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Some Remarks on the Hildebrandt-Graves Theorem

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Abstract. This paper is concerned with a generalization of the Hildebrandt-Graves implicit function theorem building on a weaker acting condition for the derivative of the nonlinear operator involved. The abstract theorem is illustrated by means of an application to nonlinear singular integral equations where classical implicit function theorems do not apply.

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The most prominent example of an implicit function theorem is the *Hildebrandt-Graves theorem* (see [6: p. 133]). In a standard formulation this theorem reads as follows (see [7: p. 671]).

Theorem 1 (Hildebrandt-Graves). *Let (Λ,ρ) be a metric space, X and Y two Banach spaces, and $\Phi = \Phi(\lambda, x)$ some operator which is defined in the neighbourhood of some point $(\lambda_0, x_0) \in \Lambda \times X$ and takes values in Y. Suppose that the following conditions are satisfied:*

(a) *$\Phi(\lambda_0, x_0) = 0$ and $\Phi(\cdot, x_0)$ is continuous at λ_0.*

(b) *The operator Φ admits a partial (Fréchet or Gâteaux) derivative Φ'_x which is continuous at (λ_0, x_0).*

(c) *The bounded linear operator $\Phi'_x(\lambda_0, x_0) \in \mathcal{L}(X, Y)$ is invertible.*

Then for any sufficiently small $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) \le \delta$, the equation $\Phi(\lambda, x) = 0$ has a unique solution $x = x_*(\lambda)$ in the ball $||x - x_0|| \le \varepsilon$.

It is well known that the hypotheses of Theorem 1 may be very hard to verify when applying this theorem to nonlinear operator equations with parameters in infinite dimensional Banach spaces. This is mainly due to the *differentiability requirement* in

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Condition (b): For instance, already rather simple operators like nonlinear integral operators of Hammerstein or Uryson type are usually differentiable only at single points, but not on open subsets, even if the generating kernel functions are very smooth (see, e.g., [1, 9] for a discussion of this phenomenon). Several authors have therefore considered certain *modifications* of the Hildebrandt- Graves theorem building on weaker notions of differentiability (see, e.g., [4, 8, 10 - 12]). In this paper we propose such a new modification whose formulation is almost the same as that of Theorem 1 above, but which applies quite naturally to Hammerstein integral equations and, in particular, to *singular* nonlinear integral equations.

We mention that a similar theorem may be found in [4, 8, 10], but our Theorem 2 below applies more naturally to nonlinear integral equations (see Theorem 3).

Let X^0, X and Y, Y_0 be four Banach spaces such that X^0 is densely and continuously embedded in X and Y is continuously embedded in Y_0. Moreover, let (Λ, ρ) be a metric space as above.

Theorem 2. *Let $\Phi = \Phi(\lambda, x)$ some operator which is defined in a neighbourhood of some point $(\lambda_0, x_0) \in \Lambda \times X$ and takes values in Y. Suppose that the following conditions are satisfied:*

(a) *$\Phi(\lambda_0, x_0) = 0$ and $\Phi(\cdot, x_0)$ is continuous at λ_0.*

(b) *The operator $\Phi : \Lambda \times X^0 \to Y_0$ admits a partial derivative Φ'_x which takes its values in $\mathcal{L}(X, Y)$ and is continuous at (λ_0, x_0), considered as an operator from $\Lambda \times X$ into $\mathcal{L}(X, Y)$.*

(c) *The bounded linear operator $\Phi'_x(\lambda_0, x_0) \in \mathcal{L}(X, Y)$ is invertible.*

Then for any sufficiently small $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) \le \delta$, the equation $\Phi(\lambda, x) = 0$ has a unique solution $x = x_*(\lambda)$ in the ball $||x - x_0|| \le \varepsilon$.

Proof. First of all, we point out that, under the hypotheses of Theorem 2, the operator $\Phi(\lambda_0, \cdot) : X \to Y$ is differentiable at $x = x_0$, but in general is not differentiable on an open subset of X. Let us denote by $B(x_0, r)$ and $B^0(x_0, r)$ the closed ball $||x - x_0|| \le r$ in the spaces X and X^0, respectively. To prove the existence and uniqueness of the solution branch $x = x_*(\lambda)$ near (λ_0, x_0), we consider the usual Goursat operator $T(\lambda, \cdot)$ defined by

$$T(\lambda, x) = x - \Phi'_x(\lambda_0, x_0)^{-1}\Phi(\lambda, x).$$

For $x_1, x_2 \in B^0(x_0, r)$ and some suitable $\theta \in (0, 1)$ we have (see [8: p. 20])

$$\begin{aligned} &||T(\lambda, x_1) - T(\lambda, x_2)|| \\ &\qquad \le ||\Phi'_x(\lambda_0, x_0)^{-1}|| \, ||\Phi'_x(\lambda, (1-\theta)x_1 + \theta x_2) - \Phi'_x(\lambda_0, x_0)|| \, ||x_1 - x_2||. \end{aligned}$$

Consequently, for sufficiently small $r > 0$ and $\delta > 0$ we obtain in X^0 the Lipschitz condition

$$||T(\lambda, x_1) - T(\lambda, x_2)|| \le k||x_1 - x_2|| \qquad (x_1, x_2 \in B^0(x_0, r), \rho(\lambda, \lambda_0) \le \delta)$$

with any Lipschitz constant $k \in (0, 1)$. Since X^0 is dense in X, this Lipschitz condition holds on the ball $B(x_0, r)$ as well. Finally, for sufficiently small $\delta > 0$ the operator

