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A Classification of Special Riemannian 3-Manifolds with Distinct Constant Ricci Eigenvalues

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Abstract. All Riemannian 3-manifolds with distinct constant Ricci eigenvalues and satisfying some additional geometrical conditions are classified in an explicit form. One obtains locally homogeneous spaces and two different classes of locally non-homogeneous spaces in this way.

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0. Introduction

According to I. M. Singer [14] a Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for every two points $p, q \in M$, there is a linear isometry $F : T_pM \to T_qM$ between the corresponding tangent spaces such that $F^* R_q = R_p$ (where R denotes the curvature tensor of type (0,4)). Note that a (locally) homogeneous Riemannian manifold is automatically curvature homogeneous. Explicit *locally non-homogeneous examples* have been constructed by many authors (see [4 - 6], [8 - 10], [12], [16] and [17]; see especially [9] and [10] for more complete references).

For 3-dimensional Riemannian manifolds (M, g) the following simple criterion holds:

A Riemannian manifold (M, g) is curvature homogeneous if and only if all Ricci eigenvalues of (M, g) are constant.

Thus the problem to classify all 3-dimensional Riemannian manifolds with prescribed constant Ricci eigenvalues is of considerable interest. This problem was investigated already in 1916 by L. Bianchi [1] who made a classification under a strong additional hypothesis of "normality". He found only some homogeneous Riemannian spaces as solutions. On the other hand, the following conclusion follows from an observation by J. Milnor [11] and a result by K. Sekigawa [13]:

For a homogeneous Riemannian 3-manifold (M, g), the signature of the Ricci tensor is never equal to $(+, +, -)$ and to $(+, 0, -)$.

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Let us describe shortly what is known about the problem at the present. Let ρ_1, ρ_2, ρ_3 denote the constant Ricci eigenvalues of (M, g). The case in which all Ricci eigenvalues are equal is trivial – we obtain only spaces of constant curvature. The case $\rho_1 = \rho_2 \neq \rho_3$ was solved completely by the first author in [4 - 5]. The answer is that the local isometry classes of the corresponding metrics always depend on two arbitrary functions of one variable. (Notice that the local isometry classes of locally homogeneous spaces in dimension three depend only on a finite number of parameters.) In the same papers, some quasi-explicit formulas were given for all the solutions.

The most difficult and most interesting case is that of distinct constant Ricci eigenvalues. In 1991, K. Yamato [17] presented a broad class of *explicit* examples - these are complete Riemannian metrics given on $\mathbb{R}^3$. Such metrics exist only if the Ricci eigenvalues satisfy certain system of inequalities. In [6] the author gave an easy modification of the Yamato examples. The new metrics are not necessarily complete but the range of the possible Ricci eigenvalues is considerably larger (this includes all Ricci tensors with the signatures $(+,+,-)$ and $(+,0,-)$, but still not all of the possible cases).

Recently, A. Spiro and F. Tricerri [15] proved the following results:

a) For each triplet of prescribed distinct constant Ricci eigenvalues, there exists at least one corresponding Riemannian metric (1992).

b) The local isometry classes of all such metrics depend on an infinite number of parameters (1993).

(In 1981, DeTurck [2] proved the local existence of a Riemannian metric with the prescribed non-singular Ricci tensor R_{ij}. Let us mention that his method cannot be directly applied to the problem of prescribed Ricci eigenvalues.)

In a previous paper [7], the present authors gave *explicit* examples of locally non-homogeneous metrics on $\mathbb{R}^3$ with any prescribed distinct constant Ricci eigenvalues. These examples can be considered as "generalized Yamato examples".

In this article we use the same method for solving a special classification problem. In this classification we obtain locally homogeneous spaces, all generalized Yamato examples and also a new class of locally non-homogeneous examples.

1. A classification theorem

In the following we assume that (M, g) is a Riemannian manifold of class C^∞ and ∇ will denote the Levi-Civita connection of (M, g). Suppose that all Ricci eigenvalues (i.e. principal Ricci curvatures) ρ_1, ρ_2, ρ_3 are distinct and constant. Locally, one can always find a smooth orthonormal frame $\{E_1, E_2, E_3\}$ consisting of the eigenvectors of the Ricci tensor ρ, respectively. We get with respect to this moving frame

$$\rho_{ii} = \rho_i \quad (i = 1,2,3) \qquad \text{and} \qquad \rho_{ij} = 0 \quad (i \neq j).$$

For the corresponding components of the Riemannian curvature tensor R we have

$$\begin{aligned} &R_{1212} = \lambda_3,\ R_{1313} = \lambda_2,\ R_{2323} = \lambda_1 \quad (\lambda_1, \lambda_2, \lambda_3 \text{ - constants}) \\ &R_{ijkl} = 0 \quad \text{if at least three indices are distinct.} \end{aligned} \tag{1.1}$$

Moreover

$$\lambda_i - \lambda_j = -(\rho_i - \rho_j) \qquad (i,j = 1,2,3)$$

because

$$\rho_1 = \lambda_2 + \lambda_3, \qquad \rho_2 = \lambda_1 + \lambda_3, \qquad \rho_3 = \lambda_1 + \lambda_2.$$

Furthermore we put

$$\alpha = \frac{\rho_1 - \rho_3}{\rho_3 - \rho_2}.$$

We see at once that $\alpha \neq 0$ and $\alpha \neq -1$. We shall limit ourselves to a sufficiently small connected neighbourhood U_p of a fixed point $p \in M$.

In [7] an *adapted* system of local coordinates (w, x, y) was introduced on U_p such that $E_3 = \frac{\partial}{\partial y}$. Denoting the connection coefficients by $a^i_{jk} = g(\nabla_{E_k} E_j, E_i)$, we get in such a coordinate system

$$a^2_{32} = \alpha\, a^1_{31}, \quad a^2_{33} = (\alpha+1)\, a^1_{21}, \quad a^1_{33} = -\left(\frac{\alpha+1}{\alpha}\right) a^1_{22}. \tag{1.2}$$

We call (M, g) a *generalized Yamato space* on U_p if (with respect to some adapted coordinate system) the a^i_{jk} do not depend on y and satisfy the following conditions on U_p:

(Y1) $a^1_{31} = 0, \;\; a^1_{23} + a^2_{31} = 0, \;\; a^1_{22} = 0.$

(Y2) a^1_{23} is an arbitrary non-constant function of class C^∞ on U_p such that $a^1_{23}(q) \neq 0$ for $q \in U_p$.

(Y3) a^1_{21} is a smooth function on U_p satisfying
(a) $(\alpha+1)\left(a^1_{21}\right)^2 + \left(a^1_{23}\right)^2 = \lambda_2$
(b) $da^1_{21}(q) \neq 0$ for all $q \in U_p$

(Y4) a^1_{32} is a smooth function on U_p satisfying $-\alpha\, a^1_{23} a^1_{32} = (\alpha+1)\lambda_3 + \lambda_2$.

