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On the Existence of Closed Orbits for a Differential System

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Abstract. Some criteria which verify the existence or non-existence of closed orbits for a non-linear differential system are given.

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1. Introduction

In this paper, we consider the existence of closed orbits for the differential system

$$\begin{aligned} x'(t) &= y \\ y'(t) &= x + x^2 - (\varepsilon_1 + \varepsilon_2 x)y \end{aligned} \tag{1.1}$$

where ε_1 and ε_2 are real constants. It is easy to see that this system has two singular points: $O(0,0)$ (saddle point) and $A(-1,0)$ (focus or node). In [1: p. 215] the following result has been obtained:

1. *If $\varepsilon_1\varepsilon_2 \le 0$, then there are no closed orbits in the system* (1.1).

2. *If $\varepsilon_1\varepsilon_2 > 0$ and $\varepsilon_1/\varepsilon_2 \ge 3/2$, then there are again no closed orbits in the system* (1.1).

3. *If $\varepsilon_1\varepsilon_2 > 0$ and $\varepsilon_1/\varepsilon_2 < 3/2$, then the existence of closed orbits of the system* (1.1) *remains an open question.*

In [2], the following theorem has been given.

Theorem [2: p. 94]: *If $\varepsilon_1 = \varepsilon_2$, then there exists a neighbourhood of the point* $(-1,0)$, *such that there are infinitely many limit cycles within this neighbourhood with* $(-1,0)$ *in their interiors.*

The system (1.1) is a special form of quadratic differential systems. A considerable number of papers have been written in connection with limit cycles of quadratic systems (see, e.c., [3]).

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2. The main results

Below f' for a function $f = f(t)$ ($t \in \mathbb{R}$) always mean the derivation for t. In this paper, we shall prove the following theorems.

Theorem 2.1: *Suppose $\varepsilon_1\varepsilon_2 > 0$. If $\varepsilon_1 \le \varepsilon_2 - 2$ or $\varepsilon_1 \ge \varepsilon_2 + 2$, then there are no closed orbits in the system* (1.1).

Proof: Setting

$$F(x,y) = y - k(x+1) = 0 \tag{2.1}$$

we have

$$\begin{aligned} F'|_{(1\cdot1)} = y' - kx' &= x + x^2 - (\varepsilon_1 + \varepsilon_2 x)y - ky \\ &= (1 - k\varepsilon_2)x^2 + (1 - k\varepsilon_1 - k\varepsilon_2 - k^2)x - k\varepsilon_1 - k^2. \end{aligned} \tag{2.2}$$

Consider the equation

$$k^2 - k(\varepsilon_2 - \varepsilon_1) + 1 = 0. \tag{2.3}$$

Clearly, if

$$(\varepsilon_2 - \varepsilon_1)^2 - 4 \ge 0, \tag{2.4}$$

then there is one real root k_1 which is a solution of the equation (2.3). When $k = k_1$, from (2.2) we obtain

$$F'|_{(1\cdot1)} = (1 - \varepsilon_2 k_1)(x+1)^2. \tag{2.5}$$

It is easy to see that $1 - \varepsilon_2 k_1 \neq 0$ when $\varepsilon_1\varepsilon_2 > 0$. Therefore the straight line $y = k_1(x+1)$ is a line without contact, i.e., the trajectory passing through any point P on the line (except for the point A) must cross the line in the same sense. Thus there are no closed orbits around the popint A in the system (1.1) ∎

Theorem 2.2: *Suppose $\varepsilon_1\varepsilon_2 > 0$. If $\varepsilon_1 = \varepsilon_2$, then there are no closed orbits and singular closed orbits in the system* (1.1).

Proof: When $\varepsilon_1 = \varepsilon_2 \neq 0$, the system (1.1) can be written as

$$\begin{aligned} x' &= y \equiv P(x,y) \\ y' &= x + x^2 - \varepsilon_2(x+1)y \equiv Q(x,y). \end{aligned} \tag{1.1$'$}$$

