# On some Nonlinear Equations
# Generated by Fueter Type Operators

## V. G. Kravchenko and V. V. Kravchenko

**Abstract.** Let $I\!H(C)$ be the set of complex quaternions, $i_k$ the standart basic quaternions and, for $q \in I\!H(C)$, denote $\vec{q} = \sum_{k=1}^{3} q_k i_k$ and $\bar{q} = q_0 - \vec{q}$. In the present work some procedure of factorization with reference to the Fueter type equations $(\partial_t - aD)u = 0$ for $u = u(t,x)$ where $D = \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}$ is discussed.

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Let $I\!H(C)$ be the set of complex quaternions and $i_k$ the standart basic quaternions. Let us denote for $q \in I\!H(C)$

$$\vec{q} = \sum_{k=1}^{3} q_k i_k, \qquad \mathrm{Sc}\,(q) = q_0, \qquad \bar{q} = q_0 - \vec{q}.$$

In the present work the procedure of factorization (in the sense of [9] and [10: Section 8.3]) with reference to the Fueter type equations

$$(\partial_t - aD)u = 0 \tag{1}$$

is discussed where $u \in C^1(\Omega)$, $\Omega = I \times G$ for some interval $I$ and some domain $G \subset I\!R^3$, $u(t,x) \in I\!H(C)$ for $(t,x) \in \Omega$,

$$D = \sum_{k=1}^{3} i_k \partial_k = \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k} \qquad (x = (x_1, x_2, x_3) \in G)$$

and $a \in C$ is some constant.

Remark that here the factorization is not a decomposition of something into factors but is defined in terms of invariants of classical symmetries of an initial equation. For instance, the result of such factorization of the heat equation is the Burger equation (see [6: Example 2.42]).

As is known (see, for example, [2, 4, 5, 8]) equation (1) is closely related to the Maxwell equations. In this connection we note the following result.

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**Proposition 1:** *Let $\varphi : \Omega \to \boldsymbol{C}, \vec{A} : \Omega \to \boldsymbol{C}^3, f = (i\varphi, \vec{A}) \in C^2(\Omega) \cap \mathrm{Ker}\,(\partial_t - iD)$.*
*Then $\varphi, \vec{A}$ are electromagnetic potentials in the Lorentz gauge, i.e. the equalities*

$$\mathrm{rot}\,\mathrm{rot}\,\vec{A} + \partial_t^2 \vec{A} + \partial_t\,\mathrm{grad}\,\varphi = 0 \tag{2}$$

$$\partial_t\,\mathrm{div}\,\vec{A} + \Delta\varphi = 0 \tag{3}$$

$$\mathrm{div}\,\vec{A} + \partial_t\varphi = 0 \tag{4}$$

*are valid, where $f$ is introduced by Minkowski vector of 4-dimensional world (see [7 : p. 114]) and $\Delta$ is the 3-dimensional Laplace operator.*

**Proof:** The scalar part of the condition $f \in \mathrm{Ker}\,(\partial_t - iD)$ gives the Lorentz gauge condition (4). After applying to the vector part of $f \in \mathrm{Ker}\,(\partial_t - iD)$ the operators div and $\partial_t$ we obtain (2) and (3)  ∎

The above mentioned factorization with respect to subgroups of the symmetry group to the equation (1) allows to point out some systems of nonlinear differential equations, solutions of which can be obtained proceeding from solutions of equation (1). We present here an example connected with the factorization of equation (1) with respect to the extension group $u \to \lambda u, \lambda \in \boldsymbol{C}$ (which obviously is the symmetry subgroup of equation (1)).

Let us consider the following equation for an $I\!H(\boldsymbol{C})$-valued function $v \in C^1(\Omega)$:

$$\partial_t v = aDv + a\vec{h}v - ab^{-1}(\mu, v)v \tag{5}$$

where $\vec{h} = \mathrm{grad}\,\psi : G \to \boldsymbol{C}^3$  $(\psi \in C^2(G))$ and

$$\Delta\psi + \langle\mathrm{grad}\,\psi, \mathrm{grad}\,\psi\rangle = 0. \tag{6}$$

Here $\langle\cdot, \cdot\rangle$ is the scalar product in $\boldsymbol{C}^3$, $b \in \boldsymbol{C}$ is a constant, $(\mu, v) = \sum_{k=0}^3 \mu_k v_k$ with $\mu_k \in \boldsymbol{C}$, and $\mu = \sum_{k=0}^3 \mu_k i_k$ is not zero or a zero divisor in $I\!H(C)$. As it will be shown later equation (5) generalizes the Jackiw-Nohl-Rebbi-'t Hooft equation describing a class of instantons. The condition (6) is satisfied for any function $\psi$ connected with an arbitrary harmonic function $\varphi_0$ by the equality

$$\mathrm{grad}\,\psi = \frac{\mathrm{grad}\,\varphi_0}{\varphi_0}. \tag{7}$$

**Proposition 2:** *For any function $\vec{h} \in C^1(\Omega)$ and $\mu \in I\!H(\boldsymbol{C})$ satisfying the above conditions there exists a function $u \in C^2(\Omega) \cap Ker(\partial_t - aD)$ such that*