$T(\lambda, \cdot)$ maps the ball $||x - x_0|| \leq r$ into itself: in fact, by Conditions (a) and (b) we know that

$$||\Phi'_x(\lambda_0, x_0)^{-1}|| \, ||\Phi(\lambda, x_0)|| \leq (1-k)r$$

for $\rho(\lambda, \lambda_0) \leq \delta$, hence

$$\begin{aligned} ||T(\lambda, x) - x_0|| &\leq ||T(\lambda, x) - T(\lambda, x_0)|| + ||T(\lambda, x_0) - x_0|| \\ &\leq kr + ||\Phi'_x(\lambda_0, x_0)^{-1}|| \, ||\Phi(\lambda, x_0)|| \\ &\leq kr + (1-k)r \\ &= r \end{aligned}$$

for $x \in B(x_0, r)$. The assertion follows now from the classical Banach-Caccioppoli fixed point principle. ∎

The proof of Theorem 2 almost litterally repeats the usual proof of Theorem 1. However, Condition (b) in Theorem 2 is *weaker* than Condition (b) in Theorem 1, since the differentiability of $\Phi(\lambda, \cdot)$ between X^0 and Y_0 is less restrictive than its differentiability between X and Y (even if the derivative is supposed to belong to $\mathcal{L}(X, Y) \subset \mathcal{L}(X^0, Y_0)$). Formally this difference may appear rather harmless, but it considerably enlarges the applicability to nonlinear integral equations. In fact, our Theorem 3 below covers more general integral equations than those which may be treated by applying the Hildebrandt-Graves theorem (see, e.g., [5: Chapter V, §7]).

To illustrate this, let us consider a *Hammerstein integral equation* with parameter $\lambda \in \mathbb{R}$, i.e.

$$x(t) = \int_\Omega k(t, s) f(\lambda, s, x(s)) \, ds. \tag{1}$$

Equation (1) may be written as operator equation $x = KF(\lambda, x)$, where K is the linear integral operator

$$Ky(t) = \int_\Omega k(t, s) y(s) \, ds \tag{2}$$

generated by the kernel k, and $F(\lambda, \cdot)$ is the nonlinear superposition operator

$$F(\lambda, x)(s) = f(\lambda, s, x(s)) \tag{3}$$

generated by the function f. We suppose that both operators (2) and (3) map some Hölder space C^α $(0 < \alpha < 1)$ into itself, and that the integral operator (2) acts also from some Lebesgue space L_p into some Lebesgue space L_q $(1 \leq q \leq p \leq \infty)$. Moreover, we assume that the function $\lambda \mapsto f(\lambda, s, u)$ is continuous for all $(s, u) \in \Omega \times \mathbb{R}$, the partial derivative $g(\lambda, s, u) = f'_u(\lambda, s, u)$ of the nonlinearity f with respect to the last argument exists, and the superposition operator $G(\lambda, \cdot)$ defined by

$$G(\lambda, x)(s) = g(\lambda, s, x(s)) \tag{4}$$

also maps the space C^α into itself.

Theorem 3. *Suppose that the above hypotheses on k, f, and g are satisfied, and that*

$$x_0(t) = \int_\Omega k(t,s) f(\lambda_0, s, x_0(s))\, ds$$

for some $(\lambda_0, x_0) \in \mathbb{R} \times C^\alpha$. Assume that the operator (4) is continuous at (λ_0, x_0), and 1 does not belong to the spectrum of the linear integral operator L defined by

$$Lh(t) = \int_\Omega k(t,s) g(\lambda_0, s, x_0(s)) h(s)\, ds \tag{5}$$

in the space C^α. Then for any sufficiently small $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) \leq \delta$, equation (1) has a unique solution $x = x_*(\lambda)$ in the ball $||x - x_0|| \leq \varepsilon$.

Proof. To apply Theorem 2 we put $X = Y = C^\alpha$, $X^0 = C^\alpha$ and $Y_0 = L_q$, and consider the operator Φ defined by

$$\Phi(\lambda, x) = x - KF(\lambda, x). \tag{6}$$

The continuity requirement on the function $f(\cdot, s, u)$ implies that Condition (a) of Theorem 2 is true. Moreover, it is not hard to see that Φ, considered as an operator from $\mathbb{R} \times X^0$ into Y_0, has a partial derivative Φ'_x which is given by

$$\Phi'_x(\lambda, x)h(t) = \int_\Omega k(t,s) g(\lambda, s, x(s)) h(s)\, ds. \tag{7}$$

Consequently, Condition (b) is satisfied. Finally, our assumption on the spectrum of the operator $L = \Phi'_x(\lambda_0, x_0)$ implies Condition (c). Thus the assertion follows from Theorem 2. ∎

We make some comments on Theorem 3. Although the differentiability of the operator (6) is required only between the Hölder space C^α and the Lebesgue space L_q, the linear operator (7) should map C^α into itself, by Condition (b) of Theorem 2. A typical example for such a situation is when the kernel k is *singular*, e.g. of the form

$$k(t,s) = \frac{l(t,s)}{|t-s|^m},$$

where l is a function bounded on $\Omega \times \Omega$, and the number m is equal to the dimension of Ω. Interestingly, in this case it suffices to assume just the *existence* of the partial derivative $g = f'_u$ of f, but no *regularity* of g. For example, the operator (4) may map $\mathbb{R} \times C^\alpha$ into C^α even if the function g is *discontinuous* in u [3]. Moreover, even if the operator (4) maps C^α into itself and is bounded on bounded sets, it may be discontinuous in the norm of C^α [2] (a detailed account of the "pathological" properties of superposition operators in Hölder spaces is contained in Chapter 7 of the monograph [1]). Roughly speaking, Theorem 3 may be viewed as a continuation theorem for Hölder continuous solutions of singular nonlinear integral equations under "formal" differentiability hypotheses on the nonlinearity involved.

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