# On the Existence of Solution of a System of Partial Differential Equations

#### A. A. Andrian

Abstract. Let  $\Pi_{\alpha} = \{t | 0 < \arg t < \alpha\}$  for  $\alpha < \pi$  and denote by *M* the class of m-dimensional vector functions  $u = u(x,t)$  of  $C^{\infty}(I\!\!R^n \times \bar{\Pi}_{\alpha})$  analytic in  $t \in \Pi_{\alpha}$  and having polynomial growth in  $(x, t)$ . Let  $A(\xi)$  ( $\xi \in \mathbb{R}^n$ ) be a square matrix of order m with polynomial elements. In the paper we define regularity and strictly regularity of the system  $\frac{\partial u}{\partial t} = A(D_x)u + f$  and prove its solvability in *M* for all  $f \in M$ .

Keywords: *Partial differential equations, regularity, strictly regularity, solvability*  AMS subject classification: 35A99

#### **1. Introduction**

In the following let  $\mathbb{N}^T$  be the set of natural numbers,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\},$   $\Pi$  a complex plane and, for  $0 < \alpha < \pi$ ,

**introduction**

\ne following let 
$$
N
$$
 be the set of natural numbers,  $N_0 = N \cup \{0\}$ ,  $\Pi$  a complex for  $0 < \alpha < \pi$ ,

\n
$$
\Pi_{\alpha} = \left\{ \lambda \middle| 0 < \arg \lambda < \alpha \right\}
$$
 and 
$$
\Pi_{\alpha}^* = \left\{ \lambda \middle| \frac{1}{2} \pi \leq \arg \lambda \leq \frac{3}{2} \pi - \alpha \right\}.
$$

Let  $A(\xi)$  ( $\xi \in \mathbb{R}^n$ ) be a square matrix of order *m* with polynomial elements such that, for all  $\xi \in \mathbb{R}^n$ , the roots  $\lambda_1(\xi), \ldots, \lambda_m(\xi)$  (some of them can coincide) of the characteristic equation  $\det(\lambda E_m - A(\xi)) = 0$  satisfy the conditions  $\mathbf{V} \cup \{0\}, \Pi \text{ a}$ <br>  $\leq \arg \lambda \leq \frac{3}{2}$ <br>
polynomial<br>
them can conditions<br>  $\frac{1}{\alpha}$ <br>
efined as the

 $\lambda_{r+1}(\xi), \ldots, \lambda_m(\xi) \in C\Pi_{\alpha}^* := \Pi \setminus \Pi_{\alpha}^*$  $(1.1)$  $\lambda_1(\xi), \ldots, \lambda_r(\xi) \in \Pi_{\alpha}^*$  (1.1)<br>  $\lambda_{r+1}(\xi), \ldots, \lambda_m(\xi) \in C\Pi_{\alpha}^* := \Pi \setminus \Pi_{\alpha}^*$  (1.1)<br>
e unit matrix of order *m*. The class *M* is defined as the set of vector<br>  $u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$  of  $C^{\infty}(\mathbb{R}^n \times \bar{\Pi}_{\alpha})$ <br>

where  $E_m$  is the unit matrix of order m. The *class* M is defined as the set of vector functions

$$
u(x,t)=(u_1(x,t),\ldots,u_m(x,t)) \text{ of } C^{\infty}(\mathbb{R}^n\times\bar{\Pi}_{\alpha})
$$

analytic in  $t \in \Pi_{\alpha}$  and satisfying the inequality

$$
|D_z^j D_t^k u(x,t)| \leq c_{jk} (1+|x|)^{\gamma} (1+|t|)^{\beta} \qquad (|j|, k \in \mathbb{N}_0)
$$
 (1.2)

for all  $(x, t) \in \mathbb{R}^n \times \overline{\Pi}_{\alpha}$ , where

$$
y_{i,j} = c_{jk}(1 + |x|) (1 + |x|)
$$
  
where  

$$
j = (j_1, ..., j_n), \quad |j| = j_1 + ... + j_n
$$

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ISSN 0232-2064 **/** \$ 2.50 © Heldermann Verlag Berlin

and

$$
D_x^j = \left(i\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(i\frac{\partial}{\partial x_n}\right)^{j_n}
$$
  
denotes the complex analysis d

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and<br>  $D_x^j = \left(i\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(i\frac{\partial}{\partial x_n}\right)^{j_n}$ <br>
Here  $i = \sqrt{-1}$ ,  $D_t = \frac{\partial}{\partial t}$  denotes the complex analysis differentiation,  $c_{jk}$  are non-<br>
negative constants and  $\gamma, \beta \in \mathbb{R}$ .<br>
If in (1.2 negative constants and  $\gamma, \beta \in \mathbb{R}$ .

If in (1.2)  $\gamma$  is fixed, then the corresponding class we shall denote by  $M_{\gamma} \subset M$ , and if  $u(x,t) \equiv u(t)$ , then we have the classes *N* or  $N_g$ .

In the paper the problem of solvability of the system

ndrian  
\n
$$
D_x^j = \left(i\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(i\frac{\partial}{\partial x_n}\right)^{j_n}
$$
\n
$$
D_t = \frac{\partial}{\partial t} \text{ denotes the complex analysis differentiation, } c_{jk} \text{ are non-\nnts and } \gamma, \beta \in \mathbb{R}.
$$
\nis fixed, then the corresponding class we shall denote by  $M_\gamma \subset M$ , and  
\nthen we have the classes  $N$  or  $N_\beta$ .  
\n
$$
F = \frac{\partial u(x,t)}{\partial t} = A(D_x)u(x,t) + f(x,t) \qquad ((x,t) \in \mathbb{R}^n \times \Pi_\alpha) \qquad (1.3)
$$
\n
$$
u \in M \text{ and given } f \in M \text{ is considered.}
$$
\n
$$
(1.3) \text{ satisfying condition (1.1) will be called *strictly regular* and the
$$

with unknown  $u \in M$  and given  $f \in M$  is considered.

The system (1.3) satisfying condition (1.1) will be called *Strictly regular* and the number *r* will be called its *order of regularity*. When the condition (1.1) is violated only in a finite number of points, then the system (1.3) is said to be *regular.* 

The main result of the paper looks as follows.

**Theorem** 1.1. *Consider the system (1.8). We have the following statements:* 

*a)* The system (1.3) (regular or strictly regular) admits a solution  $u \in M$  for all  $f \in M$ .

*b)* If the system (1.3) is strictly regular and  $f \in M_{\gamma}$ , then for its solution u the *inclusion*  $u \in M_{\gamma}$  *is true. M*.<br>*b) If the system* (1.9) is strictly regular and  $f \in M_{\gamma}$ , then for its usion  $u \in M_{\gamma}$  is true.<br>*c) If the system* (1.9) is regular and  $f \in M_{\gamma}$ , then there exists  $\gamma_1 \ge \gamma$  olution u the inclusion  $u \in M_{$ 

*c)* If the system (1.3) is regular and  $f \in M_{\gamma}$ , then there exists  $\gamma_1 \geq \gamma$  such that for *its solution u the inclusion*  $u \in M_{\gamma_1}$  *is true.* 

Boundary value problems for system (1.3) with  $f \equiv 0$ ,  $t \in \mathbb{R}^+ = \{t | t > 0\}$  are studied in [2, 3] and for  $t \in \Pi_{\alpha}$  in [1]. It should be noted that the operators which are not regular in the sence of [2, 3] can become regular in our case  $(t \in \Pi_{\alpha})$ . Par example the Helmholtz operator  $\Delta + k^2$  with  $k > 0$ , where  $\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$  is the Laplacean. Indeed, we have  $\lambda_{1,2}(\xi) = \pm \sqrt{\xi^2 - k^2}$  which implies  $r = 2$  for  $|\xi| \leq k$  and  $r = 1$  for  $|\xi| > k$ . Now, by taking  $t \in \Pi_{\alpha}$   $(0 < \alpha < \frac{\pi}{2})$  one can see that  $r=1$  for all  $\xi \neq \pm k$ . Another motivation for taking  $t \in \Pi_{\alpha}$  is the fact that the solutions of many boundary value problems in reality, can be extended analytically with respect to *t* to some angle;. Par example the solution,  $u = u(x, t)$  of the Dirichlet problem  $u(x, 0) = f(x)$  in the half plane  $\mathbb{R} \times \mathbb{R}^+$  for the Laplace operator is analytic in *t* from the set  $\{t \mid |\arg t| < \frac{\pi}{2}\}.$ 

### **2. Some auxiliary propositions**

Denote by  $F_{\theta}(p)$  the Laplace transform of a function  $f(t)$  from the class N when arg  $t =$ *9,* i.e.

$$
F_{\theta}(p) = \int\limits_{0}^{\infty} e^{-p\tau} f(\tau e^{i\theta}) d\tau.
$$

We have the following statement.

