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Recurrence Coefficients of Orthogonal Polynomials with Respect to Some Self-Similar Singular Distributions

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Abstract. The limiting behaviour of arithmetic and geometric means of the coefficients of three term recurrence relations satisfied by orthogonal polynomials is investigated. The measure of orthogonality is not assumed to be absolutely continuous, but it must guarantee regular limit distribution of the zeros of the orthogonal polynomials. Some examples of self-similar distributions satisfying this condition are given.

Keywords: *Orthogonal polynomials, recurrence relations, zero distribution*

AMS subject classification: 43 C 05

1. Introduction

In the last years it has been realized that singular measures are not only pathological examples but have important applications in such areas as theoretical physics or others (see [1] and the references cited there). Consequently, there has been some interest in the study of orthogonal polynomials with respect to these measures, especially of their three term recurrence relation.

First, we introduce some notations. Let μ be a finite Borel measure on the real line, with *support* (or *spectrum*)

$$S = S(\mu) = \left\{x \in \mathbf{R} : \ \mu([x-\delta, x+\delta]) > 0 \quad \text{for all} \ \ \delta > 0\right\}.$$

We assume S to be an infinite compact set. Then there exist unique orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \ \dots \qquad (\gamma_n = \gamma_n(\mu) > 0)$$

with orthogonality relation

$$\int p_m(x) p_n(x)\, d\mu(x) = \delta_{mn} \qquad (m, n \geq 0).$$

The corresponding *monic* (i.e. with leading coefficient 1) polynomial of degree n we denote by P_n. Obviously, $P_n = \frac{1}{\gamma_n} p_n$. The recurrence relation takes the form

$$P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x) \qquad (n \geq 1)$$

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where we assume $P_{-1} = 0$ and $P_0 = 1$. In this relation the coefficient λ_1 would be arbitrary, but it is often convenient to assume $\lambda_1 = \mu(S)$, since with this convention the normalization constant simply writes

$$\frac{1}{\gamma_n} = (\lambda_1 \ldots \lambda_{n+1})^{1/2} \tag{1}$$

(for these and other simple facts about general orthogonal polynomials we refer to [2: Chapter I]). The asymptotic behaviour of the coefficients c_n and λ_n is very interesting. A well known result of Rakhmanov states the following:

If $S \subset [0,1]$ and $\frac{d\mu(x)}{dx} > 0$ a.e. on $[0,1]$, then

$$\lim_{n\to\infty} c_n = \frac{1}{2} \qquad \textit{and} \qquad \lim_{n\to\infty} \lambda_n = \frac{1}{16} \tag{2}$$

(the assumptions, of course, imply $S = [0,1]$). For the singular measures considered in [1], the coefficients do not converge. However, in [4] it is suggested to investigate the behaviour of the arithmetic means

$$\frac{1}{n-1} \sum_{k=2}^{n} \lambda_k \tag{3}$$

though in view of (1) the geometric mean would be more natural. In the case of an almost periodic sequence the arithmetic means, of course, converge.

In the present note, we show that under appropriate assumptions the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} c_k, \qquad \lim_{n\to\infty} \frac{1}{n} \left(\sum_{k=1}^{n} c_k^2 + 2 \sum_{k=2}^{n} \lambda_k \right), \qquad \lim_{n\to\infty} (\lambda_2 \ldots \lambda_n)^{1/n} \tag{4}$$

exist. In the case $S = [0,1]$, we can prove a weaker version of (2), which we could call *convergence in density.*

Our results in some sense complement the investigations of Mercer [5], where the reverse problem is investigated: If we are given the mean behaviour of the sequences (c_n) and (λ_n) – what can we say about the limiting distribution of the zeros of the polynomials p_n?

2. Connection with zero distribution

There is a simple relation between two of the above expressions in (4) and the zeros of the polynomial p_n.

Lemma 1. *Let $x_{n1}, \ldots, x_{nn}$ be the zeros of p_n. Then*

$$\sum_{k=1}^{n} x_{nk} = \sum_{k=1}^{n} c_k \,. \tag{5}$$

and

$$\sum_{k=1}^{n} x_{nk}^2 = \sum_{k=1}^{n} c_k^2 + 2\sum_{k=2}^{n} \lambda_k. \tag{6}$$

Proof. Since the zeros of the polynomials p_n and P_n coincide, we have

$$P_n(x) = (x - x_{n1}) \ \ldots \ (x - x_{nn}).$$

Thus the coefficient of x^{n-1} in $P_n(x)$ is $-(x_{n1} + \ldots + x_{nn})$. On the other hand, this coefficient is equal to $-(c_1 + \ldots + c_n)$ (see [2: p. 19]), and this proves (5).

