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The Grünwald-Letnikov Difference Operator and Regularization of the Weyl Fractional Differentiation

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Abstract. The limit for $h \to 0$ of the Grünwald-Letnikov difference operator

$$h^{-\alpha}(\Delta_h^\alpha f)(x) = h^{-\alpha}\sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} f(x+jh) \qquad (h>0)$$

is proved to be the inversion of the Weyl integral operator of the form

$$(I_-^\alpha u)(x) = \int_x^\infty \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} u(t)\,dt = f(x) \qquad (0<x<\infty)$$

under the assumptions $\alpha \geq 1/p$, $1 \leq p < \infty$ and $(x^\alpha + 1)u \in L_p(0,\infty)$. This result is then applied to obtain a regularized approximate solution u_ϵ to the Weyl integral equation. Under some additional conditions on u error estimates are obtained.

Keywords: *Grünwald-Letnikov difference operator, Weyl integral operator, regularization, stability estimates*

AMS subject classification: 65R30, 45E10, 45L05, 45L10

1. Introduction

In [1, 7] the limit for $h \to 0$ of the Grünwald-Letnikov difference operator

$$h^{-\alpha}(\Delta_h^\alpha f)(x) = h^{-\alpha}\sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} f(x+jh) \qquad (h>0) \tag{1}$$

is proved to be the inversion of the Riemann-Liouville fractional integral operator of periodic functions in $L_p(0,2\pi)$, $1 \leq p < \infty$. In [5] this result is generalized to the Weyl fractional integral operator of non-periodic functions in $L_p(\mathbb{R})$

$$(I_-^\alpha u)(x) = \int_x^\infty \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} u(t)\,dt = f(x) \qquad (0<x<\infty) \tag{2}$$

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with the order α, $0 < \alpha < 1/p$, $1 < p < \infty$. We prove that this generalization is still valid for the Weyl fractional integral operator with the order $\alpha \geq 1/p$, provided that $(x^\alpha + 1)u \in L_p(0,\infty)$, $1 \leq p < \infty$. This result is then applied to regularize the Weyl integral equation (2), that is in general ill-posed. Indeed, suppose

$$(I_-^\alpha u_n)(x) = n^{\beta-\alpha}e^{-nx} \qquad (1/p < \beta < \alpha + 1/q,\ 1 \leq p,q \leq \infty).$$

Then

$$u_n(x) = n^\beta e^{-nx} \tag{3}$$

and we have

$$\|u_n\|_{L_p(0,\infty)} = p^{-1/p} n^{\beta-1/p} \longrightarrow \infty \qquad \text{as } n \to \infty$$

although

$$\|I_-^\alpha u_n\|_{L_q(0,\infty)} = q^{-1/q} n^{\beta-\alpha-1/q} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Suppose instead of $I_-^\alpha u$ the data f is known with some noise and the noise level is given by

$$\|I_-^\alpha u - f\|_p \leq \varepsilon. \tag{4}$$

In the work [2] this problem has been considered by applying the Marchaud integral (see [5: p.101]). Here the Grünwald-Letnikov difference operator is applied as the regularizing operator. Together with [6] they seem to be the first works where the Grünwald-Letnikov difference operator is used as the regularizing operator (for other regularization methods for analogous Abel integral equations see [3]). Since the Grünwald-Letnikov difference operator is an operator depending on a step length h, our method can be used as a discretization method, and moreover, is a local method, that means for recovering the function u at point x one needs the given data f only in the neighbourhood of x, therefore is convenient for numerical treatment. Under the additional assumptions

$$\alpha > 1 \qquad \text{and} \qquad u' \in L_p(0,\infty)$$

the error estimate

$$\|u_\varepsilon - u\|_p \leq C\varepsilon^{1/(\alpha+1)}(1 + \|x^\alpha u\|_p + \|u'\|_p) \qquad (1 \leq p \leq \infty)$$

is obtained. This estimate is shown to have optimal order of approximation. If the function u belongs to the Hölder space $H_p^\lambda(0,\infty)$ of order λ [5: p.200], then the error estimate

$$\|u_\varepsilon - u\|_p \leq C\varepsilon^{\lambda/(\alpha+\lambda)}(1 + \|x^\alpha u\|_p + \|u\|_{H_p^\lambda}) \qquad (1 \leq p \leq \infty)$$

is valid. Here $\|u\|_{H_p^\lambda}$ is the norm of u in the Hölder space $H_p^\lambda(0,\infty)$.