Conversely, if the a^i_{jk} are given by (Y1) - (Y4) in our coordinate system, then we can find an explicit Riemannian metric g on U_p with the constant Ricci eigenvalues ρ_1, ρ_2, ρ_3 and not being locally homogeneous (see [7] for the motivation and the next theorem for more details).

Each generalized Yamato space satisfies, for one of the basic vector fields E_γ ($\gamma \in \{1,2,3\}$), the following two *geometrical* conditions (in which without loss of generality $\gamma = 3$ can be assumed):

(G1) The connection coefficients a^i_{jk} are constant along the trajectories of E_γ, i.e. $E_\gamma(a^i_{jk}) = 0$.

(G2) The local group of local diffeomorphismus defined by E_γ is volume-preserving, i.e. $\mathrm{div} E_\gamma = 0$.

In the following, we shall also consider the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ corresponding to $\{E_1, E_2, E_3\}$ and call it an *adapted* orthonormal coframe.

Now we are able to formulate our main result.

Theorem 1.1. *Let (M,g) be a C^∞-Riemannian manifold of dimension three with distinct constant Ricci eigenvalues. If (M,g) satisfies the conditions* (G1) *and* (G2) *(with $\gamma = 3$) in a neighbourhood of each point $p \in M$, then there is a dense open subset $U \subset M$ such that, for some neighbourhood V_q of any point $q \in U$, one of the following three cases* (i) - (iii) *occur:*

(i) *(M,g) restricted to V_q is locally isometric to a 3-dimensional Lie group with a left-invariant metric.*

(ii) *(M,g) restricted to V_q is locally isometric to a generalized Yamato space. This means that the adapted orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ is given with respect to an adapted system (w,x,y) of local coordinates by the formulas*

$$\left.\begin{aligned} \omega^1 &= A\,dw + B\,dx \\ \omega^2 &= C\,dw + D\,dx \\ \omega^3 &= G\,dw + H\,dx + dy \end{aligned}\right\} \tag{1.3}$$

and

$$\begin{aligned} A &= C(\varphi_3 - \varphi_1)y + A_0 & B &= D(\varphi_3 - \varphi_1)y + B_0 \\ C &= \frac{(\varphi_1)'_w}{(\alpha\varphi_1 + (\alpha+2)\varphi_3)\varphi_2} & D &= \frac{(\varphi_1)'_x}{(\alpha\varphi_1 + (\alpha+2)\varphi_3)\varphi_2} \\ G &= (\alpha+1)C\varphi_2 y + G_0 & H &= (\alpha+1)D\varphi_2 y + H_0 \end{aligned}$$

where φ_1 is an arbitrary non-constant smooth function on $\mathbb{R}^2[w,x]$, $\varphi_1(w,x) \neq 0$, and the functions φ_2, φ_3 are defined by

$$(\alpha+1)(\varphi_2)^2 + (\varphi_1)^2 = \lambda_2, \quad -\alpha\varphi_1\varphi_3 = (\alpha+1)\lambda_3 + \lambda_2, \quad d\varphi_2(w,x) \neq 0.$$

Further, A_0, B_0, G_0 and H_0 are any smooth functions of w and x satisfying the partial differential equations

$$\begin{aligned} (A_0)'_x - (B_0)'_w &= (DA_0 - CB_0)\varphi_2 + (DG_0 - CH_0)(\varphi_1 - \varphi_3) \\ (G_0)'_x - (H_0)'_w &= (DA_0 - CB_0)(\varphi_1 + \varphi_3) - (\alpha+1)(DG_0 - CH_0)\varphi_2. \end{aligned}$$

(iii) *(M,g) restricted to V_q has, in an adapted system of local coordinates, the following form: an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ is again defined by* (1.3) *and*

$$A = A(w,x), \quad B = B(w,x), \quad C = \varphi A y + C_0, \quad D = \varphi B y + D_0$$

$$G = \frac{1}{2}(\alpha+1)\sqrt{-\lambda_3}\varphi A y^2 + (\alpha+1)\sqrt{-\lambda_3} C_0 y + G_0$$

$$H = \frac{1}{2}(\alpha+1)\sqrt{-\lambda_3}\varphi B y^2 + (\alpha+1)\sqrt{-\lambda_3} D_0 y + H_0$$

where $A, B, C_0, D_0, G_0, H_0, \varphi$ are arbitrary smooth functions of w and x satisfying the system of quasilinear partial differential equations

$$\begin{aligned} A'_x - B'_w &= \sqrt{-\lambda_3}(AD_0 - BC_0) \\ (C_0)'_x - (D_0)'_w &= \varphi(AH_0 - BG_0) \\ (G_0)'_x - (H_0)'_w &= -\varphi(AD_0 - BC_0) + (\alpha+1)\sqrt{-\lambda_3}(C_0H_0 - D_0G_0) \\ A\varphi'_x - B\varphi'_w &= (AD_0 - BC_0)\alpha\sqrt{-\lambda_3}\varphi. \end{aligned}$$

Here $\varphi = \varphi(w,x)$ is a non-constant function, and the equality $\lambda_1\lambda_3 = (\lambda_2)^2$ and the inequalities $\lambda_1 < 0$ and $\lambda_3 < 0$ must be satisfied.

The spaces in the items (i) – (iii) are never locally isometric to each other; in particular, the cases (ii) and (iii) are never locally homogeneous.

Remark 1.2. 1. In the case (iii) of Theorem 1.1 the Ricci eigenvalues satisfy the equality $\rho_1^2 + \rho_3^2 = \rho_2(\rho_1 + \rho_3)$ and the inequalities $\rho_2 + \rho_3 < \rho_1$ and $\rho_1 + \rho_2 < \rho_3$.

2. If the condition (G2) is not satisfied, one can find a system of five independent algebraic equations for the six independent connection coefficients a^i_{jk}. Here a^1_{21}, a^1_{22} and a^1_{31} can be easily eliminated but the remaining two algebraic equations for a^1_{23}, a^1_{32} and a^2_{31} (including the parameters α and $\lambda_1, \lambda_2, \lambda_3$) are very complicated and they were printed on six pages by a computer. (We are obliged to B. Fiedler for the computer test.)

3. As it is well-known (see, e.g., [13]), each locally homogeneous Riemannian 3-manifold is either locally symmetric or locally isometric to a Lie group with a left-invariant metric. In the first case, the Ricci eigenvalues cannot be distinct.

4. Obviously, a Riemannian manifold (M,g) with three distinct constant Ricci eigenvalues is locally homogeneous if and only if all connection coefficients a^i_{jk} from condition (G1) are constant.