The coefficient of x^{n-2} is $\sum_{1\le i<j\le n} x_{ni}x_{nj}$ and must be equal to

$$\sum_{1\le i<j\le n} c_i c_j - \sum_{k=2}^{n} \lambda_k$$

(see [2: p. 24]). From this we have the elementary identity

$$\begin{aligned}\sum_{k=1}^{n} x_{nk}^2 &= \left(\sum_{k=1}^{n} x_{nk}\right)^2 - 2\sum_{1\le i<j\le n} x_{ni}x_{nj} \\ &= \left(\sum_{k=1}^{n} c_k\right)^2 - 2\left(\sum_{1\le i<j\le n} c_i c_j - \sum_{k=2}^{n} \lambda_k\right) \\ &= \sum_{k=1}^{n} c_k^2 + 2\sum_{k=2}^{n} \lambda_k\end{aligned}$$

and (6) is verified. ∎

Remark. It would be difficult (though possible) to write down the corresponding identities for $\sum_{k=1}^{n} x_{nk}^m$ with $m \ge 3$. In fact, we do not need the exact equalities to study limits of mean values, since the following fact of P. Nevai holds (this is [6: Lemma 5.1]).

If the support $S(\mu)$ is compact and the function f is continuous, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left\{ f(x_{nk}) - \int f(x) p_{k-1}^2(x)\, d\mu(x) \right\} = 0.$$

This could replace (with $f(x) = x$ or $f(x) = x^2$) the equations (5) and (6), but the above derivation is simpler.

The existence of the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{nk} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{nk}^2$$

or, more general,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_{nk})$$

for any continuous function f on $S(\mu)$ depends on wether the zeros of the polynomial p_n have a regular limiting behaviour in the sense of Ullman [9], i.e. wether μ is a *regular measure* in the sense of [9] or [8]. Some sufficient conditions for regularity will be cited in the next section. We only mention that absolute continuity of the measure μ (and moreover positivity a.e. of the density) is not necessary. In fact already in [9] an example of an atomic regular measure is given.

3. Regular measures

For the following we need some facts from potential theory (see [8] or [7]). For a probability measure ν with compact support $S(\nu)$ in the complex plane $\mathbf{C}$ the *logarithmic energy* of ν is defined as

$$I(\nu) = \iint \log \frac{1}{|z-t|}\, d\nu(z)d\nu(t).$$

If K is a compact subset of $\mathbf{C}$, then with

$$V_K = \inf_{S(\nu)\subset K} I(\nu) \tag{7}$$

the *capacity* of K is defined as

$$\operatorname{cap} K = \exp(-V_K).$$

If cap $K > 0$, then there exists a unique probability measure ω_K, called the *equilibrium distribution* of K, at which the infimum in (7) is attained. The notion of capacity can be extended to any Borel set B: $\operatorname{cap} B$ denotes the supremum of the capacities of compact subsets of B. We say that a property holds q.e. (quasi everywhere) on a set $S \subset \mathbf{C}$ if it holds on S with possible exceptions on a subset of capacity zero.

Now we are able to define the notion of regularity of the measure: A measure μ with support $S = S(\mu)$ is called *regular* if

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\operatorname{cap} S(\mu)}. \tag{8}$$

As it is shown in [8], there are several equivalent formulations of regularity. We list only a few of them:

1. $\limsup_{n\to\infty} |p_n(x)|^{1/n} = 1$ for quasi-every $x \in S(\mu)$ (see [8: Theorem 3.1/(iii)]; this is essentially the notion of regularity in [9]).
2. $\limsup_{n\to\infty} |p_n(x)|^{1/n} \le 1$ for quasi-every $x \in S(\mu)$ (see [8: Theorem 3.2/(iv)]).

3. $\lim_{n\to\infty}\left(\sup_{x\in S(\mu)}|p_n(x)|\right)^{1/n} \le 1$ (if $S(\mu)$ is regular with respect to Dirichlet problems in $\mathbf{C}\setminus S(\mu)$, see [8: Theorem 3.3/(iv)]).

Of course, the definition (8) is just what we would like to prove for a measure, i.e. we need some easy to verify criteria for the regularity of a measure. Some of them can be found in [8]. For our purposes their "criterion Λ" suffices: Let $\Delta_r(x) = \{z \in \mathbf{R} : |z-x| \le r\}$. Then

$$\operatorname{cap}\left\{x \in \mathbf{R} : \limsup_{r\to 0+} \frac{\log 1/\mu(\Delta_r(x))}{\log 1/r} < \infty\right\} = \operatorname{cap} S(\mu) \tag{9}$$

implies the regularity of the measure μ (see [8: Theorem 4.8]). Since regularity also implies regular limiting zero distribution, we can state our first result.

