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On a Comparison Theorem for Second Order Nonlinear Ordinary Differential Equations

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Abstract. We present a comparison theorem for second order nonlinear differential equations of the form

$$\big(R(t)w(x(t))x'(t)\big)' + p(t)f(x(t)) = 0 \qquad (t \in [t_0,\beta), \beta \le \infty)$$

where p is a continuous function on $[t_0,\beta)$ without any restriction on its sign.

Keywords: *Second order nonlinear differential equations, oscillation theory*

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0. Introduction

Consider the second order differential equation

$$\big(R(t)x'(t)\big)' + p(t)x(t) = 0, \qquad \big(t \in [t_0,\infty)\big) \tag{D1$_1$}$$

with given functions p and R on $[t_0,\infty)$. A function defined on an interval $[t_0,\beta), \beta \le +\infty$, is said to be *oscillatory* at β if for every $a \in (t_0,\beta)$ it has an infinite number of zeros on the interval (a,β), and otherwise it is said to be *non-oscillatory* at β. A differential equation of the form $(\mathrm{D1})_1$ is called oscillatory at β if all its solutions are oscillatory at β, and otherwise such an equation is called non-oscillatory at β. In the following we set $I_\infty = [t_0,\infty)$ and $I_\beta = [t_0,\beta)$.

A fundamental problem concerning the oscillation theory of second order linear or nonlinear ordinary differential equations may be posed as follows. Suppose R and p are functions on I_∞ which make the differential equation $(\mathrm{D1})_1$ oscillatory at ∞. Are there relations between the functions R and r as well as functions p and q which ensure that the differential equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = 0 \qquad (t \in I_\infty) \tag{D1$_2$}$$

is also oscillatory at ∞ ? A well-known relation for this linear case is the classical Sturm comparison theorem [6: p. 2]. Hille [3] and Wintner [8] extended Sturm's result in the following way.

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Theorem 1 (Hille-Wintner comparison theorem). *Suppose* $R \equiv 1$ *and* $r \equiv 1$ *in equations* $(\mathrm{D1})_1$ *and* $(\mathrm{D1})_2$*, respectively. Let* $p, q \in C(I_\infty)$ *be functions such that* $P(t) = \int_t^\infty p(s)\,ds$ *and* $Q(t) = \int_t^\infty q(s)\,ds$ *exist with* $0 \le P(t) \le Q(t)$*, for all* $t \in I_\infty$*. Then if the differential equation* $(\mathrm{D1})_1$ *is oscillatory at* ∞*, then also the differential equation* $(\mathrm{D2})_2$ *is oscillatory at* ∞*.*

Taam [7] proved the following generalization of the Hille-Wintner theorem.

Theorem 2. *Let* $p, q, r, R \in C(I_\infty)$ *be functions such that* r *is bounded from above on* I_∞*,* $P(t) = \int_t^\infty p(s)\,ds$ *and* $Q(t) = \int_t^\infty q(s)\,ds$ *exist,* $|P(t)| \le Q(t)$ *and* $0 < r(t) \le R(t)$*, for all* $t \in I_\infty$*. Then if the differential equation* $(\mathrm{D1})_1$ *is oscillatory at* ∞*, so also the differential equation* $(\mathrm{D2})_2$ *is oscillatory at* ∞*.*

These and other Sturm-type comparison theorems hold for very general second order linear and nonlinear differential equations. Butler [1] obtained such a nonlinear extension of Theorem 2 for a certain class of equations.

We consider the differential equations

$$\big(R(t)w(x(t))x'(t)\big)' + p(t)f(x(t)) = 0 \qquad (t \in I_\beta) \tag{D2$_1$}$$

$$\big(r(t)w(x(t))x'(t)\big)' + q(t)f(x(t)) = 0 \qquad (t \in I_\beta). \tag{D2$_2$}$$

Theorem 3 (see Butler [1]). *Suppose* $w \equiv 1$ *and* $\beta = \infty$ *in equations* $(\mathrm{D2})_1$ *and* $(\mathrm{D2})_2$*. Let* $p, q, r, R \in C(I_\infty)$ *be functions such that* $P(t) = \int_t^\infty p(s)\,ds$ *and* $Q(t) = \int_t^\infty q(s)\,ds$ *exist,* $|P(t)| \le Q(t)$ *and* $0 < r(t) \le R(t)$*, for all* $t \in I_\infty$*. Assume that* f *is a function on* $\mathbb{R}$ *satisfying the following conditions:*

($\mathbf{a}_0$) $f \in C'(\mathbb{R})$*,* $uf(u) > 0$ *and* $f'(u) > 0$*, for all* $u \neq 0$

and either

($\mathbf{b}_0$) f' *is non-increasing on* $(-\infty, 0]$ *and non-decreasing on* $[0, \infty)$

or

($\mathbf{c}_0$) $\liminf_{|u| \to +\infty} f'(u) > 0$ *and* $\int_{\pm 1}^{\pm\infty} \frac{du}{f(u)}\,du < \infty$.

