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# Gauss' and Related Inequalities

**S. Varošanec and J. Pečarić**

**Abstract.** Let $g : [a,b] \to \mathbb{R}$ be a non-negative increasing differentiable function and $f : [a,b] \to \mathbb{R}$ a non-negative function such that the quotient $f/g'$ is non-decreasing. Then the function

$$Q(r) = (r+1)\int_a^b g(x)^r f(x)\,dx$$

is log-concave. If $g(a) = 0$, $b \in (a,\infty]$ and the quotient $f/g'$ is non-increasing, then the function $Q$ is log-convex.

**Keywords:** *Gauss' inequality, Popoviciu's inequality, Hölder's inequality, log-concave functions, log-convex functions*

**AMS subject classification:** 26 D 10

## 1. Introduction

The following result was mentioned by Gauss [4]:

If $f$ is a non-negative and decreasing function, then

$$\left(\int_0^\infty x^2 f(x)\,dx\right)^2 \le \frac{5}{9}\left(\int_0^\infty f(x)\,dx\right)\left(\int_0^\infty x^4 f(x)\,dx\right). \tag{1}$$

This inequality can be extended in different ways. For example, some generalization of (1) can be found in [1, 2, 8, 9, 12, 13]. Here, we will restrict our attention to Pólya's inequality. We wish to investigate its connection with inequality (1) and give some new improvements.

In [5: p. 166] or [11: Vol II, p. 114] one can find the following statement.

**Theorem 1** (Pólya' s inequality:) *Let $f : [0,\infty) \to \mathbb{R}$ be a non-negative and decreasing function. If $a$ and $b$ are non-negative real numbers, then*

$$\left(\int_0^\infty x^{a+b} f(x)\,dx\right)^2$$

S. Varošanec: Univ. Zagreb, Dep. Math., Bijenička cesta 30, 41000 Zagreb, Croatia
J. Pečarić: Univ. Zagreb, Fac. Textile Techn., Pierottijeva 6, 41000 Zagreb, Croatia

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$$\leq \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\left(\int\limits_0^\infty x^{2a}f(x)\,dx\right)\left(\int\limits_0^\infty x^{2b}f(x)\,dx\right) \tag{2}$$

*if all the integrals exist.*

It is clear that for $a = 0$ and $b = 2$ we obtain Gauss' inequality (1). Also in the same book [11: Vol I, p. 94] the following reverse statement is given.

**Theorem 2:** *Let $f : [0,1] \to I\!R$ be a non-negative and increasing function. If $a$ and $b$ are non-negative real numbers, then*

$$\begin{aligned}\left(\int\limits_0^1 x^{a+b}f(x)\,dx\right)^2 \\ \geq \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\left(\int\limits_0^1 x^{2a}f(x)\,dx\right)\left(\int\limits_0^1 x^{2b}f(x)\,dx\right).\end{aligned} \tag{3}$$

A. M. Fink and M. Jodeit Jr. [3] showed that inequality (3) is valid not only for non-negative numbers $a$ and $b$, but for $a, b \geq -1/2$. It is obvious that inequalities (2) and (3) are related results and in this paper we shall give a unified treatment and extention with simple proofs of such results.

## 2. Main results

In this section we shall give an extension of Theorems 1 and 2.

**Theorem 3:** *Let $g : [a,b] \to I\!R$ be a non-negative increasing differentiable function and let $f : [a,b] \to I\!R$ be a non-negative function such that the quotient $f/g'$ is non-decreasing. Let $p_i$ $(i = 1,\ldots,n)$ be positive real numbers such that $\sum_{i=1}^n 1/p_i = 1$. If $a_i$ $(i = 1,\ldots,n)$ are real numbers such that $a_i > -1/p_i$, then*

$$\int\limits_a^b g(x)^{a_1+\cdots+a_n}f(x)\,dx \geq \frac{\prod_{i=1}^n (a_ip_i+1)^{1/p_i}}{1+\sum_{i=1}^n a_i}\prod_{i=1}^n\left(\int\limits_a^b g(x)^{a_ip_i}f(x)\,dx\right)^{1/p_i}. \tag{4}$$

*If $g(a) = 0$, then equality holds in (4) if and only if $f/g'$ is a constant function.*

*If $g(a) = 0$ and if the quotient function $f/g'$ is non-increasing, then the reverse inequality in (4) holds, with equality if and only if $f/g'$ is the characteristic function of an interval $[a, b_1]$, $a < b_1 \leq b$.*