Theorem 1. *Let μ be a regular measure, S its support and ω_S its equilibrium distribution. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} c_k = \int x\, d\omega_S(x) \tag{10}$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n} c_k^2 + 2\sum_{k=2}^{n}\lambda_k\right) = \int x^2 d\omega_S(x) \tag{11}$$

$$\lim_{n\to\infty}\left(\lambda_2\ldots\lambda_n\right)^{1/n} = (\operatorname{cap} S)^2. \tag{12}$$

Proof. For any $n > 0$ let ν_n denote the measure that places mass 1 to every zero of p_n. The regularity of μ implies (see [8: Theorem 5.8/(a)])

$$\frac{1}{n}\nu_n \xrightarrow{*} \omega_S \qquad \text{for } n\to\infty$$

and this means

$$\lim_{n\to\infty}\frac{1}{n}\int f(x)\, d\nu_n(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_{nk}) = \int f(x)\, d\omega_S(x)$$

for any continuous function f on S. Now for $f(x) = x$ with (5) we obtain (10), for $f(x) = x^2$ with (6) we obtain (11). The third limit (12) is an immediate consequence of (1) and (8). ∎

Let us illustrate this theorem by two examples.

3.1 Bessis example. In his article [1], D. Bessis investigates orthogonal polynomials with respect to the equilibrium distribution of the Julia set of a polynomial $T = T(z)$. In general, this is a set in the complex plane. In the special case $T(z) = z^2 - \lambda$ with λ real and $\lambda > 2$, he shows that the Julia set is a Cantor type set on the real line $K \subset [-\xi, \xi]$, where

$$\xi = \frac{1+\sqrt{1+4\lambda}}{2}$$

(one easily shows $2 < \xi < \lambda$). Here $\mu = \omega_K$ is symmetric and invariant under the transformation $T(x) = x^2 - \lambda$, and we obtain

$$\int x\, d\omega_K(x) = 0 \qquad \text{and} \qquad \int (x^2 - \lambda)\, d\omega_K(x) = \int x\, d\omega_K(x) = 0.$$

Consequently,

$$\int x^2\, d\omega_K(x) = \int \lambda\, d\omega_K(x) = \lambda.$$

The regularity of μ follows from the explicit bound

$$|p_n(x)| < \frac{1}{2}\left(\frac{4\lambda^{1/2}}{\sqrt{1+4\lambda}-1}\right)^k \quad \text{for} \quad 2^{k-1} \le n \le 2^k,$$

when $x \in K$ (this is [1: eq. IV.141]), since this implies

$$\limsup_{n\to\infty} |p_n(x)|^{1/n} \le 1 \qquad \text{for} \quad x \in K,$$

one of the equivalent formulations of regularity. Thus our Theorem 1 is applicable, but we need the capacity of K. Implicitely, it is calculated in [1, eq. II.35]: if we compare this with the general formula for the Green function (e.g. [8: p. 397]), we see that $\log \operatorname{cap} K = 0$, i.e. $\operatorname{cap} K = 1$. For the symmetric measure μ, of course, all c_n are zero, so (10) is trivial, and (11) becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \lambda_k = \frac{\lambda}{2}. \tag{13}$$

Equation (12) takes the form

$$\lim_{n\to\infty} (\lambda_2 \ldots \lambda_n)^{1/n} = 1. \tag{14}$$

We can now compare our results with the recurrences derived by Bessis: his system of equations [1: IV.105] in our notations reads

$$\lambda_{2k-1}\lambda_{2k-2} = \lambda_k \tag{15}$$

$$\lambda_{2k-1} + \lambda_{2k} = \lambda \tag{16}$$

$$\lambda_2 = \lambda \tag{17}$$

for $k \ge 2$. Equation (14) follows from (15) easily (if the existence of the limit is known), (16) immediately gives (13). Observe that for a measure satisfying the conditions of Rakhmanov on the interval $[-\xi, \xi]$ we would have (after an obvious rescaling) $\lim_{n\to\infty} \lambda_n = \xi^2/4$, which is strictly between 1 and $\lambda/2$ (since $\xi^2 = \xi + \lambda$ and $2 < \xi < \lambda$).

3.2 Cantor measure. In the article [4] moments of the Cantor measure (together with interesting asymptotics) are investigated and the first recurrence coefficients λ_n are generated. We will not give a detailed construction of the Cantor set and the Cantor measure here, but only recall a few facts sufficient to prove the regularity of the measure.

