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On Lower Semicontinuity and Metric Upper Semicontinuity of Nemytskiĭ Set-Valued Operators

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Abstract. Sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskiĭ set-valued operators N_F generated by a set-valued function $F: \Omega \times X \to 2^Y$, where X and Y are Orlicz-Musielak F-spaces are presented.

Keywords: *Nemytskiĭ set-valued operators, lower semicontinuity, metric upper semicontinuity, superposition measurable set-valued operators*

AMS subject classification: 47H04, 28C20, 54C60

In years 1933 - 1934 V. Nemytskiĭ (see [10] and [11]) considered the operator $F: L^2[a,b] \to L^2[a,b]$ defined by $y(\cdot) = N_F(x(\cdot))$, where $y(t) = F(t, x(t))$. From that time the operator N_F was generalized in several way and there is a lot of papers devoted to such subjects. Operators of this type are now called *Nemytskiĭ operators*.

In last years new important applications of Nemytskiĭ set-valued operators in the theory of differential and integral inclusions appear (see [1 - 4]). For these applications lower semicontinuity and metric upper semicontinuity of Nemytskiĭ set-valued operators are important.

In [1] Appell, Ngyen and Zabrejko give sufficient conditions of lower semicontinuity of Nemytskiĭ set-valued operators acting in so-called *ideal spaces*. We shall not give the definition here. We want only to mention that ideal spaces are some spaces of functions defined on a measure space Ω admitting values in finite-dimensional spaces and it can be shown that each Orlicz space admitting values in a finite-dimensional space is an ideal space.

Thus the natural problem arose to give sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskiĭ set-valued operators for spaces consisting of functions admitting values in infinite dimensional spaces.

In this paper sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskiĭ set-valued operators acting in Orlicz-Musielak F-spaces are

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given.

Let X be a separable F-space (i.e. a complete linear metric space; basic properties can be found in [12]) with an F-norm $\|\cdot\|_X$ and (Ω, Σ, μ) a measure space with complete and σ-finite measue μ. A function $x = x(t)$ mapping Ω into X is called *measurable* if for every open set $Q \subset X$ the inverse image $x^{-1}(Q) = \{t \in \Omega : x(t) \in Q\}$ is measurable, i.e. if $x^{-1}(Q) \in \Sigma$. The set of all measurable functions defined on Ω with values in X we shall denote by $S(\Omega, X)$.

A closed-valued set-valued function $F = F(t)$ mapping Ω into subsets of X is called *measurable* if for every open set $Q \subset X$ the inverse image $F^{-1}(Q) = \{t \in \Omega : F(t) \cap Q \neq \emptyset\}$ is measurable, i.e. if $F^{-1}(Q) \in \Sigma$. By a *measurable selection* of the set-valued function F we mean a (single-valued) function x_F such that $x_F(t) \in F(t)$ for all $t \in \Omega$.

Let $F = F(t,x)$ be a closed-valued set-valued function mapping $\Omega \times X$ into subsets of an F-space Y, i.e. into 2^Y. We say that F is *sup-measurable* if for any measurable function $x(\cdot) : \Omega \to X$ the set-valued function $s \to F(s, x(s)) : \Omega \to 2^Y$ is measurable.

Given a sup-measurable closed-valued set-valued function $F : \Omega \times X \to 2^Y$. This set-valued function induces a set-valued operator $N_F : S(\Omega, X) \to S(\Omega, Y)$ defined by $y(\cdot) = N_F(x(\cdot))$, where $y(t) = F(t, x(t))$. The set-valued operator N_F is called *superposition operator* (or *Nemytskiĭ operator*) generated by the set-valued function F.

