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On the Existence of Solution of a System of Partial Differential Equations

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Abstract. Let $\Pi_\alpha = \{t \mid 0 < \arg t < \alpha\}$ for $\alpha < \pi$ and denote by M the class of m-dimensional vector functions $u = u(x,t)$ of $C^\infty(\mathbb{R}^n \times \bar{\Pi}_\alpha)$ analytic in $t \in \Pi_\alpha$ and having polynomial growth in (x,t). Let $A(\xi)$ $(\xi \in \mathbb{R}^n)$ be a square matrix of order m with polynomial elements. In the paper we define regularity and strictly regularity of the system $\frac{\partial u}{\partial t} = A(D_x)u + f$ and prove its solvability in M for all $f \in M$.

Keywords: *Partial differential equations, regularity, strictly regularity, solvability*

AMS subject classification: 35A99

1. Introduction

In the following let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, Π a complex plane and, for $0 < \alpha < \pi$,

$$\Pi_\alpha = \left\{\lambda \,\middle|\, 0 < \arg\lambda < \alpha\right\} \qquad \text{and} \qquad \Pi_\alpha^* = \left\{\lambda \,\middle|\, \frac{1}{2}\pi \le \arg\lambda \le \frac{3}{2}\pi - \alpha\right\}.$$

Let $A(\xi)$ $(\xi \in \mathbb{R}^n)$ be a square matrix of order m with polynomial elements such that, for all $\xi \in \mathbb{R}^n$, the roots $\lambda_1(\xi), \ldots, \lambda_m(\xi)$ (some of them can coincide) of the characteristic equation $\det(\lambda E_m - A(\xi)) = 0$ satisfy the conditions

$$\begin{aligned} \lambda_1(\xi), \ldots, \lambda_r(\xi) &\in \Pi_\alpha^* \\ \lambda_{r+1}(\xi), \ldots, \lambda_m(\xi) &\in C\Pi_\alpha^* := \Pi \setminus \Pi_\alpha^* \end{aligned} \tag{1.1}$$

where E_m is the unit matrix of order m. The *class* M is defined as the set of vector functions

$$u(x,t) = (u_1(x,t), \ldots, u_m(x,t)) \quad \text{of } C^\infty(\mathbb{R}^n \times \bar{\Pi}_\alpha)$$

analytic in $t \in \Pi_\alpha$ and satisfying the inequality

$$|D_x^j D_t^k u(x,t)| \le c_{jk}(1 + |x|)^\gamma (1 + |t|)^\beta \qquad (|j|, k \in \mathbb{N}_0) \tag{1.2}$$

for all $(x,t) \in \mathbb{R}^n \times \bar{\Pi}_\alpha$, where

$$j = (j_1, \ldots, j_n), \quad |j| = j_1 + \ldots + j_n$$

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and

$$D_x^j = \left(i\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(i\frac{\partial}{\partial x_n}\right)^{j_n}.$$

Here $i = \sqrt{-1}$, $D_t = \frac{\partial}{\partial t}$ denotes the complex analysis differentiation, c_{jk} are non-negative constants and $\gamma, \beta \in I\!R$.

If in (1.2) γ is fixed, then the corresponding class we shall denote by $M_\gamma \subset M$, and if $u(x,t) \equiv u(t)$, then we have the classes N or N_β.

In the paper the problem of solvability of the system

$$\frac{\partial u(x,t)}{\partial t} = A(D_x)u(x,t) + f(x,t) \qquad ((x,t) \in I\!R^n \times \Pi_\alpha) \tag{1.3}$$

with unknown $u \in M$ and given $f \in M$ is considered.

The system (1.3) satisfying condition (1.1) will be called *strictly regular* and the number r will be called its *order of regularity*. When the condition (1.1) is violated only in a finite number of points, then the system (1.3) is said to be *regular*.

The main result of the paper looks as follows.

Theorem 1.1. *Consider the system (1.3). We have the following statements:*

a) The system (1.3) (regular or strictly regular) admits a solution $u \in M$ for all $f \in M$.

b) If the system (1.3) is strictly regular and $f \in M_\gamma$, then for its solution u the inclusion $u \in M_\gamma$ is true.

c) If the system (1.3) is regular and $f \in M_\gamma$, then there exists $\gamma_1 \geq \gamma$ such that for its solution u the inclusion $u \in M_{\gamma_1}$ is true.