The Cantor set C is defined by

$$C = \bigcap_{n=0}^{\infty} \bigcup_{k=1}^{2^n} I_{nk}$$

where for any $n \geq 0$ the sets $I_{nk} \subset [0,1]$ are disjoint intervals of length 3^{-n}. The Cantor measure μ is defined by

$$\mu(I_{nk}) = \frac{1}{2^n}.$$

Let $3^{-n} \leq r \leq 3^{-n+1}$ and $x \in C$. This means that there is a $k = k(x)$ with $x \in I_{nk}$. Since the length of I_{nk} is $3^{-n} \leq r$, we have

$$\Delta_r(x) \supset I_{nk} \qquad \text{and} \qquad \mu(\Delta_r(x)) \geq \mu(I_{nk}) = \frac{1}{2^n} \geq \left(\frac{r}{3}\right)^{\alpha}$$

where $\alpha = \log 2/\log 3$. From this

$$\limsup_{r \to 0+} \frac{\log 1/\mu(\Delta_r(x))}{\log 1/r} \leq \alpha < \infty$$

follows for all $x \in C$ (not only q.e.), and criterion Λ (9) is satisfied. Thus we have proved the following

Lemma 2. *The Cantor measure μ is regular.*

The measure μ is symmetric about $\frac{1}{2}$, hence $c_n \equiv \frac{1}{2}$ and (10) is again trivial. The statements (11) and (12) of our Theorem 1 now give the following

Corollary 1. *The recurrence coefficients λ_n for the Cantor measure satisfy the relations*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \lambda_k = \frac{1}{2} \left(\int x^2 \, d\omega_C(x) - \frac{1}{4} \right) \tag{18}$$

and

$$\lim_{n\to\infty} (\lambda_2 \dots \lambda_n)^{1/n} = (\operatorname{cap} C)^2. \tag{19}$$

Unfortunately, this is far less explicit than (13) and (14), since $\operatorname{cap} C$ and ω_C are unknown (to the author).

4. Dilation, translation and symmetrization of a measure

To prove stronger results in the case $S = [a,b]$, we need a device to reduce this situation to a simple standard case (e.g. $S = [0,1]$). If μ is a measure with support $S(\mu) = [a,b]$, then we can define a new measure μ_1 via

$$\int f(a+(b-a)x)\,d\mu_1(x) = \int f(x)\,d\mu(x)$$

for any measurable bounded function f on $[a,b]$. Obviously, $S(\mu_1) = [0,1]$, and if P_n $(n \geq 0)$ are the monic orthogonal polynomials with respect to μ, then

$$P_n(a+(b-a)x) \qquad (n \geq 0)$$

must be orthogonal with respect to μ_1. Since $P_n(a+(b-a)x)$ has leading coefficient $(b-a)^n$, the monic orthogonal polynomials with respect to μ_1 must be

$$Q_n(x) = (b-a)^{-n} P_n(a+(b-a)x).$$

Solving for P_n and substituting this into the recurrence relation (with x replaced by $a+(b-a)x$) immediately gives

$$Q_n(x) = \left(x - \frac{c_n(\mu)-a}{b-a}\right) Q_{n-1}(x) - \frac{\lambda_n(\mu)}{(b-a)^2} Q_{n-2}(x)$$

for $n \geq 1$ (this is essentially [2: Chapter I, Exercise 4.4/(a)]). Consequently, the recurrence coefficients corresponding to the measures μ and μ_1 are connected via the equations

$$c_n(\mu_1) = \frac{c_n(\mu)-a}{b-a} \qquad \text{and} \qquad \lambda_n(\mu_1) = \frac{\lambda_n(\mu)}{(b-a)^2} \tag{20}$$

where the second equation, of course, is valid only for $n \geq 2$ (since $Q_{-1} = 0$).