Then if the differential equation $(\mathrm{D2})_1$ *is oscillatory at* β*, so also the differential equation* $(\mathrm{D2})_2$ *is oscillatory at* β*.*

Butler also showed that Theorem 2 holds without the restriction that r is bounded.

1. Preliminaries

To obtain the main result of the paper we need the following well-known three theorems.

Theorem 4 (see Rudek [5: Theorem 1]). *Consider the nonlinear differential equation*

$$\big(R(t)x'(t)\big)' + a(t)g(x'(t)) + p(t)f(x(t)) = 0 \qquad (t \in I_\infty) \tag{D3}$$

where a, p, R *are functions on* I_∞ *satisfying the following conditions:*

(V1) $a, p, R \in C(I_\infty)$, $a(t) \geq 0$ *and* $R(t) > 0$ *for all* $t \in I_\infty$, R *decreasing on* I_∞, $P(t) = \int_t^\infty p(s)\,ds$ *existing for all* $t \in I_\infty$ *and* $\liminf_{T\to+\infty} \int_t^T \frac{P(s)}{R(s)}\,ds > -\infty$.

(V2) $f \in C'(\mathbb{R})$, $uf(u) > 0$ *and* $f'(u) > 0$, *for all* $u \neq 0$, *and* $0 \leq g \in C(I_\infty)$.

(V3) $\liminf_{|u|\to+\infty} f'(u) > 0$ *and* $\int\limits_{\pm 1}^{\pm\infty} \frac{du}{f(u)} < \infty$.

Then the differential equation (D3) *is oscillatory at* ∞ *if the condition*

$$\textbf{(V4)} \quad \int\limits_t^\infty \frac{1}{R(s)} \left(P(s) + \int\limits_s^\infty \frac{(P_+(u))^2}{R(u)}\,du \right) ds = \infty$$

is satisfied.

Theorem 5 (see Butler [1: Lemma 2.3]). *Consider the differential equation*

$$x''(s) + p(s)F(x(s)) = 0 \qquad (s \in [0,\infty)). \tag{D4}$$

Let $P(t) = \int_t^\infty p(s)ds$ *exists on* I_∞, F *be continuously differentiable with* $uF(u) > 0$ *and* $F'(u) > 0$ *for all* $u \neq 0$. *Suppose further that* $\liminf_{|x|\to\infty} F'(x) > 0$ *and* $\int_{\pm 1}^{\pm\infty} \frac{du}{F(u)} < \infty$. *Then the relation*

$$\int\limits^\infty \left(|P(s)| + \int\limits_s^\infty P^2(u)\,du \right) ds = \infty$$

is a necessary condition for the differential equation (D4) *to be oscillatory at* ∞.

Theorem 6 (see Rudek [5: Theorem 2]). *Consider the differential equation* $(\mathrm{D2})_1$. *Let* $p, R \in C(I_\beta)$, $R(t) > 0$ $(t \in I_\beta)$, *and assume that* $\int^\beta \frac{du}{R(u)}$ *and* $\overline{P}(t) = \int_t^\beta |p(s)|\,ds$ *exist on* I_β. *Let further the following conditions be satisfied:*

The functions f *and* w *are continuous, the product* $(fw)(u) = f(u)w(u)$ $(u \in \mathbb{R})$ *is continuously differentiable,* $uf(u) > 0$ *and* $(w(u)f(u))' > 0$ *for all* $u \neq 0$. *Let there exist* $s_0, S \in \mathbb{R}$ *such that* $0 < s_0 \leq w(u) \leq S$ *holds for all* $u \in \mathbb{R}$.

Then the differential equation $(\mathrm{D2})_1$ *is non-oscillatory at* β.

2. Main result

Let $\beta < \infty$ or $\beta = \infty$.

