

Preservation of the Exponential Stability under Perturbations of Linear Delay Impulsive Differential Equations

L. Berezansky and E. Braverman

Abstract. Exponential stability of an impulsive functional-differential equation under perturbations is studied by means of a new method. We transform a differential equation into an operator equation. The method is based on the equivalence of exponential stability and solvability of the operator equation in certain function spaces.

Keywords: *Impulsive delay differential equations, exponential stability, perturbations*

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1. Introduction

The paper deals with the preservation of the exponential stability under small perturbations for the equation

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \end{aligned} \quad (1)$$

satisfying the impulsive conditions

$$x(\tau_j) = B_j x(\tau_j - 0) \quad (2)$$

with $\lim_{j \rightarrow \infty} \tau_j = \infty$.

We consider perturbations of equation (1), precisely, of the functions A_k and h_k and caused by addition of new terms containing delay in the left-hand side of (1). It turns out that the changing of parameters of problem (1),(2) on any finite segment does not affect its exponential stability. In particular, the removal or the addition of a finite number of impulses does not influence the stability.

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The stability preservation under various perturbations is one of the central problems in stability. According to R. Bellman [4] the stability theory is a theory studying whether properties of an equation preserve under perturbations. The basic instrument in the investigation of the stability preservation is the developed theory of differential and integral inequalities (see [4, 10, 19]) as well as the method of Lyapunov functions and functionals (see [8, 15, 17, 20]).

If the functions h_k in equation (1) change, then the application of the above methods is not so efficient. The method we propose here is quite different. It is based on the Bohl-Perron theorem (see [1, 6]) for impulsive delay differential equations. This theorem connects the exponential stability and the solvability of the equation in certain function spaces. So the preservation of the exponential stability can be reduced to the preservation of the solvability of a linear equation. This problem is standard for linear analysis. It may be reduced to the estimation of the norm of the difference of corresponding operators. In fact it is better to estimate the norm not of differential but of transformed integral operators. This method for the stability investigation for differential equations without impulses is developed in [5]. It is to be emphasized that results obtained by the method in the present work are new for delay differential equations without impulses as well.

In conclusion we note the growing role of different solution representations in stability theory (see [3, 9]). We use the solution representation formula obtained in [1].

2. Preliminaries

Let \mathbb{R}^n be the space of n -dimensional column vectors $x = \text{col}(x_1, \dots, x_n)$ with the norm $\|x\| = \max_{1 \leq i \leq n} |x_i|$ (by the same symbol $\|\cdot\|$ we shall denote the corresponding matrix norm), E_n the $n \times n$ unit matrix and $\chi_e : [0, \infty) \rightarrow \mathbb{R}$ the characteristic function of the set e : $\chi_e(t) = 1$ if $t \in e$ and $\chi_e(t) = 0$ otherwise.

We consider the problem (1), (2) under the following assumptions:

- (a1) $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ are fixed points with $\lim_{j \rightarrow \infty} \tau_j = \infty$
- (a2) f and columns of A_k ($k = 1, \dots, m$) are integrable on each interval $[0, b]$
- (a3) $h_k : [0, \infty) \rightarrow \mathbb{R}$ are Lebesgue measurable with $h_k(t) \leq t$ ($k = 1, \dots, m$)
- (a4) $\varphi : (-\infty, 0) \rightarrow \mathbb{R}^n$ is Borel measurable and bounded
- (a5) $B = \sup_j \|B_j\| < \infty$
- (a6) $K = \sup_{t, s > 0} \left\{ \frac{i(t, s)}{t-s} \mid i(t, s) > 1 \right\} < \infty$

In hypothesis (a6) $i(t, s)$ is a number of points τ_j belonging to the interval (s, t) . Hypothesis (a6) is satisfied, for instance, if $\tau_{j+1} - \tau_j \geq \rho > 0$. Denote $M = \max\{B, 1\}$ and $I = \max\{K, 1\}$.

The solution of the problem (1), (2) is a function $x = x(t)$ absolutely continuous on the interval $[\tau_{j-1}, \tau_j)$, right continuous in the points τ_i , satisfying equation (1) almost everywhere and satisfying the impulsive conditions (2).

By [1] under the hypotheses (a1) - (a6) the problem (1), (2) with $x(0) = \alpha$ has one and only one solution that can be presented as

$$x(t) = X(t, 0)x(0) + \int_0^t X(t, s)f(s) ds - \int_0^t X(t, s) \sum_{k=1}^m A_k(s)\varphi[h_k(s)] ds \quad (3)$$

where $\varphi(\zeta) = 0$ if $\zeta > 0$. The matrix $X(t, s)$ is said to be a *fundamental matrix* of the equation (1). For fixed s the matrix $X(t, s)$ as a function of t is a solution of the problem

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= 0 & (t \geq s, x(t) \in \mathbb{R}^{n \times n}) \\ x(\xi) &= 0 & (\xi < s, x(s) = E_n) \\ x(\tau_j) &= B_j x(\tau_j - 0) & (\tau_j > s). \end{aligned}$$

Definition: Problem (1), (2) is said to be *exponentially stable* if there exist positive constants N_0 and β_0 such that for any solution x of the corresponding homogeneous problem

$$\begin{aligned} \dot{x}(t) + \sum_{i=1}^m A_i(t)x[h_i(t)] &= 0 & (t \geq 0) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \\ x(\tau_j) &= B_j x(\tau_j - 0) & (j \in \mathbb{N}) \end{aligned}$$

the estimate

$$\|x(t)\| \leq N_0 \exp(-\beta_0 t) \left(\sup_{s < 0} \|\varphi(s)\| + \|x(0)\| \right) \quad (4)$$

holds.