Boundary value problems for system (1.3) with $f \equiv 0$, $t \in I\!R^+ = \{t \mid t > 0\}$ are studied in [2, 3] and for $t \in \Pi_\alpha$ in [1]. It should be noted that the operators which are not regular in the sence of [2, 3] can become regular in our case ($t \in \Pi_\alpha$). Par example the Helmholtz operator $\Delta + k^2$ with $k > 0$, where $\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$ is the Laplacean. Indeed, we have $\lambda_{1,2}(\xi) = \pm\sqrt{\xi^2 - k^2}$ which implies $r = 2$ for $|\xi| \leq k$ and $r = 1$ for $|\xi| > k$. Now, by taking $t \in \Pi_\alpha$ $(0 < \alpha < \frac{\pi}{2})$ one can see that $r = 1$ for all $\xi \neq \pm k$. Another motivation for taking $t \in \Pi_\alpha$ is the fact that the solutions of many boundary value problems in reality can be extended analytically with respect to t to some angle. Par example the solution $u = u(x,t)$ of the Dirichlet problem $u(x,0) = f(x)$ in the half plane $I\!R \times I\!R^+$ for the Laplace operator is analytic in t from the set $\{t \mid |\arg t| < \frac{\pi}{2}\}$.

2. Some auxiliary propositions

Denote by $F_\theta(p)$ the Laplace transform of a function $f(t)$ from the class N when $\arg t = \theta$, i.e.

$$F_\theta(p) = \int_0^\infty e^{-p\tau} f(\tau e^{i\theta})d\tau.$$

We have the following statement.

Lemma 2.1. *The function $F_0(p)$ being analytic in the half plane $\operatorname{Re} p > 0$ extends analytically to the domain $C\Pi^*_\alpha$.*

Proof. From Cauchy integral theorem there follows

$$\int\limits_{\Gamma_0 \cup \Gamma_\alpha} e^{-pt} f(t)\, dt = 0 \qquad \left(\arg p = -\frac{\alpha}{2}\right) \tag{2.1}$$

where Γ_β denotes the ray $\arg t = \beta$, $0 \le \beta \le \alpha$. Rewrite (2.1) in the form

$$F_0(p) = e^{i\alpha} F_\alpha(pe^{i\alpha}) \qquad \left(\arg p = -\frac{\alpha}{2}\right).$$

The functions $F_0(p)$ and $F_\alpha(pe^{i\alpha})$ being the Laplace transforms of functions of polynomial growth are analytic in $\operatorname{Re} p > 0$ and $\operatorname{Re}(pe^{i\alpha}) > 0$, respectively. So the function

$$\Phi(p) = \begin{cases} F_0(p) & \text{if } \operatorname{Re} p > 0 \\ e^{i\alpha} F_\alpha(pe^{i\alpha}) & \text{if } \operatorname{Re}(pe^{i\alpha}) > 0 \end{cases}$$

is analytic in $C\Pi^*_\alpha$ ∎

From the representation

$$f(t) = P_s(t) + g(t) \qquad \text{with } P_s(t) = f(0) + tf'(0) + \ldots + \frac{t^s}{s!} f^{(s)}(0) \tag{2.2}$$

for any function $f \in N$ and number $s \in \mathbb{N}$ we deduce that $g \in N$ and $g^{(j)}(0) = 0$ for any $j \le s$. Let $\beta \in \mathbb{N}$ be such that

$$\left|D_t^j f(t)\right| + \left|D_t^j g(t)\right| \le c_j(1 + |t|)^\beta \qquad (t \in \bar{\Pi}_\alpha). \tag{2.3}$$

Introduce the function

$$q(t) = g(t)(1+t)^{-2-\beta} \qquad (q^{(j)}(0) = 0 \text{ for all } j \le s). \tag{2.4}$$