Remark 1.3. A converse to Theorem 1.1/Part(i) also holds in the sense that a 3-dimensional Lie group G with a left-invariant metric g (and with three distinct Ricci eigenvalues) satisfies the conditions (G1) and (G2) for some $\gamma \in \{1,2,3\}$. First, condition (G1) is trivially satisfied because all connection coefficients a^i_{jk} are constant. As concerns condition (G2), let us mention as obvious fact that a left-invariant vector field X on (G,g) satisfies $\operatorname{div} X = 0$ if and only if $\mathcal{L}_X(\omega^1 \wedge \omega^2 \wedge \omega^3) = 0$, i.e., if and only if $\operatorname{trace} ad(X) = 0$ holds on the Lie algebra $\mathcal{G}$. If G is unimodular, then this condition is satisfied for all $X \in \mathcal{G}$. If G is not unimodular, then the Lie algebra $\mathcal{G}$ contains the 2-dimensional *unimodular kernel* $\mathcal{U} = \{X \in \mathcal{G} \mid \operatorname{trace} ad(X) = 0\}$. This kernel is spanned by two vector fields consisting of Ricci eigenvectors, say E_2 and E_3 (see [11] for more details). Hence $\operatorname{div}E_2 = \operatorname{div}E_3 = 0$ on (G,g).

The rest of this paper contains the proof of the previous theorem.

2. The basic differential equations under the assumptions (G1) and (G2)

First it is obvious (cf. [7]) that an adapted orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ can be always expressed in the form (1.3) with respect to the adapted local coordinates. Because $\omega^1 \wedge \omega^2 \neq 0$ we see that $AD - BC \neq 0$.

From (1.1) we obtain at once, using the standard formulas (see [3])

$$\left.\begin{aligned} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= \lambda_3 \omega^1 \wedge \omega^2 \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 &= \lambda_2 \omega^1 \wedge \omega^3 \\ d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 &= \lambda_1 \omega^2 \wedge \omega^3 \end{aligned}\right\} \qquad \text{where } \omega^i_j = \sum_k a^i_{jk}\omega^k.$$

This leads to a *first order* partial differential equation system for the functions a^i_{jk} and the coefficients A, B, C, D, G and H from (1.3) (see [7: Chapter 2]).

Now we shall use the conditions (G1) and (G2). From condition (G1) we get at once

$$a^i_{jk} = a^i_{jk}(w, x) \qquad (1 \le i, j, k \le 3).$$

Using (1.2) and the definition of the functions a^i_{jk} we get $\operatorname{div} E_3 = \sum_j a^j_{3j} = (1+\alpha)a^1_{31}$, and hence condition (G2) means

$$a^1_{31} = 0 \quad \text{on } U_p. \tag{2.1}$$

Therefore our partial differential equation system has a more special form than in [7]. We obtain the nine equations

$$\left.\begin{aligned} A'_y &= C(a^1_{32} - a^1_{23}) \\ B'_y &= D(a^1_{32} - a^1_{23}) \\ C'_y &= A(a^1_{23} + a^2_{31}) \\ D'_y &= B(a^1_{23} + a^2_{31}) \\ G'_y &= (\alpha+1)Ca^1_{21} - \frac{\alpha+1}{\alpha}Aa^1_{22} \\ H'_y &= (\alpha+1)Da^1_{21} - \frac{\alpha+1}{\alpha}Ba^1_{22} \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} G'_x - H'_w &= \mathcal{D}(a^1_{32} - a^2_{31}) - \frac{\alpha+1}{\alpha}\mathcal{E}a^1_{22} + (\alpha+1)\mathcal{F}a^1_{21} \\ C'_x - D'_w &= \mathcal{D}a^1_{22} + \mathcal{E}(a^1_{23} + a^2_{31}) \\ A'_x - B'_w &= \mathcal{D}a^1_{21} + \mathcal{F}(a^1_{32} - a^1_{23}) \end{aligned}\right\} \tag{2.3}$$

and additional nine equations:

$$\left.\begin{aligned} A(a^1_{21})'_x - B(a^1_{21})'_w + C(a^1_{22})'_x - D(a^1_{22})'_w + G(a^1_{23})'_x & \\ -H(a^1_{23})'_w - \mathcal{D}(U_3 - \lambda_3) - \mathcal{E}V_3 - \mathcal{F}W_3 &= 0 \\ (a^1_{23})'_w + AV_3 + CW_3 &= 0 \\ (a^1_{23})'_x + BV_3 + DW_3 &= 0 \end{aligned}\right\} \tag{2.4}$$

$$\left.\begin{aligned} C(a^1_{32})'_x - D(a^1_{32})'_w + G(a^1_{33})'_x - H(a^1_{33})'_w - \mathcal{D}U_2 & \\ -\mathcal{E}(V_2 - \lambda_2) - \mathcal{F}W_2 &= 0 \\ (a^1_{33})'_w + A(V_2 - \lambda_2) + CW_2 &= 0 \\ (a^1_{33})'_x + B(V_2 - \lambda_2) + DW_2 &= 0 \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} A(a^2_{31})'_x - B(a^2_{31})'_w + G(a^2_{33})'_x - H(a^2_{33})'_w & \\ -\mathcal{D}U_1 - \mathcal{E}V_1 - \mathcal{F}(W_1 - \lambda_1) &= 0 \\ (a^2_{33})'_w + AV_1 + C(W_1 - \lambda_1) &= 0 \\ (a^2_{33})'_x + BV_1 + D(W_1 - \lambda_1) &= 0. \end{aligned}\right\} \tag{2.6}$$

Here we use the abbreviations

$$\mathcal{D} = AD - BC, \qquad \mathcal{E} = AH - BG, \qquad \mathcal{F} = CH - DG$$

and

$$\left.\begin{aligned} U_1 &= \alpha a_{21}^1 a_{31}^2 - (\alpha+2) a_{21}^1 a_{32}^1 \\ V_1 &= \frac{(\alpha+1)(\alpha+2)}{\alpha} a_{21}^1 a_{22}^1 \\ W_1 &= \frac{\alpha+1}{\alpha}(a_{22}^1)^2 - (\alpha+1)^2 (a_{21}^1)^2 + a_{23}^1 a_{31}^2 - a_{32}^1 a_{31}^2 + a_{32}^1 a_{23}^1 \end{aligned}\right\} \qquad (2.7)$$

$$\left.\begin{aligned} U_2 &= \frac{1}{\alpha} a_{22}^1 a_{32}^1 - \frac{2\alpha+1}{\alpha} a_{22}^1 a_{31}^2 \\ V_2 &= (\alpha+1)(a_{21}^1)^2 - \left(\frac{\alpha+1}{\alpha}\right)^2 (a_{22}^1)^2 - a_{32}^1 a_{23}^1 - a_{32}^1 a_{31}^2 - a_{23}^1 a_{31}^2 \\ W_2 &= \frac{(2\alpha+1)(\alpha+1)}{\alpha} a_{22}^1 a_{21}^1 \end{aligned}\right\} \qquad (2.8)$$

$$\left.\begin{aligned} U_3 &= -(a_{21}^1)^2 - (a_{22}^1)^2 + a_{23}^1 a_{31}^2 - a_{23}^1 a_{32}^1 + a_{32}^1 a_{31}^2 \\ V_3 &= \frac{1}{\alpha} a_{22}^1 a_{23}^1 - \frac{2\alpha+1}{\alpha} a_{22}^1 a_{31}^2 \\ W_3 &= -\alpha a_{21}^1 a_{23}^1 - (\alpha+2) a_{21}^1 a_{32}^1 . \end{aligned}\right\} \qquad (2.9)$$