Consider the function which is similar to that in [3: Section 12]

$$B(x,y) = (y - \alpha x)^{-\alpha\varepsilon_2} \exp(-\alpha\varepsilon_2(x+1)) \tag{2.6}$$

where $\alpha > 0$ is a constant that will be defined below. We have

$$\begin{aligned} \frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) &= (y - \alpha x)^{-\alpha\varepsilon_2 - 1} \\ &\quad \times \Big(-\alpha\varepsilon_2 y^2 + (\alpha^2\varepsilon_2 + \alpha\varepsilon_2^2 - \varepsilon_2)y(x+1)\Big) \\ &\quad \times \exp(-\alpha\varepsilon_2(x+1)). \end{aligned} \tag{2.7}$$

Set

$$\alpha^2 + \varepsilon_2\alpha - 1 = 0. \tag{2.8}$$

Let $\alpha_1 > 0$ be a root of the equation (2.8). Then, from (2.7) we obtain

$$\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = (y - \alpha_1 x)^{-\alpha_1\varepsilon_2 - 1}(-\alpha_1\varepsilon_2 y^2)\exp(-\alpha_1\varepsilon_2(x+1)). \tag{2.9}$$

Obviously, the straight line

$$\Phi(x,y) \equiv y - \alpha_1 x = 0 \tag{2.10}$$

passes through two points $(0,0)$ and $(-1,-\alpha_1)$ in the (x,y)-plane, and we have

$$\Phi'|_{(1.1)} = y' - \alpha_1 x' = x + x^2 - \varepsilon_2(x+1)y - \alpha_1 y = (1 - \alpha_1\varepsilon_2)x^2. \tag{2.11}$$

Therefore the straight line $y = \alpha_1 x$ is a line without contact, i.e., the trajectory passing through any point P on the line (except for the point O) must cross the line in the same sense. Thus each closed orbit can not intersect the line $\Phi = 0$. Moreover, the function $\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}$ does not change sign in the region $y - \alpha_1 x > 0$, and it does not vanish identically in any subregion of this region. By Dulac's theorem (see [1: p. 205]) it follows that there are no closed orbits and singular closed orbits in the region $y - \alpha_1 x > 0$. Note that the singular point $A(-1,0)$ lies in the region $y - \alpha_1 x > 0$, thus there are no closed orbits of the system (1.1) in the (x,y)-plane ∎

Remark: A singular closed orbit means a simple closed curve which is the union of alternating non-closed whole orbits and singular points (see [1: p. 205]).

Theorem 2.3: *Suppose $\varepsilon_1\varepsilon_2 > 0$. If $\varepsilon_2(\varepsilon_1 - \varepsilon_2) > 0$, then there are no closed orbits in the system* (1.1).

Proof: For convenience, we set $x = x_1 - 1$ and $y = y_1$. By (1.1), we get the system

$$\begin{aligned} x_1' &= y_1 \\ y_1' &= -x_1 + (\varepsilon_2 - \varepsilon_1)y_1 + x_1^2 - \varepsilon_2 x_1 y_1. \end{aligned} \tag{2.12}$$

Obviously, the system (2.12) in the (x_1, y_1)-plane has two singular points: $O_1(0,0)$ (focus or node) and $A_1(1,0)$ (saddle point). When $\varepsilon_1 = \varepsilon_2$, then the system (2.12) can be written as

$$\begin{aligned} x_1' &= y_1 \\ y_1' &= -x_1 + x_1^2 - \varepsilon_2 x_1 y_1. \end{aligned} \tag{2.13}$$

At first we prove that the singular point O_1 is a focus of the system (2.13). We use a classical method (see [4: p. 180]). Set $x_1 = r\cos\theta$ and $y_1 = r\sin\theta$. Then (2.13) yields

$$\begin{aligned} r' &= r^2\sin\theta\cdot(\cos^2\theta - \varepsilon_2\sin\theta\cos\theta) \\ \theta' &= -1 + r\cos\theta\cdot(\cos^2\theta - \varepsilon_2\sin\theta\cos\theta). \end{aligned}$$

Hence we obtain by division the relation

$$\frac{dr}{d\theta} = \frac{r^2\sin\theta\cdot(\cos^2\theta - \varepsilon_2\sin\theta\cos\theta)}{-1 + r\cos\theta\cdot(\cos^2\theta - \varepsilon_2\sin\theta\cos\theta)} \equiv R(r,\theta) \tag{2.14}$$

and

$$R(r,\theta) = r^2 R_2(\theta) + r^3 R_3(\theta) + \ldots + r^n R_n(\theta) + \ldots$$

where the series converges for r sufficiently small and any θ. In fact, we have

$$R_n(\theta) = R_{n-1}(\theta)\cos\theta \cdot (\cos^2\theta - \varepsilon_2 \sin\theta\cos\theta) \qquad (n \geq 3;\, R_1 = -\sin\theta).$$