Let $N = N(t, u)$ be a real-valued measurable function defined on $\Omega \times R$ such that for every $t \in \Omega$ the function $N(t, \cdot)$ is increasing and moreover $N(t, 0) = 0$ for all $t \in \Omega$. Then we can define on $S(\Omega, X)$ a *metrizing modular*

$$\rho_{N,\mu}(x(\cdot)) = \int_\Omega N\big(t, \|x(t)\|_X\big)\, d\mu \tag{1}$$

(see, e.c., Nakano ([7], [8: p. 153] and [9: p. 204]), Musielak [5: p. 1] and Rolewicz [12: p. 6]). The set of those measurable functions $x(\cdot) \in S(\Omega, X)$ that there is a positive k such that $\rho_{N,\mu}(kx(\cdot)) < \infty$ we shall denote by $N\big(L(\Omega, \Sigma, \mu; X)\big)$. Recall that a *metrizing modular* on a linear space X is a function $\rho : X \to [0, \infty]$ having the following properties:

(md1) $\rho(x) = 0$ if and only if $x = 0$

(md2) $\rho(ax) = \rho(x)$ provided $|a| = 1$

(md3) $\rho(ax + by) \leq \rho(x) + \rho(y)$ provided $a, b > 0$ and $a + b = 1$

(md4) $\rho(a_n x) \to 0$ provided $a_n \to 0$ and $\rho(x) < +\infty$

(md5) $\rho(ax_n) \to$ provided $\rho(x_n) \to 0$.

We shall denote by (X, ρ) a linear space with a modular ρ and we shall call it *modular space*. Let (X, ρ) be a modular space with metrizing modular ρ. It is known that ρ induces in the space X an F-norm $\|\cdot\|_X$ by

$$\|x\|_X = \inf\left\{\varepsilon > 0 \,\middle|\, \rho\left(\frac{x}{\varepsilon}\right) < \varepsilon\right\}. \tag{2}$$

The norm $\|\cdot\|_X$ is equivalent to the moduler ρ in the sense that, for any sequence $\{x_n\}$, $\|x_n\|_X \to 0$ if and only if $\rho(x_n) \to 0$ (see Musielak and Orlicz [6] and Musielak [5: p. 2]; see also Rolewicz [12: p. 8]). Observe that if the functions $x_n(\cdot) \in N(L(\Omega,\Sigma,\mu;X))$ $(n \in I\!N)$ have disjoint supports, then

$$\sum_{n=1}^{\infty} \rho_{N,\mu}(x_n(\cdot)) = \rho_{N,\mu}\left(\sum_{n=1}^{\infty} x_n(\cdot)\right). \tag{3}$$

Let (X,ρ_X) and (Y,ρ_Y) be two modular spaces. A set-valued operator $\Gamma = \Gamma(x)$ mapping (X,ρ_X) into subsets of (Y,ρ_Y) will be called *lower semicontinuous at a point* $(x_0,y_0) \in X \times Y$ if it is lower semicontinuous at this point in the metric induced by the F-norms introduced above. In other words Γ is lower semicontinuous at (x_0,y_0) if for every $r > 0$ there is a number $q_{(x_0,y_0)}(r) > 0$ with the following property: for every $x \in X$ such that $\rho_{N,\mu}(x_0(\cdot)-x(\cdot)) < q_{(x_0,y_0)}(r)$ there is an $y \in \Gamma(x)$ such that $\rho_{M,\mu}(y_0(\cdot)-y(\cdot)) < r$. This is equivalent with the property that $\rho_{N,\mu}(x_0(\cdot) - x(\cdot)) < q_{(x_0,y_0)}(r)$ implies

$$\Gamma(x) \cap Q \neq \emptyset \tag{4}$$

where $Q = \{y : \ \rho_{M,\mu}(y_0(\cdot) - y(\cdot)) < r\}$.

A set-valued operator Γ mapping (X,ρ_X) into subsets of (Y,ρ_Y) will be called *lower semicontinuous at a point* $x_0 \in X$ if it is lower semicontinuous at all the points (x_0,y_0) $(y_0 \in \Gamma(x_0))$.