The following formula is a direct consequence of our notation for λ_i, ρ_i and α:

$$\lambda_1 + \alpha\lambda_2 = (\alpha+1)\lambda_3.$$

Due to (1.2) and (2.1) we have only five basic coefficient functions, namely

$$a_{21}^1, \qquad a_{22}^1, \qquad a_{23}^1, \qquad a_{32}^1, \qquad a_{31}^2.$$

Substituting the second and the third equations of (2.4) - (2.6) into the first equation (2.4) and using (2.1) we obtain

$$(W_1 - \lambda_1) + \alpha(V_2 - \lambda_2) + (\alpha+1)(U_3 - \lambda_3) = 0. \qquad (2.10)$$

By means of the notations (2.7) - (2.9) and formula (2.10), this can be written in the form

$$a_{23}^1 a_{31}^2 - \alpha\, a_{23}^1 a_{32}^1 - (\alpha+1)(a_{21}^1)^2 - (\alpha+1)(a_{22}^1)^2 = (\alpha+1)\lambda_3. \qquad (2.11)$$

The differentiation of the second and the third equations of (2.4) - (2.6) with respect to y gives

$$\begin{matrix} V_3 A'_y + W_3 C'_y = 0 & (V_2 - \lambda_2)A'_y + W_2 C'_y = 0 & V_1 A'_y + (W_1 - \lambda_1)C'_y = 0 \\ V_3 B'_y + W_3 D'_y = 0 & (V_2 - \lambda_2)B'_y + W_2 D'_y = 0 & V_1 B'_y + (W_1 - \lambda_1)D'_y = 0. \end{matrix} \qquad (2.12)$$

This is a system of linear algebraic equations for A'_y, B'_y, C'_y and D'_y with the matrix

$$\mathcal{M} = \begin{pmatrix} V_3 & V_2 - \lambda_2 & V_1 \\ W_3 & W_2 & W_1 - \lambda_1 \end{pmatrix}.$$

Once given a numeration of the Ricci eigenvalues, we see easily from (2.7) - (2.9) that rank $\mathcal{M}$ is a *geometrical invariant.* (In fact, the only freedom in the choice of an adapted frame is taking the opposite of any of the vector fields E_1, E_2, E_3.)

3. Proof of the main result

In the following we always suppose that U_p is a connected and sufficiently small neighbourhood of $p \in M$ such that we can construct our adapted coordinate system (w, x, y) with $E_3 = \frac{\partial}{\partial y}$ on U_p. Furthermore, we assume that the restriction of g on U_p satisfies the conditions (G1) and (G2) for $\gamma = 3$.

Suppose first that (M, g) is locally homogeneous on U_p. According to the Remark 1.2/Item 3, (U_p, g) is locally isometric to a Lie group with a left-invariant metric; this is the case (i) of Theorem 1.1.

Thus we have to deal only with the cases when the metric is not locally homogeneous. The following Lemma will be very useful in the sequel.

Lemma 3.1. *Suppose that $a^1_{23} + a^2_{31} = 0$ and $a^1_{23} - a^1_{32} = 0$ on U_p. Then all connection coefficients a^i_{jk} are constant on U_p and the matrix $\mathcal{M}$ must have rank zero on U_p.*

Proof. From (2.11) we obtain on U_p

$$(a^1_{22})^2 + (a^1_{21})^2 + (a^1_{23})^2 = -\lambda_3. \tag{3.1}$$

Differentiate (3.1) with respect to x and w. Using (1.2) and (2.12) we get (because of $\mathcal{D} = AD - BC \neq 0$ on U_p)

$$\frac{\alpha}{\alpha+1}a^1_{22}(V_2 - \lambda_2) - \frac{1}{\alpha+1}a^1_{21}V_1 - a^1_{23}V_3 = 0$$
$$\frac{\alpha}{\alpha+1}a^1_{22}W_2 - \frac{1}{\alpha+1}a^1_{21}(W_1 - \lambda_1) - a^1_{23}W_3 = 0.$$

Substituting from (2.7) - (2.9) we obtain on U_p

$$\begin{aligned}\frac{\alpha}{\alpha+1}a^1_{22}\lambda_2 = a^1_{22}\Big\{&-\frac{\alpha+1}{\alpha}(a^1_{22})^2 + \frac{\alpha^2-\alpha-2}{\alpha}(a^1_{21})^2 \\ &+ \left(\frac{\alpha}{\alpha+1} - \frac{2(\alpha+1)}{\alpha}\right)(a^1_{23})^2\Big\}\end{aligned} \tag{3.2}$$

$$\begin{aligned}-\frac{1}{\alpha+1}a^1_{21}\lambda_1 = a^1_{21}\Big\{&\left(2\alpha+1-\frac{1}{\alpha}\right)(a^1_{22})^2 + (\alpha+1)(a^1_{21})^2 \\ &+ \left(2(\alpha+1) - \frac{1}{\alpha+1}\right)(a^1_{23})^2\Big\}.\end{aligned} \tag{3.3}$$

For the further considerations we define the following four sets:

$$\mathcal{M}_1 = \left\{q \in U_p \,\middle|\, a^1_{22}(q)a^1_{21}(q) \neq 0\right\}$$
$$\mathcal{M}_2 = \left\{q \in U_p \,\middle|\, a^1_{22}(q) = 0,\ a^1_{21}(q) \neq 0\right\}$$
$$\mathcal{M}_3 = \left\{q \in U_p \,\middle|\, a^1_{22}(q) \neq 0,\ a^1_{21}(q) = 0\right\}$$
$$\mathcal{M}_4 = \left\{q \in U_p \,\middle|\, a^1_{22}(q) = a^1_{21}(q) = 0\right\}.$$

We see that $U_p = \bigcup_{j=1}^4 \mathcal{M}_j$ and $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for $i \neq j$.

In the first step we prove that all connection coefficients a^i_{jk} are constant on $\mathcal{M}_j$ ($j = 1, \dots, 4$).

Let $q \in \mathcal{M}_1$. We can delete the factors $a^1_{22}(q), a^1_{21}(q)$ in (3.2), (3.3) and (3.1) - (3.3) give a system of linear equations for $(a^1_{22})^2(q), (a^1_{21})^2(q), (a^1_{23})^2(q)$ $(q \in \mathcal{M}_1)$. The determinant of this system is equal to $-\frac{(\alpha-1)^2}{\alpha}$. If $\alpha = 1$, then from (3.2) and (3.3) there follows $\lambda_1 = \lambda_2$ and this is a contradiction. From $\alpha \neq 1$ we find that the solution of our system is independent of $q \in \mathcal{M}_1$. Therefore, $(a^1_{22})^2, (a^1_{21})^2$ and $(a^1_{23})^2$ are constant on $\mathcal{M}_1$.