**Proof:** We will denote $F = f/g'$. First, suppose that $F$ is a non-decreasing function. The inequality (4) reduces to an equality for $F \equiv 0$ and thus we may assume without loss of generality that $F(b) > 0$. Applying integration by parts we conclude

$$\left(1+\sum_{i=1}^{n} a_i\right) \int_a^b g(x)^{a_1+\cdots+a_n} f(x)\,dx$$

$$= F(b)g(b)^{a_1+\cdots+a_n+1} - F(a)g(a)^{a_1+\cdots+a_n+1} - \int_a^b g(x)^{a_1+\cdots+a_n+1} dF(x)$$

$$= F(b)g(b)^{a_1+\cdots+a_n+1} - F(a)g(a)^{a_1+\cdots+a_n+1} - \int_a^b \prod_{i=1}^{n} \left(g(x)^{a_ip_i+1}\right)^{1/p_i} dF(x)$$

$$\geq F(b)g(b)^{a_1+\cdots+a_n+1} - F(a)g(a)^{a_1+\cdots+a_n+1} - \prod_{i=1}^{n} \left(\int_a^b g(x)^{a_ip_i+1} dF(x)\right)^{1/p_i}$$

where in the last inequality we use Hölder's inequality.

Let us consider the Popoviciu inequality (see [7: p. 118])

$$\sum_{i=1}^{m} w_i a_{i1} \cdots a_{in} \geq \prod_{j=1}^{n} \left(\sum_{i=1}^{m} w_i a_{ij}^{p_j}\right)^{1/p_j}$$

where

$$w_1 > 0 \quad \text{and} \quad w_2, \ldots, w_m \leq 0$$
$$a_{ij} \geq 0 \quad \text{and} \quad p_i > 0 \quad (i = 1, \ldots, m; \; j = 1, \ldots, n)$$
$$\sum_{i=1}^{n} 1/p_i = 1 \quad \text{and} \quad \sum_{i=1}^{m} w_i a_{ij}^{p_j} \geq 0 \quad (j = 1, \ldots, n).$$

Set $m = 3$, $w_1 = F(b) > 0$, $w_2 = -F(a)$, $w_3 = -1$ and

$$a_{1i} = \left(g(b)^{a_ip_i+1}\right)^{1/p_i}, \quad a_{2i} = \left(g(a)^{a_ip_i+1}\right)^{1/p_i}, \quad a_{3i} = \left(\int_a^b g(x)^{a_ip_i+1} dF(x)\right)^{1/p_i}$$

for $i = 1, \ldots, n$. Using the Popoviciu inequality we conclude

$$F(b)g(b)^{a_1+\cdots+a_n+1} - F(a)g(a)^{a_1+\cdots+a_n+1} - \prod_{i=1}^{n} \left(\int_a^b g(x)^{a_ip_i+1} dF(x)\right)^{1/p_i}$$

$$\geq \prod_{i=1}^{n} \left(F(b)g(b)^{a_ip_i+1} - F(a)g(a)^{a_ip_i+1} - \int_a^b g(x)^{a_ip_i+1} dF(x)\right)^{1/p_i}$$

$$= \prod_{i=1}^{n} \left((a_ip_i+1) \int_a^b g(x)^{a_ip_i} f(x)\,dx\right)^{1/p_i}.$$

and so, inequality (4) is proven.

If $g(a) = 0$ and if the quotient function $f/g'$ is non-increasing, then we can use Hölder's inequality for discrete case instead of Popoviciu's inequality and the proof is similar to the previous one. ∎

When we know results of the previous theorem, it is easy to check the following statement.

**Theorem 4:** *Let $f$ and $g$ be functions satisfying the assumptions of Theorem 3. If the quotient $f/g'$ is non-decreasing, then the function*

$$Q(r) = (r+1)\int_a^b g(x)^r f(x)\,dx \tag{5}$$

*is log-concave, i.e. $\log Q$ is concave.*

*If $g(a) = 0$, $b \in (a, \infty]$ and the quotient $f/g'$ is non-increasing, then the function $Q$ is log-convex, i.e. $\log Q$ is convex.*

Now, using some well-known properties of convex and concave functions we have the following statement.

