# **On a Comparison Theorem for Second Order Nonlinear Ordinary Differential Equations**

#### **Th. Rudek**

**-Abstract.** We present a comparison theorem for second order nonlinear differential equations of the form

 $(R(t)w(x(t))x'(t))' + p(t)f(x(t)) = 0$   $(t \in [t_0, \beta), \beta \leq \infty)$ 

where p is a continuous function on  $[t_0,\beta)$  without any restriction on its sign.

**Keywords:** *Second order nonlinear differential equations, oscillation theory*  **AMS subject classification: 34 C 10** 

#### **0. Introduction**

Consider the second order differential equation

$$
(R(t)x'(t))' + p(t)x(t) = 0, \qquad (t \in [t_0, \infty))
$$
 (D1)<sub>1</sub>

 $p(x(t))x'(t)'+p(t)f(x(t)) = 0$   $(t \in [t_0, \beta), \beta \leq \infty)$ <br>
function on  $[t_0, \beta)$  without any restriction on its sign.<br> *der nonlinear differential equations, oscillation theory*<br>
cation: 34 C 10<br> *R(t)x'(t)'* + *p(t)x(t)* = 0,  $(t \in [t_$ with given functions p and R on  $[t_0, \infty)$ . A function defined on an interval  $[t_0, \beta)$ ,  $\beta \leq$  $+\infty$ , is said to be *oscillatory* at  $\beta$  if for every  $a \in (t_0, \beta)$  it has an infinite number of zeros on the interval  $(a, \beta)$ , and otherwise it is said to be *non-oscillatory* at  $\beta$ . A differential equation of the form  $(D1)_1$  is called oscillatory at  $\beta$  if all its solutions are oscillatory at  $\beta$ , and otherwise such an equation is called non-oscillatory at  $\beta$ . In the following we set  $I_{\infty} = [t_0, \infty)$  and  $I_{\beta} = [t_0, \beta)$ . ( $R(t)x'(t)$ )' +  $p(t)x(t) = 0$ ,<br>
(with given functions p and R on  $[t_0, \infty)$ . A function c<br>
+ $\infty$ , is said to be oscillatory at  $\beta$  if for every  $a \in \{0\}$ <br>
of zeros on the interval  $(a, \beta)$ , and otherwise it is said<br>
differen *nd R* on  $[t_0, \infty)$ . A function defined on an interva<br>*latory* at  $\beta$  if for every  $a \in (t_0, \beta)$  it has an infi<br> $(a, \beta)$ , and otherwise it is said to be *non-oscillat*<br>he form  $(D1)_1$  is called oscillatory at  $\beta$  if all

A fundamental problem concerning the oscillation theory of second order linear or nonlinear ordinary differential equations may be posed as follows.. Suppose *R* and p are functions on  $I_{\infty}$  which make the differential equation  $(D1)_1$  oscillatory at  $\infty$ . Are there relations between the functions  $R$  and  $r$  as well as functions  $\dot{p}$  and  $q$  which ensure that the differential equation

$$
(r(t)x'(t))' + q(t)x(t) = 0 \qquad (t \in I_{\infty})
$$
 (D1)<sub>2</sub>

is also oscillatory at  $\infty$  ? A well-known relation for this linear case is the classical Sturm comparison theorem  $[6: p. 2]$ . Hille  $[3]$  and Wintner  $[8]$  extended Sturm's result in the following way.

Th. Rudek: Pädag. Hochschule, Inst. Math. und Inform., PF 307, D - 99006 Erfurt

ISSN 0232-2064 */* \$ 2.50 **©** Heldermann Verlag Berlin

**Theorem 1** (Hille-Wintner comparison theorem). *Suppose*  $R \equiv 1$  and  $r \equiv 1$  *in equations* (D1)<sub>1</sub> and (D1)<sub>2</sub>, *respectively. Let*  $p, q \in C(I_\infty)$  *be functions such that*  $\frac{1}{2}$  $P(t) = \int_t^{\infty} p(s) ds$  and  $Q(t) = \int_t^{\infty} q(s) ds$  exist with  $0 \le P(t) \le Q(t)$ , for all  $t \in I_{\infty}$ . Then if the differential equation  $(D1)_1$  is oscillatory at  $\infty$ , then also the differential *equation*  $(D2)_2$  *is oscillatory at*  $\infty$ .

Taam [7] proved the following generalization of the Hille-Wintner theorem.

**Theorem 2.** Let  $p, q, r, R \in C(I_\infty)$  be functions such that r is bounded from above **Theorem 2.** Let  $p, q, r, R \in C(I_{\infty})$  be functions such that  $r$  is bounded from above<br>on  $I_{\infty}$ ,  $P(t) = \int_{t}^{\infty} p(s) ds$  and  $Q(t) = \int_{t}^{\infty} q(s) ds$  exist,  $|P(t)| \leq Q(t)$  and  $0 < r(t) \leq$ <br> $P(t)$  for all  $t \in I$ . Then if the differen  $R(t)$ , for all  $t \in I_{\infty}$ . Then if the differential equation  $(D1)_1$  is oscillatory at  $\infty$ , so also *the differential equation*  $(D2)_2$  *is oscillatory at*  $\infty$ . *d* the following generalization of the Hille-Wintner theorem<br> *Let*  $p, q, r, R \in C(I_{\infty})$  *be functions such that*  $r$  *is bounded fri*<br>  $p'(s) ds$  and  $Q(t) = \int_{t}^{\infty} q(s) ds$  exist,  $|P(t)| \leq Q(t)$  and 0<br> *n.* Then if the different *(r)*  $f(x) = \int_{0}^{\infty} f(x) dx$  (r)  $f(x) = \int_{0}^{\infty} f(x) dx$  (r)  $f(x) = \int_{0}^{\infty} f(x) dx$  and  $Q(t) = \int_{0}^{\infty} f(x) dx$  and  $Q(t) = \int_{0}^{\infty} f(x) dx$ . Then if the differential equation (D1)<sub>1</sub> is oscillatory at  $\infty$ , so also ition (D2)<sub>2</sub> is o

These and other Sturm-type comparison theorems hold for very general second order linear and nonlinear differential equations. Butler [1] obtained such a nonlinear extension of Theorem 2 for a certain class of equations.