2. Inversion of the Weyl fractional integral operator

The main result of this section is the following

Theorem 1. *Let* $\alpha \geq 1/p$ *and* $(x^\alpha + 1)u \in L_p(0,\infty)$, $1 \leq p < \infty$. *Then*

$$\lim_{h\to 0+} \left\|u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x)\right\|_p = 0, \tag{5}$$

that means $\lim_{h\to 0+} h^{-\alpha}\Delta_h^\alpha$ *is the inversion of the Weyl fractional integral operator.*

Proof. First we note that the condition $(x^\alpha + 1)u \in L_p(0,\infty)$ implies $I_-^\alpha u \in L_p(0,\infty)$. This follows from the Hardy-Littlewood inequality (see [4: p. 583])

$$\|(I_-^\alpha u)(x)\|_p \leq C\|x^\alpha u(x)\|_p.$$

Consequently,

$$\sum_{j=k+1}^{\infty} \left\|\binom{\alpha}{j}(I_-^\alpha u)(x)\right\|_p = O(k^{-\alpha})\|x^\alpha u(x)\|_p$$

since

$$\sum_{j=k+1}^{\infty}\left|\binom{\alpha}{j}\right| = \sum_{j=k+1}^{\infty} O(j^{-\alpha-1}) = O(k^{-\alpha}). \tag{6}$$

Therefore, in the expression $h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x)$ one can change the order of summation and integration to obtain

$$h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x) = \int_0^\infty p_\alpha(t)u(x+ht)\,dt, \tag{7}$$

where

$$p_\alpha(t) = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\frac{(t-j)_+^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{with } \varphi_+(x) = \begin{cases}\varphi(x) & \text{for } \varphi(x) > 0\\ 0 & \text{for } \varphi(x) \leq 0\end{cases}.$$

Since (see [7])

$$p_\alpha \in L_p(0,\infty) \qquad \text{and} \qquad \int_0^\infty p_\alpha(t)\,dt = 1 \tag{8}$$

we have

$$\left\|u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x)\right\|_p = \left\{\int_0^\infty \left|\int_0^\infty p_\alpha(t)[u(x) - u(x+ht)]\,dt\right|^p dx\right\}^{1/p}.$$

Applying the Minkovski inequality [5: p. 26] we obtain

$$\left\|u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x)\right\|_p \leq \int_0^\infty |p_\alpha(t)|\left\{\int_0^\infty |u(x+ht) - u(x)|^p\,dx\right\}^{1/p} dt.$$

From $u \in L_p(0,\infty)$ we conclude that for all $\delta > 0$ there exists $\eta_\delta > 0$ such that $\eta_\delta \to 0$ as $\delta \to 0$ and

$$\left\{ \int_0^\infty |u(x+t) - u(x)|^p \, dx \right\}^{1/p} \le \eta_\delta \qquad \text{as } 0 < t < \delta.$$

Now we have, by a standard splitting technique,

$$\begin{aligned} \left\| u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x) \right\|_p &\le \eta_\delta \int_0^{\delta/h} |p_\alpha(t)| \, dt + 2\|u\|_p \int_{\delta/h}^\infty |p_\alpha(t)| \, dt \\ &\le \eta_\delta \int_0^\infty |p_\alpha(t)| \, dt + 2\|u\|_p \int_{\delta/h}^\infty |p_\alpha(t)| \, dt. \end{aligned}$$

From (8) we see that

$$\lim_{h \to 0} \left\| u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x) \right\|_p = 0$$

as $\delta/h \to 0$ and $h \to 0$. Hence the theorem is proved ∎

3. A regularized solution

Suppose now instead of $I_-^\alpha u$ we know only the data f with some noise level (4). From now throughout the paper we assume that

$$\begin{aligned} u &\in L_p(0,\infty) \quad \text{if } 0 < \alpha < 1/p \\ (x^\alpha + 1)u &\in L_p(0,\infty) \quad \text{if } \alpha \ge 1/p, \; 1 \le p < \infty. \end{aligned} \tag{9}$$

First we shall show that $h^{-\alpha}\Delta_h^\alpha f$, with an appropriate choice of the step length h, is a regularizing scheme for the ill-posed problem (2) under the conditions (4) and (9). We have

$$\begin{aligned} |u(x) - h^{-\alpha}(\Delta_h^\alpha f)(x)| &\le |u(x) - h^{-\alpha}(\Delta_h^\alpha I_-^\alpha u)(x)| + h^{-\alpha}|(\Delta_h^\alpha (I_-^\alpha u - f))(x)| \\ &= I_h(x) + I\!I_h(x). \end{aligned}$$