Now we shall introduce a notion of global lower semicontinuity at x_0, which is nothing else than an uniformization of lower semicontinuity on $\Gamma(x_0)$. More precisely, we say that a set-valued mapping Γ is *globally lower semicontinuous at the point* x_0, if for every $r > 0$ there is a $q_{x_0}(r) > 0$ with the following property: for every $y_0 \in \Gamma(x_0)$ and for every $x \in X$ such that $\rho_{N,\mu}(x_0(\cdot)-x(\cdot)) < q_{x_0}(r)$ there is an $y \in \Gamma(x)$ such that $\rho_{M,\mu}(y_0 - y) < r$. This is equivalent with the property that $\rho_{N,\mu}(x_0(\cdot) - x(\cdot)) < q_{x_0}(r)$ implies

$$\Gamma(x_0) \subset B(\Gamma(x), r) \tag{5}$$

where

$$B(A,s) = \left\{ y(\cdot) : \inf_{y_1(\cdot)\in A} \rho_{M,\mu}(y_1(\cdot) - y(\cdot)) < s \right\}.$$

The essential difference between lower semicontinuity and global lower semicontinuity is that in the first case $q_{(x_0,y_0)} > 0$ depends on x_0 and $y_0 \in \Gamma(x_o)$, while in the second case it depends on x_0 only.

Changing the role of x and x_0 in formula (5) we obtain a notion of metric upper semicontinuity. We say that the set-valued mapping Γ is *metric upper semicontinuous* (*Hausdorff upper semicontinuous*) at a point x_0 if for every $r > 0$ there is a $q_{x_0}(r) > 0$ such that $\rho_{N,\mu}(x_0(\cdot) - x(\cdot)) < q_{x_0}(r)$ implies

$$\Gamma(x) \subset B(\Gamma(x_0), r). \tag{6}$$

The both notions are not equivalent. Indeed, it is easy to give an example of a set-valued mapping Γ which is globally lower semicontinuous at a point x_0, but which is not metric upper semicontinuous at this point. Conversely, an example of a set-valued mapping Γ which is metric upper semicontinuous at a point x_0 but which is not globally lower semicontinuous at this point can be easily given too.

In the sequel we shall add some assumptions about the functions $N(t,u)$ and $M(t,u)$. Namely, we assume that the function $N = N(t,u)$ satisfies the following condition

(A) For every $\varepsilon > 0$ there are numbers $\alpha > 0$ and $\delta > 0$ such that for every measurable set $E \subset \Omega$ with $\mu(E) > \delta$ we have $\int_E N(t,\varepsilon)\,dt \geq \alpha$.

Observe that in Orlicz spaces, i.e. in the case when $N(t,u) = N_0(u)$ depends only on u, the condition (A) is satisfied. Further, we assume that the function $M(t,u)$ satisfies the following condition

(Δ_2) There are a constant $k \geq 1$ and a non-negative function δ such that $\int_E M(t,\delta(t))\,d\mu < +\infty$ and for almost all t, if $u \geq \delta(t)$, then $M(t,2u) \leq k\,M(t,u)$.

The condition (Δ_2) plays an essential role in the theory of Orlicz-Musielak spaces. Observe that if the function $M(t,u)$ satisfies condition (Δ_2), then from the fact that $\rho_{M,\mu}(x) < +\infty$ and $\rho_{M,\mu}(y) < +\infty$ it follows that $\rho_{M,\mu}(ax+by) < +\infty$ for all real a and b. We conclude that $M\big(L(\Omega,\Sigma,\mu;Y)\big)$ is the set of those measurable functions $y(\cdot)$ with values in the space X, that $\rho_{M,\mu}(y) < +\infty$ (see [5: p. 52] and [6]).