We consider the differential equations

$$
(R(t)w(x(t))x'(t))' + p(t)f(x(t)) = 0 \t (t \in I_{\beta})
$$
 (D2)<sub>1</sub>

$$
(r(t)w(x(t))x'(t))' + q(t)f(x(t)) = 0 \qquad (t \in I_\beta).
$$
 (D2)<sub>2</sub>

**Theorem 3** (see Butler [1]). *Suppose*  $w \equiv 1$  and  $\beta = \infty$  in equations  $(D2)_1$  and (D2)<sub>2</sub>. Let  $p, q, r, R \in C(I_{\infty})$  be functions such that  $P(t) = \int_{t}^{\infty} p(s) ds$  and  $Q(t) =$  $\int_t^{\infty} q(s) ds$  exist,  $|P(t)| \leq Q(t)$  and  $0 < r(t) \leq R(t)$ , for all  $t \in I_{\infty}$ . Assume that f is a *function on JR satisfying the following conditions:*  rem 3 (see Butler [1]). Supp<br>
et p,q,r,R  $\in C(I_{\infty})$  be funct<br>
s exist,  $|P(t)| \leq Q(t)$  and  $0 <$ <br>
on  $\mathbb R$  satisfying the following<br>
f  $\in C'( \mathbb R),$  uf(u) > 0 and f'(<br>
r<br>
f' is non-increasing on  $(-\infty, 0]$ <br>  $\liminf_{u|\to+\infty} f'(u) >$ 

(a<sub>0</sub>) 
$$
f \in C'(R)
$$
,  $uf(u) > 0$  and  $f'(u) > 0$ , for all  $u \neq 0$ 

*and either* 

 $(b_0)$  *f'* is non-increasing on  $(-\infty, 0]$  and non-decreasing on  $(0, \infty)$ *or*

$$
(c_0)\liminf_{|u|\to+\infty}f'(u)>0\ \text{and}\ \int_{\pm 1}^{\pm \infty}\frac{du}{f(u)}\,du<\infty.
$$

*Then if the differential equation*  $(D2)$  *is oscillatory at*  $\beta$ *, so also the differential equation*  $(D2)_2$  *is oscillatory at*  $\beta$ *.* 

Butler also showed that Theorem 2 holds without the restriction that *r is* bounded.

## **1. Preliminaries**

To obtain the main result of the paper we need the following well-known three theorems.

**Theorem 4** (see Rudek [5: Theorem 1]). *Consider the nonlinear differential equation (Retail equation (D2)*1 *is oscillatory at*  $\beta$ *, so also the differential equatory at*  $\beta$ *.*<br>
showed that Theorem 2 holds without the restriction that r is boundaries<br>
ain result of the paper we need the following well-k

$$
(R(t)x'(t))' + a(t)g(x'(t)) + p(t)f(x(t)) = 0 \qquad (t \in I_{\infty})
$$
 (D3)

where  $a, p, R$  are functions on  $I_{\infty}$  satisfying the following conditions:

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\n(V1) 
$$
a, p, R \in C(I_{\infty}), a(t) \ge 0
$$
 and  $R(t) > 0$  for all  $t \in I_{\infty}, R$  decreasing on  $I_{\infty}$ ,  
\n
$$
P(t) = \int_{t}^{\infty} p(s) ds \text{ existing for all } t \in I_{\infty} \text{ and } \liminf_{T \to +\infty} \int_{t}^{T} \frac{P(s)}{R(s)} ds > -\infty.
$$
\n(V2)  $f \in C'(R)$  at  $f(x) > 0$  and  $f'(x) > 0$  for all  $x \ne 0$  and  $f(0) > 0$  for all  $x \ne 0$ .

(V2) 
$$
f \in C'(R)
$$
,  $uf(u) > 0$  and  $f'(u) > 0$ , for all  $u \neq 0$ , and  $0 \leq g \in C(I_{\infty})$ .

$$
\begin{aligned}\n\text{(V2)} \quad & \int \in C \text{ (H)}, \, u_f(u) > 0 \text{ and } f'(u) > 0, \\
\text{(V3)} \quad & \liminf_{|u| \to +\infty} f'(u) > 0 \text{ and } \int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < \infty. \\
\text{In the differential equation (D3) is oscillation} \\
\text{(V4)} \quad & \int_{t}^{\infty} \frac{1}{R(s)} \left( P(s) + \int_{s}^{\infty} \frac{(P_+(u))^2}{R(u)} du \right) ds\n\end{aligned}
$$

(V3) 
$$
\liminf_{|u| \to +\infty} f'(u) > 0
$$
 and  $\int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < \infty$ .  
\nThen the differential equation (D3) is oscillatory at  $\infty$  if the condition  
\n(V4)  $\int_{t}^{\infty} \frac{1}{R(s)} \left( P(s) + \int_{s}^{\infty} \frac{(P_+(u))^2}{R(u)} du \right) ds = \infty$ 

