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Derived Sets in Multiobjective Optimization

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Abstract. In the present paper the theory of derived sets established by M. R. Hestenes and J. W. Nieuwenhuis for optimization problems with a real-valued objective function and a finite number of constraints is extended to multiobjective optimization problems. The main result asserts that local solutions of weak multiobjective optimization problems satisfy a multiplier rule.

Keywords: *Multiobjective optimization, necessary optimality conditions, derived set*

AMS subject classification: 90C29, 49K27

1. Introduction

Consider the following optimization problem:

(P$_1$) Maximize $f_1(x)$ subject to $x \in X$, $f_2(x) \geq 0$, $f_3(x) = 0$

where X is a non-empty set and $f_1 \colon X \to R$, $f_2 \colon X \to R^{m_2}$, $f_3 \colon X \to R^{m_3}$. The notation $f_2(x) \geq 0$ means that each component of $f_2(x)$ is non-negative.

Let $m = 1 + m_2 + m_3$. By making use of so-called derived sets in the image space R^m of the vector function $f = (f_1, f_2, f_3) : X \to R^m$, Hestenes has shown in [8 - 10] that the solutions of problem (P$_1$) satisfy a certain multiplier rule which generalizes the classical Lagrange multiplier rule concerning the optimization of real-valued functions subject to equality constraints. Hestenes' multiplier rule played an important role in optaining necessary optimality conditions in optimization theory, variational theory and control theory. It also has been one of the starting points for the present paper.

According to the definition given in [10: p. 368], a non-empty subset $D \subseteq R^m$ is said to be a *derived set* for the function f at the point $x_0 \in X$ if for every $n \in N$ and every n-tuple $(d^1, \ldots, d^n)$ of points belonging to D there exist a number $r > 0$ and a function $\omega : [0, r]^n \to X$ such that the following conditions are satisfied:

(i) $\lim_{t \to 0} \frac{1}{\|t\|}\big(f(\omega(t)) - f(x_0) - A(t)\big) = 0$, where $A(t) = t_1 d^1 + \ldots + t_n d^n$ for all $t = (t_1, \ldots, t_n) \in R^n$.

(ii) $\omega(0) = x_0$.

(iii) $f \circ \omega$ (the composite function) is continuous.

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A convex cone in R^m that is a derived set for f at $x_0 \in X$ is said to be a *derived convex cone* for f at x_0.

The above-mentioned multiplier rule by Hestenes (see [10: Theorem 10.1]) asserts that, if $x_0 \in X$ is a solution to problem (P_1) and $D \subseteq R \times R^{m_2} \times R^{m_3}$ is a derived set for f at x_0, then there exists a vector

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in R \times R^{m_2} \times R^{m_3} \setminus \{(0,0,0)\}$$

such that

$$\lambda_1^* \geq 0 \qquad \text{and} \qquad \lambda_2^* \geq 0 \tag{1.1}$$

$$\sup \left\{ d_1 \lambda_1^* + \langle d_2, \lambda_2^* \rangle + \langle d_3, \lambda_3^* \rangle \,\middle|\, (d_1, d_2, d_3) \in D \right\} \leq 0 \tag{1.2}$$

$$\langle f_2(x_0), \lambda_2^* \rangle = 0. \tag{1.3}$$

Nieuwenhuis [14] has investigated a generalization of problem (P_1), namely the following problem:

($\mathbf{P}_2$) Maximize $f_1(x)$ subject to $x \in X$, $f_2(x) \in K_2$, $f_3(x) \in K_3$

where K_2 is a closed convex cone in the space R^{m_2} with int $K_2 \neq \emptyset$, while K_3 is a closed convex cone in the space R^{m_3}. Associating with this problem a suitable concept of a derived set, he succeeded to prove a multiplier rule (see [14: Theorem 3.1]) for the solutions of problem (P_2). But, in contrast to Hestenes' multiplier rule, in this new multiplier rule an orthogonality condition of type (1.3) is missing.

In the present paper we carry on Hestenes' and Nieuwenhuis' researches concerning derived sets. Three substantial improvements will be achieved. Firstly, we introduce derived sets for multiobjective optimization problems, i.e. problems in which it is required to maximize a vector function $f_1 : X \to R^{m_1}$ on a non-empty subset X of a topological space subject to constraints of the form $f_2(x) \in K_2$ and $f_3(x) \in K_3$, where f_2, f_3, K_2 and K_3 have the same meaning as in problem (P_2). Secondly, in our concept of a derived set the functions f_1, f_2 and f_3, occurring in the formulation of the optimization problem, cease to play the same role. This manifests itself through the fact that the conditions imposed on f_1 and f_2 are weaker than those imposed on f_3. Thirdly, by using this generalized concept of a derived set we state a multiplier rule which holds even for local solutions of weak multiobjective optimization problems and in which an orthogonality condition of type (1.3) also occurs. All these improvements of the theories known till now are obtained due to a subtle modification of the ideas applied by Nieuwenhuis in his research. Nevertheless a sophistication of the proofs was inevitable.