Theorem 1. *Let X, Y be two separable F-spaces, (Ω,Σ,μ) a measure space with complete non-atomic and σ-finite measure μ, and N_F a Nemytskiĭ set-valued operator mapping of a modular space $\big(N\big(L(\Omega,\Sigma,\mu;X)\big),\rho_{N,\mu}\big)$ into subsets of a modular space $\big(M\big(L(\Omega,\Sigma,\mu;Y)\big),\rho_{M,\mu}\big)$ induced by a sup-measurable set-valued function $F(t,u): \Omega\times X \to 2^Y$. Suppose that*

(i) *the function $N = N(t,u)$ satisfies condition (A)*

(ii) *the function $M = M(t,u)$ satisfies condition (Δ_2).*

If the function $= F(t,u)$ is globally lower semicontinuous with respect to u for almost all $t \in \Omega$, then the operator N_F is globally lower semicontinuous.

Proof. Suppose that the operator N_F is not globally lower semicontinuous. This implies that there are a number $r > 0$ and sequences $\{x_n(\cdot)\}, \{y_n(\cdot)\}$ with $y_n(\cdot) \in N_F(x_0(\cdot))$ such that

$$\begin{aligned}
&\rho_{N,\mu}\big(x_n(\cdot) - x_0(\cdot)\big) \to 0\\
&\inf_{z(\cdot)\in N_F(x_n(\cdot))} \rho_{M,\mu}\big(z(\cdot) - y_n(\cdot)\big) > r \quad (n \in \mathbb{N}).
\end{aligned}$$

By [2: Theorem 8.24 and Corollary 8.23], for arbitrary $\eta > 0$ there is a sequence $\{z_n(\cdot)\}$ of measurable selections $z_n(\cdot) \in N_F(x_n(\cdot))$ such that, for almost all $t \in \Omega$,

$$\big\|y_n(t) - z_n(t)\big\|_Y < (1+\eta)d_Y\big(y_n(t), F(t,x_n(t))\big) \qquad (7)_1$$

where $d_Y(y, A) = \inf_{z \in A} \|z - y\|_Y$ denotes the distance of a point y to a set A in the norm $\|\cdot\|_Y$. We denote $u_n(t) = \|y_n(t) - z_n(t)\|_Y$. Since $z_n(\cdot) \in N_F(x_n(\cdot))$, then

$$\int_\Omega M(t, u_n(t))d\mu \geq \inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M,\mu}(z(\cdot) - y_n(\cdot)) > r \qquad (n \in \mathbb{N}). \qquad (8)_1$$

The convergence $\rho_{N,\mu}(x_n(\cdot) - x_0(\cdot)) \to 0$ implies that the sequence $\{x_n(\cdot) - x_0(\cdot)\}$ contains a subsequence $\{x_{n_k}(\cdot) - x_0(\cdot)\}$ such that

$$\sum_{k=1}^{\infty} \int_\Omega N(t, \|x_{n_k}(t) - x_0(t)\|_X)d\mu < +\infty. \qquad (9)_1$$

Thus replacing the sequence $\{x_n(\cdot) - x_0(\cdot)\}$ by the subsequence $\{x_{n_k}(\cdot) - x_0(\cdot)\}$ we can assume without loss of generality that

$$\sum_{n=1}^{\infty} \int_\Omega N(t, \|x_n(t) - x_0(t)\|_X)\, d\mu < +\infty. \qquad (10)_1$$

Now we have the following two possibilities: either (1) $\mu(\Omega)$ is finite or (2) $\mu(\Omega)$ is infinite.

Case (1): $\mu(\Omega)$ is finite. We shall construct by induction a sequence of positive numbers $\{\varepsilon_k\}$, a sequence of measurable sets $\{\Omega_k\}$ $(\Omega_k \subset \Omega)$ and a subsequence $\{x_{n_k} - x_0\}$ such that the following conditions are satisfied:

(a) $\varepsilon_{k+1} < \frac{\varepsilon_k}{2}$

(b) $\mu(\Omega_k) \leq \varepsilon_k$

(c) $\int_{\Omega_k} M(t, u_{n_k}(t))\, d\mu > \frac{2}{3}r$

(d) $\int_D M(t, u_{n_k}(t))\, d\mu < \frac{1}{3}r$ for any set $D \subset \Omega_k$ such that $\mu(D) \leq 2\varepsilon_{k+1}$.