Now, let us suppose $q \in \mathcal{M}_2$. Then, in a similar way, (3.1) and (3.3) give a linear equation system for $(a^1_{21})^2(q), (a^1_{23})^2(q)$ $(q \in \mathcal{M}_2)$ with determinant $\frac{\alpha(\alpha+2)}{\alpha+1}$ which is non-zero (indeed, substituting $\alpha = -2$ in this system we get $\lambda_1 = \lambda_3$, i.e. $\rho_1 = \rho_3$ and this is a contradiction). We obtain again that $(a^1_{22})^2, (a^1_{21})^2$ and $(a^1_{23})^2$ are constant on $\mathcal{M}_2$.

If $q \in \mathcal{M}_3$, then (3.1) and (3.2) give a linear equation system for $(a^1_{22})^2(q), (a^1_{23})^2(q)$ with the determinant $-\frac{(2\alpha+1)}{\alpha(\alpha+1)}$ which is again non-zero. (Substituting $\alpha = -\frac{1}{2}$ in this system we get $\lambda_2 = \lambda_3$, i.e. $\rho_2 = \rho_3$ and this is a contradiction.) This shows that $(a^1_{22})^2, (a^1_{21})^2$ and $(a^1_{23})^2$ are constant on $\mathcal{M}_3$.

Similarly we find that $(a^1_{22})^2, (a^1_{21})^2$ and $(a^1_{23})^2$ are constant on $\mathcal{M}_4$.

Because $U_p = \bigcup_{j=1}^4 \mathcal{M}_j$ and a^i_{jk} are smooth functions on U_p we get that $(a^1_{22})^2$, $(a^1_{21})^2$ and $(a^1_{23})^2$ are global constants on U_p. This and (1.2) show that a^1_{33}, a^2_{33} and a^1_{23} are constant on U_p. Using the second and the third equation of (2.4)-(2.6) and the inequality $AD - BC \neq 0$ we see that all entries of the matrix $\mathcal{M}$ are zero. ∎

Corollary 3.2. *The matrix $\mathcal{M}$ has nowhere rank two.*

Proof. Let p be a point of M with rank $\mathcal{M}(p) = 2$. Hence there is some neighbourhood U_p of p with rank $\mathcal{M} = 2$ on U_p. Now the system (2.12) implies

$$A'_y = B'_y = C'_y = D'_y = 0 \quad \text{on } U_p.$$

Then, because $AD - BC \neq 0$ on U_p, the system (2.2) gives

$$a^1_{32} - a^1_{23} = a^1_{23} + a^2_{31} = 0 \quad \text{on } U_p.$$

Now, Lemma 3.1 shows that rank $\mathcal{M} = 0$ on U_p and this is a contradiction. ∎

Let $\mathcal{N}_1$ be the (open) set of all points of M such that rank $\mathcal{M} = 1$.

Proposition 3.3. *There is an open dense subset $\widetilde{\mathcal{N}}_1 \subset \mathcal{N}_1$ such that, for each point $p \in \widetilde{\mathcal{N}}_1$ there exists a neighbourhood U_p of p such that the restriction of g to U_p is isometric to a generalized Yamato metric (i.e., the case* (ii) *of Theorem 1.1 occurs).*

Proof. Choose $p \in \mathcal{N}_1$. Substituting from (2.2) into (2.12), we get easily in a neighbourhood U_p of p

$$\left.\begin{aligned} (a^1_{23} + a^2_{31})W_3 &= 0 & (a^1_{32} - a^1_{23})V_3 &= 0 \\ (a^1_{23} + a^2_{31})W_2 &= 0 & (a^1_{32} - a^1_{23})(V_2 - \lambda_2) &= 0 \\ (a^1_{23} + a^2_{31})(W_1 - \lambda_1) &= 0 & (a^1_{32} - a^1_{23})V_1 &= 0 \end{aligned}\right\} \qquad (3.4)$$

and we can also suppose that rank $\mathcal{M} = 1$ on U_p.

Because, rank $\mathcal{M} > 0$, then (2.12) shows that $A'_y D'_y - B'_y C'_y = 0$, and using (2.2) we get

$$(a^1_{23} - a^1_{32})(a^1_{23} + a^2_{31}) = 0$$

on U_p. Let us suppose that there exist sequences $\{q_m\}_{m\in N}$ and $\{\bar{q}_m\}_{m\in N}$ with $q_m, \bar{q}_m \in U_p$ $(m \in N)$ such that $q_m, \bar{q}_m \to p$ and

$$(a^1_{23} + a^2_{31})(q_m) \neq 0 \quad \text{and} \quad (a^1_{32} - a^1_{23})(\bar{q}_m) \neq 0 \qquad \text{for all } m \in N.$$

Then we get from (3.4)

$$V_1(p) = (V_2 - \lambda_2)(p) = V_3(p) = W_3(p) = W_2(p) = (W_1 - \lambda_1)(p) = 0$$

because the functions are continuous. But this is a contradiction to rank $\mathcal{M}(p) = 1$. Therefore we have only the following two possibilities:

a) There is a neighbourhood $\tilde{U}_p$ of p, $\tilde{U}_p \subset U_p$ such that $a^1_{23} - a^1_{32} = 0$ on $\tilde{U}_p$.

b) There is a neighbourhood $\tilde{U}_p$ of p, $\tilde{U}_p \subset U_p$ such that $a^1_{23} + a^2_{31} = 0$ on $\tilde{U}_p$.

If we interchange the notation in the Case a) (i.e., the basic frame $\{E_1, E_2, E_3\}$ is replaced by $\{\bar{E}_1, \bar{E}_2, \bar{E}_3\}$ with $\bar{E}_1 = E_2, \bar{E}_2 = E_1, \bar{E}_3 = E_3$), the new connection coefficients $\bar{a}^i_{jk}$ satisfy the equality $\bar{a}^2_{32} = 0$ (and hence $\bar{a}^1_{31} = 0$ due to (1.2)) and the conditions $\bar{a}^1_{23} + \bar{a}^2_{31} = 0$ and $\bar{a}^1_{23} - \bar{a}^1_{32} \neq 0$ on $\tilde{U}_p$. Therefore, we can limit ourselves to the Case b).

Thus, let us suppose $a^1_{23} + a^2_{31} = 0$ on U_p. If $(a^1_{32} - a^1_{23})(q) \neq 0$ for some $q \in U_p$, then $V_1(q) = (V_2 - \lambda_2)(q) = V_3(q) = 0$. On the other hand, if $(a^1_{32} - a^1_{23})(q) = 0$, then there is a sequence $q_m \to q$, $q_m \in U_p$ with $(a^1_{32} - a^1_{23})(q_m) \neq 0$ for all $m \in N$ (otherwise, due to Lemma 3.1, we get rank $\mathcal{M} = 0$ in a neighbourhood of q, which is a contradiction). This gives $V_3(q_m) = (V_2 - \lambda_2)(q_m) = V_1(q_m)$ and from the continuity we get $V_3(q) = (V_2 - \lambda_2)(q) = V_1(q) = 0$, too.