Lemma 2.1. The function  $F_0(p)$  being analytic in the half plane  $Rep > 0$  extends analytically to the domain  $C\Pi_{\alpha}^*$ .

Proof. From Cauchy integral theorem there follows

On the Solution of Partial Differential Equations  
\nfunction 
$$
F_0(p)
$$
 being analytic in the half plane  $Rep > 0$  extends  
\nain  $C\Pi_{\alpha}^*$ .  
\nby integral theorem there follows  
\n
$$
\int_{\Gamma_0 \cup \Gamma_{\alpha}} e^{-pt} f(t) dt = 0 \qquad \left(\arg p = -\frac{\alpha}{2}\right)
$$
\n(2.1)  
\nay arg  $t = \beta$ ,  $0 \le \beta \le \alpha$ . Rewrite (2.1) in the form  
\n
$$
F_0(p) = e^{i\alpha} F_{\alpha}(pe^{i\alpha}) \qquad \left(\arg p = -\frac{\alpha}{2}\right).
$$

where  $\Gamma_{\beta}$  denotes the ray  $\arg t = \beta$ ,  $0 \leq \beta \leq \alpha$ . Rewrite (2.1) in the form

$$
F_0(p) = e^{i\alpha} F_\alpha(pe^{i\alpha}) \qquad \left(\arg p = -\frac{\alpha}{2}\right).
$$

The functions  $F_0(p)$  and  $F_\alpha(p e^{i\alpha})$  being the Laplace transforms of functions of polynomial growth are analytic in  $\text{Re } p > 0$  and  $\text{Re}(pe^{i\alpha}) > 0$ , respectively. So the function

$$
\int_{c_0 \cup \Gamma_a}^{c} f(t) dt = 0 \quad \text{(where } P = -\frac{1}{2} \text{)}
$$
\n
$$
\text{By } \arg t = \beta, \ 0 \le \beta \le \alpha. \text{ Rewrite (2.1) is}
$$
\n
$$
F_0(p) = e^{i\alpha} F_\alpha(p e^{i\alpha}) \quad \left(\arg p = -\frac{\alpha}{2}\right)
$$
\n
$$
\text{If } F_\alpha(p e^{i\alpha}) \text{ is the Laplace transform}
$$
\n
$$
\text{or } \text{in } \text{Re } p > 0 \text{ and } \text{Re}(p e^{i\alpha}) > 0, \text{ respect}
$$
\n
$$
\Phi(p) = \begin{cases} F_0(p) & \text{if } \text{Re } p > 0 \\ e^{i\alpha} F_\alpha(p e^{i\alpha}) & \text{if } \text{Re}(p e^{i\alpha}) > 0 \end{cases}
$$

is analytic in  $C\Pi_{\alpha}^*$ 

 $\sim$ 

$$
\Gamma_0 \cup \Gamma_a
$$
\n
$$
\Gamma_0 \cup \Gamma_a
$$
\n
$$
\Gamma_0 \cup \Gamma_a
$$
\n
$$
F_0(p) = e^{i\alpha} F_\alpha(p e^{i\alpha}) \qquad \left(\arg p = -\frac{\alpha}{2}\right).
$$
\n
$$
\text{functions } F_0(p) \text{ and } F_\alpha(p e^{i\alpha}) \text{ being the Laplace transforms of functions of polynomials}
$$
\n
$$
\Gamma_0(p) \text{ and } F_\alpha(p e^{i\alpha}) \text{ being the Laplace transforms of functions of polynomials}
$$
\n
$$
\Phi(p) = \begin{cases} F_0(p) & \text{if } \text{Re } p > 0 \\ e^{i\alpha} F_\alpha(p e^{i\alpha}) & \text{if } \text{Re}(p e^{i\alpha}) > 0 \end{cases}
$$
\n
$$
\text{and} \text{which implies } \Gamma_0 \cap \Gamma_1 \text{ and } \Gamma_2 \cap \Gamma_2 \text{ and } \Gamma_3 \cap \Gamma_4 \text{ and } \Gamma_5 \cap \Gamma_5 \text{ and } \Gamma_6 \cap \Gamma_7 \text{ and } \Gamma_8 \cap \Gamma_7 \text{ and } \Gamma_8 \cap \Gamma_8 \text{ and } \Gamma_9 \cap \Gamma_9 \text{ and } \
$$

for any function  $f \in N$  and number  $s \in \mathbb{N}$  we deduce that  $g \in N$  and  $g^{(j)}(0) = 0$  for any  $j \leq s$ . Let  $\beta \in \mathbb{N}$  be such that

and give in the 
$$
p > 0
$$
 and the  $(pc^{-1}) > 0$ , respectively. So the function

\n
$$
\Phi(p) = \begin{cases} F_0(p) & \text{if } \text{Re } p > 0 \\ e^{i\alpha} F_\alpha(pe^{i\alpha}) & \text{if } \text{Re }(pe^{i\alpha}) > 0 \end{cases}
$$
\nresentation

\n
$$
P_s(t) + g(t) \quad \text{with } P_s(t) = f(0) + tf'(0) + \ldots + \frac{t^2}{s!}f^{(s)}(0) \qquad (2.2)
$$
\n
$$
f \in N \text{ and number } s \in \mathbb{N} \text{ we deduce that } g \in N \text{ and } g^{(j)}(0) = 0 \text{ for } \in \mathbb{N} \text{ be such that}
$$
\n
$$
\left| D_t^j f(t) \right| + \left| D_t^j g(t) \right| \le c_j (1 + |t|)^{\beta} \qquad (t \in \bar{\Pi}_\alpha). \tag{2.3}
$$
\naction

\n
$$
q(t) = g(t)(1 + t)^{-2-\beta} \qquad (q^{(j)}(0) = 0 \text{ for all } j \le s). \tag{2.4}
$$
\nthe Laplace transform of  $q(t)$   $(t \in \mathbb{R}^+)$ , then thanks to the relations

\nwe have

\n
$$
|Q(p)| \le c(1 + |p|)^{-s-1} \qquad (p \in C\bar{\Pi}_\alpha^*). \tag{2.5}
$$
\n
$$
= \frac{1}{\alpha/2}, \text{ where } \gamma_0 \text{ is the boundary of } \Pi_\alpha^* \text{ and } \varepsilon > 0. \text{ Using the estimation}
$$
\nof  $\alpha$  is the boundary of  $\Pi_\alpha^*$  and  $\varepsilon > 0$ .

