Q_1 \equiv Q_{1,s} := Q_1A_2^{-1}B_1$.
(ii) In (3.2), $D_0 = D_0P\bar{P}_1(0)$, $D_1 = D_1P\bar{P}_1(1)$, rank $\begin{pmatrix} D_0 & D_1 \\ I-P\bar{P}_1(0) & 0 \end{pmatrix} = m$.
(iii) Problem (3.1), (3.2) has a unique solution.

Theorem 3.1: *For all sufficiently small* $\varepsilon > 0$, *equation (3.3) is transferable.*

Proof: According to the definition of transferability we consider the pair $(A_\varepsilon, B_\varepsilon)$ with $A_\varepsilon = A + \varepsilon B_0P$ and $B_\varepsilon = B_0$. Trivially, $\mathcal{N}(A(t)) \subseteq \mathcal{N}(A\varepsilon(t))$. Let now $t \in [0,1]$ be fixed and $z \in \mathcal{N}(A_\varepsilon(t))$. Then

$$0 = A_\varepsilon z = (A_2 + \varepsilon B_0P - B_0Q - B_0PQ_1)z.$$

Multiplying this equation by $Q_1A_2^{-1}$ and regarding $Q_1 = Q_1A_2^{-1}B_0P$, $A_2^{-1}B_0Q = Q$ leads to $\varepsilon Q_1 z = 0$. On the other hand, by multiplying with PA_2^{-1} we obtain $Pz + \varepsilon PA_2^{-1}B_0Pz = 0$. Hence, for sufficiently small $\varepsilon > 0$, $Pz = 0$. But this is equivalent to $z \in \mathcal{N}(A(t))$. Therefore, $\mathcal{N}(A(t)) = \mathcal{N}(A_\varepsilon(t))$. A simple calculation shows that, for $\varepsilon > 0$ sufficiently small,

$$(A_\varepsilon + B_\varepsilon Q)^{-1} = (A_2 + \varepsilon B_1)^{-1}\left(I - (A_2 + (\varepsilon B_1 - B_1Q_1)\frac{1}{\varepsilon}Q_1A_2^{-1}\right) + \frac{1}{\varepsilon}Q_1A_2^{-1}.$$

hence the assertion ∎

The analogue of Lemma 2.2 is the following

Lemma 3.2: *Let assumption (LP2) hold. If x is a solution of problem (3.3), (3.2), then*

$$\begin{aligned}
&(I + \varepsilon C_1)z' + (C_1 - (PP_1)')z - ((PP_1)' + \varepsilon C_1')y = PP_1A_2^{-1}q\\
&\varepsilon y' + (I - \varepsilon(PQ_1)')y - \varepsilon(PQ_1)'PP_1z = PQ_1A_2^{-1}q\\
&v = QP_1A_2^{-1}q - C_2z - \varepsilon C_2(z' + y') - QP_1P(z' + y')\\
&D_0z(0) + D_1z(1) = \gamma, (I - PP_1(0))z(0) = 0
\end{aligned}\tag{3.4}$$

where

$$C_1 = PP_1A_2^{-1}B_1, \qquad C_2 = QP_1A_2^{-1}B_1$$

and

$$z = PP_1x, \qquad y = PQ_1x, \qquad v = Qx.$$

Conversely, if (z, y, v) is a solution of equation (3.4) and $y(0) \in \mathcal{R}(PQ_1(0))$, then

$$PP_1z = z, \qquad PQ_1y = y, \qquad Qv = v$$

and $x = z + y + v$ is a solution of problem (3.3), (3.2).

Proof: Equation (3.3) ist equivalent to

$$A_1\{P(Px)' + Qx\} + \varepsilon B_1(Px)' + B_0Px = q$$

and

$$A_2[P_1\{P(Px)' + Qx\} + Q_1x] + B_0PP_1x + \varepsilon B_1(Px)' = q.$$

Multiplying by $PP_1A_2^{-1}$, $PQ_1A_2^{-1}$, and $QP_1A_2^{-1}$, respectively, yields

$$\begin{aligned}
PP_1(Px)' + PP_1A_2^{-1}B_0PP_1x + \varepsilon PP_1A_2^{-1}B_1(Px)' &= PP_1A_2^{-1}q\\
PQ_1x + \varepsilon PQ_1A_2^{-1}B_1(Px)' &= PQ_1A_2^{-1}q\\
-Q_1(Px)' + Qx + QP_1A_2^{-1}B_0PP_1x + \varepsilon QP_1A_2^{-1}B_1(Px)' &= QP_1A_2^{-1}q.
\end{aligned}\tag{3.5}$$

It holds

$$A_2^{-1}B_1Q_1 = A_2^{-1}(A_1 + B_1Q_1)Q_1 = Q_1 \qquad \text{and} \qquad PQ_1A_2^{-1}B_1 = PQ_1$$

$(Q_1 \equiv Q_{1,s}!)$. With $C_1 = PP_1A_2^{-1}B_1$ and $C_2 = QQ_1A_2^{-1}B_1$ we have

$$\begin{aligned} C_1(Px)' &= C_1(PP_1x + PQ_1x)' \\ &= C_1(PP_1x)' + C_1'PQ_1x - (C_1PQ_1x)' \\ &= C_1(PP_1x)' + C_1'PQ_1x \end{aligned}$$

and

$$\begin{aligned} PQ_1(Px)' &= PQ_1(PP_1x + PQ_1x)' \\ &= -(PQ_1)'PP_1x + (PQ_1x)' - (PQ_1)'PQ_1x \end{aligned}$$

such that system (3.5) is equivalent with system (3.4) where $z = PP_1x$, $y = PQ_1x$ and $v = Qx$.