From (4) and the inequality

$$\sum_{j=0}^\infty \left| \binom{\alpha}{j} \right| \le 2^{\alpha+1}$$

[5: p.279] we get

$$\|I\!I_h\|_p \le h^{-\alpha} \sum_{j=0}^\infty \left| \binom{\alpha}{j} \right| \; \|f(x+jh) - I_-^\alpha u(x+jh)\|_p \le 2^{\alpha+1} h^{-\alpha} \varepsilon.$$

For the function I_h from [5: p. 287] for the case $0 < \alpha < 1/p$ and Theorem 1 for the case $\alpha \geq 1/p$ we have $\lim_{h\to 0} \|I_h\|_p = 0$. Therefore, if we choose h such that $h = o(1)$ and $h^{-\alpha} = o(\varepsilon^{-1})$ (this can be satisfied, for example, by $h = \varepsilon^{1/(\alpha+1)}$), then $h^{-\alpha}\Delta_h^\alpha f$ tends towards u in p norm as $\varepsilon \to 0$. Hence $h^{-\alpha}\Delta_h^\alpha f$ is a regularizing scheme for the equation (2).

Putting now

$$u_\varepsilon = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(x+jh) \tag{10}$$

where $k \to \infty$ as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\big\|u_\varepsilon(x) - h^{-\alpha}(\Delta_h^\alpha f)(x)\big\|_p \\
&= \left\| h^{-\alpha} \sum_{j=k+1}^{\infty} (-1)^j \binom{\alpha}{j} f(x+jh) \right\|_p \\
&\leq h^{-\alpha} \left\| \sum_{j=k+1}^{\infty} (-1)^j \binom{\alpha}{j} (I_-^\alpha u - f)(x+jh) \right\|_p \\
&\qquad + h^{-\alpha} \left\| \sum_{j=k+1}^{\infty} (-1)^j \binom{\alpha}{j} I_-^\alpha u(x+jh) \right\|_p \\
&\leq 2^{\alpha+1} h^{-\alpha}\varepsilon + \sum_{j=k+1}^{\infty} \left|\binom{\alpha}{j}\right| \left\{ \int_{jh}^{\infty} x^{\alpha p} |u(x)|^p dx \right\}^{1/p} \\
&\leq 2^{\alpha+1} h^{-\alpha}\varepsilon + Ck^{-\alpha}\|x^\alpha u\|_p
\end{aligned} \tag{11}$$

because of (6). Therefore, since

$$\|u - u_\varepsilon\|_p \leq \|u - h^{-\alpha}(\Delta_h^\alpha f)\|_p + \|u_\varepsilon - h^{-\alpha}(\Delta_h^\alpha f)\|_p,$$

we obtain the following

Theorem 2. *Let $u \in L_p(0,\infty)$ if $0 < \alpha < 1/p$ and $(x^\alpha + 1)u \in L_p(0,\infty)$ if $\alpha \geq 1/p$, $1 \leq p < \infty$. Assume that (4) holds. Then u_ε according to (10) is a regularized solution of the equation (2) if*

$$h = o(1), \quad h^{-\alpha} = o(\varepsilon^{-1}), \quad k^{-1} = o(1) \qquad \text{as } \varepsilon \to 0. \tag{12}$$

Remark. The scheme (10) means "regularization by discretization" , the step length h playing the role of the parameter of regularization. Note that k can be chosen so that $kh \to 0$ as $\varepsilon \to 0$. Then u_ε will depend only on local values of f and therefore we obtain a local regularization scheme.

4. Regularization estimate

In practice sometimes we have a priori some information about the solution u, for example, knowledge on the smoothness of the solution. In this case more precise regularization estimates are expected to be obtained. Here we consider the following two cases:

(i) u' exists and belongs to $L_p(0,\infty)$, $1 \le p \le \infty$
(ii) u is a Hölder function, $u \in H_p^\lambda(0,\infty)$ [5] with the norm

$$\|u\|_{H_p^\lambda} = \|u\|_p + \sup_{\delta>0} \delta^{-\lambda} \left\{ \int_0^\infty |u(x+\delta) - u(x)|^p dx \right\}^{1/p}. \tag{13}$$

Consider the first case (i). Suppose that $u' \in L_p(0,\infty)$, $\alpha > 1$. We have

$$\|u - h^{-\alpha}(\Delta_h^\alpha f)\|_p \le \|I_h\|_p + \|I\!I_h\|_p \qquad \text{where } \|I\!I_h\|_p \le 2^{\alpha+1} h^{-\alpha}\varepsilon.$$