Introduce the function

 $\mathcal{L}^{\text{max}}$  and  $\mathcal{L}^{\text{max}}$ 

$$
q(t) = g(t)(1+t)^{-2-\beta} \qquad (q^{(j)}(0) = 0 \text{ for all } j \leq s). \tag{2.4}
$$

Introduce the function<br>  $q(t) = g(t)(1 + t)^{-2-\beta}$   $(q^{(j)}(0) = 0$  for all  $j \le s$ ). (2.4)<br>
Now, if  $Q(p)$  is the Laplace transform of  $q(t)$   $(t \in I\!\!R^+)$ ; then thanks to the relations<br>
(2.3) and (2.4) we have<br>  $|Q(p)| \le c(1 + |p|)^{-s-1}$   $(2.3)$  and  $(2.4)$  we have

$$
|Q(p)| \le c(1+|p|)^{-s-1} \qquad (p \in C\bar{\Pi}_\alpha^*). \tag{2.5}
$$

Let  $\gamma_{\epsilon} = \gamma_0 + \epsilon e^{-i\alpha/2}$ , where  $\gamma_0$  is the boundary of  $\Pi_{\alpha}^{*}$  and  $\epsilon > 0$ . Using the estimation (2.5) it is easy to get

$$
+ \varepsilon e^{-i\alpha/2}, \text{ where } \gamma_0 \text{ is the boundary of } \Pi_{\alpha}^* \text{ and } \varepsilon > 0. \text{ Using the}
$$
\n
$$
q(t) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{pt} Q(p) \, dp = \frac{1}{2\pi i} \int_{\gamma_0}^{\varepsilon} e^{pt} Q(p) \, dp \qquad (t \in I\!\!R^+).
$$

Evidently this function  $q(t)$  is analytic in  $\Pi_{\alpha}$  and bounded in  $\bar{\Pi}_{\alpha}$ . Finally we obtain the following

**Lemma 2.2.** *Every function*  $f \in N$  can be presented in the form

$$
a_n
$$
\n2. Every function  $f \in N$  can be presented in the form

\n
$$
f(t) = P_s(t) + \frac{(1+t)^{\beta+2}}{2\pi i} \int_{\tau_0} e^{pt} Q(p) dp \qquad (t \in \Pi_\alpha).
$$
\nconsider the differential equation

\n
$$
\frac{\partial u(t)}{\partial t} - \lambda u(t) = f(t) \qquad (t \in \Pi_\alpha) \qquad (2.6)
$$
\n
$$
\in N \text{ is given and } u \in N \text{ is the unknown function. When } t \in \mathbb{R}^+, \text{ then}
$$
\nequation (2.6) can be written immediately in function of sign (Re A). In

Now let us consider the differential equation

$$
\frac{\partial u(t)}{\partial t} - \lambda u(t) = f(t) \qquad (t \in \Pi_{\alpha})
$$
\n(2.6)

where  $\lambda \in \Pi, f \in N$  is given and  $u \in N$  is the unknown function. When  $t \in \mathbb{R}^+$ , then the solution of equation (2.6) can be written immediately in function of sign(Re  $\lambda$ ). In our case the form of the solution depends not only of  $\lambda \in \Pi_{\alpha}^{*}$  or  $\lambda \in C\Pi_{\alpha}^{*}$ , but also of  $\in \Pi_{\alpha}$  *(*2.6)<br> **n** function. When *t* ∈ *R*<sup>+</sup>, then<br>
ely in function of sign(Re λ). In<br>
λ ∈ Π<sup>*t*</sup></sup> or λ ∈ *C* Π<sup>*t*</sup><sub>*n*</sub>, but also of<br>
btain a unique form of solution.<br>
uation (2.6) admits a polynomial<br>
ith *f*

but disc the form of the solution depends not only of  $\lambda \in \Pi_{\alpha}$  or  $\lambda \in CH_{\alpha}$ , but also c<br>a position of  $\lambda \in CH_{\alpha}^*$ . For applications we prefer to obtain a unique form of solution<br>At first, if  $f(t) = P_s(t)$  (see (2.2)) At first, if  $f(t) = P_s(t)$  (see (2.2)), then evidently equation (2.6) admits a polynomial solution  $u_0$ . So it remains to consider equation (2.6) with  $f = g$  and  $\lambda \neq 0$ . It is clear that the function

$$
u_1(t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} e^{pt} \frac{C_s(p)}{p - \lambda} dp \qquad (t \in \Pi_{\alpha})
$$
 (2.7)

where  $\varepsilon$  is such that  $\lambda$  and 0 are placed on different sides of  $\gamma_{\varepsilon}$  in the case of  $\lambda \in C\Pi_{\alpha}^*$ , is a solution of equation (2.6) satisfying the estimate  $|u_1(t)| \leq ce^{\epsilon|t|}$   $(t \in \bar{\Pi}_{\alpha})$ . Let us show that  $u_1 \in N$ . Since  $f \in N$  and  $u_1$  satisfies equation (2.6) it is sufficient to verify inequality (1.2) only for  $k = 0$ . From (2.4) we have  $G(p) = (1 - \frac{d}{dp})^{\beta+2}Q(p)$ . Then by integrating by parts we get the solution of equation (2.6) can be writted our case the form of the solution depend<br>
a position of  $\lambda \in C\Pi_{\alpha}^*$ . For applications v<br>
At first, if  $f(t) = P_s(t)$  (see (2.2)), the<br>
solution  $u_0$ . So it remains to conside  $=\frac{1}{2\pi i}\int_{\gamma_e} e^{pt} \frac{C_s(p)}{p-\lambda} d\theta$ <br>are placed on differe<br>) satisfying the estin<br>*N* and  $u_1$  satisfies<br>0. From (2.4) we had<br> $\frac{1}{\pi i} \int_{\gamma_e} \frac{e^{pt}}{p-\lambda} \left(1-\frac{d}{dp}\right)$ 

$$
2\pi i \int_{\gamma_e}^{\gamma_e} p - \lambda^{ap} \qquad (c \in \Omega_\alpha) \qquad (2.1)
$$
\n
$$
\lambda \text{ and 0 are placed on different sides of } \gamma_e \text{ in the case of } \lambda \in C\Pi_\alpha^*,
$$
\n
$$
\lambda \text{ into (2.6) satisfying the estimate } |u_1(t)| \le ce^{\epsilon|t|} (t \in \overline{\Pi}_\alpha). \text{ Let us}
$$
\nSince  $f \in N$  and  $u_1$  satisfies equation (2.6) it is sufficient to verify  
\nfor  $k = 0$ . From (2.4) we have  $G(p) = (1 - \frac{d}{dp})^{\beta+2} Q(p)$ . Then by  
\nwe get\n
$$
u_1(t) = \frac{1}{2\pi i} \int_{\gamma_e} \frac{e^{pt}}{p - \lambda} \left(1 - \frac{d}{dp}\right)^{\beta+2} Q(p) dp
$$
\n
$$
= \frac{1}{2\pi i} \int_{\gamma_0}^{\gamma_e} Q(p) \left(1 + \frac{d}{dp}\right)^{\beta+2} \left(\frac{e^{pt}}{p - \lambda}\right) dp
$$
\n
$$
= \sum_{j \le \beta+2} t^j \int_{\gamma_0}^{\gamma_g} q_j(p) Q(p) e^{pt} dp \qquad (t \in \overline{\Pi}_\alpha)
$$
\n(2.8)

where the functions  $q_j(p)$  are well-defined and bounded, i.e.  $u_1(t)$  is of polynomial growth. Hence the following lemma is proved.

**Lemma 2.3.** *The equation (2.6) admits a solution*  $u \in N$  *for all functions*  $f \in N$ *.*<br>
Now, if we have a system<br>  $\frac{\partial u(t)}{\partial t} - Au(t) = f(t)$   $(t \in \Pi_{\alpha})$ 

Now, if we have a system

$$
\frac{\partial u(t)}{\partial t} - Au(t) = f(t) \qquad (t \in \Pi_{\alpha})
$$

where *A* is a constant matrix of order  $m, f \in N$  is a given vector function and  $u \in N$  is the unknown vector function, we can solve it by transforming the matrix *A* into Jordan form and using Lemma *2.3.*

#### **3. Construction of a solution of system (1.3)**

**Strictly regular case.** Let the system (1.3) be strictly regular. Introduce the following polynoms in *A:*

On the Solution of Partial Differential Equat  
\nuction of a solution of system (1.3)  
\nular case. Let the system (1.3) be strictly regular. Introduce  
\n
$$
\lambda:
$$
\n
$$
Q(\xi, \lambda) = \prod_{j=1}^{r} (\lambda - \lambda_j(\xi)) = \lambda^r + \sum_{j=1}^{r} a_j(\xi) \lambda^{r-j}
$$
\n
$$
R(\xi, \lambda) = \prod_{j=1}^{m-r} (\lambda - \lambda_{r+j}(\xi)) = \lambda^{m-r} + \sum_{j=1}^{m-r} a_{r+j}(\xi) \lambda^{m-r-j}.
$$
\nwn (see [4: Lemma 3.1/p. 194]) that  
\n
$$
|\lambda_j(\xi) \le c_j(1 + |\xi|)^{n_j}, \quad a_j \in C^{\infty}(I\!\!R^n)
$$
\n
$$
|D_{\xi}^{k} a_j(\xi)| \le c_k(1 + |\xi|)^{n_{jk}} \quad (|k| \in N_0).
$$
\na solution of the Cauchy problem  
\n
$$
\sum_{j=1}^{m} a_j(\xi) \frac{\partial^{r-j} v(\xi, t)}{\partial t^{r-j}} = 0 \quad (t \in I\!\!R^n)
$$
\n
$$
D_{\xi}^{j} v(\xi, 0) = b_j(\xi) \quad (0 \le j \le r - 1)
$$