**is** *satisfied.* 

**Theorem** 5 (see Butler [1: Lemma 2.3]). *Consider the differential equation* 

$$
x''(s) + p(s)F(x(s)) = 0 \t (s \in [0, \infty)). \t (D4)
$$

*x* (D3) is oscillatory at  $\infty$  if the condition<br>  $s$ ) +  $\int_s^{\infty} \frac{(P_+(u))^2}{R(u)} du$   $ds = \infty$ <br>  $B$ <br>  $B$ <br>  $B$ <br>  $x''(s) + p(s)F(x(s)) = 0$  (s  $\in [0, \infty)$ ). (D4)<br>  $x$  ists on  $I_{\infty}$ ,  $F$  be continuously differentiable with  $uF(u) > 0$  *Let*  $P(t) = \int_{t}^{\infty} p(s)ds$  exists on  $I_{\infty}$ , F be continuously differentiable with  $uF(u) > 0$  and (V4)  $\int_{t}^{\infty} \frac{1}{R(s)} \left( P(s) + \int_{s}^{\infty} \frac{(P_+(u))^2}{R(u)} du \right) ds = \infty$ <br>
is satisfied.<br>
Theorem 5 (see Butler [1: Lemma 2.3]). Consider the differential equat<br>  $x''(s) + p(s)F(x(s)) = 0$  (s  $\in [0, \infty)$ ).<br>
Let  $P(t) = \int_{t}^{\infty} p(s)ds$  exists 00 *du*   $(x)$ ).<br>  $entiable with uF(u) > 0$ <br>  $F'(x) > 0$  and  $\int_{\pm 1}^{\pm \infty} \frac{dx}{F(x)}$ r;y *<*  00. *Then the relation*

$$
\int\limits_{0}^{\infty}\left(|P(s)|+\int\limits_{s}^{\infty}P^{2}(u)\,du\right)\,ds=\infty
$$

*is a necessary condition for the differential equation (D4) to be oscillatory at* $\infty$ .

**Theorem** 6 (see Rudek [5: Theorem 2]). *Consider the differential equation* (D2)1. **Let p, R**  $\in C(I_\beta)$ ,  $R(t) > 0$  ( $t \in I_\beta$ ), and assume that  $\int_{\beta}^{\beta} \frac{du}{R(u)}$  and  $\overline{P}(t) = \int_{t}^{\beta} |p(s)| ds$ *exist on Ig. Let further the following conditions be satisfied:* 

The functions *f* and *w* are continuous, the product  $(fw)(u) = f(u)w(u)$   $(u \in \mathbb{R})$ *is continuously differentiable,*  $uf(u) > 0$  *and*  $(w(u)f(u))' > 0$  *for all*  $u \neq 0$ *. Let there exist*  $s_0, S \in \mathbb{R}$  such that  $0 < s_0 \leq w(u) \leq S$  holds for all  $u \in \mathbb{R}$ .

Then the differential equation  $(D2)_1$  is non-oscillatory at  $\beta$ .

### **2. Main result**

Let  $\beta < \infty$  or  $\beta = \infty$ .

*or*

**Main result**<br>  $\beta < \infty$  or  $\beta = \infty$ .<br> **Theorem 7.** Let  $p, q, r, R \in C(I_{\beta})$  be such that there exist  $\overline{Q}(t) = \int_{t}^{\beta} q(s) ds$  and<br>  $\beta = \int_{t}^{\beta} \ln(s) ds$   $\overline{P}(t) < \overline{O}(t)$  and  $0 < r(t) < P(t)$  for all  $t \in I_{\beta}$ . Further let finit  $\overline{P}(t) = \int_t^\beta |p(s)| ds$ ,  $\overline{P}(t) \leq \overline{Q}(t)$  and  $0 < r(t) \leq R(t)$ , for all  $t \in I_\beta$ . Further let  $f, w$  be functions on  $\boldsymbol{R}$  satisfying the following conditions: Th. Rudek<br>
in result<br>  $\infty$  or  $\beta = \infty$ .<br>
orem 7. Let  $p, q, r, R \in C(I_{\beta})$  be such<br>  $\binom{\beta}{t} |p(s)| ds$ ,  $\overline{P}(t) \leq \overline{Q}(t)$  and  $0 < r(t) \leq$ <br>
s on  $R$  satisfying the following condition<br>  $f, w \in C^1(R), u f(u) > 0, f'(u) > 0$  and<br>  $v$  exi

(a) 
$$
f, w \in C^1(\mathbb{R}), u f(u) > 0, f'(u) > 0
$$
 and  $(w(u)f(u))' > 0$ , for all  $u \neq 0$ .

Let there exist  $s_0, S \in \mathbb{R}$  such that  $0 < s_0 \leq w(u) \leq S$  for all  $u \in \mathbb{R}$  and either

(b) 
$$
\frac{f'(u)}{w(u)}
$$
 be non-increasing on  $(-\infty, 0]$  and non-decreasing on  $[0, \infty)$ 

(c) 
$$
\liminf_{|u|\to+\infty} (w(u)f(u))' > 0
$$
 and  $\int_{\pm 1}^{\pm \infty} \frac{du}{f(u)} < \infty$ .

*Then if the differential equation*  $(D2)$  *is oscillatory at*  $\beta$ *, so also the differential equation*  $(D2)_2$  *is oscillatory at*  $\beta$ *.* 

**Proof.** It will be convenient to separate the proof into the following three cases:

(i) Conditions (a), (b) and 
$$
\int^{\beta} \frac{du}{r(u)} = \infty \text{ are fulfilled}
$$

**Proof.** It will be convenient to separate the proof into  
\n(i) Conditions (a), (b) and 
$$
\int^{\beta} \frac{du}{r(u)} = \infty
$$
 are fulfilled  
\n(ii) Conditions (a), (c) and  $\int^{\beta} \frac{du}{r(u)} = \infty$  are fulfilled

(iii) Condition  $\int \frac{du}{r(u)} < \infty$  is fulfilled.