Theorem 7. *Let $p, q, r, R \in C(I_\beta)$ be such that there exist $\overline{Q}(t) = \int_t^\beta q(s)\,ds$ and $\overline{P}(t) = \int_t^\beta |p(s)|\,ds$, $\overline{P}(t) \leq \overline{Q}(t)$ and $0 < r(t) \leq R(t)$, for all $t \in I_\beta$. Further let f, w be functions on $\mathbb{R}$ satisfying the following conditions:*

(a) $f, w \in C^1(\mathbb{R})$, $uf(u) > 0, f'(u) > 0$ *and* $\big(w(u)f(u)\big)' > 0$, *for all* $u \neq 0$.

Let there exist $s_0, S \in \mathbb{R}$ such that $0 < s_0 \leq w(u) \leq S$ for all $u \in \mathbb{R}$ and either

(b) $\dfrac{f'(u)}{w(u)}$ *be non-increasing on* $(-\infty, 0]$ *and non-decreasing on* $[0, \infty)$

or

(c) $\liminf\limits_{|u|\to+\infty} \big(w(u)f(u)\big)' > 0$ *and* $\displaystyle\int_{\pm 1}^{\pm\infty} \frac{du}{f(u)} < \infty$.

Then if the differential equation $(\mathrm{D2})_1$ *is oscillatory at* β, *so also the differential equation* $(\mathrm{D2})_2$ *is oscillatory at* β.

Proof. It will be convenient to separate the proof into the following three cases:

(i) Conditions (a), (b) and $\displaystyle\int^\beta \frac{du}{r(u)} = \infty$ are fulfilled

(ii) Conditions (a), (c) and $\displaystyle\int^\beta \frac{du}{r(u)} = \infty$ are fulfilled

(iii) Condition $\displaystyle\int^\beta \frac{du}{r(u)} < \infty$ is fulfilled.

Case (i): Let the differential equation $(\mathrm{D2})_1$ be oscillatory at β. Suppose on the contrary, that the differential equation $(\mathrm{D2})_2$ is not oscillatory at β. Then there is a solution x of the differential equation $(\mathrm{D2})_2$ which is non-oscillatory at β. Without loss of generality, we may assume that

$$x(t) > 0 \qquad \textit{for all} \quad t \in I_\beta. \tag{1}$$

In this case we show all assumptions of a corollary of Tychonov's theorem [2: p. 405] are satisfied. Setting

$$z(t) = \frac{r(t)w(x(t))x'(t)}{f(x(t))} \qquad (t \in I_\beta) \tag{2}$$

we obtain from $(\mathrm{D2})_2$

$$z'(t) = -\frac{f'(x(t))z^2(t)}{r(t)w(x(t))} - q(t) \qquad (t \in I_\beta).$$

Then we have

$$z(t) = z(T) + \int_t^T q(s)ds + \int_t^T \frac{f'(x(s))z^2(s)}{r(s)w(x(s))}ds \qquad (t_0 \le t \le T < \beta) \tag{3}$$

where, for $T \to \beta$,

$$\int_t^T q(s)\,ds \longrightarrow \overline{Q}(t) \qquad \text{and} \qquad \int_t^T \frac{f'(x(s))z^2(s)}{r(s)w(x(s))}ds \longrightarrow k$$

with $0 < k \le \infty$. Hence we have

$$\lim_{T\to\beta} z(T) = b \qquad \text{where} \quad -\infty \le b < \infty.$$

Now we show that $0 \le b < \infty$. Suppose that $-\infty \le b < 0$. Then choosing T sufficiently large, say $T \ge \overline{T}$, we have $r(T)x'(T) < 0$. If there exists an $\varepsilon > 0$ such that $r(T)x'(T) \le -\varepsilon$ for all $T \ge \overline{T}_1$, then we obtain

$$x(T) - x(\overline{T}_1) \le -\varepsilon \int_{\overline{T}_1}^T \frac{ds}{r(s)}.$$

The right-hand side tends to $-\infty$ if $T \to \beta$. This contradicts $x(t) > 0$ $(t \in I_\beta)$. Thus we have $\limsup_{T\to\beta} r(T)x'(T) = 0$.