So far as we know Hestenes' image space technique for deriving necessary optimality conditions has not been used in multiobjective optimization hitherto. However, multiplier rules for diverse multiobjective optimization problems (especially for Pareto optimization problems) have already been obtained by other methods (see, e.g., [7, 11 - 13, 15 - 18]).

The image space technique is very useful in optimization, because it merely requires that the space in which the solutions of the optimization problem are sought is a topological one. This assumption can always be satisfied. But not any of algebraical structure of the underlying space is needed.

Finally, it should be mentioned that for ordinary (scalar) optimization problems not only the above-mentioned extension of Hestenes' concept of a derived set owing to Nieuwenhuis is known. There also exist other enlargements (see [2] and [4]). For some interesting applications of the derived sets the reader is referred to [1, 5, 6, 8, 9, 20].

2. Notations and preliminaries

Throughout this paper, N is the set of all positive integers, R is the set of all real numbers, and R^m is, for every $m \in N$, the usual m-dimensional Euclidean space of all m-tuples $x = (x_1, \ldots, x_m)$ of real numbers. The inner product of two vectors $x, y \in R^m$ is denoted by $\langle x, y \rangle$. If $x \in R^m$, then $\|x\|$ marks its Euclidean norm.

The subset of R^m, consisting of all vectors $x = (x_1, \ldots, x_m)$ with $x_j \geq 0$ for each $j \in \{1, \ldots, m\}$, is denoted by R^m_+. In particular, R_+ designates the set of all non-negative real numbers. Given any number $r > 0$, we put

$$B^m_+(r) = \left\{x \in R^m_+ \,\middle|\, \|x\| \leq r\right\}.$$

If M is a subset of the space R^m, then we denote by $\operatorname{int} M$ its interior. A subset K of the space R^m is said to be a:

i) *cone* if it is not empty and if $ax \in K$ whenever $a \in R_+$ and $x \in K$

ii) *convex cone* if it is both a convex set and a cone.

If M is a non-empty subset of the space R^m, then the sets

$$\mathcal{K}(M) = \left\{y \in R^m \,\middle|\, y = a_1 x^1 + \ldots + a_k x^k \quad (k \in N, a \in R^k_+,\ x^1, \ldots, x^k \in M)\right\}$$

and

$$M^* = \left\{y \in R^m \,\middle|\, \langle x, y \rangle \geq 0 \ \text{ for all } \ x \in M\right\}$$

are convex cones. They are called the *convex cone generated* by M and the *dual cone* of M, respectively. It is well-known (see, for instance, [3: Satz 2.42]) that M is a closed convex cone if and only if $(M^*)^* = M$.

In our investigations we shall use the following two results referring to convex cones.

Proposition 2.1. *Let K and L be convex cones in the space R^m, of which L is closed. Then $K^* \cap L \neq \{0\}$ if and only if the condition*

$$\{x^j \mid j = 1, \ldots, m+1\}^* \cap L \neq \{0\} \quad \textit{whenever } x^1, \ldots, x^{m+1} \in K$$

is satisfied.

Proposition 2.2. *Let t^0 be a vector in the space R^m, let K be a closed convex cone in the space R^p, and let $F : R^m \to R^p$ be a function satisfying the following conditions:*

(i) *F is continuous*

(ii) *F is differentiable at* 0

(iii) $F(0) \in K$, $F'(0)(t^0) \in K$ *and* $F'(0)(R^m) + K = R^p$.

Then there exist a sequence (a_n) *of positive numbers and a sequence* (t^n) *of vectors in the space* R^m *such that*

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} t^n = t^0, \qquad F(a_n t^n) \in K \quad \textit{for all } n \in N.$$

The first of these two propositions has been stated in [14: Lemma 3.1], while the last one is a special case of [19: Theorem 1].

3. The weak multiobjective optimization problem and K-derived sets

Let m_1, m_2 and m_3 be positive integers and $m = m_1 + m_2 + m_3$. In what follows the corresponding space R^m will always be conceived as the product space $R^{m_1} \times R^{m_2} \times R^{m_3}$, i.e. any vector $y \in R^m$ is identified with a certain triple $(y_1, y_2, y_3) \in R^{m_1} \times R^{m_2} \times R^{m_3}$. In particular, the zero-vector in R^m is $0 = (0_1, 0_2, 0_3)$, where 0_i $(i \in \{1,2,3\})$ is the zero-vector in R^{m_i}.

In accord with the above made convention concerning R^m, any function ϕ from a non-empty set M to R^m will be interpreted as a triple (ϕ_1, ϕ_2, ϕ_3), where $\phi_i : M \to R^{m_i}$ $(i \in \{1,2,3\})$ are functions such that

$$\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x)) \quad \text{for all } x \in M.$$

Let X be a non-empty subset of a topological space $\mathcal{X}$, and let $f\colon X \to R^m$. Further, let K_1, K_2 and K_3 be convex cones in the spaces R^{m_1}, R^{m_2} and R^{m_3}, respectively, satisfying the following assumptions:

i) $\operatorname{int} K_1 \neq \emptyset$ and $\operatorname{int} K_2 \neq \emptyset$

ii) K_2 and K_3 are closed.

Obviously, $K = K_1 \times K_2 \times K_3$ is a convex cone in the space R^m.