Now, if $Q(p)$ is the Laplace transform of $q(t)$ $(t \in \mathbb{R}^+)$, then thanks to the relations (2.3) and (2.4) we have

$$|Q(p)| \le c(1 + |p|)^{-s-1} \qquad (p \in C\bar{\Pi}^*_\alpha). \tag{2.5}$$

Let $\gamma_\varepsilon = \gamma_0 + \varepsilon e^{-i\alpha/2}$, where γ_0 is the boundary of Π^*_α and $\varepsilon > 0$. Using the estimation (2.5) it is easy to get

$$q(t) = \frac{1}{2\pi i} \int\limits_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{pt} Q(p)\, dp = \frac{1}{2\pi i} \int\limits_{\gamma_0} e^{pt} Q(p)\, dp \qquad (t \in \mathbb{R}^+).$$

Evidently this function $q(t)$ is analytic in Π_α and bounded in $\bar{\Pi}_\alpha$. Finally we obtain the following

Lemma 2.2. *Every function $f \in N$ can be presented in the form*

$$f(t) = P_s(t) + \frac{(1+t)^{\beta+2}}{2\pi i} \int_{\gamma_0} e^{pt} Q(p)\, dp \qquad (t \in \Pi_\alpha).$$

Now let us consider the differential equation

$$\frac{\partial u(t)}{\partial t} - \lambda u(t) = f(t) \qquad (t \in \Pi_\alpha) \tag{2.6}$$

where $\lambda \in \Pi, f \in N$ is given and $u \in N$ is the unknown function. When $t \in I\!R^+$, then the solution of equation (2.6) can be written immediately in function of sign(Re λ). In our case the form of the solution depends not only of $\lambda \in \Pi_\alpha^*$ or $\lambda \in C\Pi_\alpha^*$, but also of a position of $\lambda \in C\Pi_\alpha^*$. For applications we prefer to obtain a unique form of solution.

At first, if $f(t) = P_s(t)$ (see (2.2)), then evidently equation (2.6) admits a polynomial solution u_0. So it remains to consider equation (2.6) with $f = g$ and $\lambda \neq 0$. It is clear that the function

$$u_1(t) = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} e^{pt} \frac{C_s(p)}{p-\lambda}\, dp \qquad (t \in \Pi_\alpha) \tag{2.7}$$

where ε is such that λ and 0 are placed on different sides of γ_ε in the case of $\lambda \in C\Pi_\alpha^*$, is a solution of equation (2.6) satisfying the estimate $|u_1(t)| \leq ce^{\varepsilon|t|}$ $(t \in \bar{\Pi}_\alpha)$. Let us show that $u_1 \in N$. Since $f \in N$ and u_1 satisfies equation (2.6) it is sufficient to verify inequality (1.2) only for $k = 0$. From (2.4) we have $G(p) = (1 - \frac{d}{dp})^{\beta+2} Q(p)$. Then by integrating by parts we get

$$\begin{aligned} u_1(t) &= \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{e^{pt}}{p-\lambda} \left(1 - \frac{d}{dp}\right)^{\beta+2} Q(p)\, dp \\ &= \frac{1}{2\pi i} \int_{\gamma_0} Q(p) \left(1 + \frac{d}{dp}\right)^{\beta+2} \left(\frac{e^{pt}}{p-\lambda}\right) dp \\ &= \sum_{j \leq \beta+2} t^j \int_{\gamma_0} q_j(p) Q(p) e^{pt} dp \qquad (t \in \bar{\Pi}_\alpha) \end{aligned} \tag{2.8}$$

where the functions $q_j(p)$ are well-defined and bounded, i.e. $u_1(t)$ is of polynomial growth. Hence the following lemma is proved.

Lemma 2.3. *The equation (2.6) admits a solution $u \in N$ for all functions $f \in N$.*

Now, if we have a system

$$\frac{\partial u(t)}{\partial t} - Au(t) = f(t) \qquad (t \in \Pi_\alpha)$$

where A is a constant matrix of order m, $f \in N$ is a given vector function and $u \in N$ is the unknown vector function, we can solve it by transforming the matrix A into Jordan form and using Lemma 2.3.