Next we choose a sequence $(T_n)_{n\ge1} \subseteq I_\beta$ such that, for sufficiently large n and for all $T \in [\overline{T}, T_n)$, we have

$$-\frac{1}{n} = r(T_n)w(x(T_n))x'(T_n) > r(T)w(x(T))x'(T)$$

and therefore $\lim_{n\to\infty} T_n = \beta$. Let n be sufficiently large. Integrating the differential equation $(D2)_2$ from T to T_n, we have

$$0 > \int_T^{T_n} q(s)f(x(s))\,ds.$$

Integration by parts yields

$$V(T) < \int_T^{T_n} H(s)V(s)ds \qquad (\overline{T} \le T < T_n)$$

where

$$V(T) = f(x(T)) \int_T^{T_n} q(u)du \qquad \text{and} \qquad H(T) = -\frac{x'(T)f'(x(T))}{f(x(T))}.$$

From (1), condition (a) and $x'(T) < 0$ we get $H(t) > 0$ $(t \in [\overline{T}, T_n))$. Setting

$$W(T) = \int_T^{T_n} H(s)V(s)ds \qquad (T \in [\overline{T}, T_n))$$

we obtain $W'(T) = -H(T)V(T) > -H(T)W(T)$. Then we have

$$\frac{d}{dT}\left(W(T)exp\left(\int_{\overline{T}}^{T} H(s)\,ds\right)\right) > 0 \qquad (T \in [\overline{T}, T_n))$$

which implies that

$$W(T)exp\left(\int_{\overline{T}}^{T} H(s)\,ds\right)$$

is strongly increasing on $[\overline{T}, T_n)$. Considering $W(T_n) = 0$ we obtain that the function W, and hence the function V, is negative for every $t \in [\overline{T}, T_n)$, and, consequently, it is easy to see that

$$\int_{\overline{T}}^{\beta} q(s)\,ds < 0,$$

contradicting the non-negativity of $\overline{Q}(t)$ for all $t \in I_\beta$. Now we have

$$0 \leq b = \lim_{T \to \beta} z(T) < \infty. \tag{4}$$

Letting $T \to \beta$ it follows from (3) that

$$z(t) = b + \overline{Q}(t) + \int_t^{\beta} \frac{f'(x(s))z^2(s)}{r(s)w(x(s))}\,ds \qquad (t \in I_\beta). \tag{5}$$

By (2) and the substitution $x(s) = u$ we have

$$\int_c^{x(t)} \frac{w(u)}{f(u)}\,du = \int_{t_0}^{t} \frac{z(s)}{r(s)}\,ds \qquad (t \in I_\beta) \tag{6}$$

where $c = x(t_0) > 0$. It follows from (1), (4) and (5) that $z(t) \geq 0$ for $t \in I_\beta$. Thus it follows from (2) that the function x is increasing on I_β. We define

$$\Omega(x) = \int_c^{x} \frac{w(u)}{f(u)}du \qquad (x \geq c). \tag{7}$$

The function Ω is strongly increasing on $D(\Omega) = [c, \infty)$. Let Γ be the inverse function of Ω which is also monotone increasing. $D(\Omega)$ is an interval, and therefore the function Γ is continuous. From (6) and (7) we get

$$x(t) = \Gamma(\Omega(x(t))) = \Gamma\left(\int_c^{x(t)} \frac{w(u)}{f(u)}\,du\right) = \Gamma\left(\int_{t_0}^{t} \frac{z(s)}{r(s)}\,ds\right) \qquad (t \in I_\beta)).$$

Using

$$G(z, r; t) = f'\left(\Gamma\left(\int_{t_0}^{t} \frac{z(\tau)}{r(\tau)}\,d\tau\right)\right) \Big/ w\left(\Gamma\left(\int_{t_0}^{t} \frac{z(\tau)}{r(\tau)}\,d\tau\right)\right)$$

it follows by (5) that

$$z(t) = b + \overline{Q}(t) + \int_t^{\beta} \frac{z^2(s)}{r(s)} G(z, r; s)\,ds \qquad (t \in I_\beta).$$

The right-hand side defines a map M by

$$(Mu)(t) = b + \overline{Q}(t) + \int_t^{\beta} \frac{u^2(s)}{r(s)} G(u, r; s)\,ds \qquad (t \in I_\beta). \tag{8}$$

Let the domain of this map M be the set C_z of all continuous functions u on I_β with the restriction $0 \le u(t) \le z(t)$ $(t \in I_\beta)$. Then M is a map of C_z into itself. Since Γ and $f'w^{-1}$ (condition (b)) are increasing it is easy to see that $0 \le (Mu)(t) \le z(t) = (Mz)(t)$ for every $u \in C_z$.

