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On a Property of Harmonic Functions

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Abstract. If we divide the space $\mathbb{R}^n$ into two disjoint areas with one common hypersurface and define a harmonic function in each part of these areas such that their gradients vanish at infinity and the normal components of their gradients are equal on the hypersurface, then for some hypersurfaces such as a circle in $\mathbb{R}^2$ or a hyperplane in $\mathbb{R}^n$ the sum of the tangential components of the gradients is zero. We investigate for which hypersurfaces we have this property and prove that such hypersurfaces in $\mathbb{R}^2$ are only circles and straight lines. We also give an application of this property to an ideal plane flow through a porous surface.

Keywords: *Laplace equation, flows through porous media*

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1. Introduction

In this work we denote by Γ a piecewise smooth hypersurface in $\mathbb{R}^n$ which divides $\mathbb{R}^n$ into two domains $\mathbb{R}^n_-$ and $\mathbb{R}^n_+$. In other words we have

$$\mathbb{R}^n = \mathbb{R}^n_- \cup \mathbb{R}^n_+ \cup \Gamma \qquad \text{and} \qquad \mathbb{R}^n_- \cap \mathbb{R}^n_+ = \mathbb{R}^n_- \cap \Gamma = \mathbb{R}^n_+ \cap \Gamma = \emptyset.$$

By ν we denote the unit normal of Γ and by $\operatorname{Grad}\Phi$ we denote the tangential projection of $\operatorname{grad}\Phi$ on Γ. In $\mathbb{R}^2$ we denote the unit tangent to the curve by σ.

Definition 1.1: We say that the hypersurface Γ has the *linear property* if for each pair (Φ_-, Φ_+) of harmonic functions $\Phi_\pm \in C^1(\mathbb{R}^n_\pm \cup \Gamma)$ the identities

$$\frac{\partial \Phi_-}{\partial \nu}(\mathbf{x}) = \frac{\partial \Phi_+}{\partial \nu}(\mathbf{x}) \quad (\mathbf{x} \in \Gamma) \tag{1}$$

and

$$\lim_{|\mathbf{x}|\to\infty} \operatorname{grad}\Phi_\pm(\mathbf{x}) = 0 \quad (\mathbf{x} \in \mathbb{R}^n_\pm) \tag{2}$$

imply

$$\operatorname{Grad}\Phi_-(\mathbf{x}) + \operatorname{Grad}\Phi_+(\mathbf{x}) = 0 \qquad (\mathbf{x} \in \Gamma). \tag{3}$$

In $\mathbb{R}^2$ we speak of *curves* with the linear property and the equation (3) turns into

$$\frac{\partial \Phi_-}{\partial \sigma}(\mathbf{x}) = \frac{\partial \Phi_+}{\partial \sigma}(\mathbf{x}) \qquad (\mathbf{x} \in \Gamma).$$

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The aim of this article is to investigate which curves in $\mathbb{R}^2$ have this property. This property can in case $n = 2$ be equivalently defined for holomorphic functions in $\mathbb{C}$. This is considered in Section 2. In Section 3 we give some examples of hypersurfaces and curves with and without the linear property. In Section 4 we prove that the only curves with the linear property in $\mathbb{R}^2$ are circles and straight lines. In the last, Section 5 we show an application of this property in the flow of an ideal fluid through a porous surface.

2. Equivalent definition for holomorphic functions

If we identify $\mathbb{R}^2$ with $\mathbb{C}$, we can prove the following theorem.