Conversely, let (z, y, v) be a solution of system (3.4) and $y(0) \in \mathcal{R}(PQ_1(0))$. Multiply the second equation by $(I - PQ_1)$:

$$\varepsilon(I - PQ_1)y' + (I - PQ_1)(I - \varepsilon(PQ_1)')y + \varepsilon(I - PQ_1)(PQ_1)'PP_1z = 0.$$

With

$$(I - PQ_1)y' = ((I - PQ_1)y)' + (PQ_1)'y$$

we obtain

$$\varepsilon((I - PQ_1)y)' + (I - PQ_1)y + \varepsilon PQ_1(PQ_1)'y + \varepsilon(I - PQ_1)(PQ_1)'PP_1z = 0.$$

Furthermore,

$$PQ_1(PQ_1)' = (PQ_1)' - (PQ_1)'PQ_1$$

such that

$$\varepsilon((I - PQ_1)y)' + (I - PQ_1)y + \varepsilon(PQ_1)'(I - PQ_1)y = 0.$$

This is a homogeneous linear differential equation with respect to $(I - PQ_1)y$ subject to the initial condition $(I - PQ_1(0))y(0) = 0$. Hence, $(I - PQ_1(t))y(t) \equiv 0$.

Now, multiply the first equation by $(I - PP_1)$:

$$(I - PP_1)z' - (I - PP_1)(PP_1)'z - (I - PP_1)((PP_1)' + \varepsilon C_1')y = 0.$$

Using $(I - PQ_1)y = 0$ we obtain by partial integration

$$((I - PP_1)z)' - (PP_1)'(I - PP_1)z = 0.$$

Now $(I - PP_1(0))z(0) = 0$ implies $(I - PP_1)z = 0$. Finally, the third equation yields $(I - Q)v = 0$, hence, the desired equivalence ∎

A careful examination of system (3.4) and system (2.12) shows that Theorem 2.6 is applicable to the first two equations of both systems provided that additional initial conditions for y are given, since in system (3.4) the relevant matrix is $W(t) = -I$. Thus, we obtain the following

Theorem 3.2: *Let assumption (LP2) be fulfilled. Let x_ε be the solution of problem (3.3), (3.2) and $PQ_1x(0) = y^0$. Then the asymptotic expansion*

$$x_\varepsilon(t) \simeq \bar{x}_{-1}(\tau)\varepsilon^{-1} + \sum_{j=0}^{N}(x_j(t) + \bar{x}_j(\tau))\varepsilon^j \tag{3.6}$$

holds. Here,

$$\tau = t/\varepsilon, \quad |\bar{x}_j(\tau)| \leq C\exp(-\tau) \ (j = -1, 0, 1, \ldots) \qquad \text{and} \qquad P\bar{x}_{-1} = 0.$$

Moreover, $\bar{x}_{-1}(\tau) \equiv 0$ if and only if $y^0 = PQ_1x_0(0)$.

The proof is simply given by applying Theorem 2.6 to system (3.4). The asymptotic expansion of v_ε is obtained if the asymptotics of z_ε and y_ε are inserted into the third equation of (3.4). The representation (3.6) shows that it is important to find consistent initial values for integrating (3.3). Unless $y^0 = PQ_1x_0(0)$ (the exact initial condition given by problem (3.1), (3.2)), $x_\varepsilon(0)$ grows unboundedly for $\varepsilon \to 0$. Usually it is very hard to compute these additional conditions [17].

Example: Consider the semiexplicit system (2.5) and its parametrization (2.18). A simple calculation shows

$$PQ_1 = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad PP_1 = \begin{pmatrix} I - R & 0 \\ 0 & 0 \end{pmatrix}$$

with the projection $R = B_{12}(B_{21}B_{12})^{-1}B_{21}$. Inserting this expression into the second equation of (2.5) yields $Ru(0) = -B_{12}(B_{21}B_{12})^{-1}q_2(0)$, which is equivalent to

$$B_{21}u(0) = -q_2(0). \tag{3.7}$$

Hence, in order to avoid the unbounded boundary layer, one should choose the additional boundary condition (3.7) besides (3.2). Obviously, condition (3.7) is easily available from (2.5). Note that, because of assumption (LP2), only boundary conditions for some components of u are allowed in (3.2).