Now we consider the map L defined by

$$(Lm)(t) = b + \overline{P}(t) + \int_t^{\beta} \frac{m^2(s)}{R(s)} G(m, R; s)\,ds \qquad (t \in I_\beta). \tag{9}$$

Let the domain of L be the set D_z of all continuous functions u on I_β with $\overline{P}(t) \le u(t) \le z(t)$ $(t \in I_\beta)$. Then L is a map of D_z into itself.

In the following we need the Fréchet space $C_\rho(I_\beta)$, i.e. the linear, locally convex, compact space $C_\rho(I_\beta)$ of continuous real-valued functions on I_β. The corresponding topology ρ is defined by the metric

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{p_i(x-y)}{1 + p_i(x-y)}\right) \qquad (x, y \in C_\rho(I_\beta)). \tag{10}$$

Here $\{p_i\}_{i \ge 1}$ is a family of seminorms with

$$p_i(x - y) = \sup_{t \in [t_0, t_i]} |x(t) - y(t)|$$

where

$$t_i = \begin{cases} \beta - \dfrac{\beta - t_0}{i+1} & \text{for } \beta < \infty \\ t_0 + i & \text{for } \beta = \infty. \end{cases} \tag{11}$$

From (10) and (11) it follows that D_z is a closed, convex subset of the Fréchet space $C_\rho(I_\beta)$. In the following we show that the functions belonging to $L(D_z)$ are uniformly bounded and equicontinuous on compact subintervals of I_β. Let $T \in (t_0, \beta)$ and $m \in D_z$. There exists a constant $k \in \mathbb{R}$ with $|z(t)| \le k$ for all $t \in [t_0, T]$, and therefore we have $|(Lm)(t)| \le k$ for all $t \in [t_0, T]$ and all $m \in D_z$. Thus there follows the uniform boundedness of $L(D_z)$ on compact subintervals of I_β. From $m(t) \le z(t)$ $(t \in I_\beta)$, (8) and (9) it follows for $s, t \in [t_0, T]$ that, for all $m \in D_z$,

$$\Big|(Lm)(t) - (Lm)(s)\Big| \le \int_s^t |p(u)|\,du + \left|\int_s^t q(u)\,du\right| + |z(s) - z(t)|. \tag{12}$$

Let $\varepsilon > 0$ and $|t - s| < \delta(\varepsilon)$. By virtue of the estimations

$$\int_s^t |p(u)|\,du \le \frac{\varepsilon}{3}, \qquad \left|\int_s^t q(u)\,du\right| \le \frac{\varepsilon}{3}, \qquad |z(t) - z(s)| \le \frac{\varepsilon}{3}$$

for $|t - s| < \delta(\varepsilon)$ and (12) we obtain that the functions in $L(D_z)$ are equicontinuous on compact subintervals of I_β. By the Arzela-Ascoli theorem, it follows that $L(D_z)$ is pre-compact on closed subintervals of I_β. Now we show that the map L is continuous on D_z. Using

$$\Phi(m; s) = \frac{m^2(s)}{R(s)} G(m, R; s) \qquad (s \in I_\beta)$$

it follows by (9) that

$$(Lm)(t) = b + \overline{P}(t) + \int_t^\beta \Phi(m; s)\,ds.$$

Let the sequence $(m_n)_{n \ge 1} \subseteq D_z$ be such that $m_n \to m$ $(m \in D_z)$, uniformly on compact subintervals of I_β. Let $\varepsilon > 0$ be given. There exists a $T \in (t_1, \beta)$ such that

$$\int_T^\beta \frac{z^2(s)}{r(s)} G(z, r; s)\,ds < \frac{\varepsilon}{3}.$$

On the other hand, there exists an $n_0(\varepsilon)$ such that

$$max\Big|\Phi(m_n; t) - \Phi(m; t)\Big| \le \frac{\varepsilon}{3|T - t_0|} \qquad (n \ge n_0).$$

Then for all $t \in [t_0, T]$ and $n \geq n_0$ we obtain

$$\begin{aligned}
&\Big|(Lm_n)(t) - (Lm)(t)\Big| \\
&\quad = \left|\int_t^\beta \Big(\Phi(m_n; s) - \Phi(m; s)\Big)\, ds\right| \\
&\quad \leq \left|\int_t^T \Big(\Phi(m_n; s) - \Phi(m; s)\Big)\, ds\right| + \int_T^\beta \Big(|\Phi(m_n; s)| + |\Phi(m; s)|\Big)\, ds \\
&\quad \leq \left|\int_t^T \Big(\Phi(m_n; s) - \Phi(m; s)\Big)\, ds\right| + 2\int_T^\beta \frac{z^2(s)}{r(s)} G(z, r; s)\, ds \\
&\quad < \varepsilon.
\end{aligned}$$

Thus $Lm_n \to Lm$ as $n \to \infty$, uniformly on compact subintervals of I_β.