This shows $V_1 = V_2 - \lambda_2 = V_3 = 0$ on U_p. From here we get

$$(\alpha + 2)a^1_{21}a^1_{22} = 0 \tag{3.5}$$

$$a^1_{23}a^1_{22} = 0 \tag{3.6}$$

$$(\alpha + 1)(a^1_{21})^2 - \left(\frac{\alpha + 1}{\alpha}\right)^2 (a^1_{22})^2 + (a^1_{23})^2 = \lambda_2 \tag{3.7}$$

on U_p.

Next, let us suppose that there exists a point $q \in U_p$ with $a^1_{23}(q) = 0$. We have two different possibilities:

Case 1: $a^1_{23} = 0$ on a connected neighbourhood $\bar{U}$ of q, $\bar{U} \subset U_p$. Then we get $a^2_{31} = 0$ on $\bar{U}$. From (3.7) it follows on $\bar{U}$

$$(\alpha + 1)(a^1_{21})^2 - \left(\frac{\alpha + 1}{\alpha}\right)^2 (a^1_{22})^2 = \lambda_2$$

and from (2.11) we obtain

$$(a_{21}^1)^2 + (a_{22}^1)^2 = -\lambda_3$$

on $\bar{U}$. Solving this equation system, we see that a_{21}^1 and a_{22}^1 are constant on $\bar{U}$ and hence all coefficient functions a_{jk}^i are constant. Using (2.4) - (2.6) we find that $W_3 = W_2 = W_1 - \lambda_1 = 0$ on $\bar{U}$, but this is a contradiction to rank $\mathcal{M} = 1$ on U_p.

Case 2. There exists a sequence $\{q_m\}_{m\in N}$, $q_m \in U_p$ $(m \in N)$ and $q_m \to q$, with $a_{23}^1(q_m) \neq 0$ for all $m \in N$. From (3.6) we get $a_{22}^1(q_m) = 0$ and hence $a_{22}^1(q) = 0$. Now we know that $a_{23}^1(q) = -a_{31}^2(q) = a_{22}^1(q) = 0$. Differentiating (3.7) with respect to x and w at the point q we obtain

$$(a_{21}^1)(q)(a_{21}^1)'_w(q) = 0 \qquad \text{and} \qquad (a_{21}^1)(q)(a_{21}^1)'_x(q) = 0.$$

Here $a_{21}^1(q) \neq 0$ and hence $(a_{21}^1)'_x(q) = (a_{21}^1)'_w(q) = 0$ (indeed, if $a_{21}^1(q) = 0$, then (3.1) and (3.7) imply $\lambda_3 = \lambda_2 = 0$ and hence $\rho_2 = \rho_3$, which is a contradiction).

Now, from (1.2) we see that $(a_{33}^2)'_x(q) = (a_{33}^2)'_w(q) = 0$, and from (2.6) it follows $(W_1 - \lambda_1)(q) = 0$. Furthermore, from (2.8) we find $W_2(q) = 0$. Using (2.5), (1.2) and $W_2(q) = 0$ we see that $(a_{22}^1)'_x(q) = (a_{22}^1)'_w(q) = 0$. Now the differentiation of (2.11) at q gives

$$(a_{23}^1)'_x(q)(a_{32}^1)(q) = 0 \qquad \text{and} \qquad (a_{23}^1)'_x(q)(a_{32}^1)(q) = 0.$$

If $a_{32}^1(q) = 0$, then $W_3(q) = 0$ follows from (2.9). On the other hand, if $(a_{23}^1)'_x(q) = (a_{23}^1)'_w(q) = 0$, we get $W_3(q) = 0$ again using (2.4). Summarizing, we see that rank $\mathcal{M}(q) = 0$, which is a contradiction.

So, the both Cases 1) and 2) lead to a contradiction and we must have $a_{23}^1(q) \neq 0$ for all $q \in U_p$. Then (3.6) gives $a_{22}^1 = 0$ on the whole U_p.

Summarizing all our considerations we have

$$a_{22}^1 = 0, \quad a_{23}^1 + a_{31}^2 = 0, \quad a_{23}^1 \neq 0 \qquad \text{on} \quad U_p. \tag{3.8}$$

On the other hand (2.8) and (3.8) gives $W_2 = 0$ on U_p. The formula (3.7) is reduced to

$$(\alpha + 1)(a_{21}^1)^2 + (a_{23}^1)^2 = \lambda_2. \tag{3.9}$$

Using (3.8), (3.9) and (2.11) we obtain

$$-\alpha\, a_{23}^1 a_{32}^1 = (\alpha + 1)\lambda_3 + \lambda_2. \tag{3.10}$$

Let us suppose that $a_{23}^1 = const \neq 0$ on some open subset $V \subset U_p$. Then (2.4) shows $W_3 = 0$ on V, i.e., more explicitly

$$a_{21}^1(\alpha a_{23}^1 + (\alpha + 2)a_{32}^1) = 0 \tag{3.11}$$

on V. Furthermore, (2.10) implies that $W_1 + (\alpha+1)U_3$ is constant on U_p. In the explicit form,

$$((\alpha + 1)^2 + \alpha + 1)(a_{21}^1)^2 + (\alpha + 2)(a_{23}^1)^2 + 2\alpha a_{23}^1 a_{32}^1 = const. \tag{3.12}$$

Differentiating (3.11) with respect to x and w we get

$$\begin{aligned}(a^1_{21})'_x(\alpha a^1_{23}+(\alpha+2)a^1_{32})+a^1_{21}(\alpha+2)(a^1_{32})'_x&=0\\(a^1_{21})'_w(\alpha a^1_{23}+(\alpha+2)a^1_{32})+a^1_{21}(\alpha+2)(a^1_{32})'_w&=0\end{aligned}\tag{3.13}$$

and differentiating (3.12) we obtain

$$\begin{aligned}2((\alpha+1)^2+\alpha+1)(a^1_{21})'_x a^1_{21}+2\alpha a^1_{23}(a^1_{32})'_x&=0\\2((\alpha+1)^2+\alpha+1)(a^1_{21})'_w a^1_{21}+2\alpha a^1_{23}(a^1_{32})'_w&=0.\end{aligned}\tag{3.14}$$

If $(\alpha a^1_{23}+(\alpha+2)a^1_{32})(q)\neq 0$ for some $q\in V$, then $a^1_{21}(q)=0$ due to (3.11). Then (3.14) and $a^1_{23}\neq 0$ gives $(a^1_{32})'_x(q)=(a^1_{32})'_w(q)=0$. From (3.13) we see that $(a^1_{21})'_x(q)=(a^1_{21})'_w(q)=0$. Using (1.2) and (2.6) we get $(W_1-\lambda_1)(q)=0$. Summarizing we obtain $\operatorname{rank}\mathcal{M}(q)=0$, which is a contradiction.

Therefore $\alpha a^1_{23}+(\alpha+2)a^1_{32}=0$ on V, i.e. $a^1_{32}=const$ on V. It follows from (3.12) that $a^1_{21}=const$ on V. Then all a^i_{jk} are constant functions and (2.4) - (2.6) show that $\operatorname{rank}\mathcal{M}=0$ on V, which is a contradiction.