We put $\varepsilon_1 = \mu(\Omega), x_{n_1} - x_0 = x_1 - x_0$ and $\Omega_1 = \Omega$. Suppose that $\varepsilon_k, x_{n_k} - x_0$ and Ω_k have been constructed. Since N_F is a Nemytskii set-valued operator mapping of the modular space $(N(L(\Omega, \Sigma, \mu; X)), \rho_{N,\mu})$ into subsets of the modular space $(M(L(\Omega, \Sigma, \mu; Y)), \rho_{M,\mu})$, by property (Δ_2),

$$\int_{\Omega_k} M(t, u_{n_k}(t))\, d\mu < +\infty.$$

Thus the function $M(t, u_{n_k}(t))$ is absolutely continuous. Hence it is easy to find ε_{k+1} satisfying conditions (a) and (d). Since the function N $= N(t, u)$ satisfies condition (A) and $\rho(x_n(\cdot) - x_0(\cdot)) \to 0$, the functions x_n tend to x_0 in measure. Replacing eventually the sequence $\{x_{n_k} - x_0\}$ by its subsequence, we can assume without loss of generality that $x_{n_k}(t)$ tends to $x_0(t)$ almost everywhere. By the global lower semicontinuity of $F(t, x)$, we obtain that $d_Y(y_n(t), F(t, x_n(t)))$ tends to 0 almost everywhere.

Since $\mu(\Omega) < +\infty$, for the sequence $\{z_n(\cdot)\}$ of measurable selections chosen at the beginning of this proof such that $(7)_1$ holds, there are an index n_{k+1} and a set $E_{k+1} \subset \Omega$ such that, for $t \in E_{k+1}$,

$$M(t, u_{n_k}(t)) < \frac{r}{3\mu(\Omega)} \quad (11)_1$$

and

$$\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1}. \quad (12)_1$$

Let $\Omega_{k+1} = \Omega \setminus E_{k+1}$. Observe that $(12)_1$ implies condition (b). By $(8)_1$ and $(11)_1$, we obtain

$$\int_{\Omega_{k+1}} M(t, u_{n_{k+1}}(t))\, d\mu = \int_{\Omega} M(t, u_{n_{k+1}}(t))\, d\mu - \int_{E_{k+1}} M(t, u_{n_{k+1}}(t))\, d\mu > \frac{2}{3}r. \quad (13)_1$$

By properties (a) and (b), we get

$$\mu\Big(\bigcup_{j=k+1}^{\infty} \Omega_j\Big) \le \sum_{j=k+1}^{\infty} \mu(\Omega_j) \le \sum_{j=k+1}^{\infty} \varepsilon_j < 2\varepsilon_{k+1}.$$

Thus we have constructed a sequence of positive numbers $\{\varepsilon_k\}$, a sequence of measurable sets $\{\Omega_k\}$, $\Omega_k \subset \Omega$, and a subsequence $\{x_{n_k} - x_0\}$ such that conditions (a) - (d) hold.

Now we shall continue the proof. Let

$$D_k = \Omega_k \setminus \Big(\bigcup_{j=k+1}^{\infty} \Omega_j\Big) \qquad (k \in \mathbb{N}).$$

Define functions ψ, ψ_0, y_0, z in the following way:

$$\psi(s) = \begin{cases} x_{n_k}(s) & \text{if } s \in D_k \ \ (k \in \mathbb{N}) \\ 0 & \text{if } s \notin \cup_{j=1}^{\infty} D_j \end{cases} \quad (14)_1$$

$$\psi_0(s) = \begin{cases} x_0(s) & \text{if } s \in D_k \ \ (k \in \mathbb{N}) \\ 0 & \text{if } s \notin \cup_{j=1}^{\infty} D_j \end{cases} \quad (15)_1$$

$$y_0(s) = \begin{cases} y_{n_k}(s) & \text{if } s \in D_k \ \ (k \in \mathbb{N}) \\ w(s) & \text{if } s \notin \cup_{j=1}^{\infty} D_j \end{cases} \quad (16)_1$$