Finally we need that $L(D_z)$ is pre-compact on I_β. Here we show that every sequence $(L(m_j))_{j\geq 1} \subseteq L(D_z)$ has a subsequence being a Cauchy sequence. Choosing $\varepsilon_1 > 0$ we set $\mu_1 > 0$ such that

$$\frac{\mu_1}{1 + \mu_1} \leq \frac{\varepsilon_1}{2}.$$

There exists a number $\overline{T}_1 \in I_\beta$ such that, for all $t \in [\overline{T}_1, \beta)$,

$$\Big|(Lm_n)(t) - (Lm_k)(t)\Big| \leq \int_t^\beta \Big(|\Phi(m_n; s)| + |\Phi(m_k; s)|\Big)\, ds \leq 2\int_t^\beta \Phi(z; s)\, ds < \mu_1$$

independently of n and k. Therefore we obtain the inequality

$$\sup_{t\in[\overline{T}_1, \bar{t}_i]} \Big|(Lm_n)(t) - (Lm_k)(t)\Big| \leq \mu_1$$

where

$$\bar{t}_i = \begin{cases} \beta - \dfrac{\beta - t_0}{i + j_1} & \text{for } \beta < \infty \\[2mm] t_0 + i + j_1 - 1 & \text{for } \beta = \infty \text{ and } j_1 = \min\{i : \ t_i > \overline{T}_1\}. \end{cases}$$

Thus we have for all n, k

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup_{t\in[\overline{T}_1, \bar{t}_i]} \Big|(Lm_n)(t) - (Lm_k)(t)\Big|}{1 + \sup_{t\in[\overline{T}_1, \bar{t}_i]} \Big|(Lm_n)(t) - (Lm_k)(t)\Big|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mu_1}{1 + \mu_1} \leq \frac{\varepsilon_1}{2}.$$

Since $L(D_z)$ is pre-compact on the interval $[t_0, \overline{T}_1]$, the sequence $(Lm_j)_{j\geq 1}$ has a convergent subsequence $(Lm_j^{(1)})_{j\geq 1}$ on $[t_0, \overline{T}_1]$ being a Cauchy sequence. It follows that

$$\sup_{t\in[t_0, \overline{T}_1]} \Big|(Lm_n^{(1)})(t) - (Lm_k^{(1)})(t)\Big| \leq \frac{\varepsilon_1}{2}$$

holds if $n, k \geq n_1$. We obtain

$$\rho\left(Lm_n^{(1)}, Lm_k^{(1)}\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup\limits_{t\in[t_0,\bar{t}_i]} \left|(Lm_n^{(1)})(t) - (Lm_k^{(1)})(t)\right|}{1 + \sup\limits_{t\in[t_0,\bar{t}_i]} \left|(Lm_n^{(1)}(t) - (Lm_k^{(1)})(t)\right|}$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup\limits_{t\in[t_0,\overline{T}_1]} \left|(Lm_n^{(1)})(t) - (Lm_k^{(1)})(t)\right|}{1 + \sup\limits_{t\in[t_0,\overline{T}_1]} \left|(Lm_n^{(1)}(t) - (Lm_k^{(1)})(t)\right|}$$

$$+ \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup\limits_{t\in[\overline{T}_1,\bar{t}_i]} \left|(Lm_n^{(1)})(t) - (Lm_k^{(1)})(t)\right|}{1 + \sup\limits_{t\in[\overline{T}_1,\bar{t}_i]} \left|(Lm_n^{(1)})(t) - (Lm_k^{(1)})(t)\right|}$$

$$\leq \varepsilon_1$$

for all $n, k \geq n_1$. This process can be repeated infinitely often setting $\varepsilon_m = \varepsilon_{m-1}/2$ and using successively μ_m, T_m, j_m and n_m $(n = 2, 3, \ldots)$. Then the diagonal sequence

