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Analyticity of some Kernels

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Abstract. The object here is to prove that the series $\sum_{n\in\mathbb{N}} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is analytic if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is such that there exist constants $\rho > 2$ and $C > 0$ with $|\lambda_n| \le \frac{C}{\rho^n}$ for all $n \in \mathbb{N}$. This is important in the study of eigenvalues of integral operators in relation to their kernels and is a partial affirmative response to a question raised by A.L. Brown.

Keywords: *Hermite functions, Hermite polynomials, analytic functions*

AMS subject classification: 26E05, 26B99, 45C05, 47B10

1. Introduction

The object of this paper is to prove a theorem, which is a partial affirmative answer to the question asked A.L. Brown

whether $\sum_{n\in\mathbb{N}} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is analytic for all sequences $(\lambda_n)_{n\in\mathbb{N}} \in \Gamma^-$

where Γ^- is some set of sequences defined below. This question is important in the study of eigenvalues of integral operators in relation to their kernels.

We prove below that the series $\sum_{n\in\mathbb{N}} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is analytic for some sequences $(\lambda_n)_{n\in\mathbb{N}}$ forming a subclass of the sequence space Γ^-. Namely, the following theorem will be true.

Theorem. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence such that, for some constants $\rho > 2$ and $C > 0$, $|\lambda_n| \le \frac{C}{\rho^n}$ for all $n \in \mathbb{N}$. Then the series $\sum_{n\in\mathbb{N}} \lambda_n \mathcal{H}_n(x)\mathcal{H}_n(y)$ is an analytic function in $\mathbb{R}^2$ where $\mathcal{H}_n$ are the Hermite functions as defined in* [2 : *p.* 261].

Before we proceed to prove this theorem we give some preliminaries.

2. Preliminaries

The following definitions of Hermite functions and Hermite polynomials are taken from [2: p. 261]. The m^{th} *Hermite polynomial* (m a non-negative integer) is defined as

$$H_m(x) = \frac{1}{c_m} e^{2\pi x^2} \frac{d^m}{dx^m} e^{-2\pi x^2} \qquad \text{where} \qquad c_m = (-1)^m \sqrt{m!}\, 2^{m-1/4} \pi^{m/2}.$$

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The m^{th} *Hermite function* (m a non-negative integer) is defined as

$$\mathcal{H}_m(x) = H_m(x)e^{-\pi x^2}.$$

The following recurrence formula is well known and can be easily proved using the definitions of Hermite functions:

$$\sqrt{m}\mathcal{H}_m = 2\sqrt{\pi}x\mathcal{H}_{m-1} - \sqrt{(m-1)}\mathcal{H}_{m-2} \qquad \text{for all } m \geq 2.$$

Consequently, we have for Hermite polynomials the recurrence formula

$$\sqrt{m}H_m = 2\sqrt{\pi}xH_{m-1} - \sqrt{(m-1)}H_{m-2}. \tag{1}$$

Definition 1. A sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is said to be *rapidly decreasing* if for all $k \in \mathbb{N}$ the inequality

$$\sup_{n\in\mathbb{N}} n^k|\mu_n| < +\infty$$

is fulfilled.

It is proved in [2: pp. 261 - 262] that, for a sequence $(a_n)_{n\in\mathbb{N}} \in l^2$, the series $\sum_{n=1}^{m} a_n\mathcal{H}_n(x)$ converges in L^2 as $m \to \infty$ to a function of $\mathcal{S}(\mathbb{R})$, the Schwartz class of functions, if and only if $(a_n)_{n\in\mathbb{N}}$ is rapidly decreasing. Moreover, using Sobolev's lemma it can be proved that if the sequence $(a_n)_{n\in\mathbb{N}}$ is rapidly decreasing, then the series $\sum_{n=1}^{m} a_n\mathcal{H}_n(x)$ converges in $\mathcal{S}(\mathbb{R})$ as $m \to \infty$.