In the sequel we use the following function spaces on the half-line:

L_p ($1 \leq p \leq \infty$) is the Banach space of Lebesgue-measurable functions $x : [0, \infty) \rightarrow \mathbb{R}^n$ such that x^p is integrable (x is essentially bounded on $[0, \infty)$ for $p = \infty$) on the semi-axis with the usual norm. The same notation will be used for matrix-valued functions.

D_p ($1 \leq p \leq \infty$) is the space of functions absolutely continuous on the interval $[\tau_{j-1}, \tau_j)$, right continuous in the points τ_j satisfying (2) and satisfying the inclusions $x \in L_p$ and $\dot{x} \in L_p$. This is a Banach space with the norm $\|\dot{x}\|_{D_p} = \|x\|_{L_p} + \|\dot{x}\|_{L_p}$ (see [1, 6]).

Consider the following semi-homogeneous problem of problem (1), (2):

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= 0 & (\xi < 0; x(0) = 0) \\ x(\tau_j) &= B_j x(\tau_j - 0). \end{aligned} \quad (5)$$

Theorem 1 (see [1, 6]): *Suppose the hypotheses (a1) - (a6) hold and there exists a number $\delta > 0$ such that $t - h_k(t) < \delta$. If there exists a number $p, 1 \leq p \leq \infty$, such that for each function $f \in L_p$ the solution x of problem (5) is in D_p , then the fundamental matrix of problem (1), (2) has the exponential estimate*

$$\|X(t, s)\| \leq N \exp[-\beta(t - s)] \tag{6}$$

with positive constants N and β .

Conversely, the condition $t - h_k(t) < \delta$ and the inequality (6) imply exponential stability of the problem (1), (2) (see [1, 6]).

We also need the following result from the papers [1, 6]. Consider the problem

$$\begin{aligned} \dot{x}(t) + ax(t) &= z(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(0) &= 0 \\ x(\tau_j) &= B_j x(\tau_j - 0). \end{aligned} \tag{7}$$

Lemma 1 (see [1, 6]): *Suppose the hypotheses (a5) and (a6) hold and $\nu = a - I \ln M > 0$. Then for any function $z \in L_p$ the solution $x = x(t)$ of the problem (7) is in D_p and can be presented as*

$$x(t) = (Wz)(t) = \int_0^t \exp[-a(t - s)] \prod_{s < \tau_j \leq t} B_j z(s) ds \tag{8}$$

(we assume $\prod_{s < \tau_j \leq t} B_j = E_n$ if the interval $(s, t]$ does not contain points τ_j). Besides, the fundamental matrix $X(t, s)$ of problem (7) has the estimate (6) with $N = 1$ and $\beta = \nu$.

3. Perturbation on a finite interval or addition of new terms

Suppose the parameters of problem (1), (2) change on a finite interval. We consider the problem

$$\begin{aligned} \dot{\tilde{x}}(t) + \sum_{k=1}^m \tilde{A}_k(t)x[\tilde{h}_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \\ x(\tilde{\tau}_j) &= \tilde{B}_j(\tilde{\tau}_j - 0). \end{aligned} \tag{9}$$

Theorem 2: *Suppose there exists a number $b > 0$ such that*

- a) \tilde{h}_k is measurable and $\tilde{h}_k(t) = h_k(t)$ for $t \in [b, \infty)$
- b) \tilde{A}_k is integrable on $[0, b]$ and $\tilde{A}_k(t) = A_k(t)$ for $t \in [b, \infty)$

c) $\bar{\tau}_j = \tau_j$ and $\bar{B}_j = B_j$ for $\tau_j > b$.

Let the hypotheses (a1) - (a6) hold and let exist a number $\delta > 0$ such that $t - h_k(t) < \delta$.

If the fundamental function $X(t, s)$ of the problem (1), (2) has the exponential estimate (6), then the fundamental function $\tilde{X}(t, s)$ of the problem (9) has a similar estimate with positive constants \bar{N} and $\bar{\beta}$.

Proof: Let $s < b$. The matrix $\tilde{X}(t, s)$ as a function of t for fixed s is a solution of the problem

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m \bar{A}_k(t)x[\bar{h}_k(t)] &= 0 & (t \in [s, \infty), x(t) \in \mathbb{R}^{n \times n}) \\ x(\xi) &= 0 & (\xi < s; \xi(s) = \mathbb{E}_n) \\ x(\bar{\tau}_j) &= \bar{B}_j(\bar{\tau}_j - 0) & (\bar{\tau}_j > s). \end{aligned} \tag{10}$$

Let $t > b > s$. Then the problem (10) under the hypotheses of the theorem can be rewritten as

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x(h_k(t)) &= - \sum_{k=1}^m A_k(t)\bar{x}(h_k(t)) & (t \in [b, \infty)) \\ x(\xi) &= 0 & (\xi < b; x(b) = \bar{x}(b)) \\ x(\tau_j) &= B_j(\tau_j - 0) & (\tau_j > b) \end{aligned}$$

where \bar{x} is a solution of the problem (10) for $t \leq b$ and $\bar{x}(\xi) = 0$ if $\xi > b$ (here \bar{x} is treated as initial function). The solution of this problem can be presented as

$$x(t) = X(t, b)\bar{x}(b) - \int_b^t X(t, s) \sum_{k=1}^m A_k(s)\bar{x}(h_k(s)) ds$$

where $X(t, s)$ is the fundamental function of problem (1), (2). Since $\bar{x}[h_k(t)] = 0$ for $t > b + \delta$, then

$$\|x(t)\| \leq Q\|X(t, b)\| + \int_b^{b+\delta} \|X(t, s)\|Q ds$$

where

$$Q = \sup_{0 < t < b} \|\bar{x}(t)\| \left(1 + \sup_{t \in [b, b+\delta]} \sum_{k=1}^m \|A_k(t)\| \right).$$