$$\vec{h} = \frac{grad\,(\mu, u)}{(\mu, u)}. \tag{8}$$

**Proof:** The function $\varphi_0 = \exp(\psi + c)$ with $c \in \boldsymbol{C}$ is a solution of equation (7) and condition (6) guarantees the inclusion $\varphi_0 \in \mathrm{Ker}\Delta$. It remains to show that for any function $\varphi_0 \in \mathrm{Ker}\Delta$ there exists a function $u \in \mathrm{Ker}\,(\partial_t - aD)$ such that $\varphi_0 = (\mu, u)$. As $(\mu, u) = \mathrm{Sc}\,(u\bar{\mu})$ we can write down

$$u\bar{\mu} = \varphi \tag{9}$$

where $\varphi_1, \varphi_2, \varphi_3$ are arbitrary functions from $\mathrm{Ker}(\partial_t^2 + a^2\Delta)$. As $\mu$ and, consequently, $\bar{\mu}$ are not zero divisors equation (9) is equivalent to $u = \varphi(\bar{\mu})^{-1}$. Hence the condition $u \in \mathrm{Ker}(\partial_t - aD)$ is equivalent to the inclusion $\varphi \in \mathrm{Ker}(\partial_t - aD)$. Thus, the question is if proceeding from one component $\varphi_0 \in \mathrm{Ker}\Delta$ we can construct three others so that $\varphi \in \mathrm{Ker}(\partial_t - aD)$. Reasoning as in [3], we can any solution $\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ of equation $(\partial_t - aD)\varphi = 0$ express by the formulas

$$\varphi_0 = \partial_2 + \partial_3\omega, \quad \varphi_1 = -\partial_2\omega + \partial_3\vartheta, \quad \varphi_2 = a\partial_t\vartheta + \partial_1\omega, \quad \varphi_3 = -\partial_1\vartheta + a\partial_t\omega$$

through two arbitrary functions $\{\vartheta, \omega\} \subset \mathrm{Ker}(\partial_t^2 + a^2\Delta)$. Thus we can choose $\omega \in \mathrm{Ker}(\partial_t^2 + a^2\Delta)$ as desired and $\vartheta$ to satisfy the equality $\partial_2\vartheta = \varphi_0 - \partial_3\omega$ ∎

Hence one succeeded in constructing a family of solutions for equation (5) as the following assertion is true.

**Theorem 1:** Let the function $u = C^2(\Omega) \cap \mathrm{Ker}(\partial_t - aD)$ satisfy condition (8). Then the function $v = bDu/(\mu, u)$ (which is an invariant of the extension group) is a solution of equation (5).

**Proof:** Calculating

$$Dv = -\frac{\mathrm{grad}(\mu, u)}{(\mu, u)}v + \frac{bD^2u}{(\mu, u)} \quad \text{and} \quad \partial_t v = -\frac{a}{b}(\mu, v)v + ab\frac{D^2u}{(\mu, u)}$$

and taking into account (8), we obtain that $v$ satisfies equation (5) ∎

Introducing another invariant of the extension group:

$$w = \frac{bD_r u}{(\mu, u)} \quad \text{where} \quad D_r u = \sum_{k=1}^{3} \partial_k u i_k$$

and restricting ourselves by the special case $(\mu, u) = u_0$ we obtain the following

**Theorem 2:** Let $u \in C^2(\Omega) \cap \mathrm{Ker}(\partial_t - aD)$ and $u_0 \neq 0$. Then the function $g = v + w = bu_0^{-1}(D + D_r)(u)$ belonging to $C^1(\Omega)$ is a solution of equation

$$\partial_t g = aDg - (2b)^{-1}a|g|^2 \tag{10}$$

where $|g|^2 = g\bar{g}$.

**Proof:** Let us consider $Dv = b(D(u_0^{-1})Du + u_0^{-1}D^2u)$. Then taking into account

$$D(u_0^{-1}) = -\frac{1}{2}u_0^{-2}(Du - \overline{D_r u}) = -(2bu_0)^{-1}(v - \bar{w})$$

we obtain

$$Dv = -(2b)^{-1}(v - \bar{w})v + bu_0^{-1}D^2u.$$

Then

$$\partial_t v = b\left\{-\frac{a\mathrm{Sc}(Du)}{bu_0}v + \frac{aD^2u}{u_0}\right\} = -\frac{a}{b}v_0v + abu_0^{-1}D^2u$$

i.e.

$$\partial_t v = aDv - \frac{1}{2}\frac{a}{b}(\bar{v} + \bar{w})v. \tag{11}$$

By analogy

$$\partial_t w = aDw - \frac{1}{2}\frac{a}{b}(\bar{v} + \bar{w})w. \tag{12}$$

Adding (11) to (12) we obtain (10) ∎

**Remark:** If in (10) $a = -1$ and $b = -\frac{1}{2}$, then (10) becomes the self-duality equations when we take for the gauge potential the Jackiw-Nohl-Rebbi-'t Hooft ansatz (see [10: p. 99]).

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