It is well known (see  $[4: Lemma 3.1/p. 194])$  that

$$
\prod_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} (4: \text{ Lemma } 3.1/p. 194]) \text{ that}
$$
\n
$$
|\lambda_j(\xi) \le c_j (1 + |\xi|)^{n_j}, \quad a_j \in C^{\infty}(I\!\!R^n)
$$
\n
$$
|D_{\xi}^{k} a_j(\xi)| \le c_k (1 + |\xi|)^{n_{jk}} \quad (|k| \in N_0).
$$
\n
$$
\text{ion of the Cauchy problem}
$$
\n
$$
v(\xi, t) := \frac{\partial^r v(\xi, t)}{\partial t^r} + \sum_{j=1}^{r} a_j(\xi) \frac{\partial^{r-j} v(\xi, t)}{\partial t^{r-j}} = 0 \quad (t \in \Pi_{\alpha}) \quad (3.2)
$$

Let  $v(\xi, t)$  be a solution of the Cauchy problem

known (see [4: Lemma 3.1/p. 194]) that  
\n
$$
|\lambda_j(\xi) \le c_j(1+|\xi|)^{n_j}, \quad a_j \in C^{\infty}(\mathbb{R}^n)
$$
\n
$$
|D_{\xi}^{k}a_j(\xi)| \le c_k(1+|\xi|)^{n_{jk}} \quad (|k| \in \mathbb{N}_0).
$$
\n(3.1)  
\n(b) be a solution of the Cauchy problem  
\n
$$
Q\left(\xi, \frac{\partial}{\partial t}\right)v(\xi, t) := \frac{\partial^r v(\xi, t)}{\partial t^r} + \sum_{j=1}^r a_j(\xi) \frac{\partial^{r-j} v(\xi, t)}{\partial t^{r-j}} = 0 \quad (t \in \Pi_{\alpha})
$$
\n(3.2)  
\n
$$
D_{\xi}^j v(\xi, 0) = b_j(\xi) \quad (0 \le j \le r-1)
$$
\n(3.3)  
\n
$$
\in C^{\infty}(\mathbb{R}^n)
$$
 satisfies inequality (3.1) and  $v(\xi, t)$  is analytic in  $t \in \Pi_{\alpha}$ . Rewrite  
\n3) in matrix form  
\n
$$
\frac{\partial w(\xi, t)}{\partial t} = B(\xi)w(\xi, t) \quad (t \in \Pi_{\alpha})
$$
\n
$$
w(\xi, 0) = b(\xi)
$$
\n
$$
w = \left(v, \frac{\partial v}{\partial t}, \dots, \frac{\partial^{r-1} v}{\partial t^{r-1}}\right) \quad \text{and} \quad b(\xi) = \left(b_0(\xi), \dots, b_{r-1}(\xi)\right)
$$
\nis a well-defined square matrix of order r with eigenvalues  $\lambda_1(\xi), \dots, \lambda_r(\xi)$ ging to  $\Pi_{\alpha}^*$ . We have  $w(\xi, t) = e^{B(\xi)t}b(\xi)$ , whence by using the well-known

 $D_t^j v(\xi, 0) = b_j(\xi)$   $(0 \le j \le r - 1)$  (3.3)

where  $b_j \in C^{\infty}(I\!\!R^n)$  satisfies inequality (3.1) and  $v(\xi, t)$  is analytic in  $t \in \Pi_{\alpha}$ . Rewrite (3.2), (3.3) in matrix form

$$
\frac{\partial w(\xi, t)}{\partial t} = B(\xi)w(\xi, t) \quad (t \in \Pi_{\alpha})
$$
  
 
$$
w(\xi, 0) = b(\xi)
$$

where

$$
w = \left(v, \frac{\partial v}{\partial t}, \ldots, \frac{\partial^{r-1} v}{\partial t^{r-1}}\right) \quad \text{and} \quad b(\xi) = \left(b_0(\xi), \ldots, b_{r-1}(\xi)\right)
$$

and  $B(\xi)$  is a well-defined square matrix of order *r* with eigenvalues  $\lambda_1(\xi), \ldots, \lambda_r(\xi)$ and  $B(\xi)$  is a well-defined square matrix of order r with eigenvalues  $\lambda_1(\xi), \ldots, \lambda_r(\xi)$ <br>all belonging to  $\Pi_{\alpha}^*$ . We have  $w(\xi, t) = e^{B(\xi)t}b(\xi)$ , whence by using the well-known estimate<br> $\|e^{B(\xi)t}\| \le c \sum_{j=1}^r e^{\text{Re}(\$ estimate  $\frac{\partial^{r-1}v}{\partial t^{r-1}}$  and  $b(\xi) = (b_0(\xi), \ldots, b_{r-1}(\xi))$ <br>
ned square matrix of order r with eigenvalues  $\lambda_1(\xi), \ldots, \lambda_r(\xi)$ <br>
We have  $w(\xi, t) = e^{B(\xi)t}b(\xi)$ , whence by using the well-known<br>  $B(\xi)$ <sup>*t*</sup>  $\|\leq c \sum_{j=1}^r e^{Re(\lambda_j(\xi)t)}$ 

$$
|e^{B(\xi)t}\| \leq c \sum_{j=1}^r e^{\text{Re}(\lambda_j(\xi)t)} (1+|t|)^{r-1} (1+|\xi|)^s
$$

(see [2: p. 222, item 2]) we shall get the estimate

$$
\left| D_{\xi}^{j} D_{t}^{k} v(\xi, t) \right| \leq c_{jk} (1 + |\xi|)^{m_{jk}} (1 + |t|)^{n_{jk}}.
$$
 (3.4)

For beginning we take  $f(x,t) = t^{\jmath}q(x) \in M$ . Every function  $u(x,t) \in M$  can be  $\text{considered as a distribution from } S'(I\!\!R^n)$  depending on parameter  $t\in \bar{\Pi}_\alpha$  and satisfying the inequality  $f: t^j q(x) \in M$ . Every function  $u(x)$ <br> $(R^n)$  depending on parameter  $t \in \overline{\Pi}$ <br> $\leq c_j(1+|t|)$ <sup> $\beta$ </sup> $\|\varphi\|$ ,  $(\varphi \in S(R^n))$ 

$$
\left|\frac{\partial^j}{\partial t^j}\langle u(x,t),\varphi(x)\rangle\right| \leq c_j(1+|t|)^\beta \|\varphi\|_s \qquad (\varphi \in S(\mathbb{R}^n))
$$

where  $S(R^n)$  and  $S'(R^n)$  are the pair of Schwartz spaces and

$$
S'(I\!\!R^n)
$$
 are the pair of Schwartz spaces and  

$$
\|\varphi\|_{s} = \sup_{x} (1 + |x|^2)^{s/2} \sum_{|\alpha| \leq s} |D_x^{\alpha} \varphi(x)| \qquad (s \in I\!\!N_0).
$$

Let  $\hat{u}(\xi, t) = F_x(u(x, t))$  denote the generalized Fourier transform of  $u \in M$  ( $F^{-1}$  will denote the inverse Fourier transform). The Fourier image of system (1.3) has the form

re the pair of Schwartz spaces and  
\n
$$
[(1+|x|^2)^{s/2} \sum_{|\alpha| \leq s} |D_x^{\alpha} \varphi(x)| \quad (s \in \mathbb{N}_0).
$$
\nnote the generalized Fourier transform of  $u \in M$  ( $F^{-1}$  will  
\nransform). The Fourier image of system (1.3) has the form  
\n
$$
\frac{\partial \hat{u}(\xi, t)}{\partial t} = A(\xi) \hat{u}(\xi, t) + t^j \hat{q}(\xi).
$$
\n(3.5)  
\nwe consider