Remark: For the pencil regularization (2.21), the third equation of system (3.4) is replaced by

$$\varepsilon v' + (I - \varepsilon(PQ_1)')v = QP_1A_2^{-1}q - C_2z - \varepsilon C_2(z' + y') - QP_1P(z' + y'). \tag{3.8}$$

Note that this result is true provided that Q is constant. Since the first two equations of system (3.4) remain unchanged, condition (3.7) holds true for $z_\varepsilon + y_\varepsilon = Px_\varepsilon$. Hence, the right-hand side of equation (3.8) has an asymptotic expansion of the form

$$\bar{w}_{-1}(\tau)\varepsilon^{-1} + \sum_{j=0}^{N}(w_j(t) + \bar{w}_j(\tau)\varepsilon^j. \tag{3.9}$$

Assume that we have chosen an additional initial condition $Qx(0) = v^0$. The expansion (3.9) suggests to look for a similar expansion of v_ε:

$$v_\varepsilon(t) \simeq \bar{v}_{-1}(\tau)\varepsilon^{-1} + \sum_{j=0}^{N}(v_j(t) + \bar{v}_j(\tau)\varepsilon^j. \tag{3.10}$$

Since $v_\varepsilon(0) = v^0$, $\bar{v}_{-1}(0) = 0$ has to be true for expansion (3.10) to hold. Now, inserting expansion (3.9) and (3.10) into equation (3.8) and equating equal powers of ε yields, in the boundary layer,

$$\varepsilon^{-1}: \qquad \bar{w}_{-1}(\tau) = \bar{v}'_{-1}(\tau) + \bar{v}_{-1}(\tau) \tag{3.11}$$

$$\varepsilon^{j}: \qquad \bar{w}(\tau) = \bar{v}'_j(\tau) + \bar{v}_j(\tau) - (PQ_1)'(0)\bar{v}_{j-1}(\tau) \qquad (j = 0, 1, \ldots) \tag{3.12}$$

and, for the regular part,

$$\varepsilon^{0}: \qquad w_0 = v_0 \tag{3.13}$$

$$\varepsilon^{j}: \qquad w_j = v'_{j-1} + v_j - (PQ_1)'v_{j-1} \qquad (j = 1, 2, \ldots) \tag{3.14}$$

Equation (3.11) together with $\bar{v}_{-1}(0) = 0$ yields $\bar{v}_{-1}$ such that $|\bar{v}_{-1}(\tau)| \leq Ce^{-\alpha t}$. Equation (3.13) yields v_0. Applying the initial condition $v_0(0) = v^0$ we obtain $\bar{v}_0(0) = v^0 - v_0(0)$. Equation (3.12) provides $\bar{v}_0$ now. Having determined these coefficients, $\tilde{v} := \bar{v}_{-1}\varepsilon^{-1} + \bar{v}_0 + v_0$ fulfils

$$\varepsilon(\tilde{v})' + (I + \varepsilon(PQ_1)')\tilde{v} = w_\varepsilon + O(\varepsilon)$$

such that the usual stability argument (see, e.g., [22: pp. 56 ff] and [24: pp. 54 ff]) proves (3.10) for $N = 0$. The proof is completed by successive computation of higher order approximations using (3.12) and (3.14).

4. The index 3 problem

Now we will develop the asymptotics for the linear problem

$$A(t)x' + B(t)x = q \qquad (t \in [0,1]) \tag{4.1}$$

$$D_0x(0) + D_1x(1) = \gamma \tag{4.2}$$

where equation (4.1) is now assumed to be tractable with index 3. Inspired by the example (2.15) we will use the modified parametrization

$$(A + \varepsilon\tilde{B}_1)(Px)' + B_0x = q. \tag{4.3}$$

We need the following assumption (cf. Theorem 2.5).

Assumption (LP3):

(i) Equation (4.1) is tractable with index 3, $Q_1 Q = 0$, $Q_2 \equiv Q_{2,s} := Q_2 \tilde{A}_3^{-1} \tilde{B}_2$.

(ii) In (4.2), $D_0 = D_0 P P_1 P_2(0)$, $D_1 = D_1 P P_1 P_2(1)$, rank $\begin{pmatrix} D_0 & D_1 \\ I - PP_1P_2(0) & 0 \end{pmatrix} = m$.

(iii) Problem (4.1), (4.2) has a unique solution x_0.

Theorem 4.1: *Let assumption (LP3) hold. Then, for sufficiently small $\varepsilon > 0$, (4.3) is a transferable differential-algebraic equation.*

Proof: Let $t \in [0,1]$ be fixed. We omit the argument t in the proof. Denote $A_\varepsilon = A + \varepsilon \tilde{B}_1$.