Set

$$S = \{x \in X \mid f_2(x) \in K_2 \text{ and } f_3(x) \in K_3\}.$$

A point $x_0 \in \mathcal{X}$ is said to be a

i) *weakly K_1-maximal point* of f_1 over S if $x_0 \in S$ and $(f_1(x_0) + \operatorname{int} K_1) \cap f_1(S) = \emptyset$;

ii) *local weakly K_1-maximal point* of f_1 over S if $x_0 \in S$ and if there is a neighbourhood V of x_0 such that $(f_1(x_0) + \operatorname{int} K_1) \cap f_1(S \cap V) = \emptyset$.

The problem of finding the weakly K_1-maximal points of f_1 over S is called a *weak multiobjective optimization problem* and shortly expressed as

(WMOP) $\{f_1(x) \mid x \in X,\ f_2(x) \in K_2,\ f_3(x) \in K_3\} \longrightarrow_{K_1}$ max weakly.

The introduction of problem (WMOP) allows to call the weakly K_1-maximal points of f_1 over S *solutions* to problem (WMOP). By analogy, the local weakly K_1-maximal points of f_1 over S can be named *local solutions* to problem (WMOP).

It should be noted that the problem (P_2), formulated in Section 1, is a special case of the problem (WMOP). Indeed, to see this we choose $m_1 = 1$ and $K_1 = R_+$, on the one hand, and endow the non-empty set X occurring in problem (P_2) with the indiscrete topology, on the other hand.

The main notion we shall use for obtaining a necessary optimality condition for the local solutions to problem (WMOP) is that of a K-derived convex cone. In order to introduce this concept let x_0 be any point in X.

An n-tuple $(d^1, \ldots, d^n)$ of points of the space R^m is said to be a *K-gradient* of f at x_0 if there exist a number $r > 0$, a function ω: $B^n_+(r) \to X$ and a function ρ: $B^n_+(r) \to R^m$ such that the following conditions are satisfied:

(A) $f(\omega(t)) - f(x_0) - (t_1 d^1 + \ldots + t_n d^n) - \|t\|\rho(t) \in K$ for all $t = (t_1, \ldots, t_n) \in B^n_+(r)$.

(B) $\omega(0) = x_0$ and ω is continuous at 0.

(C_1) There exists a point $y_1^0 \in K_1$ so that, for each number $\varepsilon > 0$, there is a number $r_\varepsilon \in (0, r]$ such that

$$\rho_1(t) + \varepsilon y_1^0 \in K_1 \quad \text{whenever } t \in B^n_+(r_\varepsilon). \tag{3.1}$$

(C_2) There exists a point $y_2^0 \in K_2$ so that, for each number $\varepsilon > 0$, there is a number $r_\varepsilon \in (0, r]$ such that

$$\rho_2(t) + \varepsilon y_2^0 \in K_2 \quad \text{whenever } t \in B^n_+(r_\varepsilon).$$

(C_3) $\rho_3(0) = 0_3$ and ρ_3 is continuous.

A subset $D \subseteq R^m$ is said to be a

i) *K-derived set* for f at x_0 if it is not empty and if every $m+1$-tuple of points belonging to D is a K-gradient of f at x_0;

ii) *K-derived convex cone* for f at x_0 if it is both a K-derived set for f at x_0 and a convex cone.

Remark 3.1. If $\rho_1(0) = 0_1$ and ρ_1 is continuous at 0, then condition (C_1) is satisfied. Indeed, let y_1^0 be any interior point of K_1. Then $K_1 - \varepsilon y_1^0$ is for each number $\varepsilon > 0$ a neighbourhood of 0_1. In consequence, in view of the assumptions on ρ_1 there must exist for each number $\varepsilon > 0$ a number $r_\varepsilon \in (0, r]$ such that inclusion (3.1) holds.

Of course a similar remark can be made concerning ρ_2 and condition (C_2).

Remark 3.2. The functions ρ_1, ρ_2 and ρ_3 that occur in our definition of the K-gradient of f are linked with the functions r_0, r_y and r_z used in [14]. An attentive analysis reveals that the proof of Theorem 3.1 in [14] employs the continuity of r_z on R_+^{p+2}, although this property has not been supposed. If we add this assumption, then it follows by Remark 3.1 that the definition of a derived convex cone suggested by Lemma 3.2 from [14] is a special case of our concept of a K-derived convex cone.

Proposition 3.1. *Let $x_0 \in X$, and let M be a non-empty subset of the space R^m such that, for all $n \in N$ and all $y^1, \ldots, y^n \in M$, the n-tuple $(y^1, \ldots, y^n)$ is a K-gradient of f at x_0. Then $\mathcal{K}(M)$ is a K-derived convex cone for f at x_0.*

Proof. Let $d^1, \ldots, d^{m+1}$ be vectors in $\mathcal{K}(M)$. Then we can select a finite number of vectors $y^1, \ldots, y^n \in M$ as well as numbers $a_{jk} \in R_+$ $(j \in \{1, \ldots, m+1\}$ and $k \in \{1, \ldots, n\})$ such that

$$d^j = \sum_{k=1}^{n} a_{jk} y^k \quad \text{for all} \quad j \in \{1, \ldots, m+1\}.$$

Since $(y^1, \ldots, y^n)$ is a K-gradient of f at x_0, there exist a number $r > 0$, a function $\omega : B^n_+(r) \to X$ and a function $\rho : B^n_+(r) \to R^m$ satisfying the condition

$$f(\omega(t)) - f(x_0) - (t_1 y^1 + \ldots + t_n y^n) - \|t\| \rho(t) \in K \tag{3.2}$$

for all $t = (t_1, \ldots, t_n) \in B^n_+(r)$ as well as the conditions (B), (C_1) - (C_3).