**Theorem 5:** *Let $f$ and $g$ be defined as in Theorem 3, the quotient $f/g'$ be non-decreasing and $p$, $q$, $r$, $s$, $t$ be real numbers from the domain of definition of the function $Q(r) = (r+1)\int_a^b g(x)^r f(x)\,dx$.*

**a)** *If $p > q > r$, then*

$$\begin{aligned}&\left((q+1)\int_a^b g(x)^q f(x)\,dx\right)^{p-r}\\ &\geq \left((r+1)\int_a^b g(x)^r f(x)\,dx\right)^{p-q}\left((p+1)\int_a^b g(x)^p f(x)\,dx\right)^{q-r}.\end{aligned} \tag{6}$$

**b)** *If $p \geq q$, $r \geq s$ and $p > r$, $q > s$, then*

$$\left(\frac{(p+1)\int_a^b g(x)^p f(x)\,dx}{(r+1)\int_a^b g(x)^r f(x)\,dx}\right)^{1/(p-r)} \leq \left(\frac{(q+1)\int_a^b g(x)^q f(x)\,dx}{(s+1)\int_a^b g(x)^s f(x)\,dx}\right)^{1/(q-s)}. \tag{7}$$

**c)** *If $r \geq 0$ and $r_1, \ldots, r_n > 0$, then*

$$\begin{aligned}&\left((r+1)\int_a^b g(x)^r f(x)\,dx\right)^{n-1}\\ &\times\left(r_1 + \ldots + r_n + r + 1\right)\int_a^b g(x)^{r_1+\ldots+r_n+r} f(x)\,dx\\ &\leq \left(r_1+r+1\right)\cdots\left(r_n+r+1\right)\prod_{i=1}^{n}\int_a^b g(x)^{r_i+r} f(x)\,dx.\end{aligned} \tag{8}$$

**d)** *If $q > s > r > p$ and $p \le t \le q$, then*

$$\left((p+1)\int_a^b g(x)^p f(x)\,dx\right)^{\frac{q-t}{q-p}} \left((q+1)\int_a^b g(x)^q f(x)\,dx\right)^{\frac{t-p}{q-p}} \le \left((r+1)\int_a^b g(x)^r f(x)\,dx\right)^{\frac{s-t}{s-r}} \left((s+1)\int_a^b g(x)^s f(x)\,dx\right)^{\frac{t-r}{s-r}}. \tag{9}$$

*If $g(a) = 0$ and the quotient $f/g'$ is non-increasing, then in all statements* a) - d) *reverse inequalities hold.*

**Proof:** a) This is a consequence of Theorem 4 and the following inequality for a concave function $H$ (see [7: p. 1]):

$$\begin{vmatrix} H(p) & H(q) & H(r) \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \le 0 \qquad \text{for } p > q > r.$$

b) For any concave function $H$ the inequality

$$\frac{H(p) - H(r)}{p - r} \le \frac{H(q) - H(s)}{q - s} \qquad \text{for } p \ge q \text{ and } r \ge s$$

holds (see [7: p. 2]). Therefore, (7) is a simple consequence of the previous inequality if we set $H = \log Q$.

c) Setting in (7) $r = s$, $p = r_1 + \dots + r_n + r$ and $q = r_i + r$ we have

$$\left(\frac{(r_1 + \dots + r_n + r + 1)\int_a^b g(x)^{r_1+\dots+r_n+r} f(x)\,dx}{(r+1)\int_a^b g(x)^r f(x)\,dx}\right)^{1/(r_1+\dots+r_n)} \le \left(\frac{(r_i + r + 1)\int_a^b g(x)^{r_i+r} f(x)\,dx}{(r+1)\int_a^b g(x)^r f(x)dx}\right)^{1/r_i}$$

i.e.

$$\left(\frac{(r_1 + \dots + r_n + r + 1)\int_a^b g(x)^{r_1+\dots+r_n+r} f(x)\,dx}{(r+1)\int_a^b g(x)^r f(x)\,dx}\right)^{r_i/(r_1+\dots+r_n)} \le \frac{(r_i + r + 1)\int_a^b g(x)^{r_i+r} f(x)\,dx}{(r+1)\int_a^b g(x)^r f(x)\,dx}$$

for $i = 1, \dots, n$. Multiplying all these inequalities we obtain statement (8).

d) This is a consequence of Narumi's inequality (see [9])

$$\frac{q-t}{q-p}H(p) + \frac{t-p}{q-p}H(q) \le \frac{s-t}{s-r}H(r) + \frac{t-r}{s-r}H(s)$$

where $H$ is a concave function, $q > s > r > p$ and $p \le t \le q$. ∎

## 3. The case $g(x) = x$

In the remainder of this paper we assume the function $g$ to be the identity. So, we have the following statement.