As a solution of system (3.5) we consider<br>  $\hat{y}(f, t) = \frac{j!}{s!(f+1)!} \int_{\mathbb{R}^n} f(x) dx$ 

$$
S'(I\!\!R^n)
$$
 are the pair of Schwartz spaces and  
\n
$$
|\varphi||_s = \sup(1+|x|^2)^{s/2} \sum_{|\alpha| \leq s} |D_x^{\alpha}\varphi(x)| \qquad (s \in I\!\!N_0).
$$
\n
$$
(x,t)
$$
 denote the generalized Fourier transform of  $u \in M$  ( $F^{-1}$  will  
\nFourier transform). The Fourier image of system (1.3) has the form  
\n
$$
\frac{\partial \hat{u}(\xi,t)}{\partial t} = A(\xi)\hat{u}(\xi,t) + t^j \hat{q}(\xi).
$$
\n
$$
s\text{term } (3.5) \text{ we consider}
$$
\n
$$
\hat{u}(\xi,t) = \frac{j!}{2\pi i} \int_{\gamma^-(\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda \hat{q}(\xi) \qquad (3.6)
$$
\n
$$
\text{closed contour containing only the roots } \lambda_1(\xi), \dots, \lambda_r(\xi) \in \Pi_{\alpha}^*
$$
 and the

where  $\gamma^-(\xi)$  is a closed contour containing only the roots  $\lambda_1(\xi), \ldots, \lambda_r(\xi) \in \Pi_{\alpha}^*$  and the point  $\lambda = 0$ . Introduce the matrix

$$
V(\xi, t) = \frac{j!}{2\pi i} \int_{\gamma^{-}(\xi)} (\lambda E_m - A(\xi))^{\gamma} \wedge \int_{\gamma^{-}(\xi)} \lambda_1(\xi), \dots, \lambda_r(\xi) \in
$$
  
Introduce the matrix  

$$
V(\xi, t) = \frac{j!}{2\pi i} \int_{\gamma^{-}(\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda \qquad (t \in \Pi_{\alpha}).
$$
  
3.1. The elements  $v_{ls}(\xi, t)$  of the matrix  $V(\xi, t)$  satisfy the inequality  
We have  

$$
V(\xi, 0) = \frac{j!}{2\pi i} \int_{\gamma^{-}(\xi)} \frac{a(\xi, \lambda)}{\lambda^{j+1} Q(\xi, \lambda) R(\xi, \lambda)} d\lambda
$$

Lemma 3.1. *The elements*  $v_{ls}(\xi,t)$  *of the matrix*  $V(\xi,t)$  *satisfy the inequality (3.4).* **Proof.** We have  $\pi i \int_{\gamma^{-}(\xi)} (\lambda D_m - A(\xi)) \lambda^{\gamma^{-}}(\xi)$ <br>
lements  $v_{ls}(\xi, t)$  of the matrix  $V(\xi, t)$  satisfy the inequality (3.4).<br>  $\pi(\xi, 0) = \frac{j!}{2\pi i} \int_{\gamma^{-}(\xi)} \frac{a(\xi, \lambda)}{\lambda^{j+1} Q(\xi, \lambda) R(\xi, \lambda)} d\lambda$ <br>  $(\xi, \lambda) = (aE_m - A(\xi))^{-1} |\lambda E_m - A(\xi)|$ .<br>  $\pi^+ Q(\xi, \lambda$ 

$$
E(t) = \frac{j!}{2\pi i} \int_{\gamma - (\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} dt
$$
  
contour containing only the roots  $\lambda_1(\xi)$ ,  
the matrix  

$$
\frac{j!}{2\pi i} \int_{\gamma - (\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda
$$

$$
e^{i\theta}
$$

$$
e^{i\theta}
$$

$$
V(\xi, 0) = \frac{j!}{2\pi i} \int_{\gamma - (\xi)} \frac{a(\xi, \lambda)}{\lambda^{j+1} Q(\xi, \lambda) R(\xi, \lambda)} d\lambda
$$

$$
a(\xi, \lambda) = (aE_m - A(\xi))^{-1} |\lambda E_m - A(\xi)|.
$$

$$
\lambda^{j+1} Q(\xi, \lambda) \text{ and } R(\xi, \lambda) \text{ are coprime, the}
$$

$$
1 = r(\xi, \lambda) \lambda^{j+1} Q(\xi, \lambda) + q(\xi, \lambda) R(\xi, \lambda)
$$

$$
\lambda \text{ are polynomials in } \lambda \text{ with coefficients so}
$$

$$
0) \text{ can be rewritten in the form}
$$

$$
V(\xi, 0) = \frac{j!}{2\pi i} \int_{\gamma - (\xi)} \frac{q(\xi, \lambda)}{\lambda^{j+1} Q(\xi, \lambda)} a(\xi, \lambda) d\lambda
$$

$$
T(\xi) \text{ encloses all roots of denominator}
$$

with

$$
a(\xi,\lambda)=(aE_m-A(\xi))^{-1}|\lambda E_m-A(\xi)|.
$$

Since the polynomials  $\lambda^{j+1}Q(\xi,\lambda)$  and  $R(\xi,\lambda)$  are coprime, then

$$
1 = r(\xi, \lambda)\lambda^{j+1}Q(\xi, \lambda) + q(\xi, \lambda)R(\xi, \lambda)
$$
\n(3.7)

where  $r(\xi, \lambda)$  and  $q(\xi, \lambda)$  are polynomials in  $\lambda$  with coefficients satisfying the inequality (3.1). The matrix  $V(\xi,0)$  can be rewritten in the form

$$
V(\xi,0) = \frac{j!}{2\pi i} \int\limits_{\gamma^{-}(\xi)} \frac{q(\xi,\lambda)}{\lambda^{j+1}Q(\xi,\lambda)} a(\xi,\lambda) d\lambda.
$$
 (3.8)

Since the contour  $\gamma^-(\xi)$  encloses all roots of denominator  $\lambda^{j+1}Q(\xi,\lambda)$ , then computing (3.8) with help of the residue theorem at the point  $\lambda = \infty$  we immediately get  $2\pi i \int \lambda^{j+1} Q(\xi, \lambda) dx$ <br> *p*<sup>-</sup>( $\xi$ ) encloses all roots of denormely of the residue theorem at the<br>  $D_{\xi}^{k} D_{i}^{j} v_{ls}(\xi, 0) \leq c_{kj} (1 + |\xi|)^{n_{kj}}$ <br>  $c$  function  $v_{ls}(\xi, t)$  satisfies the differential

$$
\left|D_{\xi}^{k}D_{t}^{j}v_{ls}(\xi,0)\right|\leq c_{kj}(1+|\xi|)^{n_{kj}}\qquad (|k|,j\in\mathbb{N}_{0}).
$$

Now notice that the function  $v_{ls}(\xi, t)$  satisfies the differential equation

$$
\frac{\partial^{j+1}}{\partial t^{j+1}}Q\left(\xi,\frac{\partial}{\partial t}\right)v_{ls}(\xi,t)=0
$$

the characteristic roots of which belong to the set  $\Pi_{\alpha}^*$ . Therefore it satisfies the estimate  $(3.4)$  (see  $(3.2)$  and  $(3.3)$ )  $\blacksquare$ 

Lemma 3.2. The inclusion  $F_{\xi}^{-1}(v_{ls}(\xi,t)\hat{q}_j(\xi)) \in M$  is true, where the vector func*tion*  $q(x) = (q_1(x), \ldots, q_m(x))$  is an element of M.

function

Lemma 3.2. The inclusion 
$$
F_{\xi}^{-1}(v_{ls}(\xi, t)\hat{q}_j(\xi)) \in M
$$
 is true, where the vector func-  
\n $q(x) = (q_1(x),..., q_m(x))$  is an element of M.  
\n**Proof.** We have  $|D_x^k q_j(x)| \le c_k(1+|x|)^{\gamma}$  (see (1.2)). Let  $k_0 \gg 1$  be such that the  
\ntion  
\n $w_{ls}(x,t) = F_{\xi}^{-1}((1+|\xi|^1)^{-k_0}v_{ls}(\xi,t)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-k_0}v_{ls}(\xi,t)e^{-i\xi x}d\xi$   
\nsfies the estimate  
\n $|w_{ls}(x,t)| \le c(1+|x|)^{-n-|\gamma|-1}(1+|t|)^{m_0}.$  (3.9)  
\nif  $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ , then we have

satisfies the estimate

$$
w_{ls}(x,t)| \le c(1+|x|)^{-n-|\gamma|-1}(1+|t|)^{m_0}.
$$
\n(3.9)