By applying the estimate (6) of $X(t, s)$ we obtain

$$\begin{aligned} \|x(t)\| &\leq QN \exp[-\beta(t-b)] + QN \int_b^{b+\delta} \exp[-\beta(t-s)] ds \\ &\leq QN \exp[-\beta(t-b)] + \frac{QN}{\beta} \exp[-\beta(t-b)] \{ \exp(\beta\delta) - 1 \} \\ &= Q_1 \exp[-\beta(t-b)] \\ &\leq Q_2 \exp[-\beta(t-s)] \end{aligned}$$

with $Q_2 = Q_1 e^{\beta b}$, where the constant Q_2 does not depend on t and s . Thus for $t > b$ and $s < b$ we have the estimate

$$\|\tilde{X}(t, s)\| \leq Q_2 \exp[-\beta(t-s)].$$

Let now $s < b$ and $t < b$. Denote

$$\tilde{N} = \sup_{0 < t, s < b} \|\tilde{X}(t, s)\| \exp[\beta(t-s)].$$

Then

$$\|\tilde{X}(t, s)\| \leq \tilde{N} \exp[-\beta(t-s)]$$

for $0 < t, s < b$. Finally, let $t, s > b$. Then $\tilde{X}(t, s)$ is a solution of the problem (10) for $s > b$. Thus in (10) $\tilde{A}_k = A_k$, $\tilde{h}_k = h_k$, $\tilde{\tau}_k = \tau_k$ and $\tilde{B}_j = B_j$. Therefore $\tilde{X}(t, s) = X(t, s)$, i.e. (6) is the exponential estimate of $\tilde{X}(t, s)$ as well. ■

Consider the following perturbation of the problem (1), (2):

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] + \sum_{k=1}^r \tilde{A}_k(t)x[\tilde{h}_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \\ x(\tau_j) &= B_j x(\tau_j - 0). \end{aligned} \tag{11}$$

We suppose that the hypotheses (a1) - (a6) hold, the columns of \tilde{A}_k are integrable on each interval $[0, b]$ ($b > 0$), the functions \tilde{h}_k are Lebesgue measurable, $\tilde{h}_k(t) \leq t$ and there exists a number $\delta > 0$ such that $t - \tilde{h}_k(t) < \delta$. By $\tilde{X}(t, s)$ we again denote a fundamental matrix of the perturbed problem (11).

Theorem 3: *Suppose that in addition to the above assumptions for the fundamental matrix $X(t, s)$ of the problem (1), (2) the estimate (6) is valid. There exists a number $\eta > 0$ such that if*

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} \sum_{k=1}^r \|\tilde{A}_k(s)\| ds < \eta, \tag{12}$$

then the fundamental matrix $\tilde{X}(t, s)$ of the perturbed problem (11) has an exponential estimate of the form (6), with certain constants $\tilde{N} > 0$ and $\tilde{\beta} > 0$.

Proof: For a fixed s the function $\tilde{X}(\cdot, s)$ is a solution of the problem

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= - \sum_{k=1}^r \tilde{A}_k(t)x[\tilde{h}_k(t)] & (t \in [0, \infty), x(t) \in \mathbb{R}^{n \times n}) \\ x(\xi) &= 0 & (\xi < s; x(s) = E_n) \\ x(\tau_j) &= B_j x(\tau_j - 0) & (\tau_j > s). \end{aligned}$$

By applying the solution representation (3) of problem (1), (2) with s being an initial point, we obtain

$$\tilde{X}(t, s) = X(t, s) - \int_s^t X(t, \tau) \sum_{k=1}^r \tilde{A}_k(\tau) \tilde{X}[\tilde{h}_k(\tau), s] d\tau.$$

Therefore the estimate (6) gives

$$\begin{aligned} \|\tilde{X}(t, s)\| &\leq N \exp[-\beta(t-s)] \\ &+ N \int_s^t \exp[-\beta(t-\tau)] \sum_{k=1}^r \|\tilde{A}_k(\tau)\| \max_{\tau-\delta \leq \xi \leq \tau} \|\tilde{X}(\xi, s)\| d\tau. \end{aligned}$$

Hence

$$\begin{aligned} \max_{t-\delta \leq \xi \leq t} \|\tilde{X}(\xi, s)\| &\leq N \exp[-\beta(t-s-\delta)] \\ &+ N \int_s^t \exp[-\beta(t-\tau-\delta)] \sum_{k=1}^r \|\tilde{A}_k(\tau)\| \max_{\tau-\delta \leq \xi \leq \tau} \|\tilde{X}(\xi, s)\| d\tau. \end{aligned}$$

If we denote $y(t) = \max_{t-\delta \leq \xi \leq t} \|\tilde{X}(\xi, s)\|$, then we obtain the inequality

$$\begin{aligned} y(t) &\leq N \exp(\beta\delta) \exp[-\beta(t-s)] \\ &+ N \exp(\beta\delta) \int_s^t \exp[-\beta(t-\tau)] \sum_{k=1}^m \|\tilde{A}_k(\tau)\| y(\tau) d\tau. \end{aligned}$$