3. Construction of a solution of system (1.3)

Strictly regular case. Let the system (1.3) be strictly regular. Introduce the following polynoms in λ:

$$Q(\xi,\lambda)=\prod_{j=1}^{r}(\lambda-\lambda_j(\xi))=\lambda^r+\sum_{j=1}^{r}a_j(\xi)\lambda^{r-j}$$
$$R(\xi,\lambda)=\prod_{j=1}^{m-r}(\lambda-\lambda_{r+j}(\xi))=\lambda^{m-r}+\sum_{j=1}^{m-r}a_{r+j}(\xi)\lambda^{m-r-j}.$$

It is well known (see [4: Lemma 3.1/p. 194]) that

$$\begin{aligned} |\lambda_j(\xi)\leq c_j(1+|\xi|)^{n_j}, \qquad & a_j\in C^\infty(I\!R^n) \\ |D_\xi^k a_j(\xi)|\leq c_k(1+|\xi|)^{n_{jk}} \qquad & (|k|\in I\!N_0). \end{aligned} \tag{3.1}$$

Let $v(\xi,t)$ be a solution of the Cauchy problem

$$Q\left(\xi,\frac{\partial}{\partial t}\right)v(\xi,t):=\frac{\partial^r v(\xi,t)}{\partial t^r}+\sum_{j=1}^{r}a_j(\xi)\frac{\partial^{r-j}v(\xi,t)}{\partial t^{r-j}}=0 \qquad (t\in\Pi_\alpha) \tag{3.2}$$

$$D_t^j v(\xi,0)=b_j(\xi) \qquad (0\leq j\leq r-1) \tag{3.3}$$

where $b_j\in C^\infty(I\!R^n)$ satisfies inequality (3.1) and $v(\xi,t)$ is analytic in $t\in\Pi_\alpha$. Rewrite (3.2), (3.3) in matrix form

$$\begin{aligned} \frac{\partial w(\xi,t)}{\partial t}&=B(\xi)w(\xi,t) \quad (t\in\Pi_\alpha) \\ w(\xi,0)&=b(\xi) \end{aligned}$$

where

$$w=\left(v,\frac{\partial v}{\partial t},\ldots,\frac{\partial^{r-1}v}{\partial t^{r-1}}\right) \qquad \text{and} \qquad b(\xi)=\Big(b_0(\xi),\ldots,b_{r-1}(\xi)\Big)$$

and $B(\xi)$ is a well-defined square matrix of order r with eigenvalues $\lambda_1(\xi),\ldots,\lambda_r(\xi)$ all belonging to Π_α^*. We have $w(\xi,t)=e^{B(\xi)t}b(\xi)$, whence by using the well-known estimate

$$\|e^{B(\xi)t}\|\leq c\sum_{j=1}^{r}e^{\mathrm{Re}(\lambda_j(\xi)t)}(1+|t|)^{r-1}(1+|\xi|)^s$$

(see [2: p. 222, item 2]) we shall get the estimate

$$\left|D_\xi^j D_t^k v(\xi,t)\right|\leq c_{jk}(1+|\xi|)^{m_{jk}}(1+|t|)^{n_{jk}}. \tag{3.4}$$

For beginning we take $f(x,t)=t^jq(x)\in M$. Every function $u(x,t)\in M$ can be considered as a distribution from $S'(I\!R^n)$ depending on parameter $t\in\bar\Pi_\alpha$ and satisfying the inequality

$$\left|\frac{\partial^j}{\partial t^j}\langle u(x,t),\varphi(x)\rangle\right|\leq c_j(1+|t|)^\beta\|\varphi\|_s \qquad (\varphi\in S(I\!R^n))$$

where $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ are the pair of Schwartz spaces and

$$\|\varphi\|_s = \sup_x (1+|x|^2)^{s/2} \sum_{|\alpha|\le s} |D_x^\alpha \varphi(x)| \qquad (s \in \mathbb{N}_0).$$

Let $\hat{u}(\xi,t) = F_x(u(x,t))$ denote the generalized Fourier transform of $u \in M$ (F^{-1} will denote the inverse Fourier transform). The Fourier image of system (1.3) has the form