Let $A = (A_1, \ldots, A_n) : R^{m+1} \to R^n$ be the mapping, whose components

$$A_k : \; R^{m+1} \to R \qquad (k \in \{1, \ldots, n\})$$

are defined by

$$A_k(u) = \sum_{j=1}^{m+1} u_j a_{jk} \quad \text{for all} \quad u = (u_1, \ldots, u_{m+1}) \in R^{m+1}.$$

For all $u = (u_1, \ldots, u_{m+1}) \in R^{m+1}_+$ we have

$$u_1 d^1 + \ldots + u_{m+1} d^{m+1} = A_1(u) y^1 + \ldots + A_n(u) y^n \tag{3.3}$$

$$A(u) \in R^n_+ \qquad \text{and} \qquad \|A(u)\| \le \sum_{k=1}^{n} A_k(u) \le a \|u\| \tag{3.4}$$

where

$$a = 1 + \sum_{j=1}^{m+1} \sum_{k=1}^{n} a_{jk}.$$

Set $\bar{r} = r/a$. In view of (3.4) it follows that A maps $B^{m+1}_+(\bar{r})$ into $B^n_+(r)$. Consequently we can define the functions

$$\bar{\omega} : B^{m+1}_+(\bar{r}) \to X \qquad \text{and} \qquad \bar{\rho} : B^{m+1}_+(\bar{r}) \to R^m$$

by

$$\bar{\omega}(u) = \omega(A(u)) \qquad \text{and} \qquad \bar{\rho}(u) = \begin{cases} \|A(u)\| / \|u\| \rho(A(u)) & \text{if } u \ne 0 \\ 0 & \text{if } u = 0, \end{cases}$$

respectively. We claim that for $\bar{r}$, $\bar{\omega}$ and $\bar{\rho}$ the following properties are valid:

($\bar{\mathrm{A}}$) $f(\bar{\omega}(u)) - f(x_0) - (u_1 d^1 + \ldots + u_{m+1} d^{m+1}) - \|u\|\bar{\rho}(u) \in K$ for all elements $u = (u_1, \ldots, u_{m+1}) \in B_+^{m+1}(\bar{r})$.

($\bar{\mathrm{B}}$) $\bar{\omega}(0) = x_0$ and $\bar{\omega}$ is continuous at 0.

($\bar{\mathrm{C}}_1$) For each number $\varepsilon > 0$ there is a number $\bar{r}_\varepsilon \in (0, \bar{r}]$ such that

$$\bar{\rho}_1(u) + \varepsilon y_1^0 \in K_1 \quad \text{whenever } u \in B_+^{m+1}(\bar{r}_\varepsilon). \tag{3.5}$$

($\bar{\mathrm{C}}_2$) For each number $\varepsilon > 0$ there is a number $\bar{r}_\varepsilon \in (0, \bar{r}]$ such that

$$\bar{\rho}_2(u) + \varepsilon y_2^0 \in K_2 \quad \text{whenever } u \in B_+^{m+1}(\bar{r}_\varepsilon).$$

($\bar{\mathrm{C}}_3$) $\bar{\rho}_3(0) = 0_3$ and $\bar{\rho}_3$ is continuous.

Indeed, from (3.2) and (3.3) it follows that property ($\bar{\mathrm{A}}$) holds. Taking into account that $A(0) = 0$ and that A is continuous at 0, we see that condition (B) implies property ($\bar{\mathrm{B}}$). Next we prove property ($\bar{\mathrm{C}}_1$). Let $\varepsilon > 0$ be given. In view of condition (C_1) there exists a number $r_1 \in (0, r]$ such that

$$\rho_1(t) + \frac{\varepsilon}{a} y_1^0 \in K_1 \quad \text{whenever } t \in B_+^n(r_1). \tag{3.6}$$

Choose $\bar{r}_\varepsilon = r_1/a$. By (3.4) we have $A(u) \in B_+^n(r_1)$ for all $u \in B_+^{m+1}(\bar{r}_\varepsilon)$, and hence (3.6) yields

$$a\rho_1(A(u)) \in K_1 - \varepsilon y_1^0 \quad \text{whenever} \quad u \in B_+^{m+1}(\bar{r}_\varepsilon). \tag{3.7}$$

Taking into account that $K_1 - \varepsilon y_1^0$ is a convex set containing 0_1 and that

$$0 \leq \frac{\|A(u)\|}{a\|u\|} \leq 1 \quad \text{for all} \quad u \in R_+^{m+1} \setminus \{0\},$$

it follows from (3.7) that (3.5) is true. A similar proof shows that property ($\bar{\mathrm{C}}_2$) is also true. Finally, observe that condition (C_3), equality $A(0) = 0$ and the continuity of A imply property ($\bar{\mathrm{C}}_3$). Consequently, all the properties ($\bar{\mathrm{A}}$), ($\bar{\mathrm{B}}$), ($\bar{\mathrm{C}}_1$) - ($\bar{\mathrm{C}}_3$) are true as claimed.