In the same way, it can be easily seen that if the sequence $(a_n)_{n\in\mathbb{N}}$ is rapidly decreasing, then the series

$$\sum_{n=1}^{m} a_n\mathcal{H}_n(x)\mathcal{H}_n(y)$$

converges in $\mathcal{S}(\mathbb{R}^2)$ as $m \to \infty$.

If $|\lambda_n| \leq \frac{C}{\rho^n}$ for some $\rho > 1$, then it is clear that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is rapidly decreasing. Hence, the series $\sum_{n=1}^{m} \lambda_n\mathcal{H}_n(x)\mathcal{H}_n(y)$ converges to a function $g \in \mathcal{S}(\mathbb{R}^2)$ in the topology of $\mathcal{S}(\mathbb{R}^2)$. Hence,

$$D^\alpha g(x,y) = \sum_{n=1}^{\infty} \lambda_n D^{\alpha_1}\mathcal{H}_n(x)D^{\alpha_2}\mathcal{H}_n(y) \tag{2}$$

for all $\alpha = (\alpha_1, \alpha_2)$ and $(x,y) \in \mathbb{R}^2$. To prove that g is analytic, we make use of the following well-known criterion.

Criterion. *A function $f \in C^\infty(\mathbb{R}^2)$ is real analytic if and only if for any $R > 0$ there exists a real number $C_R > 0$ such that the inequality*

$$\sup_{\substack{(x,y)\\ x^2+y^2\leq R^2}} |D^\alpha f(x,y)| \leq C_R^{|\alpha|+1}\alpha!$$

foe all $\alpha = (\alpha_1, \alpha_2)$ with α_1, α_2 non-negative integers is true.

For **Proof** see, for example, [1: Proposition 1.1.14]

To apply this criterion, we are therefore forced to estimate $D^\alpha g(x,y)$ in a ball $B(0,R)$. Because of (2) we in turn are forced to estimate $D^k\mathcal{H}_n(x)$ for any $k,n \in \mathbb{N}$ and all x with $|x| \le R$. Since $\mathcal{H}_n(x) = e^{-\pi x^2} H_n(x)$,

$$D^k\mathcal{H}_n(x) = \sum_{r=0}^{k} \binom{k}{r} D^r H_n(x) D^{k-r} e^{-\pi x^2}. \tag{3}$$

Thus, to estimate $D^k\mathcal{H}_n(x)$, we need estimates for $D^r H_n(x)$ and $D^l e^{-\pi x^2}$ for any integers r and l.

3. Necessary estimates

Estimating $D^r H_n$. Here we shall obtain the necessary estimates for $D^r H_n$. Remember that $H_k(x) = e^{\pi x^2}\mathcal{H}_k(x)$. Therefore,

$$\begin{aligned} H_k'(x) &= 2\pi x e^{\pi x^2}\mathcal{H}_k(x) + e^{\pi x^2}\mathcal{H}_k'(x) \\ &= e^{\pi x^2}\left(\mathcal{H}_k'(x) + 2\pi x\mathcal{H}_k(x)\right) \\ &= e^{\pi x^2}\tau_+\mathcal{H}_k(x) \end{aligned}$$

where $\tau_+\varphi = \varphi'(x) + 2\pi x\varphi$ for all $\varphi \in C^1$. It can be easily proved that $\tau_+\mathcal{H}_k = 2\sqrt{\pi k}\,\mathcal{H}_{k-1}$. Hence,

$$H_k'(x) = 2\sqrt{\pi k} H_{k-1}(x) \qquad \text{for all } \ k \ge 1.$$

Thus, by induction, we have for all $p \le k$

$$D^p H_k(x) = \left(\sqrt{\pi}\right)^p \sqrt{k(k-1)\cdots(k-p+1)} H_{k-p}(x). \tag{4}$$

Since H_k is a k^{th} degree polynomial, $D^p H_k(x) = 0$ for all $p > k$.