**Case (i):** Let the differential equation  $(D2)$  be oscillatory at  $\beta$ . Suppose on the contrary, that the differential equation  $(D2)_2$  is not oscillatory at  $\beta$ . Then there is a solution x of the differential equation  $(D2)_2$  which is non-oscillatory at  $\beta$ . Without loss of generality, we may assume that *x x (i)*  $\frac{du}{r(u)} = \infty$  are fulfilled<br> *z*  $\infty$  is fulfilled.<br> *x* (*x*) *x (D2)<sub>2</sub>* is not oscillatory at *β*. Suppose on the equation (D2)<sub>2</sub> is not oscillatory at *β*. Then there is a quation (D2)<sub>2</sub> which is (iii) Condition  $\int^B \frac{du}{r(u)} <$ <br>Case (i): Let the different<br>contrary, that the differential equalition x of the differential equality, we may assume the<br>of generality, we may assume the<br>In this case we show all assum<br>are oscillatory at  $\beta$ . Suppose on the<br>oscillatory at  $\beta$ . Then there is a<br>ion-oscillatory at  $\beta$ . Without loss<br> $I_{\beta}$ . (1)<br>f Tychonov's theorem [2: p. 405]<br> $(t \in I_{\beta})$  (2)

$$
x(t) > 0 \qquad \text{for all} \quad t \in I_{\beta}.
$$
 (1)

In this case we show all assumptions of a corollary of Tychonov's theorem [2: p. 405]

time that  
\n
$$
x(t) > 0
$$
 for all  $t \in I_{\beta}$ . (1)  
\nassumptions of a corollary of Tychonov's theorem [2: p. 405]  
\n
$$
z(t) = \frac{r(t)w(x(t))x'(t)}{f(x(t))}
$$
  $(t \in I_{\beta})$  (2)

we obtain from  $(D2)_2$ 

$$
z(t) = \frac{r(t)w(x(t))x'(t)}{f(x(t))}
$$
  $(t \in I_\beta)$   

$$
z'(t) = -\frac{f'(x(t))z^2(t)}{r(t)w(x(t))} - q(t)
$$
  $(t \in I_\beta).$ 

Then we have

On a Comparison Theorem for Differential Equations  
\n
$$
z(t) = z(T) + \int_{t}^{T} q(s)ds + \int_{t}^{T} \frac{f'(x(s))z^{2}(s)}{r(s)w(x(s))}ds \qquad (t_{0} \leq t \leq T < \beta)
$$
\n(3)

where, for  $T \rightarrow \beta$ ,

where

\n
$$
f(t) = z(T) + \int_{t}^{T} q(s)ds + \int_{t}^{T} \frac{f'(x(s))z^{2}(s)}{r(s)w(x(s))}ds \qquad (t_{0} \leq t \leq T < \beta)
$$
\n
$$
T \to \beta,
$$
\n
$$
\int_{t}^{T} q(s)ds \longrightarrow \overline{Q}(t) \qquad \text{and} \qquad \int_{t}^{T} \frac{f'(x(s))z^{2}(s)}{r(s)w(x(s))}ds \longrightarrow k
$$
\n
$$
\leq \infty. \text{ Hence we have}
$$
\n
$$
\lim_{T \to \beta} z(T) = b \qquad \text{where} \quad -\infty \leq b < \infty.
$$

with  $0 < k \leq \infty$ . Hence we have

$$
\lim_{T \to \beta} z(T) = b \quad \text{where} \quad -\infty \leq b < \infty.
$$

Now we show that  $0 \leq b < \infty$ . Suppose that  $-\infty \leq b < 0$ . Then choosing T sufficiently large, say  $T \geq \overline{T}$ , we have  $r(T)x'(T) < 0$ . If there exists an  $\varepsilon > 0$  such that  $r(T)x'(T) \leq$ large, say  $T \geq \overline{T}$ , we hav<br>  $-\varepsilon$  for all  $T \geq \overline{T}_1$ , then<br>
The right-hand side ten<br>
we have  $\limsup_{T \to \beta} r(T$ <br>
Next we choose a se<br>
all  $T \in [\overline{T}, T_n)$ , we have<br>  $-\frac{1}{n}$  $-\varepsilon$  for all  $T \geq \overline{T}_1$ , then we obtain  $\sim 100$  km s  $^{-1}$ 

$$
(T) = b \qquad \text{where} \quad -\infty \leq b < \infty.
$$
\nSuppose that  $-\infty \leq b < 0$ . Then choosing  $T$ 

\n $\int x'(T) < 0$ . If there exists an  $\varepsilon > 0$  such that  $r(0)$ 

\nchain

\n
$$
x(T) - x(\overline{T}_1) \leq -\varepsilon \int_{\overline{T}_1}^T \frac{ds}{r(s)}.
$$

The right-hand side tends to  $-\infty$  if  $T \to \beta$ . This contradicts  $x(t) > 0$   $(t \in I_\beta)$ . Thus we have  $\limsup_{T\to\beta}r(T)x'(T) = 0$ .  $x(T) - x(\overline{T}_1) \le -\varepsilon \int_{\overline{T}_1} \frac{ds}{r(s)}$ .<br>
The right-hand side tends to  $-\infty$  if  $T \to \beta$ . This contradicts  $x(t) > 0$  ( $t \in I_\beta$ ). Thus<br>
we have  $\limsup_{T \to \beta} r(T)x'(T) = 0$ .<br>
Next we choose a sequence  $(T_n)_{n \ge 1} \subseteq I_\beta$  such that, f