$$z(s) = \begin{cases} z_{n_k}(s) & \text{if } s \in D_k \ \ (k \in \mathbb{N}) \\ v(s) & \text{if } s \notin \cup_{j=1}^{\infty} D_j \end{cases} \quad (17)_1$$

where $w(\cdot)$ and $v(\cdot)$ belong to $N_F(0)$ (i.e. these are measurable selections of $F(s,0)$). From conditions (c), (d) and inequalities $(11)_1, (13)_1$ it follows that

$$\begin{aligned} \int_{D_k} M(t, \|y_0(t) - z(t)\|)\, d\mu &= \int_{D_k} M(t, u_{n_k}(t))\, d\mu \\ &= \int_{\Omega_k} M(t, u_{n_k}(t))\, d\mu - \int_{\Omega_k \setminus D_k} M(t, u_{n_k}(t))\, d\mu \quad (18)_1 \\ &> \frac{1}{3}r. \end{aligned}$$

Observe that $\psi_0 \in N(L(\Omega,\Sigma,\mu;X))$ and, by $(10)_1$, also $\psi \in N(L(\Omega,\Sigma,\mu;X))$. It is easy to see that $y_0(\cdot) \in N_F(\psi_0(\cdot))$ and $z(\cdot) \in N_F(\psi(\cdot))$.

On the other hand,

$$\int_\Omega M(t, \|y_0(t) - z(t)\|)\, d\mu \geq \sum_{k=1}^{\infty} \int_{D_k} M(t, \|y_0(t) - z(t)\|)\, d\mu = +\infty \qquad (19)_1$$

which contradicts the fact that N_F is a set-valued operator mapping of the modular space $(N(L(\Omega,\Sigma,\mu;X)), \rho_{N,\mu})$ into subsets of the space $(M(L(\Omega,\Sigma,\mu;Y)), \rho_{M,\mu})$. This finishes the proof of the case when $\mu(\Omega)$ is finite.

Case (2): $\mu(\Omega)$ is infinite. We will consider the following two subcases:

(2a) There are a subset $\Omega_0 \subset \Omega$ with finite measure $\mu(\Omega_0)$ and a number $\beta \in (0,r)$ such that $\int_{\Omega_0} M(t, u_n(t))\, d\mu \geq \beta \quad (n \in \mathbb{N})$.

(2b) There are a subsequence $\{u_{n_k}(t)\}$ and a sequence of measurable sets $\{D_k\}$ such that

(e) $\mu(D_k) < +\infty$ and $D_i \cap D_j = \emptyset$ for $i \neq j$

(f) $\int_{D_k} M(t, u_{n_k}(t))\, d\mu \geq \frac{1}{2} r \ (k \in \mathbb{N})$.

In the subcase (2a) the consederation can be reduced to that of Case (1) with replacing r by β. In the subcase (2b) we define functions ψ, ψ_0, y, z by formulae $(14)_1 - (17)_1$. As in Case (1) we obtain that $\psi, \psi_0 \in N(L(\Omega,\Sigma,\mu;X))$, and that $z(\cdot) \in N_F(\psi(\cdot))$ and simultaneously $y_0(\cdot) \in N_F(\psi_0(\cdot))$. On the other hand, by properties (e) and (f) we obtain that $z(t) - y_0(t) \notin (M(L(\Omega,\Sigma,\mu;Y)), \rho_{M,\mu})$, which leads to a contradiction ∎

Theorem 2. *Let X, Y be two separable F-spaces, (Ω,Σ,μ) a measure space with complete non-atomic and σ-finite measue μ, and N_F a Nemytskiĭ set-valued operator mapping of a modular space $(N(L(\Omega,\Sigma,\mu;X)), \rho_{N,\mu})$ into subsets of a modular space $(M(L(\Omega,\Sigma,\mu;Y)), \rho_{M,\mu})$ induced by a sup-measurable set-valued function $F = F(t,u) : \Omega \times X \to 2^Y$. Suppose that*

(i) *the function $N = N(t,u)$ satisfies condition* (A)

(ii) *the function $M = M(t,u)$ satisfies condition* (Δ_2).