Estimating $D^r\, e^{-\pi x^2}$. By the definition of the r^{th} Hermite polynomial, $D^r\, e^{-2\pi x^2} = c_r H_r(x) e^{-2\pi x^2}$. Let $y = \frac{x}{\sqrt{2}}$ and $g(x) = e^{-2\pi x^2}$. Then, $g(y) = e^{-2\pi y^2} = e^{-\pi x^2}$. We have therefore the equality

$$\frac{d}{dx} e^{-\pi x^2} = \frac{d}{dx} g(y) m \qquad \text{where} \quad \frac{d}{dx} g(y) = \frac{d\,g(y)}{dy}\frac{dy}{dx} = \frac{1}{\sqrt{2}}\frac{d}{dy} g(y).$$

Therefore, by induction,

$$\begin{aligned} \frac{d^r}{dx^r} g(y) &= \frac{d^r}{dy^r} g(y) \cdot \left(\frac{1}{\sqrt{2}}\right)^r \\ &= c_r H_r(y)\, e^{-2\pi y^2} \left(\frac{1}{\sqrt{2}}\right)^r \\ &= c_r \left(\frac{1}{\sqrt{2}}\right)^r e^{-\pi x^2} H_r\left(\frac{x}{\sqrt{2}}\right). \end{aligned}$$

Thus

$$D^r e^{-\pi x^2} = c_r \left(\frac{1}{\sqrt{2}}\right)^r e^{-\pi x^2} H_r\left(\frac{x}{\sqrt{2}}\right). \tag{5}$$

From the expressions in (2) and (3) we see that to estimate $D^k H_n(x)$ and $D^l e^{-\pi x^2}$ for $|x| \le R$, we need to estimate $H_{n-k}(x)$ and $H_l(x)$ for $|x| \le R$. This is the object of the next proposition. Before we go to that, we give the following definition.

Definition 2. By Γ^- we denote the space of all sequences $(a_n)_{n\in \mathbb{N}}$ such that, for all $R > 0$ and all $\varepsilon, 0 < \varepsilon < 1$, the inequality

$$\sup_{n\in\mathbb{N}} R^{n^{1-\varepsilon}} |a_n| < +\infty$$

is true.

As can be easily seen the inclusion $(a_n)_{n\in\mathbb{N}} \in \Gamma^-$ implies that, for all $R > 0$ and all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} R^{n^{1-\varepsilon}} |a_n| < +\infty.$$

Since $|\lambda_n| \le \frac{C}{\rho^n}$, the inclusion $(\lambda_n)_{n\in\mathbb{N}} \in \Gamma^-$ is true.

Proposition. *Let $0 < \varepsilon < \frac{1}{2}$. Then, for all $R > 1$, there exists a constant $C(R,\varepsilon)$ such that*

$$|H_n(x)| \le C(R,\varepsilon) R^{n^{1-\varepsilon}}$$

for all $n \in \mathbb{N}$ and all x such that $|x| \le R$.

Proof. Let us give the proof by induction on n, using the recurrence formula (1). Let us assume that, for some constant $C(R,\varepsilon)$ and some $n \ge 2$,

$$|H_{n-1}(x)| \le C(R,\varepsilon) R^{(n-1)^{1-\varepsilon}} \qquad \text{and} \qquad |H_{n-2}(x)| \le C(R,\varepsilon) R^{(n-2)^{1-\varepsilon}}$$

for all x with $|x| \le R$. Then we have to prove this for $H_n(x), |x| \le R$. By (1),

$$H_n(x) = \frac{2\sqrt{\pi}x}{\sqrt{n}} H_{n-1}(x) - \frac{\sqrt{n-1}}{\sqrt{n}} H_{n-2}(x).$$

It follows that

$$|H_n(x)| \le \frac{2\sqrt{\pi}R}{\sqrt{n}} C(R,\varepsilon) R^{(n-1)^{1-\varepsilon}} + \sqrt{1-\frac{1}{n}}\, C(R,\varepsilon) R^{(n-2)^{1-\varepsilon}}$$

for all x with $|x| \le R$, i.e., since $R > 1$,

$$|H_n(x)| \le C(R,\varepsilon) R^{(n-1)^{1-\varepsilon}} \left(\frac{2\sqrt{\pi}R}{\sqrt{n}} + \sqrt{1-\frac{1}{n}}\right).$$