Theorem 2.1: *Let $\Gamma \subseteq \mathbb{C}$ be a piecewise smooth curve which divides $\mathbb{C}$ into two domains, i.e. let*

$$\mathbb{C} = \mathbb{C}_+ \cup \mathbb{C}_- \cup \Gamma \qquad \text{and} \qquad \mathbb{C}_+ \cap \mathbb{C}_- = \mathbb{C}_+ \cap \Gamma = \mathbb{C}_- \cap \Gamma = \emptyset$$

hold and let $z : M \longrightarrow \mathbb{C}$, $M \subseteq \mathbb{R}$ be a parametrization of Γ. Then Γ is a curve with the linear property if and only if for each pair (Ψ_+, Ψ_-) of holomorphic functions $\Psi_\pm \in C^0(\mathbb{C}_\pm \cup \Gamma)$ the identities

$$Im\ \left(z'(\xi)\Psi_+(z(\xi))\right) = Im\ \left(z'(\xi)\Psi_-(z(\xi))\right) \tag{4}$$

and

$$\begin{aligned} &\lim_{q\to\infty} \Psi_\pm(q) = 0 && \Gamma \text{ unbounded} \\ &\lim_{q\to\infty} q\Psi_\pm(q) = 0 && \Gamma \text{ bounded} \end{aligned} \tag{5}$$

imply

$$Re\ \left(z'(\xi)\Psi_+(z(\xi))\right) = Re\ \left(z'(\xi)\Psi_-(z(\xi))\right). \tag{6}$$

Hereby -Im $\left(z'(\xi)\Psi_\pm(z(\xi))\right)$ can be interpreted as the "normal component" of the holomorphic function $\Psi_\pm$ on Γ and Re $\left(z'(\xi)\Psi_\pm(z(\xi))\right)$ as the "tangential component" of $\Psi_\pm$ on Γ.

Proof: Let the inclusion (4), (5) $\Longrightarrow$ (6) hold. For each pair (Φ_+, Φ_-) of harmonic functions $\Phi_\pm \in C^1(\mathbb{R}^2_\pm \cup \Gamma)$ satisfying (1) and (2) we define a pair of holomorphic functions $\Psi_\pm \in C^0(\mathbb{C}_\pm \cup \Gamma)$ by

$$\Psi_\pm(x + iy) = \frac{\partial \Phi_\pm}{\partial x}(x, y) - i\frac{\partial \Phi_\pm}{\partial y}(x, y).$$

It is easy to check that we have on Γ

$$\left.\begin{aligned} \frac{\partial}{\partial \nu}\Phi_\pm(z(\xi)) &= \text{Im}\left(\frac{z'(\xi)}{|z'(\xi)|}\Psi_\pm(z(\xi))\right) \\ \frac{\partial}{\partial \sigma}\Phi_\pm(z(\xi)) &= \text{Re}\left(\frac{z'(\xi)}{|z'(\xi)|}\Psi_\pm(z(\xi))\right) \end{aligned}\right. \qquad (\xi \in \mathbb{R}_0). \tag{7}$$

Thus from (1) there follows (4). If Γ is unbounded, we have (5) direct from (2). If Γ is bounded, we also have (5) because of (2) and because of

$$\int_{\partial K(0,R)} \operatorname{grad} \Phi_-(x)\, dx = 0$$

for R big enough (where $\mathbb{R}^2_-$ is the area which contains the infinity). This implies (6) which together with (7) gives (3). From this there follows that Γ ist the curve with the linear property.

Let now Γ be a curve with the linear property and let $\Psi_\pm \in C^0(\boldsymbol{C}_\pm \cup \Gamma)$ be two holomorphic functions with (4) and (5). We can define two harmonic fields by

$$\mathbf{w}_\pm(x,y) = \left(\operatorname{Re}\Psi_\pm(x+iy),\, -\operatorname{Im}\Psi_\pm(x+iy)\right).$$

If Γ is not bounded, we can write

$$\mathbf{w}_\pm = \operatorname{grad}\Phi_\pm \tag{8}$$

because $\boldsymbol{C}_\pm$ are simply connected domains, and if Γ is bonded, we can also write (8) because from (5) there follows

$$\int_{\partial K(0,R)} \Psi_-(z)\, dz = 0$$

and this implies

$$\int_{\partial K(0,R)} \mathbf{w}_-(x)\, dx = 0$$

for R big enough, whereby $\boldsymbol{C}_-$ respectively $\mathbb{R}^2_-$ is the area outside Γ. From (4) and (7) we have now (1), from (5) we have (2), so we must have (3), which implies (6) ∎

3. Examples

As already announced, in this section we will give some examples of hypersurfaces and curves with and without the linear property in the sense of Definition 1.1.