These properties show that $(d^1, \ldots, d^{m+1})$ is a K-gradient of f at x_0. Since the vectors $d^1, \ldots, d^{m+1}$ were arbitrarily chosen in $\mathcal{K}(M)$, it follows that $\mathcal{K}(M)$ is a K-derived set for f at x_0. On the other hand, as it has already been remarked in Section 2, the set $\mathcal{K}(M)$ is a convex cone. Consequently $\mathcal{K}(M)$ is a K-derived convex cone for f at x_0 ∎

Corollary 3.2. *Let x_0 be a point in X, and let M be a non-empty subset of the space R^m such that, for every $n \in N$ and every n-tuple $(y^1, \ldots, y^n)$ of points belonging to M, there exist a number $r > 0$ and a function $\omega : B_+^n(r) \to X$ satisfying the following conditions:*

(i) $\lim_{t \to 0} \frac{1}{\|t\|}\big(f(\omega(t)) - f(x_0) - A(t)\big) = 0$, *where* $A(t) = t_1 y^1 + \ldots + t_n y^n$ *for all* $t = (t_1, \ldots, t_n) \in R^n$.

(ii) $\omega(0) = x_0$ *and* ω *is continuous at* 0.

(iii) $f \circ \omega$ *(the composite function) is continuous.*

Then $\mathcal{K}(M)$ *is a* K*-derived convex cone for* f *at* x_0.

Proof. Let n be any positive integer, and let $y^1, \ldots, y^n \in M$. Then there exist a number $r > 0$ and a function $\omega : B^n_+(r) \to X$ such that the conditions (i) - (iii) are satisfied. Define $\rho : B^n_+(r) \to R^m$ by

$$\rho(t) = \begin{cases} \frac{1}{\|t\|}\big(f(\omega(t)) - f(x_0) - A(t)\big) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then we have

$$f(\omega(t)) - f(x_0) - A(t) - \|t\|\rho(t) = 0 \quad \text{for all} \quad t \in B^n_+(r),$$

and hence

$$f(\omega(t)) - f(x_0) - A(t) - \|t\|\rho(t) \in K \quad \text{for all} \quad t \in B^n_+(r). \tag{3.8}$$

On the other hand, it follows from conditions (i) and (iii) that ρ is continuous. Together with Remark 3.1 this implies that the conditions $(C_1) - (C_3)$ are satisfied.

Summing up, the number r and the functions ω and ρ satisfy (3.8), (ii), $(C_1)-(C_3)$. Consequently $(y^1, \ldots, y^n)$ is a K-gradient of f at x_0. But $n \in N$ and $y^1, \ldots, y^n \in M$ were arbitrarily chosen. Thus Proposition 3.1 is applicable and yields that $\mathcal{K}(M)$ is a K-derived convex cone for f at x_0 ∎

Corollary 3.2 points out that the concept of a K-derived convex cone introduced in this paper is a generalization of Hestenes' concept of a derived convex cone, which was recalled in Section 1.

4. The multiplier rule

Now we are in a position to state a multiplier rule for the local solutions of problem (WMOP). For this end all assumptions and notations specified in the preceding section will be kept on.

Theorem 4.1. *Let* $x_0 \in \mathcal{X}$ *be a local solution to problem* (WMOP) *and let* $D \subseteq R^m$ *be a* K*-derived convex cone for* f *at* x_0. *Then there exists a vector*

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

such that

$$\sup\Big\{ \langle d_1, \lambda_1^* \rangle + \langle d_2, \lambda_2^* \rangle + \langle d_3, \lambda_3^* \rangle \Big| \; (d_1, d_2, d_3) \in D \Big\} \leq 0 \tag{4.1}$$

and

$$\langle f_2(x_0), \lambda_2^* \rangle = 0. \tag{4.2}$$

Proof. Let $d^1, \ldots, d^{m+1}$ be vectors belonging to D. Define the mappimg $A : R^{m+1} \to R^m$ by

$$A(t) = t_1 d^1 + \ldots + t_{m+1} d^{m+1} \quad \text{for all } \ t = (t_1, \ldots, t_{m+1}) \in R^{m+1}.$$

Next introduce the sets

$$L = \left\{ u \in K_2^* \middle| \ \langle f_2(x_0), u \rangle = 0 \right\}$$

and

$$M = \left\{ y \in R^m \middle| \ A(t) \in y + K_1 \times L^* \times K_3 \ \text{ for some } \ t \in \operatorname{int} R_+^{m+1} \right\}.$$