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = A(\xi)\hat{u}(\xi,t) + t^j \hat{q}(\xi). \tag{3.5}$$

As a solution of system (3.5) we consider

$$\hat{u}(\xi,t) = \frac{j!}{2\pi i} \int_{\gamma^-(\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda \hat{q}(\xi) \tag{3.6}$$

where $\gamma^-(\xi)$ is a closed contour containing only the roots $\lambda_1(\xi), \dots, \lambda_r(\xi) \in \Pi_\alpha^*$ and the point $\lambda = 0$. Introduce the matrix

$$V(\xi,t) = \frac{j!}{2\pi i} \int_{\gamma^-(\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda \qquad (t \in \Pi_\alpha).$$

Lemma 3.1. *The elements $v_{ls}(\xi,t)$ of the matrix $V(\xi,t)$ satisfy the inequality (3.4).*

Proof. We have

$$V(\xi,0) = \frac{j!}{2\pi i} \int_{\gamma^-(\xi)} \frac{a(\xi,\lambda)}{\lambda^{j+1} Q(\xi,\lambda) R(\xi,\lambda)} d\lambda$$

with

$$a(\xi,\lambda) = (aE_m - A(\xi))^{-1} |\lambda E_m - A(\xi)|.$$

Since the polynomials $\lambda^{j+1}Q(\xi,\lambda)$ and $R(\xi,\lambda)$ are coprime, then

$$1 = r(\xi,\lambda)\lambda^{j+1}Q(\xi,\lambda) + q(\xi,\lambda)R(\xi,\lambda) \tag{3.7}$$

where $r(\xi,\lambda)$ and $q(\xi,\lambda$ are polynomials in λ with coefficients satisfying the inequality (3.1). The matrix $V(\xi,0)$ can be rewritten in the form

$$V(\xi,0) = \frac{j!}{2\pi i} \int_{\gamma^-(\xi)} \frac{q(\xi,\lambda)}{\lambda^{j+1} Q(\xi,\lambda)} a(\xi,\lambda)\, d\lambda. \tag{3.8}$$

Since the contour $\gamma^-(\xi)$ encloses all roots of denominator $\lambda^{j+1}Q(\xi,\lambda)$, then computing (3.8) with help of the residue theorem at the point $\lambda = \infty$ we immediately get

$$\left| D_\xi^k D_t^j v_{ls}(\xi,0) \right| \le c_{kj}(1+|\xi|)^{n_{kj}} \qquad (|k|, j \in \mathbb{N}_0).$$

Now notice that the function $v_{ls}(\xi,t)$ satisfies the differential equation

$$\frac{\partial^{j+1}}{\partial t^{j+1}} Q\left(\xi, \frac{\partial}{\partial t}\right) v_{ls}(\xi,t) = 0$$

the characteristic roots of which belong to the set Π_α^*. Therefore it satisfies the estimate (3.4) (see (3.2) and (3.3)) ∎

Lemma 3.2. *The inclusion $F_\xi^{-1}(v_{ls}(\xi,t)\hat{q}_j(\xi)) \in M$ is true, where the vector function $q(x) = (q_1(x), \dots, q_m(x))$ is an element of M.*

Proof. We have $|D_x^k q_j(x)| \le c_k(1+|x|)^\gamma$ (see (1.2)). Let $k_0 \gg 1$ be such that the function

$$w_{ls}(x,t) = F_\xi^{-1}\left((1+|\xi|^1)^{-k_0} v_{ls}(\xi,t)\right) = \frac{1}{(2\pi)^n}\int\limits_{I\!R^n}(1+|\xi|^2)^{-k_0} v_{ls}(\xi,t)e^{-i\xi x}d\xi$$

satisfies the estimate

$$|w_{ls}(x,t)| \le c(1+|x|)^{-n-|\gamma|-1}(1+|t|)^{m_0}. \tag{3.9}$$

And if $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, then we have

$$F_\xi^{-1}(v_{ls}(\xi,t)\hat{q}_j(\xi)) = w_{ls}(x,t) * (1-\Delta_x)^{k_0} q(x)$$

where the operation $*$ is taken in relation to the variable x, and according to the Peetre inequalities