Example 3.1 (*Hyperplane in $\mathbb{R}^n$*): Let Γ be a hyperplane in $\mathbb{R}^n$ and let $\mathbf{x}'$ be symmetric to the point $\mathbf{x}$ relative to the hyperplane Γ. For a given pair of harmonic functions (Φ_-, Φ_+) from Definition 1.1 with properties (1), (2) we define the function

$$\tilde{\Phi}_+ : \mathbb{R}^n_+ \cup \Gamma \longrightarrow \mathbb{R}^n \quad \text{by} \quad \tilde{\Phi}_+(\mathbf{x}) = -\Phi_-(\mathbf{x}'(\mathbf{x})) \quad (\mathbf{x} \in \mathbb{R}^n_+ \cup \Gamma).$$

The functions Φ_+ and $\tilde{\Phi}_+$ are both harmonic, belong to the space $C^1(\mathbb{R}^n_+ \cup \Gamma)$, their gradients vanish at infinity and they have the same normal derivatives on Γ. From this there follows $\Phi_+(\mathbf{x}) = \tilde{\Phi}_+(\mathbf{x}) + c$ with some real constant c. Because of the symmetry we have property (3) for the pair $(\Phi_-, \tilde{\Phi}_+)$, and so we must have property (3) for the pair (Φ_-, Φ_+) as well.

Example 3.2 (*Circle in $\mathbb{R}^2$*): Without loss of generality we set $\Gamma = \partial K(0,1)$. Let $\mathbb{R}^2_+$ be the area inside Γ and let $\mathbf{x} \longmapsto \mathbf{x}^*$ be the inversion $\mathbf{x}^*(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|^2$. Again, for a given pair (Φ_-, Φ_+) of harmonic functions from Definition 1.1 with properties (1), (2) we define a function

$$\tilde{\Phi}_+ : \mathbb{R}^2_+ \cup \Gamma \longrightarrow \mathbb{R}^2 \qquad \text{by} \quad \tilde{\Phi}_+(\mathbf{x}) = -\Phi_-(\mathbf{x}^*(\mathbf{x})) \quad (\mathbf{x} \in \mathbb{R}^n_+ \cup \Gamma).$$

Similary, we can conclude that $\Phi_+(\mathbf{x}) = \tilde{\Phi}_+(\mathbf{x}) + c$ for some real constant c. Now, we have property (3) for the pair $(\Phi_-, \tilde{\Phi}_+)$ which implies property (3) also for the pair (Φ_-, Φ_+). In the case of the unit circle we can consider the functions $q\Psi_+(q)$, $\frac{1}{q}\Psi_-(q)$ in Theorem 2.1 and obtain the following conclusion: Let $\Psi_+ \in C^0(\overline{K(0,1)})$, $\Psi_- \in C^0(\mathbb{C}\backslash K(0,1))$ be two holomorphic functions with $\Psi(0) = 0$, $\lim_{q\to\infty} \Psi_-(q) = 0$ and $\mathrm{Re}\ \Psi_+(q) = \mathrm{Re}\ \Psi_-(q)$, $q \in \partial K(0,1)$. Then we have $\mathrm{Im}\ \Psi_+(q) = -\ \mathrm{Im}\ \Psi_-(q)$, $z \in \partial K(0,1)$.

Example 3.3 (*Sphere in $\mathbb{R}^n$, $n \geq 3$*): For the sphere $\partial K(0,1)$ in $\mathbb{R}^n$ we define

$$\Phi_+(x_1, x_2, ..., x_n) = x_1$$
$$\Phi_-(x_1, x_2, ..., x_n) = \frac{x_1}{(n-1)(x_1^2 + x_2^2 + \ \dots\ + x_n^2)^{n/2}}.$$

We have the inclusions $\Phi_+ \in C^1(K(0,1))$ and $\Phi_- \in C^1(\mathbb{R}^3 \setminus K(0,1))$ and properties (1), (2) but not property (3). Thus the sphere in $\mathbb{R}^n$ is not a surface with the linear property.