We claim that the origin of the space R^m is not an interior point of M. To prove this assertion, we assume that the origin $(0_1, 0_2, 0_3)$ of the space R^m is an interior point of M. Then there exists a balanced neighbourhood V_3 of 0_3 such that $\{0_1\} \times \{0_2\} \times V_3 \subseteq M$. In consequence there exists for each $y_3 \in V_3$ a point $t \in \operatorname{int} R_+^{m+1}$ such that $A_3(t) \in y_3 + K_3$. Thus we have $V_3 \subseteq A_3(R^{m+1}) + K_3$. This relation implies that

$$R^{m_3} = A_3(R^{m+1}) + K_3.$$

Since $L^* \supseteq (K_2^*)^* = K_2$, it results that int $L^* \neq \emptyset$. Pick out a point

$$z = (z_1, z_2, z_3) \in \operatorname{int} K_1 \times \operatorname{int} L^* \times K_3.$$

Inasmuch as M is a neighbourhood of the origin of the space R^m, there exists a number $a > 0$ such that $az \in M$. Hence we can find an interior point t^0 of R_+^{m+1} for which

$$\begin{aligned} A_1(t^0) &\in az_1 + K_1 \subseteq \operatorname{int} K_1 + K_1 \subseteq \operatorname{int} K_1 \\ A_2(t^0) &\in az_2 + L^* \subseteq \operatorname{int} L^* + L^* \subseteq \operatorname{int} L^* \\ A_3(t^0) &\in az_3 + K_3 \subseteq K_3 + K_3 \subseteq K_3. \end{aligned}$$

We now take into consideration that $(d^1, \ldots, d^{m+1})$ is a K-gradient of f at x_0. So we can find a number $r > 0$, a function $\omega : B_+^{m+1}(r) \to X$ and a function $\rho : B_+^{m+1}(r) \to R^m$ that satisfy the following conditions:

(a) $f(\omega(t)) - f(x_0) - A(t) - \|t\|\rho(t) \in K$ for all $t \in B_+^{m+1}(r)$.

(b) $\omega(0) = x_0$ and ω is continuous at 0.

(c_1) There exists a point $y_1^0 \in K_1$ so that for each number $\varepsilon > 0$ there is a number $r_\varepsilon \in (0, r]$ such that $\rho_1(t) + \varepsilon y_1^0 \in K_1$ whenever $t \in B_+^{m+1}(r_\varepsilon)$.

(c_2) There exists a point $y_2^0 \in K_2$ so that for each number $\varepsilon > 0$ there is a number $r_\varepsilon \in (0, r]$ such that $\rho_2(t) + \varepsilon y_2^0 \in K_2$ whenever $t \in B_+^{m+1}(r_\varepsilon)$.

(c_3) $\rho_3(0) = 0_3$ and ρ_3 is continuous.

Let $\bar{\rho}_3 : R^{m+1} \to R^{m_3}$ be a continuous extension of ρ_3. Define $F : R^{m+1} \to R^{m_3}$ by

$$F(t) = f_3(x_0) + A_3(t) + \|t\|\bar{\rho}_3(t).$$

This function is continuous. In view of condition (c_3) it is differentiable at 0. Moreover, in virtue of $F'(0) = A_3$, we have

$$F'(0)(t^0) \in K_3 \qquad \text{and} \qquad F'(0)(R^{m+1}) + K_3 = R^{m_3}.$$

In addition to these properties of F we have $F(0) = f_3(x_0) \in K_3$. Summing up, all the hypotheses of Proposition 2.2 are satisfied by t^0, K_3 and F. By applying Proposition 2.2 we conclude that there exist a sequence (a_n) of positive numbers and a sequence (t^n) of vectors in the space R^{m+1} such that

$$\lim_{n\to\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} t^n = t^0 \tag{4.3}$$

and

$$F(a_n t^n) \in K_3 \quad \text{for all} \quad n \in N. \tag{4.4}$$

Since the sequence (t^n) converges to an interior point of the set R_+^{m+1} and the sequence $(a_n t^n)$ converges to the origin of the space R^{m+1}, we can assume without loss of the generality that all the terms $a_n t^n$ $(n \in N)$ lie in $B_+^{m+1}(r)$. Then all the points $\omega(a_n t^n)$ $(n \in N)$ lie in X. Moreover, for sufficiently large n, these points lie even in S.

To see this, we firstly remark that (a) and (4.4) imply

$$f_3(\omega(a_n t^n)) \in K_3 + F(a_n t^n) \subseteq K_3 + K_3 \subseteq K_3 \quad \text{for all} \quad n \in N. \tag{4.5}$$

Next we define the function $G : B_+^{m+1}(r) \to R^{m_2}$ by

$$G(t) = f_2(x_0) + A_2(t) + \|t\|\rho_2(t).$$

For this function we can find a number $p_0 \in N$ such that

$$\langle G(a_n t^n), u\rangle > 0 \quad \text{for all} \quad p_0 \leq n \in N \tag{4.6}$$

and all $u \in U$, where $U = \{u \in K_2^* |\ \|u\| = 1\}$. Indeed, if we suppose the contrary, there exist a subsequence $(a_{n_k} t^{n_k})$ of the sequence $(a_n t^n)$ and a sequence (u^k) of points belonging to the compact set U such that

$$\langle G(a_{n_k} t^{n_k}), u^k\rangle \leq 0 \quad \text{for all} \quad k \in N. \tag{4.7}$$