Suppose now that $1 \le p < \infty$. Almost everywhere in $\mathbb{R}$ we have

$$\begin{aligned} |u(x+h\tau) - u(x)| &\le \int_x^{x+h\tau} |u'(t)|\, dt \\ &\le \left\{ \int_x^{x+h\tau} |u'(t)|^p dt \right\}^{1/p} \left\{ \int_x^{x+h\tau} dt \right\}^{(p-1)/p} \\ &\le (h\tau)^{1-1/p} \left\{ \int_x^{x+h\tau} |u'(t)|^p dt \right\}^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|I_h\|_p &\le h^{1-1/p} \left\{ \int_0^\infty \left\{ \int_0^\infty \tau^{1-1/p} |p_\alpha(\tau)| \left\{ \int_x^{x+h\tau} |u'(t)|^p dt \right\}^{1/p} d\tau \right\}^p dx \right\}^{1/p} \\ &\le h^{1-1/p} \int_0^\infty \tau^{1-1/p} |p_\alpha(\tau)| \left\{ \int_0^\infty dx \int_x^{x+h\tau} |u'(t)|^p dt \right\}^{1/p} d\tau \\ &= h^{1-1/p} \int_0^\infty \tau^{1-1/p} |p_\alpha(\tau)| \left\{ \int_0^\infty |u'(t)|^p dt \int_{t-h\tau}^{t} dx \right\}^{1/p} d\tau \\ &= h \int_0^\infty \tau |p_\alpha(\tau)| \left\{ \int_0^\infty |u'(t)|^p dt \right\}^{1/p} d\tau. \end{aligned}$$

Here we have applied the Minkovski inequality and the Fubini Theorem. Since $p_\alpha(\tau) = O(\tau^{-\alpha-1})$ as $\tau \to \infty$ [1: p. 128], we have $\|\tau p_\alpha(\tau)\|_1 \le C < \infty$ when $\alpha > 1$. Consequently,

$$\|I_h\|_p \le Ch\|u'\|_p.$$

Let now $p = \infty$. Then

$$\|I_h\|_\infty \le \int_0^\infty |p_\alpha(\tau)| \int_x^{x+h\tau} |u'(t)|\, dt d\tau \le h\|u'\|_\infty \int_0^\infty \tau|p_\alpha(\tau)|\, d\tau.$$

Therefore

$$\|I_h\|_p \le Ch\|u'\|_p \qquad (1 \le p \le \infty).$$

Consequently,

$$\|u - h^{-\alpha}(\Delta_h^\alpha f)\|_p \le 2^{\alpha+1}h^{-\alpha}\varepsilon + Ch\|u'\|_p.$$

Combining with (11) we obtain

$$\|u - u_\varepsilon\|_p \le 2^{\alpha+2}h^{-\alpha}\varepsilon + Ch\|u'\|_p + Ck^{-\alpha}\|x^\alpha u\|_p.$$

Choosing $h = \varepsilon^{1/(\alpha+1)}$ and $k = [\varepsilon^{-1/(\alpha(\alpha+1))}]$, where $[y]$ is the greatest integer number less or equal y, we get

$$\|u - u_\varepsilon\|_p \le C\varepsilon^{1/(\alpha+1)}(1 + \|x^\alpha u\|_p + \|u'\|_p). \tag{14}$$

Thus we have proved the following

Theorem 3. *Let u' exist and belong to $L_p(0,\infty), \alpha > 1$. Then the regularized solution (10), where $h = \varepsilon^{1/(\alpha+1)}$ and $k = [\varepsilon^{-1/(\alpha(\alpha+1))}]$, satisfies the error estimate (14).*

Remark. Notice that $kh = O(\varepsilon^{(\alpha-1)/(\alpha(\alpha+1))}) \to 0$ as $\varepsilon \to 0$, so the regularization scheme is local.

Suppose now that $u \in H_p^\lambda(0,\infty)$ [5: p. 200], $0 < \lambda \le 1$ and $\lambda < \alpha$. Then we have

$$\begin{aligned}\|I_h\|_p &\le \left\{ \int_0^\infty \left| \int_0^\infty \left|p_\alpha(\tau)\Big(u(x+h\tau) - u(x)\Big)\right| d\tau \right|^p dx \right\}^{1/p} \\ &\le \int_0^\infty |p_\alpha(\tau)| \left\{ \int_0^\infty \Big|u(x+h\tau) - u(x)\Big|^p dx \right\}^{1/p} d\tau \\ &\le 2\|u\|_p \int_{\delta/h}^\infty |p_\alpha(\tau)|\, d\tau + h^\lambda \|u\|_{H_p^\lambda} \int_0^{\delta/h} \tau^\lambda |p_\alpha(\tau)|\, d\tau.\end{aligned}$$