Next we choose a sequence  $(T_n)_{n \geq 1} \subseteq I_\beta$  such that, for sufficiently large *n* and for all  $T \in [\overline{T}, T_n)$ , we have

$$
-\frac{1}{n} = r(T_n)w(x(T_n))x'(T_n) > r(T)w(x(T))x'(T)
$$

and therefore  $\lim_{n\to\infty} T_n = \beta$ . Let *n* be sufficiently large. Integrating the differential equation (D2)<sub>2</sub> from *T* to *T<sub>n</sub>*, we have

$$
0>\int\limits_{T}^{T}q(s)f(x(s))\,ds.
$$

Integration by parts yields

$$
0 > \int_{T} q(s) f(x(s)) ds.
$$
\n
$$
V(T) < \int_{T}^{T_n} H(s) V(s) ds \qquad (\overline{T} \leq T < T_n)
$$

where

 $\overline{\phantom{a}}$ 

$$
V(T) < \int_{T}^{T_n} H(s)V(s)ds \qquad (\overline{T} \leq T < T_n)
$$
\n
$$
V(T) = f(x(T)) \int_{T}^{T_n} q(u) du \qquad \text{and} \qquad H(T) = -\frac{x'(T)f'(x(T))}{f(x(T))}.
$$

 $\sim 1\,\mu$ 

From (1), condition (a) and  $x'(T) < 0$  we get  $H(t) > 0$  ( $t \in [\overline{T}, T_n)$ ). Setting

) and 
$$
x'(T) < 0
$$
 we get  $H(t) > 0$   $(t \in [\overline{T},$   

$$
W(T) = \int_{T}^{T_n} H(s)V(s)ds \qquad (T \in [\overline{T}, T_n))
$$

we obtain  $W'(T) = -H(T)V(T) > -H(T)W(T)$ . Then we have

$$
\tau
$$
  
\n
$$
= -H(T)V(T) > -H(T)W(T).
$$
 Then we have  
\n
$$
\frac{d}{dT}\left(W(T)exp\left(\int_{\overline{T}}^{T} H(s) ds\right)\right) > 0 \quad (T \in [\overline{T}, T_n))
$$

which implies that

$$
W(T) exp \left( \int_{\overline{T}}^{T} H(s) ds \right)
$$

is strongly increasing on  $[\overline{T}, T_n)$ . Considering  $W(T_n) = 0$  we obtain that the function *W*, and hence the function *V*, is negative for every  $t \in [\overline{T}, T_n)$ , and, consequently, it is easy to see that  $W(T)exp\left(\int_{\overline{T}}^{T} H(s) ds\right)$ <br> *b*. Considering  $W(T_n) = 0$  we obtain that the function<br> *s* negative for every  $t \in [\overline{T}, T_n)$ , and, consequently, it is<br>  $\int_{\overline{T}}^{g} q(s) ds < 0$ ,<br> *r* of  $\overline{Q}(t)$  for all  $t \in I_{\beta}$ . Now we have<br>

$$
\int\limits_{\overline{T}}^{\beta}q(s)\,ds<0,
$$

contradicting the non-negativity of  $\overline{Q}(t)$  for all  $t \in I_{\beta}$ . Now we have

$$
0 \le b = \lim_{T \to \beta} z(T) < \infty. \tag{4}
$$

Letting  $T \rightarrow \beta$  it follows from (3) that

function *V*, is negative for every 
$$
t \in [1, I_n)
$$
, and, consequently, it is  
\n
$$
\int_{\overline{T}}^{\beta} q(s) ds < 0,
$$
\nnon-negativity of  $\overline{Q}(t)$  for all  $t \in I_{\beta}$ . Now we have  
\n
$$
0 \leq b = \lim_{T \to \beta} z(T) < \infty.
$$
\n(4)\n  
\nallows from (3) that  
\n
$$
z(t) = b + \overline{Q}(t) + \int_{t}^{\beta} \frac{f'(x(s))z^{2}(s)}{r(s)w(x(s))} ds \qquad (t \in I_{\beta}).
$$
\n(5)\n  
\n*estitution*  $x(s) = u$  we have

By (2) and the substitution  $x(s) = u$  we have

$$
0 \le b = \lim_{T \to \beta} z(T) < \infty.
$$
\n(4)  
\nfrom (3) that  
\n
$$
b + \overline{Q}(t) + \int_{t}^{\beta} \frac{f'(x(s))z^{2}(s)}{r(s)w(x(s))} ds \qquad (t \in I_{\beta}).
$$
\n(5)  
\n
$$
\lim_{\epsilon} x(t) = u \text{ we have}
$$
\n
$$
\int_{c}^{x(t)} \frac{w(u)}{f(u)} du = \int_{t_{0}}^{t} \frac{z(s)}{r(s)} ds \qquad (t \in I_{\beta})
$$
\n(6)  
\nfollows from (1), (4) and (5) that  $z(t) \ge 0$  for  $t \in I_{\beta}$ . Thus it  
\nfunction x is increasing on  $I_{\beta}$ . We define  
\n
$$
\Omega(x) = \int_{c}^{x} \frac{w(u)}{f(u)} du \qquad (x \ge c).
$$
\n(7)

where  $c = x(t_0) > 0$ . It follows from (1), (4) and (5) that  $z(t) \ge 0$  for  $t \in I_\beta$ . Thus it follows from (2) that the function x is increasing on  $I_{\beta}$ . We define

$$
\Omega(x) = \int_{c}^{x} \frac{w(u)}{f(u)} du \qquad (x \ge c).
$$
 (7)

The function  $\Omega$  is strongly increasing on  $D(\Omega) = [c, \infty)$ . Let  $\Gamma$  be the inverse function of  $\Omega$  which is also monotone increasing.  $D(\Omega)$  is an interval, and therefore the function  $\Gamma$  is continuous. From (6) and (7) we get