**Theorem 6:** *Let the function $f: [a,b] \to \mathbb{R}$ be non-negative and non-decreasing. Let $p_i$ $(i = 1,\dots,n)$ be positive real numbers such that $\sum_{i=1}^n 1/p_i = 1$. If $a_i$ $(i = 1,\dots,n)$ are real numbers such that $a_i > -1/p_i$, then*

$$\int_a^b x^{a_1+\dots+a_n} f(x)\,dx \geq \frac{\prod_{i=1}^n (a_i p_i + 1)^{1/p_i}}{\sum_{i=1}^n a_i + 1} \prod_{i=1}^n \left( \int_a^b x^{a_i p_i} f(x)\,dx \right)^{1/p_i}. \tag{10}$$

*If $a = 0$, then equality holds in* (10) *if and only if $f$ is a constant function.*

*If $a = 0$ and $f$ is a non-increasing function, then the reverse inequality holds, with equality if and only if $f$ is the characteristic function of an interval $[0, b_1]$, $0 < b_1 \leq b$.*

**Remark 1:** For $n = 2$, $p_1 = p_2 = 2$, $a = 0$, $b = 1$ and $a_1, a_2 > -1/2$ we have the inequality (3).

**Remark 2:** Letting $b \to \infty$ we have the inequality

$$\int_a^\infty x^{a_1+\dots+a_n} f(x)\,dx \leq \frac{\prod_{i=1}^n (a_i p_i + 1)^{1/p_i}}{\sum_{i=1}^n a_i + 1} \prod_{i=1}^n \left( \int_a^\infty x^{a_i p_i} f(x)\,dx \right)^{1/p_i} \tag{11}$$

where $f$ is a non-negative and non-increasing function, $a_i$ and $p_i$ are real numbers satisfying the conditions from Theorem 3, and all integrals exist. Now, it is obvious that for $n = p_1 = p_2 = 2$ we have the inequality (2), but with condition $a_1, a_2 > -1/2$ what is stronger than condition "$a_1$ and $a_2$ are non-negative numbers" from Theorem 1.

**Remark 3:** A special case of (11), namely for $n = 2$ and $a_1, a_2 > 0$ was proved by V. N. Volkov (see [6: p. 269] or [12]).

**Remark 4:** If $f$ is a non-negative and non-increasing function and $\int_0^\infty f(x)\,dx = 1$, then substituting $r = s = 0$, $a = 0$, $b = \infty$ and $g(x) = x$ in the reverse of (7) we obtain

$$\left( (p+1) \int_0^\infty x^p f(x)\,dx \right)^{1/p} \geq \left( (q+1) \int_0^\infty x^q f(x)\,dx \right)^{1/q} \tag{12}$$

for $p \geq q$. This is the well-known Gauss-Winckler inequality (see [1: p. 455] or [13]). It is an improvement of (1). The first proof of (12) was due to Faber (see [2: pp. 9 - 11]).

In the following theorem monotonicity is replaced with concavity.

**Theorem 7:** *Let $f$ be a non-negative differentiable function on $[0,1]$ with non-increasing first derivative. Let $p_i$ $(i=1,\dots,n)$ be positive real numbers with $\sum_{i=1}^n 1/p_i = 1$. If $a_i$ $(i=1,\dots,n)$ are real numbers such that $a_i > 1/p_i$, then*

$$\int_0^1 x^{a_1+\dots+a_n} f(x)\,dx \geq \frac{\prod_{i=1}^n \left((a_ip_i+1)(a_ip_i+2)\right)^{1/p_i}}{\left(\sum_{i=1}^n a_i+1\right)\left(\sum_{i=1}^n a_i+2\right)} \prod_{i=1}^n \left(\int_0^1 x^{a_ip_i} f(x)\,dx\right)^{1/p_i}.$$

**Remark 5:** This theorem deals with a concave function $f$ defined on $[0,1]$, and it is still true if we replace the unit interval $[0,1]$ by any other interval $[0,b]$, $b>0$.