Let

$$r = f(\theta, r_0) = u_1(\theta) r_0 + u_2(\theta) r_0^2 + \ldots \tag{2.15}$$

be the solution of relation (2.14) satisfying $f(\theta, r_0) = r_0$. Then we must have

$$u_1(0) = 1 \qquad \text{and} \qquad u_n(0) = 0 \quad (n \geq 2). \tag{2.16}$$

By substituting (2.15) into (2.14) and identifying equal powers of r, we thus obtain a system

$$\begin{aligned} u_1' &= 0 \\ u_2' &= R_2(\theta) u_1^2 = R_2(\theta) \\ u_3' &= 2u_1 u_2 R_2(\theta) + u_1 R_3(\theta) \\ &\ldots \end{aligned} \tag{2.17}$$

By (2.17) and (2.16) we get

$$\begin{aligned} u_1(\theta) &= 1 \\ u_2(\theta) &= \frac{1}{3}(-1 + \cos^3\theta + \varepsilon_2 \sin^3\theta) \\ u_3(\theta) &= \frac{5}{18}(\cos^6\theta - 1) + \frac{1}{9}\varepsilon_2^2 \sin^6\theta + \frac{2}{9}(\cos^3\theta - 1) - \frac{2}{9}\varepsilon_2 \sin^3\theta \\ &\quad + \frac{1}{16}\varepsilon_2^2(\cos 2\theta - 1) - \frac{1}{48}\varepsilon_2^2(\cos^3 2\theta - 1) \\ &\quad - \frac{1}{3}\varepsilon_2\left(\theta + \frac{1}{2}\sin 2\theta\right) + \varepsilon_2\left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right) \\ &\quad - \frac{10}{3}\varepsilon_2\left(\frac{5}{16}\theta + \frac{15}{64}\sin 2\theta + \frac{3}{64}\sin 4\theta + \frac{1}{192}\sin 6\theta\right) \end{aligned}$$

hence

$$u_1(2\pi) = 1, \qquad u_2(2\pi) = 0, \qquad u_3(2\pi) = \frac{\pi}{4}\varepsilon_2. \tag{2.18}$$

Therefore, from (2.15) we obtain

$$r(2\pi) - r_0 = \frac{\pi}{4}\varepsilon_2 r_0^3 (1 + \mu_1 r_0 + \mu_2 r_0^2 + \ldots). \tag{2.19}$$

By this and the fact that $\theta' < 0$ for r sufficiently small and any θ, we know that when $\varepsilon_2 > 0$, then the singular point O_1 is a stable focus, and when $\varepsilon_2 < 0$, then O_1 is an unstable focus. From (2.12) we obtain by division the relation

$$\frac{dy_1}{dx_1} = \frac{-x_1 + x_1^2 - \varepsilon_2 x_1 y_1}{y_1} + (\varepsilon_2 - \varepsilon_1),$$

i.e.

$$\left.\frac{dy_1}{dx_1}\right|_{(2.12)} = \left.\frac{dy_1}{dx_1}\right|_{(2.13)} + (\varepsilon_2 - \varepsilon_1) \tag{2.20}$$

for $y_1 \neq 0$.

At first, we assume that $\varepsilon_2 < 0$ and $\varepsilon_2 - \varepsilon_1 > 0$. Then O_1 is an unstable focus of the system (2.13) and O_1 is also an unstable focus of the system (2.12) (by Theorem 2.1 we can consider the case $(\varepsilon_2 - \varepsilon_1)^2 - 4 < 0$ only). Moreover, it follows from Theorem 2.2 that the system (2.13) has no closed orbits and singular closed orbits.