Therefore, if the right-hand side can be estimated from above by $C(R,\varepsilon)R^{n^{1-\varepsilon}}$, we are through. Such estimation is true if

$$R^{(n-1)^{1-\varepsilon}}\left(\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\right)\le R^{n^{1-\varepsilon}},$$

i.e. if

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+1-\frac{1}{n}\le R^{n^{1-\varepsilon}-(n-1)^{1-\varepsilon}}.$$

Observe that

$$n^{\varepsilon}-(n-1)^{1-\varepsilon}=n^{1-\varepsilon}-n^{1-\varepsilon}\left(1-\frac{1}{n}\right)^{1-\varepsilon}\quad\text{and}\quad\left(1-\frac{1}{n}\right)^{1-\varepsilon}\le 1-\frac{1-\varepsilon}{n}.$$

This implies the inequality

$$n^{1-\varepsilon}-n^{1-\varepsilon}\left(1-\frac{1}{n}\right)^{1-\varepsilon}\ge n^{1-\varepsilon}-n^{1-\varepsilon}+\frac{1-\varepsilon}{n}n^{1-\varepsilon}=\frac{1-\varepsilon}{n^{\varepsilon}}.$$

Thus $R^{n^{1-\varepsilon}-(n-1)^{1-\varepsilon}}\ge R^{(1-\varepsilon)/n^{\varepsilon}}$. Therefore, if

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\le R^{(1-\varepsilon)/n^{\varepsilon}},$$

then automatically

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\le R^{n^{\varepsilon}-(n-1)^{1-\varepsilon}}.$$

Now

$$R^{(1-\varepsilon)/n^{\varepsilon}}=e^{\log R^{(1-\varepsilon)/n^{\varepsilon}}}=e^{(1-\varepsilon)/n^{\varepsilon}\log R}\ge 1+\frac{1-\varepsilon}{n^{\varepsilon}}\log R.$$

Therefore, if

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\le 1+\frac{1-\varepsilon}{n^{\varepsilon}}\log R,$$

then automatically

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\le R^{(1-\varepsilon)/n^{\varepsilon}}.$$

As $\sqrt{1-\frac{1}{n}}<1$, we will have

$$\frac{2\sqrt{\pi}R}{\sqrt{n}}+\sqrt{1-\frac{1}{n}}\le 1+\frac{1-\varepsilon}{n^{\varepsilon}}\log R\qquad\text{if}\quad\frac{2\sqrt{\pi}R}{\sqrt{n}}\le\frac{1-\varepsilon}{n^{\varepsilon}}\log R,$$

i.e. if

$$\frac{2\sqrt{\pi}R}{(1-\varepsilon)\log R}\le n^{1/2-\varepsilon},\qquad\text{i.e. if}\quad n\ge\left(\frac{2\sqrt{\pi}R}{(1-\varepsilon)\log R}\right)^{1/(1/2-\varepsilon)}.$$

This makes sense as $\varepsilon < 1/2$. Let

$$N(R,\varepsilon) = \left(\frac{2\sqrt{\pi}R}{(1-\varepsilon)\log R}\right)^{1/(1/2-\varepsilon)} + 1.$$

Since H_n is an n^{th} degree polynomial, there exists a constant $C_1(R,\varepsilon) > 0$ such that $|H_n(x)| \le C_1(R,\varepsilon)R^n$ for all x with $|x| \le R$ and all $n \le N(R,\varepsilon)$. This implies that

$$|H_n(x)| \le C_1(R,\varepsilon)R^{n^{1-\epsilon}} \frac{R^n}{R^{n^{1-\epsilon}}}.$$

As $\frac{R^n}{R^{n^{1-\epsilon}}}$ is bounded for all $n \le N(R,\varepsilon)$, the proposition is proved ∎