(i) Let $z \in \mathcal{N}(A)$ be given. Then

$$z = Qz \qquad \text{and} \qquad A_\varepsilon z = \varepsilon \tilde{B}_1 z = (B_0 - A(PP_1)')PQz = 0.$$

Hence, $\mathcal{N}(A) \subseteq \mathcal{N}(A_\varepsilon)$. Let now $z \in \mathcal{N}(A_\varepsilon)$, i.e., $A_\varepsilon z = 0$. By the definition of $\tilde{A}_3$,

$$A_\varepsilon = \tilde{A}_3 + \varepsilon \tilde{B}_1 - B_0 Q - \tilde{B}_1 Q_1 - \tilde{B}_2 Q_2,$$

hence

$$z = \tilde{A}_3^{-1}(B_0 Q + \tilde{B}_1 Q_1 + \tilde{B}_2 Q_2 - \varepsilon \tilde{B}_1)z. \tag{4.4}$$

Regarding

$$\tilde{A}_3^{-1} B_0 Q = Q, \qquad \tilde{A}_3^{-1} \tilde{B}_1 Q_1 = Q_1, \qquad \tilde{A}_3^{-1} \tilde{B}_2 Q_2 = Q_2,$$

equation (4.4) is equivalent to

$$z = (Q + Q_1 + Q_2 - \varepsilon Q_1 - \varepsilon \tilde{A}_3^{-1} \tilde{B}_2)z. \tag{4.5}$$

Multiplying this equation by Q_2 gives $\varepsilon Q_2 z = 0$, hence $Q_2 z = 0$. The multiplication of equation (4.5) by PP_1 implies

$$PP_1 z = -\varepsilon PP_1 \tilde{A}_3^{-1}(B_0 - A_1(PP_1)')PP_1 z.$$

Hence $PP_1 z = 0$ if $0 < \varepsilon \le \varepsilon_0$ with ε_0 independent of t. Now, equation (4.5) yields $z = (Q + (1-\varepsilon)Q_1)z$. Therefore, $Q_1 z = 0$, and $z = Qz$ implying $z \in \mathcal{N}(A)$. Summarizing, $\mathcal{N}(A) = \mathcal{N}(A_\varepsilon)$.

(ii) Consider now $A_{1,\varepsilon} = A_\varepsilon + B_0 Q$. We will show that this matrix is non-singular, which will complete the proof. Let $A_{1,\varepsilon} z = 0$. This is equivalent to

$$z = (Q_1 + Q_2 - \varepsilon Q_1 - \varepsilon \tilde{A}_3^{-1} \tilde{B}_2)z. \tag{4.6}$$

The multiplication by Q_2 yields $Q_2 z = 0$. Similarily, multiplying now by P_1 gives $P_1 z = -\varepsilon P_1 \tilde{A}_3^{-1} \tilde{B}_1 P_1 z$, implying $P_1 z = 0$ for $\varepsilon \le \varepsilon_0$. Finally, from equation (4.6) it now follows that $Q_1 z = 0$. Since $z = P_1 z + Q_1 z = 0$, $A_{1,\varepsilon}$ is non-singular ∎

Remark: Using the same techniques it is easy to show that (4.3) is a transferable differential-algebraic equation if equation (4.1) is tractable with index 2. In this case, Theorem 3.3 is valid with a slightly modified Q_1, too.

The next lemma contains a representation of parametrization (4.3) analogous to that given in Lemma 2.4 for equation (4.1).

Lemma 4.2: *Let assumption (LP3) hold. If x is a solution of problem (4.3), (4.2), then*

$$
\begin{aligned}
&(I+\varepsilon C_1)z' + (C_1 - (PP_1P_2)')z - ((PP_1P_2)' + \varepsilon C_1')y' - \varepsilon C_1' PQ_1 v \\
&\qquad = PP_1P_2\tilde{A}_3^{-1}q \\
&\varepsilon y' + (I - \varepsilon C_2' PP_1Q_2)y - \varepsilon C_2' PP_1P_2 z - \varepsilon C_2' PQ_1 v = PP_1Q_2\tilde{A}_3^{-1}q \\
&\varepsilon v' + \varepsilon C_3 z' - PQ_1Q_2 y' + (I - \varepsilon C_3' PQ_1)v + ((PQ_1Q_2)' + C_3)z - \varepsilon C_3' y \\
&\qquad = PQ_1P_2\tilde{A}_3^{-1}q \\
&w = QP_1P_2\tilde{A}_3^{-1}q + QQ_1(v'+y'+z') - QP_1P_2(z'+y') - C_4 z \\
&\qquad - \varepsilon C_4(v'+y'+z') \\
&D_0 z(0) + D_1 z(1) = \gamma, \quad (I - PP_1P_2(0))z(0) = 0
\end{aligned}
\tag{4.7}
$$

with

$$
\begin{aligned}
C_1 &= PP_1P_2\tilde{A}_3^{-1}\tilde{B}_1, & C_2 &= PP_1Q_2\tilde{A}_3^{-1}\tilde{B}_1 \\
C_3 &= PQ_1P_2\tilde{A}_3^{-1}\tilde{B}_1, & C_4 &= QP_1P_2\tilde{A}_3^{-1}\tilde{B}_1
\end{aligned}
$$

where $z = PP_1P_2x$, $y = PP_1Q_2x$, $v = PQ_1x$ and $w = Qx$.

Conversely, if (z,y,v,w) is a solution of equation (4.7) and

$$y(0) \in \mathcal{R}(PP_1Q_2(0)) \qquad \textit{and} \qquad v(0) \in \mathcal{R}(PQ_1(0)),$$

then

$$PP_1P_2z = z, \qquad PP_1Q_2y = y, \qquad PQ_1v = v, \qquad Qw = w$$

and $x = z + y + v + w$ is a solution of problem (4.3), (4.2).