From these considerations we obtain

$$a^1_{23}\quad\text{is a non-constant function on any open subset } V\subset U_p. \tag{3.15}$$

Hence and from (3.10) we see that $\alpha a^1_{23}+(\alpha+2)a^1_{32}\neq 0$ on a dense open subset of U_p.

If there exists a point $q\in U_p$ with $da^1_{21}(q)=0$, i.e. $(a^1_{21})'_x(q)=(a^1_{21})'_w(q)=0$, then $W_1(q)=\lambda_1$ from (2.6) and (1.2). But (3.9) shows that $a^1_{23}(q)(a^1_{23})'_x(q)=a^1_{23}(q)(a^1_{23})'_w(q)=0$. Because $a^1_{23}\neq 0$, we see that $(a^1_{23})'_x(q)=(a^1_{23})'_w(q)=0$ and this means that $W_3(q)=0$. This proves $\operatorname{rank}\mathcal{M}(q)=0$ and it is a contradiction to $\operatorname{rank}\mathcal{M}=1$ on U_p. Therefore we have

$$da^1_{21}\neq 0\quad\text{on}\quad U_p \text{ and thus } a^1_{21}\neq 0 \text{ on a dense open subset of } U_p. \tag{3.16}$$

Now, (3.8) - (3.10) and (3.15) are equivalent to (Y1) - (Y4).

Finally, put $\varphi_1=a^1_{23}$, $\varphi_2=a^1_{21}$ and $\varphi_3=a^1_{32}$. Then from [7] (see, especially, Proposition 9) it follows that the case (ii) of Theorem 1.1 occurs in any neighbourhood V_q on which $W_3\neq 0$, i.e. $a^1_{21}(\alpha a^1_{23}+(\alpha+2)a^1_{32})\neq 0$ holds. The last inequality has a geometrical meaning (once the numeration of ρ_i is chosen) and due to (3.15) and (3.16) it holds on a dense open subset of *any* neighbourhood $U_p\subset\mathcal{N}_1$. We can define $\widetilde{\mathcal{N}}_1=\{p\in\mathcal{N}_1 \mid W_3(p)\neq 0\}$ and then Proposition 3.3 follows. ■

Proposition 3.4. *Suppose that* $\operatorname{rank}\mathcal{M}=0$ *on a connected open neighbourhood* $U_p\subset M$ *and the restriction of* g *to* U_p *is not locally homogeneous. Then the restriction of* g *to* U_p *is isometric to a metric for which* $a^1_{22}=a^1_{23}=a^1_{32}=0$, $(a^1_{21})^2=-\lambda_3$ *on* U_p *and* a^2_{31} *is an arbitrary non-constant smooth function.*

Proof. The assumptions give

$$V_3=0,\quad V_2=\lambda_2,\quad V_1=0,\quad W_3=0,\quad W_2=0,\quad W_1=\lambda_1 \tag{3.17}$$

on U_p. From (2.10) we get on U_p

$$U_3 = \lambda_3. \tag{3.18}$$

Using the second and the third equation of (2.4) - (2.6) and also (1.2) we obtain on U_p that

$$a^1_{23} = const, \quad a^1_{33} = const, \quad a^1_{22} = const, \quad a^2_{33} = const, \quad a^1_{21} = const.$$

The equations $V_1 = 0$ and $W_2 = 0$ give

$$(\alpha + 2)a^1_{21}a^1_{22} = 0 \qquad \text{and} \qquad (2\alpha + 1)a^1_{21}a^1_{22} = 0$$

on U_p. This shows that

$$a^1_{21}a^1_{22} = 0$$

on U_p. We have the following three possible cases for the constants a^1_{22} and a^1_{21}:

(a) $a^1_{22} = a^1_{21} = 0$ **(b)** $a^1_{22} \neq 0, a^1_{21} = 0$ **(c)** $a^1_{22} = 0, a^1_{21} \neq 0.$

Case (a): Here (3.17) and (3.18) give

$$\begin{aligned} a^1_{32}a^1_{23} - a^1_{32}a^2_{31} + a^1_{23}a^2_{31} &= \lambda_1 \\ -a^1_{32}a^1_{23} - a^1_{32}a^2_{31} - a^1_{23}a^2_{31} &= \lambda_2 \\ -a^1_{32}a^1_{23} + a^1_{32}a^2_{31} + a^1_{23}a^2_{31} &= \lambda_3. \end{aligned}$$

It follows

$$a^1_{32}a^2_{31} = -\frac{1}{2}(\lambda_1 + \lambda_2), \qquad a^1_{23}a^2_{31} = \frac{1}{2}(\lambda_1 + \lambda_3), \qquad a^1_{23}a^1_{32} = -\frac{1}{2}(\lambda_2 + \lambda_3).$$

If $\lambda_1 + \lambda_3 \neq 0$ or $\lambda_2 + \lambda_3 \neq 0$, then $a^1_{23} \neq 0$. Because $a^1_{23} = const$, we get that a^1_{32} and a^2_{31} are constant on U_p. But in this case, all connection coefficients are constant on U_p and the metric would be locally homogeneous. On the other hand, $\lambda_1 + \lambda_3 = \lambda_2 + \lambda_3 = 0$ is not possible. This shows that the case (a) does not occur.

Case (b): If we interchange the notation (i.e., the basic frame $\{E_1, E_2, E_3\}$ is replaced by $\{\bar{E}_1, \bar{E}_2, \bar{E}_3\}$ with $\bar{E}_1 = E_2$, $\bar{E}_2 = E_1$ and $\bar{E}_3 = E_3$), the new connection coefficients $\bar{a}^i_{jk}$ satisfy $\bar{a}^2_{32} = 0$, $\bar{a}^1_{31} = 0$, $\bar{a}^1_{22} = 0$, $\bar{a}^1_{21} \neq 0$, $\bar{a}^1_{23} = const$ on U_p. Therefore, it is sufficient to handle the case (c).

Case (c): From (3.17) and (3.18) we obtain

$$-(\alpha + 1)^2(a^1_{21})^2 + a^2_{31}a^1_{23} - a^1_{32}a^2_{31} + a^1_{32}a^1_{23} = \lambda_1 \tag{3.19}$$

$$(\alpha + 1)(a^1_{21})^2 - a^1_{32}a^1_{23} - a^1_{32}a^2_{31} - a^1_{23}a^2_{31} = \lambda_2 \tag{3.20}$$

$$-(a^1_{21})^2 + a^2_{31}a^1_{23} - a^1_{23}a^1_{32} + a^1_{32}a^2_{31} = \lambda_3 \tag{3.21}$$

$$-\alpha a^1_{23} - (\alpha + 2)a^1_{32} = 0. \tag{3.22}$$

If $\alpha = -2$, we get $a^1_{23} = 0$ and this shows that $\lambda_1 = \lambda_2$, which is impossible. Therefore we can suppose $\alpha \neq -2$. Then (3.22) shows that $a^1_{32} = const$ on U_p. The addition of (3.19) and (3.21) gives

$$-\left(1 + (\alpha + 1)^2\right)(a^1_{21})^2 + 2a^1_{23}a^2_{31} = \lambda_1 + \lambda_3,$$

i.e. if $a^1_{23} \neq 0$, then $a^2_{31} = const$ on U_p and the restriction of g onto U_p is locally homogeneous. Therefore $a^1_{23} = 0$ on U_p. Now (3.22) gives $a^1_{32} = 0$ on U_p since $\alpha \neq -2$. Then, (3.21) shows $(a^1_{21})^2 = -\lambda_3$. ∎

For the eigenvalues in the last case we get $\lambda_2 = -(\alpha+1)\lambda_3$ and $\lambda_1 = (\alpha+1)^2\lambda_3$, and this is equivalent to the single equation $\lambda_1\lambda_3 = \lambda_2^2$.