Example 3.4 (*Square in $\mathbb{R}^2$*): Let $Q = \{(x,y) \in \mathbb{R}^2 |\ |x| < 1 \text{ and } |y| < 1\}$ be the unit square in $\mathbb{R}^2$ which we identify with $\mathbb{C}$ in the topological sence. On the boundary $\Gamma = \partial Q$ we define the function u by

$$u(a+bi) = \begin{cases} |1-a| & \text{for } a \in [-1,1], \quad b = \pm 1 \\ i|1-b| & \text{for } a = \pm 1, \quad b \in [-1,1] \end{cases}$$

and the functions $\Psi_\pm(q)$ by

$$\Psi_\pm(q) = \frac{1}{2\pi i} \int_\Gamma \frac{u(z)dz}{z-q}.$$

We have the inclusions $\Psi_- \in C^0(\mathbb{C}_- \cup \Gamma)$ and $\Psi_+ \in C^0(\mathbb{C}_+ \cup \Gamma)$ and the properties (3), (4) but not property (6). Thus the square in $\mathbb{R}^2$ is no curve with the linear property.

4. Curves with the linear property in $\mathbb{R}^2$

Now we prove that the only curves with the linear property in $\mathbb{R}^2$ are circles and straight lines. We divide the problem into the following two cases:

1. The hypersurface Γ is unbounded
2. The hypersurface Γ is closed and bounded.

Theorem 4.1: *The straight lines are the only unbounded curves in $\mathbb{R}^2$ with the linear property.*

Proof: We already know that all straight lines have the linear property. We only need to show that there are not any other unbounded curves with the linear property. Let Γ be some piecewise smooth curve in $\mathbb{C}$ with the linear property which divides $\mathbb{C}$ into two open areas $\mathbb{C}_-$ and $\mathbb{C}_+$, let Γ_0 be the smooth part of Γ (the non-smooth part is assumed to consist of finitely many points), and let $z : \mathbb{R} \longrightarrow \mathbb{C}$ be a parametrization of Γ with $|z'(\xi)| = 1$ for each $\xi \in \mathbb{R}_0$ where $\mathbb{R}_0 \subseteq \mathbb{R}$ is the set of all ξ with $z(\xi)$ smooth. For each $u \in C_0^\infty(\mathbb{R}_0)$ (i.e. the set of all $u \in C^\infty(\mathbb{R}_0)$ with compact support in $\mathbb{R}_0$) we define the functions $\Psi_\pm : \mathbb{C}_\pm \longrightarrow \mathbb{C}$ by

$$\Psi_\pm(q) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{u(\eta)}{z(\eta)-q}\,d\eta = \frac{1}{2\pi i}\int_\Gamma \frac{u(z^{-1}(\tau))}{z'(z^{-1}(\tau))(\tau-q)}\,d\tau \qquad (q \in \mathbb{C}_\pm)$$

with $z^{-1} : \Gamma \longrightarrow \mathbb{R}$ the inverse of z. These functions can be extended continuously onto Γ_0. The Plemelj formulae give the representation

$$\Psi_\pm(t) = \pm\frac{1}{2}\frac{u(z^{-1}(t))}{z'(z^{-1}(t))} + \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus K(z,\varepsilon)} \frac{u(z^{-1}(\tau))}{z'(z^{-1}(\tau))(\tau-t)}\,d\tau \qquad (t \in \Gamma_0),$$

i.e.

$$\Psi_\pm(z(\xi)) = \pm\frac{1}{2}\frac{u(\xi)}{z'(\xi)} + \frac{1}{2\pi i}\,\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)}{z(\eta)-z(\xi)}\,d\eta. \tag{9}$$