4. Proof of the theorem

We have to show that

$$\sup_{\substack{(x,y)\\ x^2+y^2\le R^2}} |D^\alpha g(x,y)| \le C_R^{|\alpha|+1}\alpha!$$

for some constant C_R and all α, where $D^\alpha g(x,y)$ is given by (2). By (3), for any $k \in \mathbb{N}$ we have

$$D^k \mathcal{H}_n(x) = D^k\left(e^{-\pi x^2}H_n(x)\right) = \sum_{r=0}^{k}\binom{k}{r} D^r H_n(x) D^{k-r}e^{-\pi x^2}.$$

By (4), we have

$$D^r H_n(x) = \begin{cases} (2\sqrt{\pi})^r\sqrt{n(n-1)\cdots(n-r+1)}H_{n-r}(x) & \text{if } r \le n \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, by the Proposition, we have for all $r, n \in \mathbb{N}$ and $|x| \le R$,

$$\begin{aligned} |D^r H_n(x)| &\le C(R,\varepsilon)(2\sqrt{\pi}^r\sqrt{\binom{n}{r} r!}\, R^{(n-r)^{1-\epsilon}} \\ &\le C(R,\varepsilon)(2\sqrt{\pi})^r\sqrt{\binom{n}{r} r!}\, R^{n^{1-\epsilon}} \\ &\le C(R,\varepsilon)(2\sqrt{\pi})^r\sqrt{r!}\sqrt{2^n}R^{n^{1-\epsilon}} \\ &\le C(R,\varepsilon)(2\sqrt{\pi})^k\sqrt{k!}\sqrt{2^n}R^{n^{1-\epsilon}} \end{aligned}$$

as $r \le k$ and $\binom{n}{r} \le 2^n$. By (5), we have

$$D^{k-r}e^{-\pi x^2} = c_{k-r}\left(\frac{1}{\sqrt{2}}\right)^{k-r} H_{k-r}\left(\frac{x}{\sqrt{2}}\right)e^{-\pi x^2}.$$

Therefore, if $|x| \le R$, then by Proposition

$$\begin{aligned}|D^{k-r}e^{-\pi x^2}| &\le C(R,\varepsilon)c_{k-r}\left|H_{k-r}\left(\frac{x}{\sqrt{2}}\right)\right| \\ &\le C(R,\varepsilon)c_{k-r}R^{(k-r)^{1-\epsilon}} \\ &\le C(R,\varepsilon)\sqrt{(k-r)!}\,2^{k-r-1/4}\pi^{(k-r)/2}R^{k^{1-\epsilon}} \\ &\le C(R,\varepsilon)\sqrt{k!}\,2^k\pi^{k/2}R^k.\end{aligned}$$

Therefore, for $|x| \le R$, we have by (3)

$$\begin{aligned}|D^k\mathcal{H}_n(x)| &\le C^2(R,\varepsilon)\sum_{r=0}^{k}\binom{k}{r}(2\sqrt{\pi})^k\sqrt{k!}\,\sqrt{2^n}R^{n^{1-\epsilon}}\sqrt{k!}\,2^k\pi^{k/2}R^k \\ &\le C^2(R,\varepsilon)2^{2k}(\sqrt{\pi})^k k!\,\pi^{k/2}R^k\sqrt{2^n}R^{n^{1-\epsilon}}\sum_{r=0}^{k}\binom{k}{r} \\ &\le C^2(R,\varepsilon)2^{2k}(\sqrt{\pi})^k k!\,\pi^{k/2}R^k\sqrt{2^n}R^{n^{1-\epsilon}}2^k \\ &= C^2(R,\varepsilon)(8\pi R)^k k!\,2^{n/2}R^{n^{1-\epsilon}}.\end{aligned}$$