Since $\tau^\lambda p_\alpha(\tau) = O(\tau^{\lambda-\alpha-1})$ as $\tau \to \infty$, we have $\int_0^{\delta/h} \tau^\lambda |p_\alpha(\tau)| d\tau \le C < \infty$. Consequently, $\|I_h\|_p \le 2C\|u\|_p(h/\delta)^\alpha + Ch^\lambda \|u\|_{H_p^\lambda}$. Therefore,

$$\|u - u_\varepsilon\|_p \le 2^{\alpha+2} h^{-\alpha}\varepsilon + 2C(h/\delta)^\alpha \|u\|_p + Ch^\lambda \|u\|_{H_p^\lambda} + Ck^{-\alpha}\|x^\alpha u\|_p.$$

Choosing $h = \varepsilon^{1/(\alpha+\lambda)}, \delta = \varepsilon^{(\alpha-\lambda)/(\alpha(\alpha+\lambda))}, k = [\varepsilon^{-\lambda/(\alpha(\alpha+\lambda))}]$ and noting that $\|u\|_p \le \|u\|_{H_p^\lambda}$, we get

$$\|u_\varepsilon - u\|_p \le C\varepsilon^{\lambda/(\alpha+\lambda)}(1 + \|x^\alpha u\|_p + \|u\|_{H_p^\lambda}) \qquad (1 \le p \le \infty). \tag{15}$$

Hence we have proved the following

Theorem 4. *Let $u \in H_p^\lambda(0,\infty)$ and $0 < \lambda \le 1$, $\lambda < \alpha$. Then the regularized solution (10), where $h = \varepsilon^{1/(\alpha+\lambda)}, \delta = \varepsilon^{(\alpha-\lambda)/(\alpha(\alpha+\lambda))}$ and $k = [\varepsilon^{-\lambda/(\alpha(\alpha+\lambda))}]$, satisfies the error estimate (15).*

Remark. If it is known a priori that $\|x^\alpha u\|_p + \|u'\|_p \le E$ or $\|x^\alpha u\|_p + \|u\|_{H_p^\lambda} \le E$, respectively, with a positive constant E, then we can by appropriate choice of h and k obtain the estimates

$$\|u_\varepsilon - u\|_p \le C\varepsilon^{1/(\alpha+1)} E^{\alpha/(\alpha+1)} \qquad \text{instead of } (14),$$
$$\|u_\varepsilon - u\|_p \le C\varepsilon^{\lambda/(\alpha+\lambda)} E^{\alpha/(\alpha+\lambda)} \qquad \text{instead of } (15).$$

5. Optimal order of approximation

We shall show that the exponent $1/(\alpha+1)$ of ε in the error estimate (14) is the best possible provided that $u' \in L_p(0,\infty)$. Without restriction of generality one may assume that $\|x^\alpha u\|_p$, $\|u'\|_p \le 1$.

Taking $\beta = 1/p - 1$ in the example considered in the Introduction we have, with $u_n(x) = p^{1/p} n^{1/p-1} e^{-nx}$,

$$\|u_n\|_p = n^{-1}, \qquad \|u_n'\|_p = 1, \qquad \|I_-^\alpha u_n\|_p = n^{-\alpha-1}.$$

Suppose that the exponent $1/(\alpha+1)$ of ε is not the best possible. Since (14) is still valid if we put $\|f - I_-^\alpha u\|_p$ instead of ε, we see that there exists a regularizing operator $A : f \to u_f$ and $\gamma > 1/(\alpha+1)$, such that

$$\|u_f - u\|_p \le C\|f - I_-^\alpha u\|_p^\gamma$$

where the constant C does not depend on f and u. Taking $u = \pm u_n$ and $f = 0$, we see that u_n when n is big enough satisfy the conditions imposed on u. Hence,

$$\|u_f \pm u_n\|_p \le C\|I_-^\alpha u_n\|_p^\gamma = Cn^{-\gamma(\alpha+1)}.$$

But

$$n^{-1} = \|u_n\|_p \le \tfrac{1}{2}\left(\|u_f + u_n\|_p + \|u_f - u_n\|_p\right).$$

Therefore $n^{-1} \le Cn^{-\gamma(\alpha+1)}$, that cannot hold for all integers n if $\gamma > 1/(\alpha+1)$. Consequently, $\gamma \le 1/(\alpha+1)$, that means the exponent $1/(\alpha+1)$ of ε is the best possible.

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