The proof can be given using the same techniques as in Lemma 2.4 (see Appendix) and is therefore omitted. Again, system (4.7) shows clearly the singular perturbation behaviour of the parametrization (4.3). We consider the first three equations of system (4.7). After some algebraic manipulations we obtain the equivalent systems

$$
\begin{pmatrix} \varepsilon y' \\ z' \\ Dy' + \varepsilon v' \end{pmatrix} = \begin{pmatrix} U_{11}(\varepsilon) & U_{12}(\varepsilon) & U_{12}(\varepsilon) \\ U_{21}(\varepsilon) & U_{22}(\varepsilon) & U_{23}(\varepsilon) \\ U_{31}(\varepsilon) & U_{32}(\varepsilon) & U_{33}(\varepsilon) \end{pmatrix} \begin{pmatrix} y \\ z \\ v \end{pmatrix} + \begin{pmatrix} f \\ g(\varepsilon) \\ h(\varepsilon) \end{pmatrix}
\tag{4.8}
$$

with coefficients and right-hand side being analytic with respect to ε:

$$
U_{jl}(\varepsilon) = \sum_{i=0}^{\infty} U_{jli}\varepsilon^i, \qquad g(\varepsilon) = \sum_{i=0}^{\infty} g_i\varepsilon^i, \qquad h(\varepsilon) = \sum_{i=0}^{\infty} h_i\varepsilon^i
\tag{4.9}
$$

for sufficiently small $\varepsilon > 0$. These sums converge uniformly with respect to $t \in [0,1]$. Moreover,

$$D = -PQ_1Q_2 \tag{4.10}$$

does not vanish in the interval $[0,1]$. In order to obtain a system in the standard form covered by Theorem 2.6 we differentiate the first equation of system (4.8) and introduce the new unknown $u = y'$. This gives finally

$$\begin{pmatrix} y' \\ z' \\ \varepsilon u' \\ \varepsilon v' \end{pmatrix} = \begin{pmatrix} 0 & 0 & I & 0 \\ V_{21}(\varepsilon) & V_{22}(\varepsilon) & V_{23}(\varepsilon) & V_{24}(\varepsilon) \\ V_{31}(\varepsilon) & V_{32}(\varepsilon) & V_{33}(\varepsilon) & V_{34}(\varepsilon) \\ V_{41}(\varepsilon) & V_{42}(\varepsilon) & V_{43}(\varepsilon) & V_{44}(\varepsilon) \end{pmatrix} \begin{pmatrix} y \\ z \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(\varepsilon) \\ f(\varepsilon) \\ h(\varepsilon) \end{pmatrix} \tag{4.11}$$

where all coefficients and right-hand sides have series expansions which converge uniformly with respect to t. Taking the formal limit $\varepsilon \to 0$ in (4.11) we obtain the system

$$\begin{pmatrix} y' \\ z' \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & I & 0 \\ (PP_1P_2)' & (PP_1P_2)' - C_1 & 0 & 0 \\ -I & -C_2'PQ_1((PQ_1Q_2)' + C_3) & -I + C_2'PQ_1Q_2 & -C_2'PQ_1 \\ -0 & -(PQ_1Q_2)' - C_3 & PQ_1Q_2 & -I \end{pmatrix} \tag{4.12}$$

$$\times \begin{pmatrix} y \\ z \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ PP_1P_2\tilde{A}_3^{-1}q \\ C_2'PQ_1P_2\tilde{A}_3^{-1}q + (PP_1Q_2\tilde{A}_3^{-1}q)' \\ PQ_1P_2\tilde{A}_3^{-1}q \end{pmatrix}.$$

This system can also be obtained from the first three equations of system (2.9) by the following transformations:

(i) substitute $PQ_1Q_2z' = -(PQ_1Q_2)'z$
(ii) differentiate the second equation
(iii) introduce $y' = u$
(iv) mulitply the third equation by $C_2'PQ_1$ and add it to the second one.

Now, for equation (4.11), the matrix W appearing in Theorem 2.6 is given by

$$W = \begin{pmatrix} -I + C_2'PQ_1Q_2 & -C_2'PQ_1 \\ PQ_1Q_2 & -I \end{pmatrix}. \tag{4.13}$$

The eigenvalues of the matrix W determine the convergence behaviour of the solutions of equation (4.11) provided that appropriate boundary conditions are given. Unfortunately, the eigenvalues of the matrix W may vary almost arbitrarily.

Example: Let A, B be given by (2.15). It holds

$$W = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\alpha t & -(1+\alpha) & 0 & \alpha & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\alpha t & -1 & 0 & -1 & 0 \\ 0 & (\alpha t)^2 & -\alpha t^2 & 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of the matrix W are -1 (fourfold) and the zeros of the polynomial $\lambda^2 + (2+\alpha)\lambda + 1 + 2\alpha = 0$. The real parts of these eigenvalues can have an arbitrary sign in dependence on α.