Proposition 3.5. *Under the assumptions of Proposition 3.4, the metric g restricted on U_p is given by the case (iii) of our Theorem.*

Proof. From $\lambda_1\lambda_3 = \lambda_2^2$ and $(a_{21}^1)^2 = -\lambda_3$ we get $a_{21}^1 \neq 0$ on U_p. Furthermore, we have $U_1 = \alpha a_{21}^1 a_{31}^2$. The equations (2.4) - (2.6) are automatically satisfied except the first equation of (2.6) which gives

$$A(a_{31}^2)'_x - B(a_{31}^2)'_w = \mathcal{D}\alpha a_{21}^1 a_{31}^2.$$

Now, the integration of the equations (2.2) gives

$$\left.\begin{aligned} &A'_y = B'_y = 0 \quad \text{on } U_p, \text{ i.e. } A = A(w,x),\ B = B(w,x) \\ &C = Aa_{31}^2 y + C_0, \quad D = Ba_{31}^2 y + D_0 \\ &G = \frac{1}{2}(\alpha+1)a_{21}^1 a_{31}^2 Ay^2 + (\alpha+1)a_{21}^1 C_0 y + G_0 \\ &H = \frac{1}{2}(\alpha+1)a_{21}^1 a_{31}^2 By^2 + (\alpha+1)a_{21}^1 D_0 y + H_0 \end{aligned}\right\}, \tag{3.23}$$

where C_0, D_0, G_0 and H_0 are functions of w and x only. From (3.23) we conclude

$$\begin{aligned} &\mathcal{D} = AD_0 - BC_0 \\ &\mathcal{E} = (AD_0 - BC_0)(\alpha+1)a_{21}^1 y + AH_0 - BG_0 \\ &\mathcal{F} = \frac{1}{2}(\alpha+1)a_{21}^1 a_{31}^2 (AD_0 - BC_0)y^2 + (AH_0 - BG_0)a_{31}^2 y + C_0H_0 - D_0G_0. \end{aligned}$$

Now, the equations (2.3) and the first equation of (2.6) can be written as a system of partial differential equations:

$$\left.\begin{aligned} A'_x - B'_w &= a_{21}^1(AD_0 - BC_0) \\ (C_0)'_x - (D_0)'_w &= a_{31}^2(AH_0 - BG_0) \\ (G_0)'_x - (H_0)'_w &= -a_{31}^2(AD_0 - BC_0) + (\alpha+1)a_{21}^1(C_0H_0 - D_0G_0) \\ A(a_{31}^2)'_x - B(a_{31}^2)'_w &= (AD_0 - BC_0)\alpha a_{21}^1 a_{31}^2. \end{aligned}\right\} \tag{3.24}$$

Substituting $a_{21}^1 = \sqrt{-\lambda_3}$ and $a_{31}^2 = \varphi$ we get the case (iii) of Theorem 1.1. ∎

Remark 3.6. (a) In the system (3.24) the functions B, D_0 and H_0 can be chosen as arbitrary functions; the remaining equations can be solved by using the Cauchy-Kowalewski theorem. Each particular solution depends on four arbitrary functions of one variable.

(b) An explicit solution of system (3.24) is given by

$$\begin{aligned} &B = H_0 = 0, \quad D_0 = 1, \quad C_0 = 0 \\ &A = \exp\left(\sqrt{-\lambda_3}x\right), \quad G_0 = -\frac{1}{2(\alpha+1)\sqrt{-\lambda_3}}\exp\left((\alpha+1)\sqrt{-\lambda_3}x\right) \\ &a_{31}^2 = \exp\left(\alpha\sqrt{-\lambda_3}x\right). \end{aligned}$$

(c) The condition $\lambda_1\lambda_3 = \lambda_2^2$ gives for the Ricci eigenvalues the relations

$$\rho_1^2 + \rho_3^2 = \rho_2(\rho_1 + \rho_3), \qquad \rho_2 + \rho_3 < \rho_1, \qquad \rho_1 + \rho_2 < \rho_3.$$

Proposition 3.7. *The rank zero examples given by the case (iii) of Theorem 1.1 are never locally isometric to the generalized Yamato examples given by the case (ii) of Theorem 1.1.*

Proof. Assume a local isometry between a generalized Yamato example and an example given by (3.23). In the second case we shall denote the corresponding geometrical quantities by bars. The Ricci eigenvalues are the same. Now, via the local isometry we must have $\bar\omega^i = \epsilon_i\omega^i$ $(i = 1,2,3;\ \epsilon_i = \pm 1)$. Especially, we get

$$\bar x = \bar x(w,x), \qquad \bar w = \bar w(w,x), \qquad \bar y = \epsilon_3 y + \Phi(w,x) \tag{3.25}$$

as a local expression for our isometry. The equation $\bar\omega^3 = \epsilon_3\omega^3$ implies $\bar G\, d\bar w + \bar H\, d\bar x = \epsilon_3(G\,dw + H\,dx) - d\,\Phi$. Now, substitute for $G, H, \bar G$ and $\bar H$ the corresponding expressions from (ii) and (iii) of Theorem 1.1 and also substitute for $\bar y$ from (3.25). According to (3.25), the differentials $d\bar x$ and $d\bar w$ are linear combinations of dx and dw with coefficients not depending on y. Then we can compare the 1-forms which are coefficients of y^2. We obtain $\bar A\, d\bar w + \bar B\, d\bar x = 0$, i.e. $\bar\omega^1 = 0$, which is a contradiction. ∎

Finally, we shall complete the proof of the main Theorem 1.1. We put

$$\begin{aligned}
\mathcal{N}_1 &= \Big\{p \in M \,\Big|\, \operatorname{rank}\mathcal{M}(p) = 1\Big\}\\
\widetilde{\mathcal{N}}_1 &= \Big\{p \in \mathcal{N}_1 \,\Big|\, W_3(p) \neq 0\Big\}\\
\mathcal{N}_2 &= \Big\{p \in M \,\Big|\, \operatorname{rank}\mathcal{M} = 0 \text{ on some open connected neighbourhood } U_p\Big\}.
\end{aligned}$$

Then $\mathcal{N} = \widetilde{\mathcal{N}}_1 \cup \mathcal{N}_2$ is an open dense subset of M and Theorem 1.1 follows from the previous propositions and arguments.

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