By applying the inequality (2.5) from [10] we obtain the estimate

$$y(t) \leq N \exp(\beta\delta) \exp[-\beta(t-s)] \exp\left(N \exp(\beta\delta) \int_s^t \sum_{k=1}^r \|\tilde{A}_k(\tau)\| d\tau \right). \tag{13}$$

Let (12) be satisfied. Then, for a certain $b > 0$,

$$\sup_{t>b} \int_t^{t+1} \sum_{k=1}^r \|\tilde{A}_k(\tau)\| d\tau < \eta. \tag{14}$$

By Theorem 2 the functions \tilde{A}_k may be changed on the segment $[0, b]$ and it does not influence the existence of the exponential estimate for $\tilde{X}(t, s)$. Therefore we assume that instead of (14) we have the inequality

$$\sup_{t>0} \int_t^{t+1} \sum_{k=1}^r \|\tilde{A}_k(\tau)\| d\tau < \eta. \tag{15}$$

Then (13) and (15) imply the estimate

$$\|\tilde{X}(t, s)\| \leq N \exp[\beta\delta + N\eta \exp(\beta\delta)] \exp\{-[\beta - N\eta \exp(\beta\delta)](t - s)\}.$$

Therefore if $0 < \eta < \frac{\beta}{N} \exp(-\beta\delta)$, then the fundamental matrix $\tilde{X}(t, s)$ of the problem (11) has an exponential estimate of type (6) with the constants

$$\tilde{N} = N \exp[\beta\delta + N\eta \exp(\beta\delta)] \quad \text{and} \quad \tilde{\beta} = \beta - N \exp(\beta\delta)\eta.$$

The proof of the theorem is complete. ■

Remark: The scheme of the proof is similar to the proofs of the same assertions for delay differential equations without impulses (see, for example, [14]).

Theorem 3 immediately implies the following two assertions.

Corollary 1: *There exists a number $\eta > 0$ such that if the inequality*

$$\limsup_{t \rightarrow \infty} \sum_{k=1}^r \|\tilde{A}_k(t)\| < \eta$$

holds, then the estimate (6) for the fundamental function of problem (1), (2) implies a similar estimate for the problem (11).

Corollary 2: *Suppose that at least one of the conditions*

$$\limsup_{t \rightarrow \infty} \sum_{k=1}^r \|\tilde{A}_k(t)\| = 0 \quad \text{and} \quad \int_0^{\infty} \sum_{k=1}^r \|\tilde{A}_k(\tau)\| d\tau < \infty$$

hold. Then an estimate of type (6) is valid either for both the fundamental matrices of problem (1), (2) and problem (11) or for none of them.

4. Stability with respect to perturbations of delay

In this section we apply Theorem 1 using the following scheme. An original and a perturbed equation are transformed into operator equations $Tz = f$ and $\tilde{T}z = f$ in L_1 , respectively. Here if the original equation is exponentially stable, then the operator $T: L_1 \rightarrow L_1$ is invertible. Then for the norm $\|T - \tilde{T}\|$ being small enough the operator $\tilde{T}: L_1 \rightarrow L_1$ is also invertible. If \tilde{T} is invertible, then Theorem 1 gives exponential stability of the perturbed equation.

Consider the following perturbation of the problem (1), (2):

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m \tilde{A}_k(t)x[\tilde{h}_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \\ x(\tau_j) &= B_j x(\tau_j - 0). \end{aligned} \tag{16}$$

We assume that the parameters of the problem (16) also satisfy the hypotheses (a1) - (a6).

Further we need a function that is often used when investigating equations with compositions $x[h(t)]$ (see [2, 7, 11 - 13, 18]). Let $h : [0, \infty) \rightarrow \mathbb{R}$ be a Lebesgue measurable function. We assume that for any $c > 0$

$$\sup_{e \subset [0, c]} \frac{\text{mes } h^{-1}(e)}{\text{mes } e} < \infty$$

where $\text{mes } e$ is the Lebesgue measure of the set e , $\text{mes } e > 0$. We define a set function $\mu_c(e) = \text{mes } \{h^{-1}(e) \cap [0, c]\}$ for a Lebesgue measurable set $e \subset [0, c]$. The measure μ_c is absolutely continuous [13] with respect to the Lebesgue measure, therefore by the Radon-Nikodým theorem there exists a function $\mu'_c \in L_1[0, c]$ such that

$$\mu_c(e) = \int_e \mu'_c(s) ds.$$

Let $e = [0, t] \subset [0, c]$. Then

$$\text{mes } \{h^{-1}[0, t] \cap [0, c]\} = \int_0^t \mu'_c(s) ds.$$

Consequently,

$$\mu'_c(t) = \frac{d}{dt} \text{mes } \{h^{-1}[0, t] \cap [0, c]\}. \tag{17}$$

The basic property of this function is expressed in the following substitution formula (see [11 - 13]) valid for $x \in L_\infty[0, c]$:

$$\int_{h^{-1}(e) \cap [0, c]} x(h(s)) ds = \int_e x(s) \mu'_c(s) ds \quad \text{for all } e \subset [0, c].$$

If h is a monotone function, then the function μ' is easily calculated and coincides with the derivative of the inverse function h^{-1} . Properties of the function μ' and its application to the investigation of delay differential equations are presented in the works of M. Drakhlin [11, 12].