And if  $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ , then we have

$$
F_{\xi}^{-1}(v_{ls}(\xi,t)\hat{q}_j(\xi)) = w_{ls}(x,t) * (1 - \Delta_x)^{k_0} q(x)
$$

where the operation  $*$  is taken in relation to the variable  $x$ , and according to the Peetre inequalities (experimentation to<br>taken in relation to<br> $(1 + |x - y|)^{-|\gamma|} \le$ 

$$
(1+|x-y|)^{-|\gamma|} \leq (1+|x|)^{\gamma}(1+|y|)^{\gamma}
$$

and

$$
(1+|x-y|)^{\gamma} \le (1+|x|)^{\gamma} (1+|y|)^{\gamma}
$$

when  $\gamma \leq 0$  or  $\gamma > 0$ , respectively, we get

$$
(1+|x-y|)^{\gamma} \le (1+|x|)^{\gamma} (1+|y|)^{\gamma}
$$
  
\n
$$
\gamma \le 0 \text{ or } \gamma > 0, \text{ respectively, we get}
$$
  
\n
$$
\left| F_{\xi}^{-1}(v_{ls}(\xi, t)\hat{q}_j(\xi)) \right| \le c(1+|x|)^{\gamma} (1+|t|)^{m_0} \int (1+|y|)^{-n-|\gamma|-1} (1+|y|)^{\gamma} dy
$$
  
\n
$$
\le c_1 (1+|x|)^{\gamma} (1+|t|)^{m_0}.
$$

The derivatives can be estimated similarly  $\blacksquare$ 

Thus, if  $f(x,t) = t^j q(x)$ , then the solution *u* of system (1.3) is constructed, namely

or 
$$
\gamma > 0
$$
, respectively, we get  
\n
$$
y_{ls}(\xi, t)\hat{q}_j(\xi)\Big| \leq c(1+|x|)^{\gamma}(1+|t|)^{m_0} \int (1+|y|)^{-n-|\gamma|-1}(1+|y|)^{\gamma}dy
$$
\n
$$
\leq c_1(1+|x|)^{\gamma}(1+|t|)^{m_0}.
$$
\nwe can be estimated similarly

\n
$$
f(x,t) = t^j q(x), \text{ then the solution } u \text{ of system (1.3) is constructed, namely}
$$
\n
$$
u(x,t) = F_{\xi}^{-1} \left( \frac{j!}{2\pi i} \int_{\gamma-(\xi)} (\lambda E_m - A(\xi))^{-1} \lambda^{-j-1} e^{\lambda t} d\lambda \cdot \hat{q}(\xi) \right).
$$
\ne representation

\n
$$
f(x,t) = f(x,0) + t \frac{\partial f}{\partial t}(x,0) + ... + \frac{t^l}{l!} \frac{\partial^l f}{\partial t^l}(x,0) + g(x,t)
$$
\ntrary  $l \in \mathbb{N}$ , we can reduce our problem to

\n
$$
\frac{\partial u(x,t)}{\partial t} = A(D_x) u(x,t) + g(x,t) \tag{3.10}
$$

So, using the representation

$$
f(x,t)=f(x,0)+t\frac{\partial f}{\partial t}(x,0)+\ldots+\frac{t^l}{l!}\frac{\partial^l f}{\partial t^l}(x,0)+g(x,t)
$$

with an arbitrary  $l \in \mathbb{N}$ , we can reduce our problem to

$$
\frac{\partial u(x,t)}{\partial t} = A(D_x)u(x,t) + g(x,t)
$$
\n(3.10)

where  $g \in M$  and  $D_i^j g(x,0) = 0$  for all  $j \leq l$ . In (1.2) assume  $\beta \in N$ . Introduce the vector function (see (2.4))  $q(x,t) = g(x,t)(1+t)^{-\beta-2}$  (3.11)<br> *q*(*x,t*) = *g*(*x,t*)(1 + *t*)<sup>- $\beta$ -2 (3.11)</sup>

$$
q(x,t) = g(x,t)(1+t)^{-\beta-2}
$$
\n(3.11)

As Fourier image of a solution of system (3.10) we consider the expression

$$
D_t^j g(x,0) = 0 \text{ for all } j \le l. \text{ In (1.2) assume } \beta \in \mathbb{N}. \text{ Introduce the}
$$
\n
$$
g(x,t) = g(x,t)(1+t)^{-\beta-2} \qquad (3.11)
$$
\n
$$
f(x,t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}(\xi)} (\lambda E_m - A(\xi))^{-1} \hat{G}(\xi, \lambda) e^{\lambda t} d\lambda \qquad (3.12)
$$
\n
$$
f(x) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}(\xi)} (\lambda E_m - A(\xi))^{-1} \hat{G}(\xi, \lambda) e^{\lambda t} d\lambda \qquad (3.13)
$$

where  $\gamma_{\epsilon}(\xi) = \gamma_0 + \epsilon(\xi)e^{-i\alpha/2}, \epsilon(\xi) > 0$ , is such that the roots  $\lambda_{r+1}(\xi), \ldots, \lambda_m(\xi) \in C\Pi_{\alpha}^*$ and the point  $\lambda = 0$  are on different sides of it (see (2.7)) and  $\hat{G}(\xi, \lambda)$  is the Laplace-Fourier transform of  $g(x,t)$   $((x,t) \in I\!\!R^n \times I\!\!R^+)$ .  $\gamma_0 + \varepsilon(\xi) e^{-i\alpha/2}$ <br>  $\lambda = 0$  are on<br>
form of  $g(x, t)$ <br>
analogous to (<br>  $\hat{u}(\xi, t) = \frac{1}{2\pi i}$ <br>  $\gamma_t$ 

A formula analogous to (3.7) permits us to rewrite (3.12) in the form

$$
q(x,t) = g(x,t)(1+t)^{-\beta-2}
$$
(3.11)  
As Fourier image of a solution of system (3.10) we consider the expression  

$$
\hat{u}(\xi,t) = \frac{1}{2\pi i} \int (\lambda E_m - A(\xi))^{-1} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda
$$
(3.12)  
where  $\gamma_{\epsilon}(\xi) = \gamma_0 + \epsilon(\xi) e^{-i\alpha/2}, \epsilon(\xi) > 0$ , is such that the roots  $\lambda_{r+1}(\xi), \dots, \lambda_m(\xi) \in C\Pi_{\alpha}^*$   
and the point  $\lambda = 0$  are on different sides of it (see (2.7)) and  $\hat{G}(\xi,\lambda)$  is the Laplace-  
Fourier transform of  $g(x,t)$  (( $x,t$ )  $\in \mathbb{R}^n \times \mathbb{R}^+$ ).  
A formula analogous to (3.7) permits us to rewrite (3.12) in the form  

$$
\hat{u}(\xi,t) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}(\xi)} \frac{q(\xi,\lambda)a(\xi,\lambda)}{Q(\xi,\lambda)} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda
$$
( $t \in \Pi_{\alpha}$ ).  

$$
+ \frac{1}{2\pi i} \int_{\gamma_{\epsilon}(\xi)} \frac{r(\xi,\lambda)a(\xi,\lambda)}{R(\xi,\lambda)} \hat{G}(\xi,\lambda) e^{\lambda t} d\lambda
$$
( $t \in \Pi_{\alpha}$ ).  
The integrals here are convergent because of  $D_t^j g(x,0) = 0$  for any  $j \le l$  with  $l \gg 1$   
taken in advance.  
Let  $l$ , be

taken in advance.