Therefore, if $x^2+y^2 \le R^2$, then

$$\begin{aligned}|D^{\alpha_1}\mathcal{H}_n(x)D^{\alpha_2}\mathcal{H}_n(y)| &\le C(R,\varepsilon)(8\pi R)^{\alpha_1}\,\alpha_1!\,2^{n/2}R^{n^{1-\epsilon}}(8\pi R)^{\alpha_2}\alpha_2!\,2^{n/2}R^{n^{1-\epsilon}} \\ &= C(R,\varepsilon)(8\pi R)^{\alpha_1+\alpha_2}\alpha_1!\,\alpha_2!\,2^n R^{2n^{1-\epsilon}}.\end{aligned}$$

Therefore, if $x^2+y^2 \le R^2$, then

$$\begin{aligned}|D^\alpha g(x,y)| &\le \sum_{n=1}^{\infty}|\lambda_n|\,|D^{\alpha_1}\mathcal{H}_n(x)D^{\alpha_2}\mathcal{H}_n(y)| \\ &\le C^2(R,\varepsilon)\left(\sum_{n=1}^{\infty}|\lambda_n|2^nR^{2n^{1-\epsilon}}\right)(8\pi R)^{\alpha_1+\alpha_2}\alpha_1!\,\alpha_2!\end{aligned}$$

Let $p>1$, to be chosen more precisely later. Let q be such that $1/p+1/q=1$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}|\lambda_n|2^nR^{2n^{1-\epsilon}} &= \sum_{n=1}^{\infty}|\lambda_n|^{1/p}\,2^n|\lambda_n|^{1/q}R^{2n^{1-\epsilon}} \\ &\le \left(\sum_{n=1}^{\infty}|\lambda_n|\,2^{np}\right)^{1/p}\left(\sum_{n=1}^{\infty}|\lambda_n|\left(R^{2n^{1-\epsilon}}\right)^q\right)^{1/q} \\ &\le \left(\sum_{n=1}^{\infty}\frac{C}{\rho^n}2^{np}\right)^{1/p}\left(\sum_{n=1}^{\infty}|\lambda_n|R^{2qn^{1-\epsilon}}\right)^{1/q} \\ &= C^{1/p}\left(\sum_{n=1}^{\infty}\left(\frac{2^p}{\rho}\right)^n\right)^{1/p}\left(\sum_{n=1}^{\infty}|\lambda_n|R^{2qn^{1-\epsilon}}\right)^{1/q}.\end{aligned}$$

Since $\rho > 2$, the number $p > 1$ can be chosen such that $\frac{2^p}{\rho} < 1$. Since $(\lambda_n)_{n\in\mathbb{N}} \in \Gamma^-$, we have $\sum_{n\in\mathbb{N}} |\lambda_n| R^{2qn^{1-\epsilon}} < +\infty$. Hence $\sum_{n\in\mathbb{N}} |\lambda_n|\, s^n R^{2n^{1-\epsilon}}$ ia a real number depending only on R. Thus, defining

$$S = S(R,\varepsilon) = C^2(R,\varepsilon) \sum_{n=1}^{\infty} |\lambda_n| 2^n R^{2n^{1-\epsilon}}$$

we have

$$\begin{aligned} |D^\alpha g(x,y)| &\le S(8\pi R)^{\alpha_1+\alpha_2} \alpha_1!\, \alpha_2! \\ &\le (S+1)^{\alpha_1+\alpha_2+1} (8\pi R)^{\alpha_1+\alpha_2+1} \alpha_1!\, \alpha_2! \\ &= (8\pi R(S+1))^{\alpha_1+\alpha_2+1}\, \alpha_1!\, \alpha_2! \end{aligned}$$

Hence the proposition is proved ∎

Remark. The above method of proof does not tell anything about the analyticity or otherwise of the series

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{H}_n(x) \mathcal{H}_n(y)$$

when the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is such that $|\lambda_n| \le \frac{C}{2^n}$ $(n \in \mathbb{N})$. When $\lambda_n = \frac{1}{2^n}$ $(n \in \mathbb{N})$, the analyticity of that series follows from Mehler's formula (see [3: Lemma 1.1.1]).

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Added in proof: Subsequently the author has settled the question of A. L. Brown in the affirmative by proving a more general result. This possibly will be the object of a forthcoming paper in this journal.

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