Denote

$$\mu'_{k, \infty}(t) = \frac{d}{dt} \text{mes } \{h_k^{-1}[0, t]\}.$$

Theorem 4: Suppose that for the problems (1), (2) and (16) the hypotheses (a1) - (a6) hold, $A_k \in L_\infty$, $\|B_j\| \geq b > 0$ and there exist numbers $\sigma, \rho, \delta \geq 0$ such that

$$\rho \leq \tau_j - \tau_{j-1} \leq \sigma, \quad t - \bar{h}_k(t) < \delta, \quad t - h_k(t) < \delta.$$

and

$$\mu' = \max_k \limsup_{t > 0} |\mu'_{k, \infty}(t)| < \infty.$$

There exists a number $\Delta > 0$ such that if

$$\max_k \limsup_{t \rightarrow \infty} \|A_k(t) - \tilde{A}_k(t)\| < \Delta \quad \text{and} \quad \max_k \limsup_{t \rightarrow \infty} |h_k(t) - \tilde{h}_k(t)| < \Delta,$$

then the exponential estimate (6) for the fundamental function of the problem (1), (2) implies a similar estimate for the fundamental function $\tilde{X}(t, s)$ of the problem (16).

Proof: Without loss of generality we can assume that $m = 1$ and, by Theorem 2,

$$\|A(t) - \tilde{A}(t)\| < \Delta \quad \text{and} \quad |h(t) - \tilde{h}(t)| < \Delta$$

for any $t \in [0, \infty)$. Let $\nu = a - I \ln M > 0$. We substitute $x = Wz$, where the operator W is defined by (8), in the semi-homogeneous problem (5) and the corresponding semi-homogeneous problem (16) ($\varphi \equiv 0$ and $x(0) = 0$). After denoting by \mathcal{L} and $\tilde{\mathcal{L}}$ the left-hand sides of the equations (5) and (16), respectively, we obtain (see Lemma 1)

$$\begin{aligned} (\mathcal{L}Wz)(t) = z(t) - a \int_0^t \exp[-a(t-s)] \prod_{s < \tau_i \leq t} B_i z(s) ds \\ + A(t) \int_0^{h^+(t)} \exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i z(s) ds \end{aligned} \tag{18}$$

and

$$\begin{aligned} (\tilde{\mathcal{L}}Wz)(t) = z(t) - a \int_0^t \exp[-a(t-s)] \prod_{s < \tau_i \leq t} B_i z(s) ds \\ + \tilde{A}(t) \int_0^{\tilde{h}^+(t)} \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i z(s) ds \end{aligned} \tag{19}$$

where $a^+ = \max\{a, 0\}$.

Consider the operator $H z = (\mathcal{L}W - \tilde{\mathcal{L}}W)z$. By Lemma 1 the operator $W : \mathbf{L}_1 \rightarrow \mathbf{D}_1^0$ is invertible, where $\mathbf{D}_1^0 = \{x \in \mathbf{D}_1 : x(0) = 0\}$. The exponential estimate of the fundamental function $X(t, s)$ of problem (1), (2) implies that the operator $\mathcal{L} : \mathbf{D}_1^0 \rightarrow \mathbf{L}_1$ is invertible (see [6]). Therefore the operator $\mathcal{L}W : \mathbf{L}_1 \rightarrow \mathbf{L}_1$ is also invertible. Thus it is sufficient to prove that

$$\lim_{\Delta \rightarrow 0} \|H\|_{\mathbf{L}_1 \rightarrow \mathbf{L}_1} = 0$$

for obtaining the invertibility of the operator $\tilde{L}W : L_1 \rightarrow L_1$ for small Δ . To this end

$$\begin{aligned} (Hz)(t) &= A(t) \int_0^{h^+(t)} \left(\exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i - \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right) z(s) ds \\ &+ A(t) \int_{\tilde{h}^+(t)}^{h^+(t)} \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i z(s) ds \\ &+ [A(t) - \tilde{A}(t)] \int_0^{\tilde{h}^+(t)} \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i z(s) ds. \end{aligned}$$

The operator H can be written as sum

$$H \doteq H_1 + H_2 + H_3$$

and we will evaluate the norms of the summands H_1, H_2 and H_3 in L_1 .

Step 1: For H_1 we have

$$\begin{aligned} \|H_1 z\|_{L_1} &\leq \|A\|_{L_\infty} \int_0^\infty \int_0^{h^+(t)} \left\| \exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i \right. \\ &\quad \left. - \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right\| \|z(s)\| ds dt \\ &\leq \|A\|_{L_\infty} \int_0^\infty \int_0^t \left\| \exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i \right. \\ &\quad \left. - \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right\| \|z(s)\| ds dt. \end{aligned}$$

By inverting the order of integrating we obtain

$$\begin{aligned} \|H_1 z\|_{L_1} &\leq \|A\|_{L_\infty} \int_0^\infty \left(\int_s^\infty \left\| \exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i \right. \right. \\ &\quad \left. \left. - \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right\| dt \right) \|z(s)\| ds. \end{aligned}$$

Let

$$\Phi(s) = \int_s^\infty \left\| \exp[-a(h(t)-s)] \prod_{s < \tau_i \leq h(t)} B_i - \exp[-a(\tilde{h}(t)-s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right\| dt.$$

We assume $\Delta < \rho$. Denote

$$e'_i = \left\{ t \in [s, \infty) \left| \begin{array}{l} \tau_{i-1} < h(t) \leq \tau_i \\ \text{and} \\ \tau_{i-1} < \tilde{h}(t) \leq \tau_i \end{array} \right. \right\}$$

$$e''_i = \left\{ t \in [s, \infty) \left| \begin{array}{l} \tau_{i-1} < \tilde{h}(t) < \tau_i < h(t) < \tau_{i+1} \\ \text{or} \\ \tau_{i-1} < h(t) < \tau_i < \tilde{h}(t) < \tau_{i+1} \end{array} \right. \right\}$$