$$(1+|x-y|)^{-|\gamma|} \le (1+|x|)^\gamma(1+|y|)^\gamma$$

and

$$(1+|x-y|)^\gamma \le (1+|x|)^\gamma(1+|y|)^\gamma$$

when $\gamma \le 0$ or $\gamma > 0$, respectively, we get

$$\begin{aligned}\left|F_\xi^{-1}(v_{ls}(\xi,t)\hat{q}_j(\xi))\right| &\le c(1+|x|)^\gamma(1+|t|)^{m_0}\int(1+|y|)^{-n-|\gamma|-1}(1+|y|)^\gamma dy\\ &\le c_1(1+|x|)^\gamma(1+|t|)^{m_0}.\end{aligned}$$

The derivatives can be estimated similarly ∎

Thus, if $f(x,t) = t^j q(x)$, then the solution u of system (1.3) is constructed, namely

$$u(x,t) = F_\xi^{-1}\left(\frac{j!}{2\pi i}\int\limits_{\gamma^-(\xi)}(\lambda E_m - A(\xi))^{-1}\lambda^{-j-1}e^{\lambda t}d\lambda\cdot\hat{q}(\xi)\right).$$

So, using the representation

$$f(x,t) = f(x,0) + t\frac{\partial f}{\partial t}(x,0) + \dots + \frac{t^l}{l!}\frac{\partial^l f}{\partial t^l}(x,0) + g(x,t)$$

with an arbitrary $l \in I\!N$, we can reduce our problem to

$$\frac{\partial u(x,t)}{\partial t} = A(D_x)u(x,t) + g(x,t) \tag{3.10}$$

where $g \in M$ and $D_t^j g(x,0) = 0$ for all $j \leq l$. In (1.2) assume $\beta \in \mathbb{N}$. Introduce the vector function (see (2.4))

$$q(x,t) = g(x,t)(1+t)^{-\beta-2} \tag{3.11}$$

As Fourier image of a solution of system (3.10) we consider the expression

$$\hat{u}(\xi,t) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\xi)} (\lambda E_m - A(\xi))^{-1} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda \tag{3.12}$$

where $\gamma_\varepsilon(\xi) = \gamma_0 + \varepsilon(\xi) e^{-i\alpha/2}, \varepsilon(\xi) > 0$, is such that the roots $\lambda_{r+1}(\xi), \dots, \lambda_m(\xi) \in C\Pi_\alpha^*$ and the point $\lambda = 0$ are on different sides of it (see (2.7)) and $\hat{G}(\xi,\lambda)$ is the Laplace-Fourier transform of $g(x,t)$ $((x,t) \in \mathbb{R}^n \times \mathbb{R}^+)$.

A formula analogous to (3.7) permits us to rewrite (3.12) in the form

$$\begin{aligned} \hat{u}(\xi,t) &= \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\xi)} \frac{q(\xi,\lambda)a(\xi,\lambda)}{Q(\xi,\lambda)} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda \\ &+ \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\xi)} \frac{r(\xi,\lambda)a(\xi,\lambda)}{R(\xi,\lambda)} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda \qquad (t \in \Pi_\alpha). \end{aligned} \tag{3.13}$$

The integrals here are convergent because of $D_t^j g(x,0) = 0$ for any $j \leq l$ with $l \gg 1$ taken in advance.

Let l_0 be such that

$$\left\| \frac{q(\xi,\lambda)a(\xi,\lambda)}{\lambda^{l_0} Q(\xi,\lambda)} \right\| \leq c(\xi) \frac{1}{|\lambda|} \qquad (|\lambda| > 1).$$

Introduce the matrix

$$\hat{K}(\xi,t) = \frac{1}{2\pi i} \int\limits_{\gamma^-(\xi)} \frac{q(\xi,\lambda)a(\xi,\lambda)}{\lambda^{l_0} Q(\xi,\lambda)} e^{\lambda t_0} d\lambda$$

with $\gamma^-(\xi)$ defined in (3.6). The first integral in (3.13) will take the form

$$\hat{u}_1(\xi,t) = \int\limits_0^t \hat{K}(\xi,\tau) \hat{g}_t^{(l_0)}(\xi, t-\tau)\, d\tau \qquad (t \in \Pi_\alpha).$$

As it was shown above (see Lemma 3.1) the elements $\hat{k}_{ls}(\xi,t)$ of the matrix $\hat{K}(\xi,t)$ satisfy the estimation (3.4), hence we can easily regularize the inverse transform $u_1(x,t)$ of $\hat{u}_1(\xi,t)$ (see Lemma 3.2) and have the inclusion $u_1 \in M$.