It follows from (9) that

$$\text{Im}\,\big(z'(\xi)\Psi_+(z(\xi))\big) = \text{Im}\,\big(z'(\xi)\Psi_-(z(\xi))\big) = \text{Im}\left(\frac{1}{2\pi i}\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right).$$

Because we have a curve with the linear property, we must have

$$\text{Re}\,\big(z'(\xi)\Psi_+(z(\xi))\big) + \text{Re}\,\big(z'(\xi)\Psi_-(z(\xi))\big) = 0.$$

This implies

$$\text{Re}\left(\frac{1}{\pi i}\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right) = -\frac{1}{\pi}\text{Im}\left(\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right) = 0$$

for each $u \in C_0^\infty(\mathbb{R})$ and for each $\xi \in \mathbb{R}_0$. Lagrange's Lemma gives

$$\operatorname{Im}\left(\frac{z'(\xi)}{z(\eta)-z(\xi)}\right) = 0 \qquad \text{for all} \quad \xi,\eta \in \mathbb{R}_0,\ \xi \neq \eta. \tag{10}$$

One can remark that for the function $pz(\xi)+q$, with $p,q \in \boldsymbol{C}$, $|p| = 1$ we have

$$\operatorname{Im}\left(\frac{(pz+q)'(\xi)}{(pz+q)(\eta)-(pz+q)(\xi)}\right) = \operatorname{Im}\left(\frac{z'(\xi)}{z(\eta)-z(\xi)}\right) = 0,$$

i.e. the function $pz+q$ satisfies the same condition as z. In the set of all functions $z:\ \mathbb{R} \longrightarrow \boldsymbol{C}$ with $|z'(\xi)| = 1$ a.e. we define an equivalence relation $\sim$ by

$$z_1 \sim z_2 \iff z_2 = pz_1 + q \quad \text{for some} \quad p,q \in \boldsymbol{C} \ \text{ with } \ |p| = 1 \tag{11}$$

and in each equivalence class we choose an element z_0 with $z_0(0) = 0$ and $z_0'(0) = 1$. We set $\xi = 0$ in (10) and obtain $\operatorname{Im} z_0(\eta)^{-1} = 0$. This implies $\operatorname{Im} z = 0$, i.e. Γ is the real line.

We go back to (11) and consider the whole class of equivalence. By $pz_0 + q$ $(p,q \in \boldsymbol{C}, |p| = 1)$ we get only straight lines in $\boldsymbol{C}$. So all unbounded curves in $\boldsymbol{C}$ with the linear property in $\boldsymbol{C}$ (and in $\mathbb{R}^2$) are straight lines. ∎

Theorem 4.2: *Circles are the only closed bounded curves in $\mathbb{R}^2$ with the linear property.*

Proof: The proof is similar to that of the previous theorem. Again, we only need to show that there are no other bounded curves with the linear property. Let $\Gamma \subseteq \boldsymbol{C}$ be some piecewise smooth bounded curve with the linear property and $z:\ M = [a,b] \longrightarrow \boldsymbol{C}$ some continuous piecewise smooth parametrization of Γ with $|z'(\xi)| = 1$ for each $\xi \in M_0 \subseteq M = [a,b]$, where M_0 describes the smooth points of z (and $M \subset M_0$ is assumed to have only finitely many points). For each $u \in C_0^\infty(M_0)$ with

$$\int_a^b u(\eta)\,d\eta = 0 \tag{12}$$

we define functions $\Psi_\pm:\ \boldsymbol{C}_\pm \longrightarrow \boldsymbol{C}$ by

$$\Psi_\pm(q) = \frac{1}{2\pi i}\int_a^b \frac{u(\eta)}{z(\eta)-q}\,d\eta = \frac{1}{2\pi i}\int_\Gamma \frac{u(z^{-1}(\tau))}{z'(z^{-1}(\tau))(\tau-q)}\,d\tau \qquad (q \in \boldsymbol{C}_\pm)$$

where $z^{-1}:\ \Gamma \longrightarrow M$ is the inverse of z. The Plemelj formulae give the representation

$$\Psi_\pm(t) = \pm\frac{1}{2}\frac{u(z^{-1}(t))}{z'(z^{-1}(t))} + \frac{1}{2\pi i}\text{V.P.}\int_a^b \frac{u(z^{-1}(\tau))}{z'(z^{-1}(\tau))(\tau-t)}\,d\tau \qquad (t \in \Gamma_0),$$

i.e.