Furthermore, we should be able to transform information about the coefficients λ_n alone into statements concerning both c_n and λ_n. In the case of symmetric (about zero) measures we have, of course, $c_n = 0$. But the general case can be reduced to this! Let μ be any measure with $S(\mu) \subset [0,\infty)$ (we always can achieve this by an appropriate shift). We define a measure ν by

$$\int x^{2n}\,d\nu(x) = \int x^n\,d\mu(x) \qquad \text{and} \qquad \int x^{2n+1}\,d\nu(x) = 0$$

for $n \geq 0$ (we assumed $S(\mu)$ to be a compact set, so ν is uniquely determined). Then the monic orthogonal polynomials S_n with respect to ν are

$$S_{2n}(x) = P_n(x^2) \qquad \text{and} \qquad S_{2n+1}(x) = xP_n^*(0;x^2)$$

for $n \geq 0$, where

$$P_n^*(0;x) = \frac{1}{x}\left(P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x)\right) \tag{21}$$

(this is [2: Chapter I, Theorem 8.1]). There is a simple relation between the recurrence coefficients, too: The coefficients $c_n(\nu)$ are identically zero by symmetry, and the $\lambda_n(\nu)$ satisfy

$$c_1(\mu) = \lambda_2(\nu) \tag{22}$$

$$c_n(\mu) = \lambda_{2n-1}(\nu) + \lambda_{2n}(\nu) \qquad (n \geq 2) \tag{23}$$

$$\lambda_n(\mu) = \lambda_{2n-2}(\nu)\lambda_{2n-1}(\nu) \qquad (n \geq 2) \tag{24}$$

(see [2: Chapter I, Theorem 9.1]).

5. The case $S = [a, b]$

We could now specify our result to the case $S = [a, b]$, where

$$\operatorname{cap} S = \frac{b-a}{4} \qquad \text{and} \qquad d\omega_S(x) = \frac{1}{\pi\sqrt{(b-x)(x-a)}}\,dx,$$

but instead we shall prove a stronger theorem in a more direct way. Using dilations and translations, we can reduce our problem to the case $S = [0, 1]$. The general case will be obtained from the equations (20). First we need some elementary estimates.

Lemma 3. *Let $S = [0, 1]$. Then for the recurrence coefficients λ_n we have the inequality*

$$\lambda_2^{1/2} + \ldots + \lambda_n^{1/2} \leq \frac{n}{4}. \tag{25}$$

Proof. It is well known that the zeros of p_n are the eigenvalues of the Jacobi matrix

$$J_n = \begin{pmatrix} c_1 & \lambda_2^{1/2} & & & 0 \\ \lambda_2^{1/2} & c_2 & \lambda_3^{1/2} & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda_{n-1}^{1/2} & c_{n-1} & \lambda_n^{1/2} \\ 0 & & & \lambda_n^{1/2} & c_n \end{pmatrix}.$$

These are all in the interval $(0, 1)$ and consequently

$$0 \leq \langle x, J_n x\rangle \leq \langle x, x\rangle \qquad \text{for any } x \in \mathbf{R}^n.$$

From the first inequality with $x = (1, -1, 1, \ldots)^T$ we obtain

$$\sum_{k=1}^{n} c_k - 2\sum_{k=2}^{n} \lambda_k^{1/2} \geq 0,$$

from the second with $x = (1, 1, 1, \ldots)^T$ we obtain

$$\sum_{k=1}^{n} c_k + 2\sum_{k=2}^{n} \lambda_k^{1/2} \leq n.$$

Subtracting these two inequalities and dividing by 4 we arrive at (25). ∎

Lemma 4. *Let* $S(\mu) = [0,1]$ *and* $M_n = \sup_{x\in[0,1]} |p_n(x)|$. *Then*

$$\frac{M_n}{\gamma_n} \geq \frac{1}{2^{2n-1}} \qquad \text{for } n \geq 1. \tag{26}$$

Proof. We have $M_n/\gamma_n = \sup_{x\in[0,1]} |P_n(x)|$ (compare the proof of [7: Proposition 3.2]) and thus the proposition is a consequence of the fact that of all monic polynomials the monic shifted Chebyshev polynomial has minimal supremum norm on $[0,1]$. ∎

Lemma 5. *If* $S(\mu) = [0,1]$ *and* $M_n = \sup_{x\in[0,1]} |p_n(x)|$, *then the inequality*

$$\sum_{k=2}^{n} \left(4\lambda_k^{1/2} - 1 - \log 4\lambda_k^{1/2}\right) \leq 1 - \log \frac{2}{\lambda_1^{1/2}} + \log M_{n-1}$$

holds for $n \geq 2$.