On a Comparison Theorem for Differential Equations  
motion 
$$
\Omega
$$
 is strongly increasing on  $D(\Omega) = [c, \infty)$ . Let  $\Gamma$  be the inverse  
which is also monotone increasing.  $D(\Omega)$  is an interval, and therefore the  
intinuous. From (6) and (7) we get  

$$
x(t) = \Gamma(\Omega(x(t))) = \Gamma\left(\int_c^{x(t)} \frac{w(u)}{f(u)} du\right) = \Gamma\left(\int_c^t \frac{z(s)}{r(s)} ds\right) \qquad (t \in I_\beta).
$$

$$
G(z, r; t) = f'\left(\Gamma\left(\int_c^t \frac{z(\tau)}{r(\tau)} d\tau\right)\right) / w\left(\Gamma\left(\int_c^t \frac{z(\tau)}{r(\tau)} d\tau\right)\right)
$$
ws by (5) that

Using

$$
G(z, r; t) = f' \left( \Gamma \left( \int_{t_0}^t \frac{z(\tau)}{\tau(\tau)} d\tau \right) \right) / w \left( \Gamma \left( \int_{t_0}^t \frac{z(\tau)}{\tau(\tau)} d\tau \right) \right)
$$
  
(5) that  

$$
z(t) = b + \overline{Q}(t) + \int_{t}^{\beta} \frac{z^2(s)}{\tau(s)} G(z, r; s) ds \qquad (t \in I_{\beta}).
$$
  
and side defines a map  $M$  by  

$$
(Mu)(t) = b + \overline{Q}(t) + \int_{t}^{\beta} \frac{u^2(s)}{\tau(s)} G(u, r; s) ds \qquad (t \in I_{\beta}).
$$
  
(8)  
sin of this map  $M$  be the set  $C_z$  of all continuous functions  $u$  on  $I_{\beta}$  with

it follows by (5) that

$$
r(t) = f'\left(\Gamma\left(\int_{t_0}^t \frac{1}{r(\tau)} d\tau\right)\right) / w \left(\Gamma\left(\int_{t_0}^t \frac{1}{r(\tau)} d\tau\right)\right)
$$
  
nat  

$$
z(t) = b + \overline{Q}(t) + \int_{t_0}^{\beta} \frac{z^2(s)}{r(s)} G(z, r; s) ds \qquad (t \in I_{\beta}).
$$

The right-hand side defines a map *M* by

$$
(Mu)(t) = b + \overline{Q}(t) + \int\limits_t^\beta \frac{u^2(s)}{r(s)} G(u,r;s) \, ds \qquad (t \in I_\beta).
$$
 (8)

Let the domain of this map M be the set  $C_z$  of all continuous functions *u* on  $I_\beta$  with the restriction  $0 \le u(t) \le z(t)$  ( $t \in I_\beta$ ). Then *M* is a map of  $C_z$  into itself. Since  $\Gamma$  and  $f'w^{-1}$  (condition (b)) are increasing it is easy to see that  $0 \leq (Mu)(t) \leq z(t) = (Mz)(t)$ for every  $u \in C_z$ . side defines a m<br>  $(Mu)(t) = b + t$ <br>  $0 \le u(t) \le z(t)$ <br>  $0 \le u(t) \le z(t)$ <br>  $\sum_{k=1}^{n}$ <br>  $(Lm)(t) = b + \overline{F}$ <br>  $0 \text{ of } L$  be the set *m*<sub>2</sub> (*t*) =  $b + \overline{Q}(t) + \int_{t}^{B} \frac{u^2(s)}{r(s)} G(u, r; s) ds$  (*t*  $\in I_{\beta}$ ). (8)<br> *m*<sup>2</sup> (*t*) =  $b + \overline{Q}(t)$  (*t*  $\in I_{\beta}$ ). Then *M* is a map of *C<sub>z</sub>* into itself. Since  $\Gamma$  and<br>
(*t*)  $\leq z(t)$  (*t*  $\in I_{\beta}$ ). Then *M* 

Now we consider the map *L* defined by

$$
\begin{aligned}\n\mathbf{F}_z \\
\mathbf{F}_z \\
\mathbf{F}_z\n\end{aligned}\n\text{subject to the map } L \text{ defined by}
$$
\n
$$
(Lm)(t) = b + \overline{P}(t) + \int_t^B \frac{m^2(s)}{R(s)} G(m, R; s) ds \qquad (t \in I_\beta).
$$
\n
$$
\text{where } \mathbf{F}_z \text{ and } \mathbf{F}_z \text{
$$

Let the domain of *L* be the set  $D_z$  of all continuous functions *u* on  $I_\beta$  with  $\overline{P}(t) \leq$  $u(t) \leq z(t)$  ( $t \in I_\beta$ ). Then *L* is a map of  $D_z$  into itself.

In the following we need the Fréchet space  $C_{\rho}(I_{\beta})$ , i.e. the linear, locally convex,<br>pact space  $C_{\rho}(I_{\beta})$  of continuous real-valued functions on  $I_{\beta}$ . The corresponding<br>ology  $\rho$  is defined by the metric<br> $\rho(x,$ compact space  $C_{\rho}(I_{\beta})$  of continuous real-valued functions on  $I_{\beta}$ . The corresponding topology  $\rho$  is defined by the metric

$$
\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \frac{p_i(x-y)}{1+p_i(x-y)} \right) \qquad (x,y \in C_{\rho}(I_{\beta})). \tag{10}
$$

Here  $\{p_i\}_{i>1}$  is a family of seminorms with

$$
p_i(x - y) = \sup_{t \in [t_0, t_i]} |x(t) - y(t)|
$$

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\nwhere  
\n
$$
t_i = \begin{cases} \beta - \frac{\beta - t_0}{i + 1} & \text{for } \beta < \infty \\ t_0 + i & \text{for } \beta = \infty. \end{cases}
$$
\n(11)  
\nFrom (10) and (11) it follows that  $D_z$  is a closed, convex subset of the Fréchet space  
\n $C(L)$ . In the following we show that the function belongs to the Tréchet space