Without loss of the generality we can assume that the sequence (u^k) converges to a point $u^0 \in U$. Passing to the limit in (4.7) when $k \to \infty$, we obtain $\langle G(0), u^0\rangle \leq 0$. In other words, we have $\langle f_2(x_0), u^0\rangle \leq 0$. But, the inequality $\langle f_2(x_0), u^0\rangle \geq 0$ is also valid, because $f_2(x_0) \in K_2$ and $u^0 \in K_2^*$. Therefore we must have the equality $\langle f_2(x_0), u^0\rangle = 0$. Consequently u^0 belongs to L. Now choose a number $\varepsilon > 0$ for which

$$A_2(t^0) - \varepsilon\,\|t^0\|\,y_2^0 - \varepsilon u^0 \in L^*. \tag{4.8}$$

Since $A_2(t^0) \in \operatorname{int} L^*$, such a choice is possible. From (4.8) and $u^0 \in L$ it follows that

$$\langle A_2(t^0) - \varepsilon\,\|t^0\|y_2^0 - \varepsilon u^0, u^0\rangle \geq 0,$$

and hence that

$$\langle A_2(t^0) - \varepsilon \|t^0\| y_2^0, u^0 \rangle \geq \varepsilon \langle u^0, u^0 \rangle > 0. \tag{4.9}$$

Because

$$\lim_{k\to\infty} \langle A_2(t^{n_k}) - \varepsilon \|t^{n_k}\| y_2^0, u^k \rangle = \langle A_2(t^0) - \varepsilon \|t^0\| y_2^0, u^0 \rangle,$$

it results from (4.9) that there is a number $k_1 \in N$ such that

$$\langle A_2(t^{n_k}) - \varepsilon \|t^{n_k}\| y_2^0, u^k \rangle > 0 \quad \text{for all} \quad k_1 \leq k \in N. \tag{4.10}$$

On the other hand, condition (c$_2$) implies the existence of a number $r_2 \in (0, r]$ such that

$$\rho_2(t) + \varepsilon y_2^0 \in K_2 \quad \text{for all} \quad t \in B_+^{m+1}(r_2). \tag{4.11}$$

To r_2 we can assign a number $k_2 \in N$ such that $\|a_{n_k} t^{n_k}\| \leq r_2$ for all $k_2 \leq k \in N$, because $\lim_{k\to\infty} a_{n_k} t^{n_k} = 0$. In virtue of (4.11) it follows that

$$\rho_2(a_{n_k} t^{n_k}) + \varepsilon y_2^0 \in K_2 \quad \text{for all } k_2 \leq k \in N,$$

and hence that the points

$$v^k = \|t^{n_k}\| (\rho_2(a_{n_k} t^{n_k}) + \varepsilon y_2^0) \quad \text{for all} \quad k_2 \leq k \in N$$

also lie in K_2. Therefore we have

$$\langle v^k, u^k \rangle \geq 0 \quad \text{for all} \quad k_2 \leq k \in N. \tag{4.12}$$

By addition (4.10) and (4.12) yield

$$\langle A_2(t^{n_k}) + \|t^{n_k}\| \rho_2(a_{n_k} t^{n_k}), u^k \rangle > 0 \quad \text{for all } \max\{k_1, k_2\} \leq k \in N.$$

This inequality implies

$$\langle G(a_{n_k} t^{n_k}), u^k \rangle = \langle f_2(x_0), u^k \rangle + a_{n_k} \langle A_2(t^{n_k}) + \|t^{n_k}\| \rho_2(a_{n_k} t^{n_k}), u^k \rangle > 0$$

for all $\max\{k_1, k_2\} \leq k \in N$, which contradicts (4.7). In conclusion there must exist a number $p_0 \in N$ such that (4.6) holds whenever $u \in U$. Therefore we have $\langle G(a_n t^n), u \rangle \geq 0$ for all $p_0 \leq n \in N$ and all $u \in K_2^*$. This means that

$$G(a_n t^n) \in (K_2^*)^* = K_2 \quad \text{for all } p_0 \leq n \in N. \tag{4.13}$$

From (a) and (4.13) it follows that

$$f_2(\omega(a_n t^n)) \in K_2 + G(a_n t^n) \subseteq K_2 + K_2 \subseteq K_2 \quad \text{for all } p_0 \leq n \in N. \tag{4.14}$$

Together (4.5) and (4.14) express that

$$\omega(a_n t^n) \in S \quad \text{for all } p_0 \leq n \in N. \tag{4.15}$$

After all we focus on f_1. Since $A_1(t^0) \in \operatorname{int} K_1$ we can select a number $\varepsilon > 0$ such that

$$A_1(t^0) - \varepsilon \|t^0\| y_1^0 \in \operatorname{int} K_1.$$

Next, according to condition (c_1), we can pick out a number $r_1 \in (0, r]$ such that

$$\rho_1(t) + \varepsilon y_1^0 \in K_1 \quad \text{for all} \quad t \in B_+^{m+1}(r_1). \tag{4.16}$$