Proof. The inequality follows from (1), (26) with $n-1$ instead of n and (25) just by elementary algebra ∎

The preceeding lemma says nothing about the c_n, so we have to investigate the symmetrization ν of μ, first. Of course, we must have a result like

Lemma 6. *If* $S(\mu) = [0,1]$ *and* μ *is regular, then* ν *is regular, too.*

Proof. Evidently, $S(\nu) = [-1,1]$. The regularity of μ by definition (8) implies

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\operatorname{cap} S(\mu)} = 4.$$

To verify the regularity of ν, we have to find a relation between $\gamma_n(\mu)$ and $\gamma_n(\nu)$. But from (1) and (23) we have

$$\frac{1}{\gamma_{2n}(\nu)} = \Big(\lambda_1(\nu) \ldots \lambda_{2n+1}(\nu)\Big)^{1/2} = \Big(\lambda_1(\mu)\ldots\lambda_{n+1}(\mu)\Big)^{1/2} = \frac{1}{\gamma_n(\mu)}$$

and (since all $\lambda_n(\nu) < 1$)

$$\frac{1}{\gamma_{2n}(\nu)} \geq \frac{1}{\gamma_{2n+1}(\nu)} \geq \frac{1}{\gamma_{2n+2}(\nu)},$$

and consequently

$$\lim_{n\to\infty} \gamma_n(\nu)^{1/n} = \lim_{n\to\infty} \gamma_{2n}(\nu)^{1/2n} = \lim_{n\to\infty} \gamma_n(\mu)^{1/2n} = \sqrt{4} = 2.$$

This means the regularity of ν, because of $\operatorname{cap}[-1,1] = \frac{1}{2}$. ∎

To formulate our next result, we need some definitions. Let M be a subset of the natural numbers. We call the limit

$$d(M) \doteq \limsup_{n\to\infty} \frac{\#\{k \leq n : k \in M\}}{n}$$

the *density* of M. We say that a sequence (a_n) converges to a *in density* ($a_n \xrightarrow{d} a$) if

$$d\big(\{n : |a_n - a| \geq \delta\}\big) = 0 \qquad \text{for all } \delta > 0.$$

This notion is weaker than convergence (in the case of convergence, the set $\{n : |a_n - a| \geq \delta\}$ is finite for all $\delta > 0$, and a finite set has density zero). But it is meaningful, as the following lemma shows.

Lemma 7. *$a_n \xrightarrow{d} a$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(a_k) = f(a) \tag{27}$$

for all bounded continuous functions f.

Proof. Suppose first $a_n \xrightarrow{d} a$. Then we have

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(a_k) - f(a)\right| = \left|\frac{1}{n}\sum_{k=1}^{n}(f(a_k) - f(a))\right| \le \frac{1}{n}\sum_{k=1}^{n}\left|f(a_k) - f(a)\right|.$$

Now if f is bounded, $|f(x)| \le K$ for all x. Let ε be any positive real number. Then there exists a $\delta > 0$ with $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$. If we write

$$N_\delta = \{n : |a_n - a| \ge \delta\}, \tag{28}$$

then we have $|f(a_k) - f(a)| < 2K$ for $k \in N_\delta$ and $|f(a_k) - f(a)| < \varepsilon$ for $k \notin N_\delta$, consequently

$$\frac{1}{n}\sum_{k=1}^{n}|f(a_k) - f(a)| \le \epsilon + \frac{2K}{n}\#\{k \le n : k \in N_\delta\}.$$

From this

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} f(a_k) - f(a)\right| \le \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|f(a_k) - f(a)\right| \le \varepsilon$$

follows, i.e.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(a_k) = f(a),$$

since $\varepsilon > 0$ was arbitrary.

To prove the "if"-part, suppose (27) holds. For $\delta > 0$ let $f(x) = \min\{\frac{1}{\delta}|x - a|, 1\}$. Obviously, f is continuous and bounded, and with the notation(28) we have

$$\frac{1}{n}\sum_{k=1}^{n} f(a_k) \ge \frac{1}{n}\#\{k \le n : k \in N_\delta\}.$$

The equation (27) for this special function f now immediately gives $d(N_\delta) = 0$. ∎

The main result of this section is the following

Theorem 2. *Let μ be regular and $S(\mu) = [a, b]$. Then*

$$\lambda_n \xrightarrow{d} \left(\frac{b-a}{4}\right)^2 \quad \text{and} \quad c_n \xrightarrow{d} \frac{a+b}{2}.$$

Proof. First we assume $S(\mu) = [0,1]$. We prove the proposition concerning λ_n. Let again $M_n = \sup_{x\in[0,1]} |p_n(x)|$. The equivalent formulation 3 of regularity (the condition for $S(\mu)$ is satisfied) gives

$$\lim_{n\to\infty} M_n^{1/n} = 1, \qquad \text{i.e.} \qquad \lim_{n\to\infty} \frac{1}{n} \log M_n = 0.$$