If the function $F = F(t,u)$ is lower semicontinuous with respect to u for almost all $t \in \Omega$, then the operator N_F is lower semicontinuous.

Proof. Suppose that the operator N_F is not lower semicontinuous at some point $(x_0(\cdot), y_0(\cdot))$ with $y_0(\cdot) \in N_F(x_0(\cdot))$. This implies that there are a number $r > 0$ and a sequence $\{x_n(\cdot)\}$ such that

$$\rho_{N,\mu}(x_n(\cdot) - x_0(\cdot)) \to 0$$
$$\inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M,\mu}(z(\cdot) - y_0(\cdot)) > r \qquad (n \in \mathbb{N}).$$

By [2: Theorem 8.24 and Corollary 8.23], for arbitrary $\eta > 0$ there is a sequence $\{z_n(\cdot)\}$ of measurable selections $z_n(\cdot) \in N_F(x_n(\cdot))$ such that, for almost all $t \in \Omega$,

$$\|y_0(t) - z_n(t)\|_Y < (1+\eta) d_Y\big(y_0(t), F(t, x_n(t))\big) \tag{7$_2$}$$

where as before $d_Y(y, A)$ denotes the distance of a point y to a set A in the norm $\|\cdot\|_Y$. We denote $u_n(t) = \|y_0(t) - z_n(t)\|_Y$. Since $z_n(\cdot) \in N_F(x_n(\cdot))$, then

$$\int_\Omega M(t, u_n(t))\, d\mu \geq \inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M,\mu}\big(z(\cdot) - y_0(\cdot)\big) > r \qquad (n \in \mathbb{N}). \tag{8$_2$}$$

Then we continue the proof step by step in the same way as in the proof of Theorem 1 replacing $y_n(\cdot)$ by $y_0(\cdot)$. The only difference is to show the existence of a subsequence $\{u_{n_k}\}$ such that the inequalities

$$M\big(t, u_{n_{k+1}}(t)\big) < \frac{r}{3\mu(\Omega)} \tag{11$_2$}$$

and

$$\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \tag{12$_2$}$$

hold. In order to do this we replace the global lower semicontinuity of Γ by its lower semicontinuity. Since the function $N = N(t, u)$ satisfies condition (A) and $\rho_{N,\mu}\big(x_n(\cdot) - x_0(\cdot)\big) \to 0$, the sequence $\{x_n(\cdot)\}$ tends to $x_0(\cdot)$ in measure. Thus by the lower semicontinuity of $F(t, x)$ with respect to x, $d_Y\big(y_0, F(t, x_n(t))\big)$ tends to 0 in measure.

Since $\mu(\Omega) < +\infty$, for the sequence of measurable selections $\{z_n(\cdot)\}$ there are an index n_{k+1} and a subset $E_{k+1} \subset \Omega$ such that, for $t \in E_{k+1}$, the inequalities

$$M\big(t, u_{n_{k+1}}(t)\big) < \frac{r}{3\mu(\Omega)} \tag{11$_2$}$$

and

$$\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \tag{12$_2$}$$

hold. The remained part of the proof can be continued in the same way as presented in the proof of Theorem 1 ∎

Theorem 3. *Let X, Y be two separable F-spaces, (Ω, Σ, μ) a measure space with complete non-atomic and σ-finite measue μ, and N_F a Nemytskiĭ set-valued operator mapping of a modular space $\big(N(L(\Omega, \Sigma, \mu; X)), \rho_{N,\mu}\big)$ into subsets of a modular space $\big(M(L(\Omega, \Sigma, \mu; Y)), \rho_{M,\mu}\big)$ induced by a sup-measurable set-valued function $F = F(t, u) : \Omega \times X \to 2^Y$. Suppose that*

(i) *the function $N = N(t, u)$ satisfies condition* (A)

(ii) *the function $M = M(t, u)$ satisfies condition* (Δ_2).