Now let us examine the second term $\hat{u}_2(\xi,t)$ in (3.13). At first as

$$\hat{G}(\xi,t) = \left(1 - \frac{d}{d\lambda}\right)^{\beta+2} \hat{Q}(\xi,\lambda)$$

(see (3.11)), where $\hat{Q}(\xi,\lambda)$ satisfies an inequality like (2.5), integrating by parts we can reduce $\hat{u}_2(\xi,t)$ to the form

$$\hat{u}_2(\xi,t) = \frac{1}{2\pi i}\int_{\gamma_0} \hat{Q}(\xi,\lambda)\psi(\xi,\lambda,t)\,d\lambda \qquad (t \in \Pi_\alpha)$$

where

$$\psi(\xi,\lambda,t) = \left(1+\frac{d}{d\lambda}\right)^{\beta+2}\frac{r(\xi,\lambda)a(\xi,\lambda)e^{\lambda t}}{R(\xi,\lambda)}.$$

As consequence of the inequality

$$|R(\xi,\lambda)| \geq c(1+|\xi|^2)^s \qquad ((\xi,\lambda)\in \mathbb{R}^n\times\gamma_0) \tag{3.14}$$

(see [1: Lemma 1.2]) we get the estimation

$$|D_\xi^k\psi(\xi,\lambda,t)| \leq c_k(1+|\xi|)^{m_k}(1+|\lambda|)^{n_k}(1+|t|)^{\beta+2} \tag{3.15}$$

for all $(\xi,\lambda,t)\in\mathbb{R}^n\times\gamma_0\times\Pi_\alpha$. Now it is clear that by the same way

$$u_2(x,t) = F_\xi^{-1}\left(\hat{u}_2(\xi,t)\right) = \frac{1}{2\pi i}F_\xi^{-1}\left(\int_{\gamma_0}\hat{Q}(\xi,\lambda)\psi(\xi,\lambda,t)\,d\lambda\right)\in M.$$

Theorem 1 for strictly regularity of system (1.3) is proved.

2. Regular case. In the considerations made above the condition (1.1) (condition of strictly regularity) was very essential (see estimation (3.1) and others). Assuming that the condition (1.1) is violated only at the point $\xi = 0$ (this does not loss the generality) we can show (see [1: Lemmas 1.2 and 1.3]) that in all inequalities crucial (i.e. in (3.1), (3.14) and (3.15)) we shall have the factor $|\xi|^{s_k}$ with $s_k \leq 0$ and this circumstance adds new difficulties (the estimation (3.9) was established thanks to smoothness of $v_{ls}(\xi,t)$ in $\xi\in\mathbb{R}^n$).

Let us demonstrate how we can surmount this obstacle. First, consider the equation $\Delta_x u(x,t) = f(x,t)$, where Δ_x is the Laplacean, $f\in M$ is a given function and $u\in M$ is the unknown function, and let us prove its solvability. If $n=1$, i.e. $\Delta_x = \frac{\partial^2}{\partial x^2}$, then obviously this equation admits a solution from M. Now we are going to show how the case $n\geq 2$ can be reduced to the case $n-1$. For this purpose we replace $f(x,t)$ with $(i\frac{\partial}{\partial x_1})^{v_1}f(x,t)$ and consider the expression $|\xi|^{-2}\xi_1^{v_1}\hat{f}(\xi,t)$. With an appropriate choice of natural v_1 we can reach any smothness order in $\mathbb{R}^n$ of the function $|\xi|^{-2}\xi_1^{v_1}$. By taking a natural $k\gg 1$ we can readly establish the inclusion