Now we can apply our Lemma 5. Observe the elementary inequality $\log x \le x - 1$ for $x > 0$. It shows that the sum in this lemma is positive! Moreover, substituting $x^{1/2}$ instead of x, we obtain

$$x - 1 - \log x \ge x - 1 - 2(x^{1/2} - 1) = (x^{1/2} - 1)^2.$$

Thus

$$\begin{aligned} \frac{1}{n}\sum_{k=2}^{n}\left(4\lambda_k^{1/2} - 1 - \log 4\lambda_k^{1/2}\right) &\ge \frac{1}{n}\sum_{k=2}^{n}\left(2\lambda_k^{1/4} - 1\right)^2 \\ &\ge \frac{\delta^2}{n}\#\left\{k \le n : |2\lambda_k^{1/4} - 1| \ge \delta\right\} \end{aligned}$$

for any $\delta > 0$. Since the left-hand side converges to zero as $n \to \infty$ by Lemma 5, we have $2\lambda_n^{1/4} \xrightarrow{d} 1$ and this immediately gives $\lambda_n \xrightarrow{d} \frac{1}{16}$. Now we are going to prove the second proposition concerning c_n. To do this we consider the symmetrization ν of μ. By Lemma 6 it is regular and we have $S(\nu) = [-1,1]$. What we have shown above implies $\lambda_n(\nu) \xrightarrow{d} \frac{1}{16}[1-(-1)]^2 = \frac{1}{4}$ in virtue of the scaling properties of recurrence coefficients (see equations (20)). From this by (22) and (23)

$$c_n(\mu) = \lambda_{2n-1}(\nu) + \lambda_{2n}(\nu) \xrightarrow{d} \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

follows and we have

$$\lambda_n(\mu) = \lambda_{2n-2}(\nu)\lambda_{2n-1}(\nu) \xrightarrow{d} \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}.$$

The general case $S = [a,b]$ follows now easily: The corresponding shifted and scaled to $[0,1]$ measure μ_1 defined in the preceeding section is regular, and we have $c_n(\mu_1) \xrightarrow{d} \frac{1}{2}$ and $\lambda_n(\mu_1) \xrightarrow{d} \frac{1}{16}$. But from the equations (20) we obtain

$$c_n(\mu) = a + (b-a)c_n(\mu_1) \xrightarrow{d} a + \frac{b-a}{2} = \frac{a+b}{2}$$

and

$$\lambda_n(\mu) = (b-a)^2\lambda_n(\mu_1) \xrightarrow{d} \frac{(b-a)^2}{16} = \left(\frac{b-a}{4}\right)^2,$$

and this is our proposition. ∎

To illustrate the result (and to justify the title of the present note) we consider two examples of self-similar distributions μ with $S(\mu) = [0,1]$.

5.1 Riesz-Nagy distributions. Another example for singular measures investigated in [4] are the Riesz-Nagy singular distributions (more precisely, a special case of them). We need more information about them than given in the cited paper [4], so we recall the construction here.

Suppose $(\tau_n)_{n\geq 1}$ is a sequence of real numbers, $0 < \tau_n < 1$. Then we define inductively a sequence of continuous functions F_n on $[0,1]$ as follows. Let $F_0(x) = x$. For $n \geq 0$ define F_{n+1} by

$$F_{n+1}\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right) \qquad (k = 0(1)2^n)$$

$$F_{n+1}\left(\frac{2k+1}{2^{n+1}}\right) = \frac{1-\tau_{n+1}}{2}F_n\left(\frac{k}{2^n}\right) + \frac{1+\tau_{n+1}}{2}F_n\left(\frac{k+1}{2^n}\right) \quad (k = 0(1)2^n - 1)$$

$$F_{n+1}(x) \text{ linear in the intervals } \left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right] \qquad (k = 0(1)2^{n+1} - 1).$$

Finally, we define the function F by $F(x) = \lim_{n\to\infty} F_n(x)$. This limit function is continuous and strictly increasing, so there is a unique measure μ with distribution function $\mu((-\infty, x)) = F(x)$. These measures are called *Riesz-Nagy distributions.* In order to apply our results, we need the following

Lemma 8. *If $\tau_n \leq 1 - \delta < 1$ for $n \geq 1$, then $S(\mu) = [0,1]$ and μ is regular.*

Proof. Let

$$I_{nk} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \qquad (k = 0, 1, \ldots, 2^n - 1)$$

for $n \geq 0$. We show inductively $\mu(I_{nk}) \geq (\delta/2)^n$ for all $0 \leq k \leq 2^n - 1$. For $n = 0$ this is trivial. Now we observe that the definition of F_{n+1} implies

$$F_m\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right) \qquad (k = 0, 1, \ldots, 2^n)$$

for all $m \geq n+1$ and thus $F\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right)$. Consequently, we have

$$\mu(I_{nk}) = F_n\left(\frac{k+1}{2^n}\right) - F_n\left(\frac{k}{2^n}\right).$$