 $\sim$ 

From (10) and (11) it follows that  $D<sub>x</sub>$  is a closed, convex subset of the Fréchet space  $C_{\rho}(I_{\beta})$ . In the following we show that the functions belonging to *L(D<sub>z</sub>*) are uniformly bounded and equicontinuous on compact subintervals of  $I_{\beta}$ . Let  $T \in (t_0, \beta)$  and  $m \in D_z$ . There exists a constant  $k \in \mathbb{$ bounded and equicontinuous on compact subintervals of  $I_\beta$ . Let  $T \in (t_0, \beta)$  and  $m \in D_z$ . There exists a constant  $k \in \mathbb{R}$  with  $|z(t)| \leq k$  for all  $t \in [t_0, T]$ , and therefore we have  $|(Lm)(t)| \leq k$  for all  $t \in [t_0, T]$  and all  $m \in D_z$ . Thus there follows the uniform

boundedness of 
$$
L(D_z)
$$
 on compact subintervals of  $I_{\beta}$ . From  $m(t) \leq z(t)$   $(t \in I_{\beta})$ , (8)

\nand (9) it follows for  $s, t \in [t_0, T]$  that, for all  $m \in D_z$ ,

\n
$$
\left| (Lm)(t) - (Lm)(s) \right| \leq \int_{s}^{t} |p(u)| \, du + \left| \int_{s}^{t} q(u) \, du \right| + |z(s) - z(t)|. \tag{12}
$$
\nLet  $\varepsilon > 0$  and  $|t - s| < \delta(\varepsilon)$ . By virtue of the estimations

\n
$$
\int_{s}^{t} |p(u)| \, du \leq \frac{\varepsilon}{3}, \qquad \left| \int_{s}^{t} q(u) \, du \right| \leq \frac{\varepsilon}{3}, \qquad |z(t) - z(s)| \leq \frac{\varepsilon}{3}
$$
\nfor  $|t - s| < \delta(\varepsilon)$  and (12) we obtain that the functions in  $L(D_s)$  are equicontinuous.

Let  $\varepsilon > 0$  and  $|t - s| < \delta(\varepsilon)$ . By virtue of the estimations

$$
\int_{s}^{t} |p(u)| du \leq \frac{\varepsilon}{3}, \qquad \left| \int_{s}^{t} q(u) du \right| \leq \frac{\varepsilon}{3}, \qquad |z(t) - z(s)| \leq \frac{\varepsilon}{3}
$$

for  $|t - s| < \delta(\varepsilon)$  and (12) we obtain that the functions in  $L(D_z)$  are equicontinuous<br>on compact subintervals of  $I_\beta$ . By the Arzela-Ascoli theorem; it follows that  $L(D_z)$  is<br>pre-compact on closed subintervals of  $I_\beta$ . on compact subintervals of  $I_\beta$ . By the Arzela-Ascoli theorem; it follows that  $L(D_x)$  is pre-compact on closed subintervals of  $I_\beta$ . Now we show that the map L is continuous on *D.* Using

$$
\Phi(m;s) = \frac{m^2(s)}{R(s)}G(m,R;s) \qquad (s \in I_\beta)
$$

it follows by (9) that

$$
(Lm)(t) = b + \overline{P}(t) + \int\limits_t^\beta \Phi(m; s) \, ds.
$$

Let the sequence  $(m_n)_{n\geq 1} \subseteq D_z$  be such that  $m_n \to m$   $(m \in D_z)$ , uniformly on compact subintervals of  $\bar{I}_{\beta}$ . Let  $\epsilon > 0$  be given. There exists a  $T \in (t_1, \beta)$  such that

$$
\int\limits_T^\beta \frac{z^2(s)}{r(s)} G(z,r;s)\,ds < \frac{\varepsilon}{3}.
$$

On the other hand, there exists an  $n_0(\varepsilon)$  such that

$$
\int_{T} \overline{r(s)} \, G(z, r; s) \, ds < \frac{\pi}{3}.
$$
\nd, there exists an  $n_0(\varepsilon)$  such that

\n
$$
\max \left| \Phi(m_n; t) - \Phi(m; t) \right| \le \frac{\varepsilon}{3|T - t_0|} \qquad (n \ge n_0).
$$

Then for all  $t \in [t_0, T]$  and  $n \geq n_0$  we obtain

$$
(Lm_n)(t) - (Lm)(t)
$$
  
\n
$$
= \left| \int_t^\beta \left( \Phi(m_n; s) - \Phi(m; s) \right) ds \right|
$$
  
\n
$$
\leq \left| \int_t^\tau \left( \Phi(m_n; s) - \Phi(m; s) \right) ds \right| + \left| \int_t^\beta \left( |\Phi(m_n; s)| + |\Phi(m; s)| \right) ds \right|
$$
  
\n
$$
\leq \left| \int_t^\tau \left( \Phi(m_n; s) - \Phi(m; s) \right) ds \right| + 2 \int_t^\beta \frac{z^2(s)}{r(s)} G(z, r; s) ds
$$
  
\n
$$
< \varepsilon.
$$

Thus  $Lm_n \to Lm$  as  $n \to \infty$ , uniformly on compact subintervals of  $I_\beta$ .