$$\left(L(m_j^{(j)})\right)_{j\geq 1} \subseteq (L(m_j))_{j\geq 1}$$

is a Cauchy sequence. Since

$$\left(L(m_j^{(j)})\right)_{j\geq 1} \subseteq C_\rho(I_\beta)$$

this Cauchy sequence is convergent. Therefore $L(D_z)$ has a compact closure on I_β. By the corollary of Tychnov's Theorem (see [2: p. 405]), L has a fixed point $\overline{n} \in D_z$, $L\overline{n} = \overline{n}$. Then

$$x(t) = \Gamma\left(\int_{t_0}^{t} \frac{\overline{n}(s)}{R(s)}\, ds\right) > 0 \qquad (t \in I_\beta)$$

is a non-oscillatory solution of the differential Equation $(D2)_1$ which contradicts the suppusition.

Case (ii): Let the differential equation $(D2)_1$ be oscillatory at β and x a solution of the differential equation $(D2)_2$. By using

$$s = \int_{t_0}^{t} \frac{d\tau}{r(\tau)w(x(\tau))}, \qquad y(s) = x(t(s)) \qquad (t \in I_\beta) \tag{13}$$

we transform the differential equation $(D2)_2$ into the equation

$$\frac{d^2y}{ds^2} + r(t(s))q(t(s))w(y)f(y) = 0 \qquad (s \in [0, \infty)).$$

We set

$$a(s) = r(t(s))q(t(s)) \qquad \text{and} \qquad F(y(s)) = w(y(s))f(y(s))$$

and obtain the differential equation

$$\frac{d^2y(s)}{ds^2} + a(s)F(y(s)) = 0 \qquad (s \in [0,\infty)). \tag{14}$$

By (13) we get

$$A(s) = \int_s^{\infty} a(\tau)\,d\tau \geq 0 \qquad (s \in [0,\infty)).$$

Hence we have

$$\begin{gathered}\int_T^{\infty} A(s)\,ds \geq \frac{1}{S^2}\int_{t(T)}^{\beta} \frac{\overline{Q}(u)}{r(u)}\,du \\ \int_T^{\infty}\left(\int_s^{\infty} A^2(\tau)\,d\tau\right) ds \geq \frac{1}{S^4}\int_{t(T)}^{\beta} \frac{1}{r(u)}\left(\int_u^{\beta} \frac{\overline{Q}^2(v)}{r(v)}\,dv\right) du.\end{gathered} \tag{15}$$

Using

$$s = \int_{t_0}^{t} \frac{d\tau}{R(\tau)w(x(\tau))} \qquad \text{and} \qquad y(s) = x(t(s)) \qquad (t \in I_\beta)$$

we transform the differential equation $(D2)_1$ into the equation

$$\frac{d^2y(s)}{ds^2} + \overline{a}(s)F(y(s)) = 0 \qquad (s \in [0,\infty)) \tag{14$'$}$$

where $\overline{a}(s) = R(t(s))p(t(s))$. The oscillation properties of the differential equations $(D2)_1$ and $(D2)_2$ are invariant under these transformations. Analogously, we have the inequalities

$$\begin{gathered}|\overline{A}(s)| = \left|\int_s^{\infty} \overline{a}(\tau)\,d\tau\right| \leq \frac{1}{s_0}\overline{P}(t(s)) \\ \int_T^{\infty} |\overline{A}(s)|\,ds \leq \frac{1}{s_0^2}\int_{t(T)}^{\beta} \frac{\overline{P}(u)}{R(u)}\,du \\ \int_T^{\infty}\left(\int_s^{\infty}(\overline{A}(\tau))^2 d\tau\right) ds \leq \frac{1}{s_0^4}\int_{t(T)}^{\beta} \frac{1}{R(u)}\left(\int_u^{\beta} \frac{(\overline{P}(v))^2}{R(v)}\,dv\right) du.\end{gathered} \tag{15$'$}$$

Concerning the differential equation (14)′ it follows from Theorem 5 that

$$\int^{\infty}\left(|\overline{A}(s)|+\int_{s}^{\infty}(\overline{A}(u))^2du\right)ds=\infty.$$

On the other hand, by relations (15)′ and, without loss of generality, $s_0\le 1$ and $S\ge 1$, we have

$$\infty=\int_{T}^{\infty}\left(|\overline{A}(s)|+\int_{s}^{\infty}(\overline{A}(u))^2du\right)ds\le\frac{1}{s_0^4}\int_{t(T)}^{\beta}\frac{1}{R(u)}\left(\overline{P}(u)+\int_{u}^{\beta}\frac{(\overline{P}(v))^2}{R(v)}\,dv\right)du.$$