We need to prove that the system (2.12) has no closed orbits. The following proof proceeds by reduction to absurdity. Suppose the system (2.12) has one closed orbit Γ. Let G be the region bounded by Γ such that $O_1 \in G$. Let the closed orbit Γ intersect the half axes $y_1 > 0, x_1 > 0, y_1 < 0$ and $x_1 < 0$ at M, N, P and Q, respectively. Clearly, the point N must lie between O_1 and A_1. Let $\gamma^+(M)$ be the positive semi-trajectory of the system (2.13), originating from M. It follows from(2.20) that $\gamma^+(M)$ must lie in G for $t > 0$ sufficiently small. We claim that $\gamma^+(M)$ lies in G for all $t > 0$. If not, there is $t = t_1 > 0$ such that $\gamma^+(M)$ meets Γ at N or Q for $t = t_1$ (since the relation (2.20) implies that $\gamma^+(M)$ can not meet Γ at other points for $t > 0$). Let $\gamma^+(M)$ meet Γ at N. Then, in the closed region $H = \{(x_1, y_1) \in \mathbb{R}^2 : x_1, y_1 \geq 0\}$, the curvilinear figure MNM made up of the trajectory arc MN of Γ, and the trajectory arc MN of $\gamma^+(M)$ encloses a region, denoted by K ($\subset G$). The positive semi-trajectory of the system (2.13) originating from any point on the open trajectory arc MN of Γ must enter K for $t > 0$ and stays in K for all $t > 0$. By the Poincaré-Bendixson theory of planar systems, it follows that there is at least one critical point in K. This is impossible because the system (2.13) has the singular points O_1 and A_1 only. Similarly, we can prove that $\gamma^+(M)$ cannot meet Γ at Q. Moreover, it follows from Theorem 2.2 that the system (2.13) has no closed orbits. Thus $\gamma^+(M)$ cannot return to M for $t > 0$, and must tend to the singular point O_1 as $t \to +\infty$. But this contradicts the fact that O_1 is an unstable focus of the system (2.13) when $\varepsilon_2 < 0$. Therefore, the system (2.12) has no closed orbits.

In the case that $\varepsilon_2 > 0$ and $\varepsilon_2 - \varepsilon_1 < 0$, the proof is similar. Instead of $\gamma^+(M)$, we consider the negative semi-trajectory $\gamma^-(M)$ of the system (2.13) originating from M, and we can prove that $\gamma^-(M)$ stays in G for all $t < 0$ (note that (2.20) implies $\frac{dy_1}{dx_1}|_{(2.12)} < \frac{dy_1}{dx_1}|_{(2.13)}$ for $y_1 \neq 0$ and $\varepsilon_2 - \varepsilon_1 < 0$). This contradicts the fact that O_1 is a stable focus of the system (2.13) when $\varepsilon_2 > 0$. Therefore, the system (2.12) also has no closed orbits in this case ∎

Theorem 2.4: *Suppose $\varepsilon_1\varepsilon_2 > 0$. If $\varepsilon_2(\varepsilon_1 - \varepsilon_2) < 0$ and $|\varepsilon_1 - \varepsilon_2|$ is sufficiently small, then there are closed orbits of the system* (1.1) *in some neighbourhood of the singular point O.*

Proof: We set $\varepsilon_2 - \varepsilon_1 = \lambda$ and write the system (2.12) in the form

$$\begin{aligned} x_1' &= y_1 \\ y_1' &= -x_1 + \lambda y_1 + x_1^2 - (\lambda + \varepsilon_1)x_1 y_1. \end{aligned} \tag{2.12$_\lambda$}$$

If $\lambda = 0$, we obtain from $(2.12)_\lambda$ the system (2.13) (note $\varepsilon_1 = \varepsilon_2$).

First, we assume that $\varepsilon_2 > 0$. Then, as proved above, the singular point O_1 is a stable focus of the system (2.13). Moreover, it is easy to see that if $\lambda > 0$ and sufficiently small, then O_1 is an unstable focus of the system $(2.12)_\lambda$. Therefore (see [5: Section 25.3]), there are a sufficiently small neighbourhood V of the point O_1 and a small $\delta_0 > 0$ such that any system $(2.12)_\lambda$ for which $\lambda < \delta_0$ has a unique closed orbit in V.

Similarly, if $\varepsilon_2 < 0$, then O_1 is an unstable focus of the system (2.13). And, when $\lambda < 0$ and sufficiently small, then O_1 is a stable focus of the system $(2.12)_\lambda$. Therefore, there are a sufficiently small neighbourhood W of O_1 and a small $\delta_1 > 0$ such that any system $(2.12)_\lambda$ for which $|\lambda| < \delta_1$ with $\lambda < 0$ has a unique closed orbit in W ∎

Remark that also the system $(2.12)_\lambda$ is a special form of quadratic differential systems.

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