Denote $e' = \cup_i e'_i$ and $e'' = \cup_i e''_i$. Then $\text{mes}\{e' \cap e''\} = 0$ and $\Delta < \rho$ implies $\text{mes}\{(s, \infty) \setminus e' \cup e''\} = 0$. Hence

$$\Phi(s) = \int_{e'} + \int_{e''}$$

If $t \in e'$, then

$$\prod_{s < \tau_i \leq h(t)} B_i = \prod_{s < \tau_i \leq \tilde{h}(t)} B_i,$$

therefore by Lemma 1

$$\begin{aligned} \int_{e'} &\leq \int_s^\infty \left| \exp[-a(h(t) - s)] - \exp[-a(\tilde{h}(t) - s)] \right| \left\| \prod_{s < \tau_i \leq h(t)} B_i \right\| dt \\ &\leq \int_s^\infty \left| \exp[a|h(t) - \tilde{h}(t)|] - 1 \right| \exp[-a(h(t) - s)] \left\| \prod_{s < \tau_j \leq h(t)} B_j \right\| dt \\ &\leq [\exp(a\Delta) - 1] \int_s^\infty \exp[-\nu(h(t) - s)] dt \tag{20} \\ &= [\exp(a\Delta) - 1] \int_s^\infty \exp[-\nu(t - s)] \exp[\nu(t - h(t))] dt \\ &\leq \frac{1}{\nu} \exp(\nu\delta) [\exp(a\Delta) - 1]. \end{aligned}$$

If $t \in e''_k$, then the products $\prod_{s < \tau_i \leq h(t)} B_i$ and $\prod_{s < \tau_i \leq \tilde{h}(t)} B_i$ may differ by one factor only, precisely, by B_k . Denote

$$p(t) = \max [h(t), \tilde{h}(t)] \quad \text{and} \quad \tilde{p}(t) = \min [h(t), \tilde{h}(t)]$$

$$c_k = \begin{cases} B_k & \text{if } h(t) > \tilde{h}(t) \\ E_n & \text{if } h(t) \leq \tilde{h}(t) \end{cases} \quad \text{and} \quad \tilde{c}_k = \begin{cases} E_n & \text{if } h(t) > \tilde{h}(t) \\ B_k & \text{if } h(t) \leq \tilde{h}(t). \end{cases}$$

Then

$$\begin{aligned}
 \int_{e''_k} &= \exp(as) \int_{e''_k} \left\| \exp[-ah(t)] \prod_{s < \tau_i \leq h(t)} B_i - \exp[-a\tilde{h}(t)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_j \right\| dt \\
 &\leq \exp(as) \int_{e''_k} \exp(-ap(t)) \left\| \exp[a(p(t) - h(t))] c_k \right. \\
 &\quad \left. - \exp[a(p(t) - \tilde{h}(t))] \tilde{c}_k \right\| \left\| \prod_{s < \tau_i \leq \tilde{p}(t)} B_i \right\| dt \\
 &\leq (B + 1) \exp(as) \exp(a\Delta) \int_{e''_k} \exp[-ap(t)] \left\| \prod_{s < \tau_j \leq \tilde{p}(t)} B_j \right\| dt.
 \end{aligned} \tag{21}$$

The set e''_k can be written as

$$e''_k = \left\{ t \in [s, \infty) \mid \tilde{h}(t) < \tau_k < h(t) \text{ or } h(t) < \tau_k < \tilde{h}(t) \right\}.$$

Since the inequality $|h(t) - \tilde{h}(t)| < \Delta$ implies $\tilde{h}(t) > h(t) - \Delta$ and $h(t) > \tilde{h}(t) - \Delta$, we have

$$\begin{aligned}
 e''_k &\subset \left\{ t \in [s, \infty) \mid h(t) - \Delta < \tau_k < h(t) \text{ or } h(t) < \tau_k < h(t) + \Delta \right\} \\
 &= \left\{ t \in [s, \infty) \mid h(t) - \Delta < \tau_k < h(t) + \Delta \right\} \\
 &= \left\{ t \in [s, \infty) \mid \tau_k - \Delta < h(t) < \tau_k + \Delta \right\}.
 \end{aligned} \tag{22}$$

The formulas (21) and (22) give

$$\begin{aligned}
 \int_{e''_k} &\leq (B + 1) \exp(as) \exp(a\Delta) \\
 &\quad \times \int_s^\infty \exp[-ap(t)] \chi_{[\tau_k - \Delta, \tau_k + \Delta]}(h(t)) \left\| \prod_{s < \tau_j \leq \tilde{p}(t)} B_j \right\| dt \\
 &\leq (B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(as) \exp(a\Delta) \\
 &\quad \times \lim_{c \rightarrow \infty} \int_s^c \exp[-ah(t)] \chi_{[\tau_k - \Delta, \tau_k + \Delta]}(h(t)) \left\| \prod_{s < \tau_j \leq h(t)} B_j \right\| dt.
 \end{aligned} \tag{23}$$

where $\chi_{[t,s]}$ is the characteristic function of the segment $[t, s]$. By applying the Radon-Nikodým theorem to the integral on the right-hand side of (23) and by Lemma 1 we

obtain (we recall that μ'_c is defined by (17))