Correspondingly, in view of inequalities $0<r(t)\le R(t)$ and $|P(t)|\le Q(t)$ for all $t\in I_\beta$ we obtain

$$\frac{1}{S^4}\int_{t(T)}^{\beta}\frac{1}{r(u)}\left(S^2\overline{Q}(u)+\int_{u}^{\beta}\frac{\overline{Q}^2(v)}{r(v)}\,dv\right)du=\infty. \tag{16}$$

Hence, by (15) and (16) we obtain

$$\int_{T}^{\infty}\left(A(s)+\int_{s}^{\infty}A^2(u)\,du\right)ds=\infty.$$

Thus, in view of Theorem 4, the differential equation (14) is oscillatory at β and hence also the differential equation $(D2)_2$ is oscillatory at β.

Case (iii): The inequalities $0<r(t)\le R(t)$ $(t\in I_\beta)$ and condition (iii) imply the existence of the integral $\int^{\beta}\frac{du}{R(u)}$. All suppositions of Theorem 6 are fulfilled for the differential equations $(D2)_1$ and $(D2)_2$. Both equations are non-oscillatory. Thus the proof of Theorem 7 is complete. ∎

Under the supposition $p(t)\ge 0$ for all $t\in[t_0,\infty)$, Theorem 7 generalizes a result of Butler [1] (Theorem 3 in that paper).

3. A corollary

Consider the nonlinear differential equations

$$x''(t)+A(t)x'(t)+B(t)f(x(t))=0 \qquad (t\in I_\beta) \tag{$D3)_1$}$$
$$x''(t)+a(t)x'(t)+b(t)f(x(t))=0 \qquad (t\in I_\beta). \tag{$D3)_2$}$$

Corollary. *Let $a,A,b,B\in C(I_\beta)$ be functions such that*

$$a(t)\le A(t) \qquad \textit{and} \qquad |B(t)|\le b(t)\exp\left(\int_{t_0}^{t}(a(s)-A(s))\,ds\right) \qquad (t\in I_\beta).$$

Suppose that there exist

$$P^*(t) = \int_t^\beta |B(s)| \exp\left(\int_{t_0}^s A(u)\,du\right) ds$$
$$Q^*(t) = \int_t^\beta b(s) \exp\left(\int_{t_0}^s a(u)\,du\right) ds \qquad (t \in I_\beta).$$

Assume further that f is a function on $\mathbb{R}$ *satisfying the condition* (a) *and either the condition* (b) *or* (c) *of Theorem 7, where* $w = 1$ *is supposed.*

Then if the differential equation $(\mathrm{D3})_1$ *is oscillatory at* β, *so also the differential equation* $(\mathrm{D3})_2$ *is oscillatory at* β.

Proof. By setting

$$r(t) = \exp\left(\int_{t_0}^t a(s)\,ds\right) \quad \text{and} \quad R(t) = \exp\left(\int_{t_0}^t A(s)\,ds\right) \qquad (t \in I_\beta),$$

the corollary immediately follows from Theorem 7. ∎

4. An example

Let $\beta < \infty$ and $t_0 > 0$. In view of Theorem 2 in [4] we can see that the differential equation

$$\left((\beta - t)^5\left(\frac{a}{e^{x(t)} + e^{-x(t)}} + b\right)x'(t)\right)' + \frac{1}{t}\left(|x(t)|^\alpha \operatorname{sign} x(t) + cx(t)\right) = 0$$

for $t \in I_\beta$ with

$$c > 0, \qquad 0 < a \le b, \qquad 1 < \alpha = \frac{2\beta + 1}{2\gamma + 1} \quad (\beta, \gamma \in \mathbb{N})$$

is oscillatory at β. Hence, by Theorem 7 it follows that the differential equation

$$\left(r(t)\left(\frac{a}{e^{x(t)} + e^{-x(t)}} + b\right)x'(t)\right)' + q(t)\left(|x(t)|^\alpha \operatorname{sign} x(t) + cx(t)\right) = 0$$

is oscillatory at β, if the functions r and q satisfy the conditions

$$r, q \in C(I_\beta), \qquad 0 < r(t) \le (\beta - t)^5, \qquad \ln\frac{\beta}{t} \le \int_t^\beta q(s)\,ds < \infty \qquad (t \in I_\beta).$$

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