Let  $l_0$  be such that

$$
T_{\gamma_{\epsilon}(\xi)} \frac{\partial}{\partial R(\xi, \lambda)} G(\xi, \lambda) e^{\lambda \tau} d\lambda
$$
\nconvergent because of  $D_{\tau}^{j} g(x, 0) = 0$  for

\n
$$
\left| \frac{q(\xi, \lambda) a(\xi, \lambda)}{\lambda^{l_{0}} Q(\xi, \lambda)} \right| \le c(\xi) \frac{1}{|\lambda|} \qquad (|\lambda| > 1).
$$
\n
$$
\hat{K}(\xi, t) = \frac{1}{2\pi i} \int \frac{q(\xi, \lambda) a(\xi, \lambda)}{\lambda^{l_{0}} Q(\xi, \lambda)} e^{\lambda t_{0}} d\lambda
$$

Introduce the matrix

$$
\hat{K}(\xi,t) = \frac{1}{2\pi i} \int_{\gamma^{-}(\xi)} \frac{q(\xi,\lambda)a(\xi,\lambda)}{\lambda^{t_0}Q(\xi,\lambda)} e^{\lambda t_0} d\lambda
$$

with  $\gamma^-(\xi)$  defined in (3.6). The first integral in (3.13) will take the form

$$
\hat{K}(\xi, t) = \frac{1}{2\pi i} \int_{\gamma^{-}(\xi)} \frac{q(\xi, \lambda)a(\xi, \lambda)}{\lambda^{l_0}Q(\xi, \lambda)} e^{\lambda t_0} d\lambda
$$
\nif in (3.6). The first integral in (3.13) will take the

\n
$$
\hat{u}_1(\xi, t) = \int_0^t \hat{K}(\xi, \tau) \hat{g}_t^{(l_0)}(\xi, t - \tau) d\tau \qquad (t \in \Pi_\alpha).
$$
\nabove (see Lemma 3.1) the elements  $\hat{k}_{ls}(\xi, t)$  of

As it was shown above (see Lemma 3.1) the elements  $\hat{k}_{ls}(\xi, t)$  of the matrix  $\hat{K}(\xi, t)$ satisfy the estimation (3.4), hence we can easily regularize the inverse transform  $u_1(x,t)$ of  $\hat{u}_1(\xi, t)$  (see Lemma 3.2) and have the inclusion  $u_1 \in M$ .

Now let us examine the second term  $\hat{u}_2(\xi,t)$  in (3.13). At first as

$$
\hat{G}(\xi,t) = \left(1 - \frac{d}{d\lambda}\right)^{\beta+2} \hat{Q}(\xi,\lambda)
$$

On the Solution of Partial Differential Equations 611<br>(see (3.11)), where  $\hat{Q}(\xi, \lambda)$  satisfies an inequality like (2.5), integrating by parts we can reduce  $\hat{u}_2(\xi,t)$  to the form

On the Solution of Partial Differential Equations  
\n
$$
\hat{Q}(\xi,\lambda)
$$
 satisfies an inequality like (2.5), integrating by par  
\nthe form  
\n
$$
\hat{u}_2(\xi,t) = \frac{1}{2\pi i} \int_{\gamma_0} \hat{Q}(\xi,\lambda) \psi(\xi,\lambda,t) d\lambda \qquad (t \in \Pi_\alpha)
$$
\n
$$
\psi(\xi,\lambda,t) = \left(1 + \frac{d}{d\lambda}\right)^{\beta+2} \frac{r(\xi,\lambda)a(\xi,\lambda)e^{\lambda t}}{R(\xi,\lambda)}.
$$
\nthe inequality  
\n
$$
|R(\xi,\lambda)| \ge c(1+|\xi|^2)^s \qquad ((\xi,\lambda) \in I\!\!R^n \times \gamma_0)
$$
\n
$$
P(\xi,\lambda) = \int_{\gamma_0}^{\gamma_1} \psi(\xi,\lambda) d\xi \, d\lambda \qquad (1 + \frac{1}{2})^s \qquad
$$

where

$$
\psi(\xi, \lambda, t) = \left(1 + \frac{d}{d\lambda}\right)^{\beta + 2} \frac{r(\xi, \lambda) a(\xi, \lambda) e^{\lambda t}}{R(\xi, \lambda)}
$$
  
inequality  

$$
R(\xi, \lambda) \ge c(1 + |\xi|^2)^s \qquad ((\xi, \lambda) \in I\!\!R^n \times \gamma)
$$

As consequence of the inequality

$$
|R(\xi,\lambda)| \geq c(1+|\xi|^2)^s \qquad ((\xi,\lambda) \in I\!\!R^n \times \gamma_0)
$$
 (3.14)

h

where  
\n
$$
\psi(\xi, \lambda, t) = \left(1 + \frac{d}{d\lambda}\right)^{\beta+2} \frac{r(\xi, \lambda)a(\xi, \lambda)e^{\lambda t}}{R(\xi, \lambda)}.
$$
\nAs consequence of the inequality  
\n
$$
|R(\xi, \lambda)| \ge c(1 + |\xi|^2)^s \qquad ((\xi, \lambda) \in \mathbb{R}^n \times \gamma_0)
$$
\n(3.14)  
\n(see [1: Lemma 1.2]) we get the estimation  
\n
$$
|D_{\xi}^{k}\psi(\xi, \lambda, t)| \le c_{k}(1 + |\xi|)^{m_{k}}(1 + |\lambda|)^{n_{k}}(1 + |t|)^{\beta+2}
$$
\n(3.15)  
\nfor all  $(\xi, \lambda, t) \in \mathbb{R}^n \times \gamma_0 \times \Pi_{\alpha}$ . Now it is clear that by the same way

for all  $(\xi, \lambda, t) \in \mathbb{R}^n \times \gamma_0 \times \Pi_\alpha$ . Now it is clear that by the same way

On the Solution of Partial Differential Equation  
\n(), where 
$$
\hat{Q}(\xi, \lambda)
$$
 satisfies an inequality like (2.5), integrating by pa  
\n
$$
\hat{u}_2(\xi, t) = \frac{1}{2\pi i} \int_{\gamma_0} \hat{Q}(\xi, \lambda) \psi(\xi, \lambda, t) d\lambda \qquad (t \in \Pi_\alpha)
$$
\n
$$
\psi(\xi, \lambda, t) = \left(1 + \frac{d}{d\lambda}\right)^{\beta+2} \frac{r(\xi, \lambda) a(\xi, \lambda) e^{\lambda t}}{R(\xi, \lambda)}
$$
\nuence of the inequality

\n
$$
|R(\xi, \lambda)| \ge c(1 + |\xi|^2)^s \qquad ((\xi, \lambda) \in I\!\!R^n \times \gamma_0)
$$
\nerman 1.2]) we get the estimation

\n
$$
|D^t_{\xi} \psi(\xi, \lambda, t)| \le c_k (1 + |\xi|)^{m_k} (1 + |\lambda|)^{n_k} (1 + |t|)^{\beta+2}
$$
\n
$$
\lambda, t) \in I\!\!R^n \times \gamma_0 \times \Pi_\alpha.
$$
 Now it is clear that by the same way  
\n
$$
u_2(x, t) = F_\xi^{-1} (\hat{u}_2(\xi, t)) = \frac{1}{2\pi i} F_\xi^{-1} \left( \int_{\gamma_0} \hat{Q}(\xi, \lambda) \psi(\xi, \lambda, t) d\lambda \right) \in M.
$$
\n11 for strictly regularity of system (1.3) is proved.

\nar case. In the considerations made above the condition (1.1) (correlation) using very essential (see estimation (3.1) and others), we

Theorem 1 for strictly regularity of system (1.3) is proved.

**2. Regular case.** In the considerations made above the condition (1.1) (condition of strictly regularity) was very essential (see estimation (3.1) and others). Assuming that the condition (1.1) is violated only at the point  $\xi = 0$  (this does not loss the generality) we can show (see [i: Lemmas 1.2 and 1.3]) that in all inequalities crucial (i.e. in (3.1),  $(3.14)$  and  $(3.15)$ ) we shall have the factor  $|\xi|^{s_k}$  with  $s_k \leq 0$  and this circumstance adds new difficulties (the estimation (3.9) was established thanks to smoothness of  $v_{ls}(\xi, t)$ in  $\xi \in \mathbb{R}^n$ ).