$$\Psi_\pm(z(\xi)) = \pm\frac{1}{2}\frac{u(\xi)}{z'(\xi)} + \frac{1}{2\pi i}\text{V.P.}\int_a^b \frac{u(\eta)}{z(\eta)-z(\xi)}\,d\eta.$$

We have

$$\int_\Gamma \Psi_-(t)\,dt = -\frac{1}{2}\int_a^b u(\xi)\,d\xi + \left(\frac{1}{2\pi i}\int_a^b\int_a^b \frac{z'(\xi)u(\eta)}{z(\eta)-z(\xi)}\,d\eta d\xi\right).$$

The integral

$$\int_a^b \frac{z'(\xi)}{z(\eta)-z(\xi)}\,d\xi$$

is constant for each $\eta \in M_0$ with value $-i\pi$. From this and from (12) we have in this case the relations

$$\int_\Gamma \Psi_-(t)\,dt = -\int_a^b u(\xi)\,d\xi = 0$$

so condition (5) is fulfilled. Further, we obtain

$$\text{Im}\,\left(z'(\xi)\Psi_+(z(\xi))\right) = \text{Im}\,\left(z'(\xi)\Psi_-(z(\xi))\right) = \text{Im}\left(\frac{1}{2\pi i}\text{V.P.}\int_a^b \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right).$$

Because we have a curve with the linear property, we must have

$$\text{Re}\,\left(z'(\xi)\Psi_+(z(\xi))\right) + \text{Re}\,\left(z'(\xi)\Psi_-(z(\xi))\right) = 0.$$

This implies

$$\text{Re}\left(\frac{1}{\pi i}\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right) = -\frac{1}{\pi}\text{Im}\left(\text{V.P.}\int_{-\infty}^{+\infty} \frac{u(\eta)z'(\xi)}{z(\eta)-z(\xi)}\,d\eta\right) = 0$$

for each $u \in C_0^\infty(M)$ with (12) and each $\xi \in M_0$. Each $w \in C^\infty(\Gamma)$ we can write in the form

$$w(\xi) = \frac{1}{b-a}\int_a^b w(\eta)\,d\eta + w_1(\xi) \qquad \text{with} \qquad \int_a^b w_1(\eta)\,d\eta = 0.$$

Then for each $w \in C^\infty(\Gamma)$ we have

$$\begin{aligned}&\text{Im}\left(\text{V.P.}\int_a^b \frac{z'(\xi)w(\eta)}{z(\eta)-z(\xi)}\,d\eta\right)\\ &= \text{Im}\left(\text{V.P.}\int_a^b \frac{z'(\xi)(b-a)^{-1}\int_a^b w(\zeta)\,d\zeta}{z(\eta)-z(\xi)}\,d\eta + \text{V.P.}\int_a^b \frac{z'(\xi)w_1(\eta)}{z(\eta)-z(\xi)}\,d\eta\right).\end{aligned}$$