Therefore, we *require* in the following that the eigenvalues $\lambda_i(t)$ of the matrix W fulfil the condition

$$\mathrm{Re}\lambda_i(t) \leq -\sigma < 0 \qquad (t \in [0,1]). \tag{4.14}$$

The appropriate additional initial conditions can be inferred from (2.13):

$$\begin{aligned} y(0) &= PP_1Q_2\tilde{A}_3^{-1}q(0) \\ v(0) &= PQ_1P_2\tilde{A}_3^{-1}q(0) + PQ_1\Big(Q_2(PP_1Q_2\tilde{A}_3^{-1}q(0))' \\ &\quad + Q_2(PP_1P_2)'z(0) - P_2\tilde{A}_3^{-1}\tilde{B}_1z(0)\Big) \end{aligned} \tag{4.15}$$

and, inserting these values into the second equation of system (4.7),

$$\begin{aligned} u(0) = C_2'P\Big\{&(P_1Q_2 + Q_1P_2)A_3^{-1}q(0) + Q_1Q_2(PP_1Q_2\tilde{A}_3^{-1}q(0))' \\ &+ \Big(P_1P_2 + Q_1Q_2(PP_1P_2)' - Q_1P_2\tilde{A}_3^{-1}\tilde{B}_1'\Big)z(0)\Big\}. \end{aligned} \tag{4.16}$$

Remark: In conditions (4.15) we assumed $y(0)$ and $v(0)$ to be given exactly. Since (4.15) results from (2.13), this assumption is nothing else but an computation of consistent initial values for the differential variables Px. If this were not so, $u(0)$ would grow unboundedly. Indeed, if $y(0)$ and $v(0)$ are perturbed by an error $O(\delta)$, then the error in $u(0)$ is $O(\delta/\varepsilon)$. This perturbation is immediately propagated onto the null space component $w = Qx$. Therefore, it is crucial to obtain very exact additional initial conditions in order to guarantee convergence near $t = 0$ when numerically approximating (4.3). In general it is not trivial to compute these exact initial conditions [17].

Theorem 2.6 implies now the desired result about the asymptotic expansion for the solution $(z_\varepsilon, y_\varepsilon, v_\varepsilon, w_\varepsilon)$ of problem (4.3), (4.2), (4.15), and $u_\varepsilon = y_\varepsilon'$.

Theorem 4.3: *Let assumption (LP3) and condition (4.14) hold. Then the parametrized problem (4.3) subject to the boundary condition*

$$D_0x(0) + D_1x(1) = \gamma$$

and the additional initial conditions

$$\begin{aligned} PP_1Q_2x(0) &= PP_1Q_2\tilde{A}_3^{-1}q(0) \\ PQ_1x(0) &= PQ_1P_2\tilde{A}_3^{-1}q(0) + PQ_1\Big(Q_2(PP_1Q_2\tilde{A}_3^{-1}q(0))' \\ &\quad + Q_2(PP_1P_2)'PP_1P_2x(0) - P_2\tilde{A}_3^{-1}\tilde{B}_1PP_1P_2x(0)\Big) \end{aligned}$$

has a unique solution x_ε for sufficiently small $\varepsilon > 0$. Moreover, the asymptotic expansion

$$x_\varepsilon(t) \simeq \sum_{j=0}^{N}(x_j(t) + \bar{x}_j(\tau))\varepsilon^j \qquad (\tau = t/\varepsilon) \tag{4.17}$$

holds, where $|\bar{x}_j(\tau)| \leq C\exp(-\alpha\tau), \alpha > 0$.

Proof: Using equation (4.12) we obtain a representation of the type (4.17) for $z_\varepsilon, y_\varepsilon, v_\varepsilon, u_\varepsilon$. Inserting this expansion into the fourth equation of system (4.7) yields the result ∎

Example: Consider the linear semiexplicit index 3 system (2.6):

$$\begin{pmatrix} u' \\ v' \\ 0 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad (t \in [0,1]). \tag{4.18}$$

Equation (4.18) is tractable with index 3 if $H := B_{32}B_{21}B_{13}$ is non-singular, which we will assume in the following. Then $R_1 := B_{13}H^{-1}B_{32}B_{21}$ is a projection for every $t \in [0,1]$. Now, parametrization (4.3) reads

$$\begin{aligned} &\begin{pmatrix} I - \varepsilon(B_{11} - R_1') & -\varepsilon B_{12} & 0 \\ -\varepsilon B_{21} & I - \varepsilon B_{22} & 0 \\ 0 & -\varepsilon B_{32} & 0 \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 0 \end{pmatrix} \\ &\qquad = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \end{aligned} \tag{4.19}$$

After some tedious calculations one obtains

$$W = \begin{pmatrix} -I & W_1 & 0 & W_3 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & W_2 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{pmatrix}$$

with some matrix functions W_1, W_2 and W_3. The block matrix W has the eigenvalue -1 only, such that Theorem 4.3 applies. A closer look at equation (4.19) shows that the term R_1' in the upper left-hand corner of the matrix multiplying the derivatives may be omitted in order to obtain results like (4.17). However, this is equivalent to the application of relation (2.20) to system (4.18).