If the function $F = F(t, u)$ is metric upper semicontinuous with respect to u for almost all $t \in \Omega$, then the operator N_F is metric upper semicontinuous.

Proof. Suppose that the operator N_F is not metric upper semicontinuous. This implies that there are a number $r > 0$ and sequences $\{x_n(\cdot)\}$ and $\{y_n(\cdot)\}$ with $y_n(\cdot) \in N_F(x_n(\cdot))$ such that

$$\rho_{N,\mu}\big(x_n(\cdot) - x_0(\cdot)\big) \to 0$$
$$\inf_{z(\cdot)\in N_F(x_0(\cdot))} \rho_{M,\mu}\big(z(\cdot) - y_n(\cdot)\big) > r \qquad (n \in \mathbb{N}).$$

By [2: Theorem 8.24 and Corollary 8.23], for arbitrary $\eta > 0$ there is a sequence $\{z_n(\cdot)\}$ of measurable selections $z_n(\cdot) \in N_F(x_0(\cdot))$ such that, for almost all $t \in \Omega$,

$$\big\|y_n(t) - z_n(t)\big\|_Y < (1+\eta) d_Y\big(y_n(t), F(t, x_0(t))\big) \tag{7$_3$}$$

where as before $d_Y(y, A)$ denotes the distance of a point y to a set A in the norm $\|\cdot\|_Y$. We denote $u_n(t) = \|y_n(t) - z_n(t)\|_Y$. Since $z_n(\cdot) \in N_F(x_0(\cdot))$, then

$$\int_\Omega M\big(t, u_n(t)\big)\, d\mu \geq \inf_{z(\cdot)\in N_F(x_0(\cdot))} \rho_{M,\mu}\big(z(\cdot) - y_n(\cdot)\big) > r \qquad (n \in \mathbb{N}). \tag{8$_3$}$$

Then we continue the proof step by step in the same way as in the proof of Theorem 1. The only difference is to show the existence of a subsequence $\{u_{n_k}\}$ such that the inequalities

$$M\big(t, u_{n_{k+1}}(t)\big) < \frac{r}{3\mu(\Omega)} \tag{11$_3$}$$

and

$$\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \tag{12$_3$}$$

hold. In order to do this we replace the global lower semicontinuity of Γ by its metric upper semicontinuity. Since the function $N = N(t,u)$ satisfies condition (A) and $\rho_{N,\mu}\big(x_n(\cdot) - x_0(\cdot)\big) \to 0$, the sequence $\{x_n(\cdot)\}$ tends to $x_0(\cdot)$ in measure. Thus by the metric upper semicontinuity of $F(t,x)$ with respect to x, $d_Y\big(y_n, F(t, x_0(t))\big)$ tends to 0 in measure.

Since $\mu(\Omega) < +\infty$, for the sequence of measurable selections $\{z_n(\cdot)\}$ there are an index n_{k+1} and a subset $E_{k+1} \subset \Omega$ such that, for $t \in E_{k+1}$, the inequalities

$$M\big(t, u_{n_{k+1}}(t)\big) < \frac{r}{3\mu(\Omega)} \tag{11$_3$}$$

and

$$\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \tag{12$_3$}$$

hold. The remained part of the proof can be continued in the same way as presented in the proof of Theorem 1 ∎

Theorem 3 generalizes Theorem 1 of [3], where it is proved for Banach spaces X, Y and for functions $N(t,u) = u^p, M(t,u) = u^q$ with $1 \leq p \leq q < +\infty$ under some estimation assumptions warranting that a Nemytskiĭ operator N_F induced by a sup-measurable set-valued function $F = F(t,u)$ maps the space $L^p(\Omega, \Sigma, \mu; X)$ into the space $L^q(\Omega, \Sigma, \mu; Y)$.

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