Because the last integral is zero we can conclude

$$\operatorname{Im}\left(\text{V.P.}\int_a^b \frac{z'(\xi)w(\eta)}{z(\eta)-z(\xi)}\,d\eta\right)$$

$$=\operatorname{Im}\left(\text{V.P.}\int_a^b \frac{z'(\xi)}{b-a}\int_a^b w(\zeta)\,d\zeta\frac{d\eta}{z(\eta)-z(\xi)}\right)=f(\xi)\int_a^b w(\zeta)\,d\zeta$$

for each $w \in C_0^\infty(M_0)$, with f some constant independent of w and η. Lagrange's Lemma gives

$$\operatorname{Im}\left(\frac{z'(\xi)}{z(\eta)-z(\xi)} - if_z(\xi)\right)=0 \tag{13}$$

for each choice $\xi, \eta \in M_0$ with $\xi \neq \eta$. One can remark that for the function $pz(\xi)+q$ with $p, q \in \boldsymbol{C}$, $|p|=1$ we have

$$\operatorname{Im}\left(\frac{(pz+q)'(\xi)}{(pz+q)(\eta)-(pz+q)(\xi)} - if_z(\xi)\right)=\operatorname{Im}\left(\frac{z'(\xi)}{z(\eta)-z(\xi)} - f_z(\xi)\right)=0,$$

i.e. $pz+q$ also satisfies the same condition as z. In the set of functions $z: I\!R \supseteq M \longrightarrow \boldsymbol{C}$ with $|z'(\xi)|=1$ a.e. we define an equivalence relation $\sim$ by

$$z_1 \sim z_2 \iff z_2 = pz_1 + q \quad \text{for some } p, q \in \boldsymbol{C} \text{ with } |p|=1$$

and from each equivalence class we choose an element z_0 with $z_0(0)=0$ and $z_0'(0)=1$. We set $\xi=0$ in equation (13) and get $\operatorname{Im} z_0(\eta)^{-1} = f_z(0) = \text{const}$, i.e. z_0^{-1} is some straight line parallel to the real axis. Then z_0 is either a circle through the origin of the coordinate system or the real axis. The last case is impossible, because the real axis is unbounded. All bounded curves which can have the linear property are now given by pz_0+q $(p,q \in \boldsymbol{C}, |p|=1)$ with z_0 a circle, and these are again only circles. ∎

5. Application to an ideal flow through a porous surface

Let Γ be some curve in $I\!R^2$ with the linear property. We consider a stationary ideal plane flow, i.e. a pair of functions $(\mathbf{v}, p): I\!R^2 \times I\!R^2 \longrightarrow I\!R^2 \times I\!R$ which satisfies the following conditions:

$$(\mathbf{v}(\mathbf{x})\cdot\nabla)\mathbf{v}(\mathbf{x}) + \frac{1}{\rho}\nabla p(\mathbf{x}) = 0 \qquad (\mathbf{x} \in I\!R^2 \setminus \Gamma) \tag{14}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x}) = 0 \quad \text{and} \quad \operatorname{rot}\mathbf{v}(\mathbf{x}) = 0 \qquad (\mathbf{x} \in I\!R^2 \setminus \Gamma) \tag{15}$$

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{v}(\mathbf{x}) = \mathbf{v}_\infty. \tag{16}$$

On Γ let

$$(\mathbf{v}\cdot\nu)_-(\mathbf{x}) = (\mathbf{v}\cdot\nu)_+(\mathbf{x}) \qquad \text{(continuity condition)} \tag{17}$$

$$(\mathbf{v}\cdot\nu)_\pm(\mathbf{x}) = \gamma(\mathbf{x})(p_-(\mathbf{x}) - p_+(\mathbf{x})) \qquad \text{(filtration law)} \tag{18}$$

hold. For given $\rho > 0$, $\mathbf{v}_\infty \in \mathbb{R}^2$ and $\gamma \in C(\Gamma)$ with $\gamma(\mathbf{x}) \geq 0$ we look for the functions $\mathbf{v}$ and p.