As in the index 2 Hessenberg case the additional initial conditions can be considerably simplified. Note that $R_2 := B_{21}B_{13}H^{-1}B_{32}$ is a projection again. We obtain

$$PP_1P_2x = \begin{pmatrix} (I - R_1)u \\ (I - R_2)v \\ 0 \end{pmatrix}, \quad PP_1Q_2x = \begin{pmatrix} 0 \\ R_2 v \\ 0 \end{pmatrix}, \quad PQ_1x = \begin{pmatrix} R_1 u \\ 0 \\ 0 \end{pmatrix}.$$

The additional initial conditions are simply

$$\begin{aligned} B_{32}v(0) + q(0) &= 0 \\ B_{32}(B_{21}u(0) + B_{22}v(0) + q_2(0)) + B_{32}'v(0) + q_3'(0) &= 0. \end{aligned}$$

A. Appendix

We will sketch the essential steps necessary to prove Lemma 2.4 now. Let us start with equation (2.7). Since $AP'P = 0, PP_1P = PP_1$ and

$$P(Px)' = PP_1(Px)' + PQ_1(Px)' = (PP_1x)' - (PP_1)'Px + PQ_1(Px)'$$

we may write equation (2.7) as

$$A_1\{(PP_1x)' + PQ_1(Px)' + Qx\} + \tilde{B}_1Px = q. \tag{A.1}$$

Clearly, this is the same as

$$\tilde{A}_2\Big\{P_1(PP_1x) + P_1PQ_1(Px)' + P_1Qx + Q_1x\Big\} + \tilde{B}_1PP_1x = q$$

or

$$\tilde{A}_3\Big\{P_2\Big(P_1(PP_1x)' + P_1PQ_1(Px)' + P_1Qx + Q_1x\Big) + Q_2x\Big\} + \tilde{B}_1PP_1P_2x = q. \tag{A.2}$$

Using the identities

$$P_2P_1PQ_1 = -QQ_1, \qquad P_2P_1Q = Q, \qquad P_2Q_1 = Q_1$$

we decompose (A.2) into two parts by multiplying with $P_2\tilde{A}_3^{-1}$ and $PP_1Q_2\tilde{A}_3^{-1}$, respectively:

$$\begin{aligned} P_2P_1(PP_1x)' - QQ_1(Px)' + Qx + Q_1x &= P_2\tilde{A}_3^{-1}(-\tilde{B}_1PP_1P_2x + q) \\ PP_1Q_2x &= PP_1Q_2\tilde{A}_3^{-1}q. \end{aligned} \tag{A.3}$$

Multiplying the first equation in (A.3) by PP_1, PQ_1 and QP_1, respectively, and taking into account the identities

$$\begin{aligned} PP_1P_2P &= PP_1P_2 & PQ_1P_2P_1PP_1 &= -PQ_1Q_2 \\ PP_1QQ_1 &= 0 & QP_1P_2P_1 &= QP_1P_2 \\ PP_1Q &= 0 & QP_1QQ_1 &= QQ_1 \\ PP_1Q_1 &= 0 & QP_1Q &= Q \\ PP_1P_2P_1PP_1 &= PP_1P_2 & QP_1Q_1 &= 0 \\ PQ_1P_2P_1 &= -PQ_1Q_2 & QP_1P_2P_1PP_1 &= -QP_1Q_2 \\ PQ_1Q &= 0 & P_2\tilde{A}_3^{-1}\tilde{B}_1PP_1P_2 &= \tilde{A}_3^{-1}\tilde{B}_1PP_1P_2 \end{aligned}$$

we obtain

$$\begin{aligned} (PP_1P_2x)' - (PP_1P_2)'PP_1x + PP_1\tilde{A}_3^{-1}\tilde{B}_1PP_1P_2x &= PP_1P_2\tilde{A}_3^{-1}q \\ -PQ_1Q_2(PP_1x)' + PQ_1P_2\tilde{A}_3^{-1}\tilde{B}_1PP_1P_2x &= PQ_1P_2\tilde{A}_3^{-1}q \\ QP_1P_2(PP_1x)' - QQ_1(Px)' + Qx + QP_1P_2\tilde{A}_3^{-1}\tilde{B}_1PP_1P_2x &= QP_1P_2\tilde{A}_3^{-1}q. \end{aligned} \tag{A.4}$$

With the abbreviations

$$z = PP_1P_2x, \qquad y = PP_1Q_2x, \qquad v = PQ_1x, \qquad w = Qx$$

equation (A.4) and the second equation of (A.3) lead to system (2.13). If (z, y, v, w) is a solution of system (2.13), then it holds obviously that

$$(I - PP_1Q_2)y = 0, \qquad (I - PQ_1)v = 0, \qquad (I - Q)w = 0.$$

Multiplying the first equation of system (2.13) by $I - PP_1P_2$ yields

$$(I - PP_1P_2)z' - (I - PP_1P_2)(PP_1P_2)'(z + y) = 0. \tag{A.5}$$

Using

$$\begin{aligned} y &= PP_1Q_2y \\ (I - PP_1P_2)z' &= ((I - PP_1P_2)z)' + (PP_1P_2)'z \\ (I - PP_1P_2)(PP_1P_2)' &= (PP_1P_2)'PP_1P_2 \\ (I - PP_1P_2)(PP_1P_2)'PP_1Q_2 &= (I - PP_1P_2)(PP_1P_2PP_1Q_2)' \\ &\quad - (I - PP_1P_2)PP_1P_2(PP_1Q_2)' \\ &= 0 \end{aligned}$$