$$\begin{aligned} u_0(x,t) &= F_\xi^{-1}\left(|\xi|^{-2}\xi_1^{v_1}(1+|\xi|^2)^{-k}(1+|\xi|^2)^k\hat{f}(\xi,t)\right)\\ &= F_\xi^{-1}\left(|\xi|^{-2}\xi_1^{v_1}(1+|\xi|^2)^{-k}\right) * (1-\Delta_x)^k f(x,t)\\ &\in M \end{aligned}$$

where the operation * is taken in relation to the variable x and $\Delta_x u_0 = (i\frac{\partial}{\partial x_1})^{v_1} f$. Find $u_1 \in M$ in a way that $(i\frac{\partial}{\partial x_1})^{v_1} u_1 = u_0$. Then we get

$$\left(i\frac{\partial}{\partial x_1}\right)^{v_1} (\Delta_x u_1 - f) = 0$$

or, by integrating with respect to x_1,

$$\Delta_x u_1 = f + \sum_{j \le v_1 - 1} a_j(x', t) x_1^j, \qquad x' = (x_2, \cdots, x_n),$$

where $a_j(x',t) \in M$ are well-defined functions. Introducing $w = u_1 - u$ we shall get

$$\Delta_x w = \sum_{j \le v_1 - 1} a_j(x', t) x_1^j. \tag{3.16}$$

Let us seek $w(x,t)$ in the form

$$w(x,t) = \sum_{j \le v_1 - 1} c_j(x', t) x_1^j$$

with unknown $c_j(x',t) \in M$. Substituting $w(x,t)$ into equation (3.16) and making use of linearly independence of the functions $1, x_1, \ldots, x_1^{v_1 - 1}$, we obtain

$$\begin{aligned} \Delta_{x'} c_{v_1 - 1}(x',t) &= a_{v_1 - 1}(x',t) \\ \Delta_{x'} c_j(x',t) &= B_j\left(c_{j+1}(x',t), \ldots, c_{v_1 - 1}(x',t)\right) + b_j(x',t) \\ j &= v_1 - 2, \ldots, 0 \end{aligned}$$

where $B_j(\cdots)$ is a linear expression and $b_j(x',t) \in M$ are certain functions. Thus the desired reduction is realized. Hence we have proved the following

Lemma 3.3. *The equation $\Delta_x u = f$ is always solvable in M.*

The solvability of the system (1.3) can be established in the same way: first in (1.3) we put $\Delta_x^v f(x,t)$ instead of $f(x,t)$ and with an appropriate choice of natural v we get solvability of the system

$$\frac{\partial u}{\partial t} = A(D_x) u + \Delta_x^v f. \tag{3.17}$$

Further, let u_0 denote a solution of system (3.17). Then according to Lemma 3.3 we can find $u_1 \in M$ in a way that $\Delta_x^v u_1 = u_0$. Now we put u_0 in (3.17) and get

$$\Delta_x^v \left(\frac{\partial u_1}{\partial t} - A(D_x) u_1 - f\right) = 0. \tag{3.18}$$

With the help of Fourier transformation from (3.18) we deduce

$$\frac{\partial u_1}{\partial t} = A(D_x) u_1 + f + \sum_{|j| \le 2v - 1} a_j(t) x^j$$

with certain coefficients $a_j \in M$. Now consider the system

$$\frac{\partial w}{\partial t} = A(D_x)w + \sum_{|j| \le 2v-1} a_j(t)x^j \tag{3.19}$$

and let us seek $w \in M$ in the form

$$w = \sum_{|j| \le 2v-1} c_j(t)x^j$$

with unknown coefficients $c_j \in N$. Substituting w into (3.19) and taking account of linearly independence of polynoms $\{x^j \mid |j| = 0, \ldots, 2v-1\}$ we get the system

$$\frac{\partial c_j(t)}{\partial t} = A(0)c_j(t) + a_j(t) \qquad (|j| = 2v-1)$$

which is solvable (see the system (2.9)). The rest c_j ($|j| \le 2v-2$) can be determined in the same manner. Evidently $u_1 - w$ is the desired solution of system (1.3). Theorem 1 is completely proved.

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