Now, taking into consideration that

$$\lim_{n \to \infty} (A_1(t^n) - \varepsilon \|t^n\| y_1^0) = A_1(t^0) - \varepsilon \| t^0 \| y_1^0$$

and that

$$\lim_{n \to \infty} a_n t^n = 0, \tag{4.17}$$

we can choose a number $p_1 \in N$ such that both properties

$$A_1(t^n) - \varepsilon \|t^n\| y_1^0 \in \operatorname{int} K_1 \qquad \text{and} \qquad \|a_n t^n\| \leq r_1 \tag{4.18}$$

hold for all $p_1 \leq n \in N$. In view of (4.18) and (4.16) we get

$$\begin{aligned} A_1(t^n) + \|t^n\| \rho_1(a_n t^n) &= A_1(t^n) - \varepsilon \|t^n\| y_1^0 + \|t^n\| (\rho_1(a_n t^n) + \varepsilon y_1^0) \\ &\in \operatorname{int} K_1 + K_1 \\ &\subseteq \operatorname{int} K_1 \end{aligned}$$

for all $p_1 \leq n \in N$. By applying (a) it follows that

$$\begin{aligned} f_1(\omega(a_n t^n)) &\in f_1(x_0) + a_n(A_1(t^n) + \|t^n\| \rho_1(a_n t^n)) + K_1 \\ &\subseteq f_1(x_0) + \operatorname{int} K_1 + K_1 \\ &\subseteq f_1(x_0) + \operatorname{int} K_1 \end{aligned} \tag{4.19}$$

for all $p_1 \leq n \in N$.

Since x_0 is a local solution to problem (WMOP), there exists a neighbourhood V of x_0 such that

$$(f_1(x_0) + \operatorname{int} K_1) \cap f_1(S \cap V) = \emptyset. \tag{4.20}$$

In virtue of (b) we can find a neighbourhood W of the origin of the space R^{m+1} such that $\omega(t) \in V$ for all $t \in W \cap B_+^{m+1}(r)$. According to (4.17), we can choose a number $p_2 \in N$ such that

$$a_n t^n \in W \quad \text{for all} \quad p_2 \leq n \in N.$$

Hence we have

$$\omega(a_n t^n) \in V \quad \text{for all} \quad p_2 \leq n \in N. \tag{4.21}$$

Choosing any number $n \in N$ which satisfies $n \geq \max\{p_0, p_1, p_2\}$, we conclude from (4.19), (4.15) and (4.21) that the point $a_n t^n$ associated with this n fulfils both inclusions

$$f_1(\omega(a_n t^n)) \in f_1(x_0) + \operatorname{int} K_1 \qquad \text{and} \qquad \omega(a_n t^n) \in S \cap V.$$

This result contradicts (4.20). So it is proved that the origin of the space R^m is not an interior point of M.

It is easily seen that M is a convex set. By applying a well-known separation theorem (see [3: Satz 2.33]) it results that there is a vector

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in R^{m_1} \times R^{m_2} \times R^{m_3} \setminus \{(0_1, 0_2, 0_3)\}$$

for which

$$\sup\{\langle y, \lambda\rangle | y \in M\} \leq 0. \tag{4:22}$$

Fix any point $t = (t_1, \ldots, t_{m+1}) \in \operatorname{int} R_+^{m+1}$. Let y_1 be any point of K_1. Then we have

$$(A_1(t) - ny_1, A_2(t), A_3(t)) \in M \quad \text{for all} \quad n \in N.$$

Therefore (4.22) implies

$$\frac{1}{n}\langle A(t), \lambda\rangle \leq \langle y_1, \lambda_1\rangle \quad \text{for all} \quad n \in N.$$

Letting $n \to \infty$ in this inequality, we get $\langle y_1, \lambda_1\rangle \geq 0$. Since y_1 was arbitrarily chosen in K_1, we have $\lambda_1 \in K_1^*$. In the same way it can be proved that $\lambda_2 \in (L^*)^* = L$ and that $\lambda_3 \in K_3^*$. In conclusion, λ belongs to $K_1^* \times L \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$.

Since $A(t) \in M$, we get from (4.22) that $\langle A(t), \lambda\rangle \leq 0$. This inequality can be rewritten as

$$\langle t_1(-d^1) + \ldots + t_{m+1}(-d^{m+1}),\ \lambda\rangle \geq 0.$$

Since t was arbitrarily chosen in $\operatorname{int} R_+^{m+1}$, we have

$$\lambda \in (\mathcal{K}(\{-d^j |\ j = 1, \ldots, m+1\}))^*,$$

and thus $\lambda \in \{-d^j |\ j = 1, \ldots, m+1\}^*$. Consequently

$$\{-d^j |\ j = 1, \ldots, m+1\}^* \cap (K_1^* \times L \times K_3^*) \neq \{(0_1, 0_2, 0_3)\}$$

holds.

But $d^1, \ldots, d^{m+1}$ were arbitrarily chosen in D. Therefore we can apply Proposition 2.1. So we obtain a vector

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in K_1^* \times K_2^* \times K_3^* \setminus \{(0_1, 0_2, 0_3)\}$$

satisfying (4.1) and (4.2) ∎

By constructing suitable K-derived convex cones we can deduce from Theorem 4.1 various practical necessary optimality conditions. In order to avoid an extension of the present paper, such applications of Theorem 4.1 will be given in subsequent papers.

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