$$\begin{aligned} \int_{e''_k} &\leq (B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(as) \exp(a\Delta) \\ &\times \lim_{c \rightarrow \infty} \int_{h(s,c) \cap [0,c]} \exp(-at) \chi_{[\tau_k - \Delta, \tau_k + \Delta]}(t) |\mu'_c(t)| \left\| \prod_{s < \tau_j \leq t} B_j \right\| dt \\ &\leq \mu'(B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \int_{\tau_k - \Delta}^{\tau_k + \Delta} \exp(-a(t - s)) \left\| \prod_{s < \tau_j \leq t} B_j \right\| dt \\ &\leq \mu'(B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \int_{\tau_k - \Delta}^{\tau_k + \Delta} \exp(-\nu(t - s)) dt \\ &\leq \mu'(B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(\nu s) \exp(a\Delta) \frac{1}{\nu} \exp(-\nu\tau_k) \left[\exp(\nu\Delta) - \exp(-\nu\Delta) \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{e''} &\leq \mu'(B + 1) \max \left\{ B, \frac{1}{b} \right\} \\ &\times \exp(a\Delta) \frac{1}{\nu} \left[\exp(\nu\Delta) - \exp(-\nu\Delta) \right] \exp(\nu s) \sum_{\tau_i > s} \exp(-\nu\tau_i). \end{aligned}$$

The inequality $\tau_i - \tau_{i-1} \geq \rho$ implies $(\tau_i - s) - (\tau_{i-1} - s) \geq \rho$, hence $\tau_i - s > (i - k)\rho$, where $\tau_{k-1} < s \leq \tau_k$. Thus

$$\begin{aligned} \sum_{\tau_i > s} \exp(-\nu\tau_i) &= \exp(-\nu s) \sum_{\tau_i > s} \exp[-\nu(\tau_i - s)] \\ &\leq \exp(-\nu s) \sum_{i=k}^{\infty} \exp[-\nu\rho(i - k)] \\ &= \exp(-\nu s) \frac{1}{1 - \exp(-\nu\rho)}. \end{aligned}$$

Hence

$$\int_{e''} \leq \frac{\mu'(B + 1) \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \left[\exp(\nu\Delta) - \exp(-\nu\Delta) \right]}{\nu \left[1 - \exp(-\nu\rho) \right]} \tag{24}$$

By comparing (20) and (24) we obtain $\|\Phi\|_{L_\infty} \rightarrow 0$ as $\Delta \rightarrow 0$. Since $\|H_1\|_{L_1 \rightarrow L_1} \leq \|A\|_{L_\infty} \|\Phi\|_{L_\infty}$, then $\lim_{\Delta \rightarrow 0} \|H_1\|_{L_1 \rightarrow L_1} = 0$.

Step 2: Now we estimate the norm of the operators H_2 . To this end

$$\begin{aligned} \|H_2 z\|_{L_1} &\leq \int_0^\infty \|A(t)\| \left\| \int_{\tilde{h}^+(t)}^{h^+(t)} \exp[-a(\tilde{h}(t) - s)] \prod_{s < \tau_i \leq \tilde{h}(t)} B_i \right\| \|z(s)\| ds dt \\ &\leq \|A\|_{L_\infty} \int_0^\infty \int_0^t \exp[-a(\tilde{h}(t) - s)] \chi_{[\tilde{p}(t), p(t)]}(s) \|z(s)\| \prod_{s < \tau_i \leq \tilde{h}(t)} B_i ds dt \end{aligned}$$

where the functions \tilde{p} and p are defined above. By inverting the order of integrating we obtain

$$\|H_2 z\|_{L_1} \leq \|A\|_{L_\infty} \int_0^\infty \left(\int_s^\infty \exp[-a(\tilde{h}(t) - s)] \chi_{[\tilde{p}(t), p(t)]}(s) \prod_{s < \tau_i \leq \tilde{h}(t)} B_i dt \right) \|z(s)\| ds.$$

Denote

$$\Psi(s) = \int_s^\infty \exp[-a(\tilde{h}(t) - s)] \chi_{[\tilde{p}(t), p(t)]}(s) \prod_{s < \tau_i \leq \tilde{h}(t)} B_i dt.$$

As $|\tilde{h}(t) - h(t)| < \Delta$, then $[\tilde{p}(t), p(t)] \subset [h(t) - \Delta, h(t) + \Delta]$. Therefore

$$\chi_{[\tilde{p}(t), p(t)]}(s) \leq \chi_{[h(t) - \Delta, h(t) + \Delta]}(s) = \chi_{[s - \Delta, s + \Delta]}(h(t)).$$

Hence

$$\begin{aligned} \Psi(s) &\leq \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \\ &\quad \times \lim_{c \rightarrow \infty} \int_s^c \exp[-a(h(t) - s)] \chi_{[s - \Delta, s + \Delta]}(h(t)) \prod_{s < \tau_i \leq h(t)} B_i dt. \end{aligned}$$

By applying the Radon-Nikodým theorem we obtain

$$\begin{aligned} \Psi(s) &\leq \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \\ &\quad \times \lim_{c \rightarrow \infty} \int_{h(\{s, c\}) \cap [0, c]} \exp[-a(t - s)] \chi_{[s - \Delta, s + \Delta]}(t) |\mu'_c(t)| \prod_{s < \tau_i \leq t} B_i dt. \end{aligned}$$

Consequently

$$\begin{aligned} \Psi(s) &\leq \mu' \max \left\{ B, \frac{1}{b} \right\} \exp(a\Delta) \int_{s - \Delta}^{s + \Delta} \exp[-\nu(t - s)] dt \\ &= \mu' \max \left\{ B, \frac{1}{b} \right\} \frac{1}{\nu} [\exp(\nu\Delta) - \exp(-\nu\Delta)] \exp(a\Delta). \end{aligned}$$

Thus

$$\|H_2\|_{L_1 \rightarrow L_1} \leq \frac{1}{\nu} \|A\|_{L_\infty} \mu' \max \left\{ B, \frac{1}{b} \right\} \left[\exp(\nu\Delta) - \exp(-\nu\Delta) \right] \exp(a\Delta).$$

Hence $\lim_{\Delta \rightarrow 0} \|H_2\|_{L_1 \rightarrow L_1} = 0$.