The definition of the measure μ now implies

$$\mu(I_{n+1\,2k}) = \frac{1+\tau_{n+1}}{2}\mu(I_{nk}) \geq \frac{1-\tau_{n+1}}{2}\mu(I_{nk}) \geq \frac{\delta}{2}\mu(I_{nk})$$

and

$$\mu(I_{n+1\,2k+1}) = \frac{1-\tau_{n+1}}{2}\mu(I_{nk}) \geq \frac{\delta}{2}\mu(I_{nk}),$$

and we are done.

If $x \in [0,1]$ and $2^{-n} \leq r \leq 2^{-n+1}$, then the interval $\Delta_r(x)$ contains at least one of the intervals I_{nk}, and it must be

$$\mu(\Delta_r(x)) \geq \left(\frac{\delta}{2}\right)^n \geq \left(\frac{r}{2}\right)^\alpha \qquad \text{where } \alpha = \frac{\log \frac{2}{\delta}}{\log 2}.$$

This proves both the inclusion $x \in S(\mu)$ and the regularity of μ. ∎

Application of our Theorem 2 gives the following

Corollary 2. *For the Riesz-Nagy distributions with $\tau_n \leq 1-\delta < 1$ for $n \geq 1$ we have*

$$\lambda_n \xrightarrow{d} \frac{1}{16} \quad \text{and} \quad c_n \xrightarrow{d} \frac{1}{2}.$$

This result is stronger than convergence of arithmetical means and it recovers the special case $\tau_n \equiv \tau \in (0,1)$ considered in [4]. But nevertheless we feel that it is not strong enough — here we should have convergence in the usual sense.

5.2 An atomic measure. Our last example shows that the recurrence coefficients may behave quiet regularly even in the case of an atomic (or discrete) measure. We define

$$\mu\left(\left\{\frac{m}{2^n}\right\}\right) = \frac{1}{3^n}$$

for $n \geq 1$, m odd and $0 < m < 2^n$, and $\mu(B) = 0$ for any Borel set B not containing a dyadic rational. The dyadic rationals are dense in $[0,1]$, so we have $S(\mu) = [0,1]$. Any interval of length 2^{-n} contains a number of the form $m2^{-n}$, and we easily prove $\mu(\Delta_r(x)) \geq (r/2)^\alpha$ with $\alpha = \log 3/\log 2$, i.e. we have the following

Lemma 9. *The atomic measure μ is regular.*

By Theorem 2 we obtain the following corollary (note that we have $c_n \equiv \frac{1}{2}$ by symmetry).

Corollary 3. *For the measure defined above the coefficients λ_n satisfy the convergence relation*

$$\lambda_n \xrightarrow{d} \frac{1}{16}.$$

Again we feel that in fact the coefficients simply converge to $\frac{1}{16}$. This is suggested by numerical experiments: The measure has a simple self-similar structure, and one easily shows that

$$\int f(x)\,d\mu(x) = \frac{1}{3}\left(f\left(\frac{1}{2}\right) + \int f\left(\frac{x}{2}\right) d\mu(x) + \int f\left(\frac{x+1}{2}\right) d\mu(x)\right)$$

for any bounded function f. From this equation we can derive simple recurrences not only for ordinary moments $\int x^n\,d\mu(x)$, but for modified moments $\int T_n^*(x)\,d\mu(x)$ too, where $T_n^*(x) = T_n(2x-1)$ are the shifted Chebyshev polynomials of first kind. With a variant of the modified Chebyshev algorithm (see [3]) we can calculate numerically some λ_n beyond the $n = 18$ in [4] (actually up to $n = 5000$), and the algorithm seems to be stable. The values

$$\varepsilon_n = \max_{2^{n-1} < k \leq 2^n} \left|\lambda_k - \frac{1}{16}\right|$$

should give some feeling for the long-range behaviour of the coefficients λ_n:

n	ε_n
4	3.05432773230336E-0002
5	2.71889842287010E-0002
6	2.04401383587083E-0002
7	1.79923599931158E-0002
8	1.46869431344498E-0002
9	1.14468494701327E-0002
10	9.76190450455761E-0003
11	7.55169329744376E-0003
12	6.23379099249632E-0003

Indeed, this figures (and rather similar results for the Riesz-Nagy distribution with $\tau_n \equiv \frac{1}{2}$) suggest convergence. Unfortunately, the author has no rigorous proof of their reliability.

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