Because of (15) we can write

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_\infty + \begin{cases} \operatorname{grad}\Phi_-(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^2_- \\ \operatorname{grad}\Phi_+(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^2_+ \end{cases}$$

and the Euler equation (14) can be replaced by the Bernoulli equation

$$\frac{1}{2}\Big(\mathbf{v}_\infty + \operatorname{grad}\Phi_\pm(\mathbf{x})\Big)^2 + \frac{1}{\rho}p(\mathbf{x}) = C_\pm$$

with C_- and C_+ two real constants. This and the continuity condition (17) taken into the filtration law (18) give

$$\begin{aligned}(\mathbf{v}_\infty \cdot \nu)(\mathbf{x}) + \frac{\partial\Phi_\pm}{\partial\nu}(\mathbf{x}) = \frac{1}{2}\rho\gamma(\mathbf{x})\Bigg(C_- - C_+ &\\ + 2(\mathbf{v}_\infty\cdot\sigma)(\mathbf{x})\left(\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x}) - \frac{\partial\Phi_-}{\partial\sigma}(\mathbf{x})\right)&\\ + \left(\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x})\right)^2 - \left(\frac{\partial\Phi_-}{\partial\sigma}(\mathbf{x})\right)^2\Bigg).&\end{aligned}$$

Because Γ is a curve with the linear property we have

$$\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x}) - \frac{\partial\Phi_-}{\partial\sigma}(\mathbf{x}) = \pm 2\frac{\partial\Phi_\pm}{\partial\sigma}(\mathbf{x}) \quad \text{and} \quad \left(\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x})\right)^2 - \left(\frac{\partial\Phi_-}{\partial\sigma}(\mathbf{x})\right)^2 = 0.$$

This gives

$$(\mathbf{v}_\infty \cdot \nu)(\mathbf{x}) + \frac{\partial\Phi_+}{\partial\nu}(\mathbf{x}) = \frac{1}{2}\rho\gamma(\mathbf{x})\left(C_- - C_+ + 4(\mathbf{v}_\infty\cdot\sigma)(\mathbf{x})\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x})\right).$$

In case when Γ is a curve with linear property, the partial derivatives of Φ_+ on Γ are connected through the Cauchy integral operator

$$\frac{\partial\Phi_+}{\partial\nu}(\xi) = \frac{z'(\xi)}{\pi i}\int_M \frac{\frac{\partial\Phi_+}{\partial\sigma}(\eta)}{z(\eta) - z(\xi)}d\xi,$$

where $z : M \longrightarrow \mathbb{C}$ is a parametrization of Γ with $|z'(\xi)| = 1$, $\xi \in M$. Now we have a singular integral equation with one free parameter $C = C_+ - C_-$

$$(\mathbf{v}_\infty \cdot \nu)(\mathbf{x}) + \frac{z'(\xi)}{\pi i}\int_M \frac{\frac{\partial\Phi_+}{\partial\sigma}(\eta)}{z(\eta) - z(\xi)}d\xi = \frac{1}{2}\rho\gamma(\mathbf{x})\left(C + 4(\mathbf{v}_\infty\cdot\sigma)(\mathbf{x})\frac{\partial\Phi_+}{\partial\sigma}(\mathbf{x})\right).$$

This equation can be explicitely solved (see [1]). After this equation is solved , we obtain Φ_+ on Γ up to a constant. We set $\Phi_- = -\Phi_+$ on Γ and then Φ_+ and Φ_- on the rest of the plane are given as solutions of the Dirichlet problem on Γ. This gives a unique solution $\mathbf{v}$ on the whole plane and p is then given by the Bernoulli equation up to a constant. If Γ is not a curve with the linear property, we also get a singular integral equation which is not linear and more difficult to solve. The similar procedure is also possible for the case when Γ is only a part of a curve with the linear property and for the non-stationary potential plane flow. The stationary flow for Γ a circle is solved in [4]. There, the problem was considered in terms of conplex velocity $\Psi_\pm = v_x - iv_y$ and the linear property in the form $\text{Im}\Psi_+ = \text{Im}\Psi_- \Longrightarrow \text{Re}\Psi_+ = -\text{Re}\Psi_-$ was used (see Example 3.2). The stationary problem for Γ a straight line and line segment was solved in [2] with the linear property in the form of Definition 1.1. The non-stationary problem for Γ a circle was considered in [5] and in [3].

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