Step 3: Now we shall evaluate the norm of the operator H_3 in L_1 . To this end by applying Lemma 1

$$\begin{aligned} \|H_3 z\|_{L_1} &\leq \int_0^\infty \|A(t) - \tilde{A}(t)\| \int_0^{\tilde{h}^+(t)} \exp[-a(\tilde{h}(t) - s)] \prod_{s < \tau_i < \tilde{h}(t)} \|B_i\| \|z(s)\| ds dt \\ &\leq \|A - \tilde{A}\|_{L_\infty} \int_0^\infty \int_0^t \exp[-\nu(\tilde{h}(t) - s)] \|z(s)\| ds dt \\ &= \|A - \tilde{A}\|_{L_\infty} \int_0^\infty \int_0^t \exp[\nu(t - \tilde{h}(t))] \exp[-\nu(t - s)] \|z(s)\| ds dt \\ &\leq \Delta \exp(\nu\delta) \int_0^\infty \int_0^t \exp[-\nu(t - s)] \|z(s)\| ds dt \\ &= \Delta \exp(\nu\delta) \int_0^\infty \left(\int_s^\infty \exp[-\nu(t - s)] dt \right) \|z(s)\| ds \\ &\leq \frac{1}{\nu} \Delta \exp(\nu\delta) \|z\|_{L_1}. \end{aligned}$$

Hence $\|H_3\|_{L_1 \rightarrow L_1} \leq \frac{1}{\nu} \Delta \exp(\nu\delta)$ and therefore $\lim_{\Delta \rightarrow 0} \|H_3\|_{L_1 \rightarrow L_1} = 0$.

Now the inequality

$$\|H\|_{L_1 \rightarrow L_1} \leq \|H_1\|_{L_1 \rightarrow L_1} + \|H_2\|_{L_1 \rightarrow L_1} + \|H_3\|_{L_1 \rightarrow L_1}$$

implies

$$\lim_{\Delta \rightarrow 0} \|H\|_{L_1 \rightarrow L_1} = 0.$$

Therefore for Δ being sufficiently small the operator $\tilde{\mathcal{L}}W : L_1 \rightarrow L_1$ is invertible. Thus for such Δ the equation $\tilde{\mathcal{L}}Wz = f$ has a solution $z \in L_1$, if $f \in L_1$. By Lemma 1 the solution of the semi-homogeneous problem (16) $x = Wz$ is in D_1 . By Theorem 1 the fundamental matrix of problem (16) has an exponential estimate. ■

Consider two special cases of the problem (1), (2). First let (1) be an ordinary differential equation, i.e. $m = 1$ and $h(t) = t$. Since in this case $\mu'(t) = 1$ Theorem 4 implies the following assertion.

Corollary 1: Consider the problems

$$\begin{aligned} \dot{x}(t) + A(t)x(t) &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\tau_j) &= B_j x(\tau_j - 0) \end{aligned} \tag{25}$$

and

$$\begin{aligned} \dot{x}(t) + \tilde{A}(t)x[\tilde{h}(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \\ x(\tau_j) &= B_j x(\tau_j - 0). \end{aligned} \tag{26}$$

Suppose that for these problems the hypotheses (a1) - (a6) hold, $A \in L_\infty$, $\|B_j\| \geq b > 0$ and $0 < \rho \leq \tau_j - \tau_{j-1} \leq \sigma$. There exists a number $\Delta > 0$ such that if

$$\limsup_{t \rightarrow \infty} \|A(t) - \tilde{A}(t)\| < \Delta \quad \text{and} \quad \limsup_{t \rightarrow \infty} (t - \tilde{h}(t)) < \Delta,$$

then the exponential estimate (6) for the fundamental function of problem (25) implies a similar estimate for the fundamental function of problem (26).

The second special case is a delay differential equation without impulses. The result obtained is new for this equation.

Corollary 2: Consider the problems

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m \tilde{A}_k(t)x[\tilde{h}_k(t)] &= f(t) & (t \in [0, \infty), x(t) \in \mathbb{R}^n) \\ x(\xi) &= \varphi(\xi) & (\xi < 0). \end{aligned} \tag{28}$$

Suppose that for these problems the hypotheses (a2) - (a4) are satisfied, $A_k \in L_\infty$, there exists a number $\delta > 0$ such that $t - h_k(t) < \delta$ and $t - \tilde{h}_k(t) < \delta$, and suppose that $\mu' < \infty$, μ' defined in Theorem 4. There exists a number $\Delta > 0$ such that if

$$\max_k \limsup_{t \rightarrow \infty} \|A_k(t) - \tilde{A}_k(t)\| < \Delta \quad \text{and} \quad \max_k \limsup_{t \rightarrow \infty} |h_k(t) - \tilde{h}_k(t)| < \Delta,$$

then the exponential estimate (6) for the fundamental function of problem (27) implies a similar estimate for the fundamental function of problem (28).

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