we obtain

$$((I - PP_1P_2)z)' + (PP_1P_2)'(I - PP_1P_2)z = 0.$$

Since $(I - PP_1P_2(0))z(0) = 0$, $(I - PP_1P_2)z \equiv 0$. Because of $I = PP_1P_2 + PP_1Q_2 + PQ_1 + Q$ the lemma is shown.

References

[1] Boyarincev, Ju. E.: *Regular and Singular Systems of Linear Ordinary Differential Equations* (in Russian). Novosibirsk: Nauka (Siberian Branch) 1980.

[2] Campbell, S. L.: *Singular Systems of Differential Equations* (Res. Notes Math.: Vol. 40). Marshfield etc.: Pitman 1980.

[3] Chua, L. O., Matsumoto, T. and S. Ichiraku: *Geometric properties of resistive nonlinear n-ports: Transversality, structural stability, reciprocity, and anti-reciprocity.* IEEE Trans. on Circuits and Systems, CAS 27 (1980), 577 - 603.

[4] Clark, K. D.: *A structural form for higher index semistate equations I: Theory and application to circuit and control theory.* Linear Alg. Appl. 98 (1988), 169 - 197.

[5] Eich, E. and M. Hanke: *Regularization methods for constraint mechanical multibody systems.* ZAMM 1994 (to appear).

[6] Engl, H. W., Hanke, M. and A. Neubauer: *Tikhonov regularization of nonlinear differenti al-algebraic equations.* In: Inverse Problems in Action (ed.: P. C. Sabatier). Berlin etc.: Springer - Verlag 1990, pp. 92 - 105.

[7] Gear, C. W.: *Simultaneous numerical solution of differential-algebraic equations.* IEEE Trans. Circuit Theory 18 (1971), 89 - 95.

[8] Gear, C. W. and L. R. Petzold: *Ode methods for the solution of differential algebraic systems.* SIAM J. Numer. Anal. 21 (1984), 716 - 728.

[9] Griepentrog, E. and R. März: *Differential-Algebraic Equations and Their Numerical Treatment* (Teubner-Texte zur Mathematik: Vol. 88). Leipzig: B. G. Teubner Verlagsges. 1986.

[10] Griepentrog, E. and R. März: *Basic properties of some differential-algebraic equations.* Z. Anal. Anw. 8 (1989), 25 - 40.

[11] Groetsch, C. W.: *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind* (Res. Notes Math.: Vol. 105). Boston etc.: Pitman 1984.

[12] Hanke, M.: *On a least-squares collocation method for linear differential-algebraic equations.* Numer. Math. 54 (1988), 79 - 90.

[13] Hanke, M.: *Linear differential-algebraic equations in spaces of integrable functions.* J. Diff. Equ. 79 (1989), 14 - 30.

[14] Hanke, M.: *On the regularization of index 2 differential-algebraic equations.* J. Math. Anal. Appl. 151 (1990), 236 - 253.

[15] Hanke, M.: *On the asymptotic representation of a regularization approach to nonlinear semiexplicit higher index differential-algebraic equations.* IMA J. Appl. Math. 46 (1991), 225 - 245.

[16] Knorrenschild, M.: *Regularisierung von differentiell-algebraischen Systemen – theoretische und numerische Aspekte.* PhD thesis. Aachen: Rhein. - Westfälische Techn. Hochschule 1988.

[17] Leimkuhler, B. J.: *Approximation methods for the consistent initialization of differential-algebraic equations.* PhD thesis. Urbana-Champaign: Univ. Illinois/Dept. Comp. Sci. 1988. Report UIUC DCS-R-88-1450.

[18] März, R.: Personal communication.

[19] März, R.: *On tractability with index 2.* Preprint. Berlin: Humboldt-Univ./Sektion Mathematik, Report 109 (1986), pp. 1 - 13.

[20] März, R.: *Index-2 differential-algebraic equations.* Results Math. 15 (1989), 149 - 171.

[21] März, R.: *Some new results concerning index-3 differential-algebraic equations.* J. Math. Anal. Appl. 140 (1989), 177 - 199.

[22] O'Malley, R. E.: *Singular Perturbation Methods for Ordinary Differential Equations* (Appl. Math. Sci.: Vol. 89). New York etc.: Springer - Verlag 1991.

[23] Petzold, L. R.: *Differential-algebraic equations are not ode's.* SIAM J. Sci. & Statist. Comp. 3 (1982), 367 - 384.

[24] Vasil'eva, A. B. and B. F. Butuzov: *Asymptotic expansions of solutions of singularly perturbed equations* (in Russian). Moskva: Nauka 1973.

Received 22.10.1993