Let us demonstrate how we can surmount this obstacle. First, consider the equation  $\Delta_x u(x,t) = f(x,t)$ , where  $\Delta_x$  is the Laplacean,  $f \in M$  is a given function and  $u \in M$ is the unknown function, and let us prove its solvability. If  $n = 1$ , i.e.  $\Delta_z = \frac{\partial^2}{\partial_z z}$ , then obviously this equation admits a solution from *M*. Now we are going to show how the case  $n \geq 2$  can be reduced to the case  $n-1$ . For this purpose we replace  $f(x, t)$  with  $(i\frac{\partial}{\partial x_1})^{v_1} f(x,t)$  and consider the expression  $|\xi|^{-2} \xi_1^{v_1} \hat{f}(\xi,t)$ . With an appropriate choice of natural  $v_1$  we can reach any smothness order in  $\mathbb{R}^n$  of the function  $|\xi|^{-2} \xi_1^{v_1}$ . By taking a natural  $k \gg 1$  we can readly establish the inclusion and consider the expression  $|\xi|^{-2} \xi_1^{v_1} f(\xi, t)$ . With an approximate can reach any smothness order in  $\mathbb{R}^n$  of the function  $u_0(x,t) = F_{\xi}^{-1} \left( |\xi|^{-2} \xi_1^{v_1} (1 + |\xi|^2)^{-k} (1 + |\xi|^2)^k \hat{f}(\xi, t) \right)$ 

$$
u_0(x,t) = F_{\xi}^{-1} \left( |\xi|^{-2} \xi_1^{v_1} (1 + |\xi|^2)^{-k} (1 + |\xi|^2)^k \hat{f}(\xi, t) \right)
$$
  
=  $F_{\xi}^{-1} \left( |\xi|^{-2} \xi_1^{v_1} (1 + |\xi|^2)^{-k} \right) * (1 - \Delta_x)^k f(x,t)$   
 $\in M$ 

where the operation \* is taken in relation to the variable x and  $\Delta_x u_0 = (i \frac{\partial}{\partial x_1})^{v_1} f$ . Find  $u_1 \in M$  in a way that  $(i\frac{\partial}{\partial x_1})^{v_1}u_1 = u_0$ . Then we get

$$
\left(i\frac{\partial}{\partial x_1}\right)^{v_1}(\Delta_x u_1 - f) = 0
$$

or, by integrating with respect to  $x_1$ ,

$$
\left(i\frac{\partial}{\partial x_1}\right)^{v_1}(\Delta_x u_1 - f) = 0
$$
  
g with respect to  $x_1$ ,  

$$
\Delta_x u_1 = f + \sum_{j \le v_1 - 1} a_j(x', t) x_1^j, \qquad x' = (x_2, \dots, x_n),
$$

where  $a_j(x',t) \in M$  are well-defined functions. Introducing  $w = u_1 - u$  we shall get

in relation to the variable 
$$
x
$$
 and  $\Delta_x u_0 = (i \frac{\partial}{\partial x_1})^{v_1} f$ . Find  
\n $u_1 = u_0$ . Then we get  
\n
$$
\left(i \frac{\partial}{\partial x_1}\right)^{v_1} (\Delta_x u_1 - f) = 0
$$
\nto  $x_1$ ,  
\n
$$
\sum_{\leq v_1 - 1} a_j(x', t) x_1^j, \qquad x' = (x_2, \dots, x_n),
$$
\n
$$
\sum_{\leq v_1 - 1} a_j(x', t) x_1^j.
$$
\n
$$
\Delta_x w = \sum_{j \leq v_1 - 1} a_j(x', t) x_1^j.
$$
\n(3.16)

Let us seek  $w(x,t)$  in the form

$$
y \le v_1 - 1
$$
  
\n
$$
w(x, t) = \sum_{j \le v_1 - 1} c_j(x', t)x_1^j
$$

with unknown  $c_j(x',t) \in M$ . Substituting  $w(x,t)$  into equation (3.16) and making use of linearly independence of the functions  $1, x_1, \ldots, x_1^{v_1-1}$ , we obtain

$$
\Delta_{x'}c_{v_{1}-1}(x',t) = a_{v_{1}-1}(x',t)
$$
  
\n
$$
\Delta_{x'}c_{j}(x',t) = B_{j}(c_{j+1}(x',t),\ldots,c_{v_{1}-1}(x',t)) + b_{j}(x',t)
$$
  
\n
$$
j = v_{1} - 2,\ldots,0
$$
  
\nis a linear expression and  $b_{j}(x',t) \in M$  are certain function  
\nis realized. Hence we have proved the following  
\n8.3. The equation  $\Delta_{x}u = f$  is always solvable in M.  
\nability of the system (1.3) can be established in the same way,  
\nt) instead of  $f(x,t)$  and with an appropriate choice of na  
\nthe system  
\n
$$
\frac{\partial u}{\partial t} = A(D_{x})u + \Delta_{x}^{v}f.
$$
  
\nthe quotient of a solution of system (3.17). Then according to M in a way that  $\Delta_{x}^{v}u_{1} = u_{0}$ . Now we put  $u_{0}$  in (3.17) and

where  $B_j(\cdots)$  is a linear expression and  $b_j(x',t) \in M$  are certain functions. Thus the desired reduction is realized. Hence we have proved the following

**Lemma 3.3.** *The equation*  $\Delta_x u = f$  *is always solvable in M.* 

The solvability of the system (1.3) can be established in the same way: first in (1.3) we put  $\Delta_x^v f(x, t)$  instead of  $f(x, t)$  and with an appropriate choice of natural *v* we get solvability of the system thus the first in  $(1.3)$ <br> *ural*  $v$  we get<br>  $(3.17)$ <br>
emma  $3.3$  we *A*  $b_j(x', t) \in M$  are certain functions. Thus the<br>have proved the following<br>*f* is always solvable in *M*.<br>an be established in the same way: first in (1.3)<br>with an appropriate choice of natural v we get<br> $A(D_x)u + \Delta_x^v f$ . (3.

$$
\frac{\partial u}{\partial t} = A(D_x)u + \Delta_x^v f. \tag{3.17}
$$

Further, let  $u_0$  denote a solution of system  $(3.17)$ . Then according to Lemma 3.3 we can find  $u_1 \in M$  in a way that  $\Delta_x^v u_1 = u_0$ . Now we put  $u_0$  in (3.17) and get

$$
\Delta_{\mathbf{x}}^{\mathbf{v}} \left( \frac{\partial u_1}{\partial t} - A(D_{\mathbf{x}})u_1 - f \right) = 0. \tag{3.18}
$$

With the help of Fourier transformation from (3.18) we deduce

$$
\Delta_{\mathbf{x}}^{\mathbf{v}} \left( \frac{\partial u_1}{\partial t} - A(D_{\mathbf{x}})u_1 - f \right) = 0.
$$
  
transformation from (3.18) we deduce  

$$
\frac{\partial u_1}{\partial t} = A(D_{\mathbf{x}})u_1 + f + \sum_{|j| \le 2v - 1} a_j(t)x^j
$$

with certain coefficients  $a_j \in M$ . Now consider the system

On the Solution of Partial Differential Equations  
\n
$$
\in M
$$
. Now consider the system  
\n
$$
\frac{\partial w}{\partial t} = A(D_x)w + \sum_{|j| \le 2v-1} a_j(t)x^j
$$
\n(3.19)  
\nThe form  
\n
$$
w = \sum_{|j| \le 2v-1} c_j(t)x^j
$$
\n(c,  $\in N$ . Substituting w into (3.19) and taking account of

and let us seek  $w \in M$  in the form

$$
w=\sum_{|j|\leq 2v-1}c_j(t)x^j
$$

with unknown coefficients  $c_j \in N$ . Substituting *w* into (3.19) and taking account of linearly independence of polynoms  $\{x^j | \; |j|=0,\ldots,2v-1\}$  we get the system

$$
v_1 \le v_1
$$
  
ients  $c_j \in N$ . Substituting  $w$  into (3.19) a:  
of polynomials  $\{x^j | |j| = 0, ..., 2v - 1\}$  we get  

$$
\frac{\partial c_j(t)}{\partial t} = A(0)c_j(t) + a_j(t) \qquad (|j| = 2v - 1)
$$

which is solvable (see the system (2.9)). The rest  $c_j$  ( $|j| \leq 2v - 2$ ) can be determined in the same manner. Evidently  $u_1 - w$  is the desired solution of system (1.3). Theorem 1 is completely proved.

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Received 13.08.1992; in revised form 11.04.1994