**Proof of Theorem 7:** By using the well-known inequality between geometric and arithmetic means we have

$$\prod_{i=1}^n \left((a_ip_i+1)(a_ip_i+2)\int_0^1 x^{a_ip_i} f(x)\,dx\right)^{1/p_i} \leq \sum_{i=1}^n \frac{1}{p_i}(a_ip_i+1)(a_ip_i+2)\int_0^1 x^{a_ip_i} f(x)\,dx.$$

Integrating by parts we get

$$\begin{aligned}
&\int_0^1 \left(\sum_{i=1}^n \frac{1}{p_i}(a_ip_i+1)(a_ip_i+2)x^{a_ip_i}\right) f(x)\,dx \\
&= f(1)\left(\sum_{i=1}^n \frac{1}{p_i}(a_ip_i+2)\right) - f'(1)\left(\sum_{i=1}^n \frac{1}{p_i}\right) + \int_0^1 \left(\sum_{i=1}^n \frac{1}{p_i} x^{a_ip_i+2}\right) df'(x) \\
&= f(1)\left(\sum_{i=1}^n a_i + 2\right) - f'(1) + \int_0^1 \left(\sum_{i=1}^n \frac{1}{p_i} x^{a_ip_i+2}\right) df'(x) \\
&\leq f(1)\left(\sum_{i=1}^n a_i + 2\right) - f'(1) + \int_0^1 \prod_{i=1}^n x^{(a_ip_i+2)/p_i} df'(x) \\
&= f(1)\left(\sum_{i=1}^n a_i + 2\right) - f'(1) + \int_0^1 x^{\left(\sum_{i=1}^n a_i+2\right)} df'(x) \\
&= \left(\sum_{i=1}^n a_i + 2\right)\left(\sum_{i=1}^n a_i + 1\right)\int_0^1 x^{\left(\sum_{i=1}^n a_i\right)} f(x)\,dx
\end{aligned}$$

where inequality between geometric and arithmetic means is again used in the last inequality. ∎

Using the result of Theorem 7 and properties of a concave function we have the following statement.

**Theorem 8:** *If $f$ is a non-negative differentiable function on $[0,1]$ with non-increasing first derivative, then the function*

$$Q_2(r)=\binom{r+2}{2}\int_0^1 x^r f(x)\,dx$$

*is log-concave and the following inequalities hold:*

**a)** *If $p>q>r$, then*

$$\left(\binom{q+2}{2}\int_0^1 x^q f(x)\,dx\right)^{p-r} \geq \left(\binom{r+2}{2}\int_0^1 x^r f(x)\,dx\right)^{p-q}\left(\binom{p+2}{2}\int_0^1 x^p f(x)\,dx\right)^{q-r}.$$

**b)** *If $p\geq q$, $r\geq s$ and $p>r$, $q>s$, then*

$$\left(\frac{\binom{p+2}{2}\int_0^1 x^p f(x)\,dx}{\binom{r+2}{2}\int_0^1 x^r f(x)\,dx}\right)^{1/(p-r)} \leq \left(\frac{\binom{q+2}{2}\int_0^1 x^q f(x)\,dx}{\binom{s+2}{2}\int_0^1 x^s f(x)\,dx}\right)^{1/(q-s)}.$$

**c)** *If $r\geq 0$ and $r_1,\dots,r_n>0$, then*

$$\left(\binom{r+2}{2}\int_0^1 x^r f(x)\,dx\right)^{n-1} \times \binom{r_1+\dots+r_n+r+2}{2}\int_0^1 x^{r_1+\dots+r_n+r} f(x)\,dx \leq \binom{r_1+r+2}{2}\cdots\binom{r_n+r+2}{2}\prod_{i=1}^n\int_0^1 x^{r_i+r} f(x)\,dx.$$

**d)** *If $q>s>r>p$ and $p\leq t\leq q$, then*

$$\left(\binom{p+2}{2}\int_0^1 x^p f(x)\,dx\right)^{\frac{q-t}{q-p}}\left(\binom{q+2}{2}\int_0^1 x^q f(x)\,dx\right)^{\frac{t-p}{q-p}} \leq \left(\binom{r+2}{2}\int_0^1 x^r f(x)\,dx\right)^{\frac{s-t}{s-r}}\left(\binom{s+2}{2}\int_0^1 x^s f(x)\,dx\right)^{\frac{t-r}{s-r}}.$$

Finally, to complete this section we shall state the following result.

**Theorem 9:** *Let $f: [0,\infty) \to \mathbb{R}$ be a non-increasing function such that $(-1)^k f^{(k)}$ are positive for $k = 1, 2, \ldots, N$. Then the function*

$$Q_k(x) = \binom{r+k}{k} \int_0^\infty x^r f(x)\,dx$$

*is log-convex for $k = 1, 2, \ldots, N$.*

The proof and consequences of this result one can find in [1: pp. 455 - 456], [8] and [10].

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