Finally we need that  $L(D_x)$  is pre-compact on  $I_\beta$ . Here we show that every sequence  $(L(m_j))_{j\geq 1} \subseteq L(D_z)$  has a subsequence being a Cauchy sequence. Choosing  $\varepsilon_1 > 0$  we set  $\mu_1 > 0$  such that

$$
\frac{\mu_1}{1+\mu_1}\leq \frac{\varepsilon_1}{2}.
$$

There exists a number  $\overline{T}_1 \in I_\beta$  such that, for all  $t \in [\overline{T}_1, \beta)$ ,

$$
1 + \mu_1 \stackrel{?}{=} 2
$$
  
re exists a number  $\overline{T}_1 \in I_\beta$  such that, for all  $t \in [\overline{T}_1, \beta)$ ,  

$$
\left| (Lm_n)(t) - (Lm_k)(t) \right| \le \int_t^\beta \left( |\Phi(m_n; s)| + |\Phi(m_k; s)| \right) ds \le 2 \int_t^\beta \Phi(z; s) ds < \mu_1
$$

$$
\sup_{t\in[\overline{T}_1,\overline{t}_i]} \Big| (Lm_n)(t) - (Lm_k)(t) \Big| \leq \mu_1
$$

where

There exists a number 
$$
1 \in I_{\beta}
$$
 such that, for an  $t \in [1], p$ ,

\n
$$
\left| (Lm_n)(t) - (Lm_k)(t) \right| \leq \int_{t}^{\beta} \left( |\Phi(m_n; s)| + |\Phi(m_k; s)| \right) ds \leq 2 \int_{t}^{\beta} \Phi(z; s).
$$
\nindependently of  $n$  and  $k$ . Therefore, we obtain the inequality:

\n
$$
\sup_{t \in [\overline{T}_1, \overline{t}_i]} \left| (Lm_n)(t) - (Lm_k)(t) \right| \leq \mu_1
$$
\nwhere:

\n
$$
\overline{t}_i = \begin{cases}\n\beta - \frac{\beta - t_0}{i + j_1} & \text{for } \beta < \infty \\
t_0 + i + j_1 - 1 & \text{for } \beta = \infty \text{ and } j_1 = \min \{i : t_i > \overline{T}_1 \}.\n\end{cases}
$$
\nThus, we have for all  $n, k$ .

\n
$$
\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup_{t \in [\overline{T}_1, \overline{t}_i]} \left| (Lm_n)(t) - (Lm_k)(t) \right|}{1 + \sup_{t \in [\overline{T}_1, \overline{t}_i]} \left| (Lm_n)(t) - (Lm_k)(t) \right|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mu_1}{1 + \mu_1} \leq \frac{\varepsilon}{2}
$$
\nSince  $L(D_z)$  is pre-compact on the interval  $[t_0, \overline{T}_1]$ , the sequence  $(Lm_j)_{j \geq 1}$  or  $[t_0, \overline{T}_1]$  being a Cauchy sequence. It follows:

Thus we have for all *n, k* 

$$
\begin{aligned} \left| \begin{array}{ll} t_0 + i + j_1 - 1 & \text{for } \beta = \infty \text{ and } j_1 = \min\left\{ i : t_i > T_1 \right\}. \end{array} \right| \\ \text{have for all } n, \ k \end{aligned}
$$
\n
$$
\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup_{t \in [\overline{T}_1, \overline{t}_i]} \left| (L m_n)(t) - (L m_k)(t) \right|}{1 + \sup_{t \in [\overline{T}_1, \overline{t}_i]} \left| (L m_n)(t) - (L m_k)(t) \right|} \le \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\mu_1}{1 + \mu_1} \le \frac{\varepsilon_1}{2}
$$

Since  $L(D_x)$  is pre-compact on the interval  $[t_0, \overline{T}_1]$ , the sequence  $(Lm_j)_{j\geq 1}$  has a convergent subsequence  $(Lm_j^{(1)})_{j\geq 1}$  on  $[t_0, \overline{T}_1]$  being a Cauchy sequence. It follows that

$$
\sup_{t\in [t_0,\overline{T}_1]} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right| \le \frac{\varepsilon_1}{2}
$$

holds if  $n, k \geq n_1$ . We obtain

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\n
$$
k \ge n_1. \text{ We obtain}
$$
\n
$$
\rho\left(Lm_n^{(1)}, Lm_k^{(1)}\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sum_{t \in [t_0, \bar{t}_i]}^{t_0} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right|}{\sum_{t \in [t_0, \bar{t}_i]}^{t_0} \left| (Lm_n^{(1)}(t) - (Lm_k^{(1)})(t) \right|)}
$$
\n
$$
\le \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sum_{t \in [t_0, \bar{T}_1]}^{t_0} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right|}{\sum_{t \in [t_0, \bar{T}_1]}^{t_0} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right|}
$$
\n
$$
+ \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sum_{t \in [\bar{T}_1, \bar{t}_i]}^{t_0} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right|}{\sum_{t \in [\bar{T}_1, \bar{t}_i]}^{t_0} \left| (Lm_n^{(1)})(t) - (Lm_k^{(1)})(t) \right|}
$$
\n
$$
\le \varepsilon_1
$$

for all  $n, k \geq n_1$ . This process can be repeated infinitely often setting  $\varepsilon_m = \varepsilon_{m-1}/2$  and using successively  $\mu_m, T_m, j_m$  and  $n_m$   $(n = 2, 3, ...)$ . Then the diagonal sequence

$$
\leq c_1
$$
\nis can be repeated infinitely on

\nand

\n
$$
n_m \quad (n = 2, 3, \ldots).
$$
\nThus

\n
$$
\left( L(m_j^{(j)}) \right)_{j \geq 1} \subseteq \left( L(m_j) \right)_{j \geq 1}
$$
\n
$$
\left( L(m_j^{(j)}) \right)_{j \geq 1} \subseteq C_{\rho}(I_{\beta})
$$

is a Cauchy sequence. Since

$$
\left(L(m_j^{(j)})\right)_{j\geq 1}\subseteq C_\rho(I_\beta)
$$

this Cauchy sequence is convergent. Therefore  $L(D_z)$  has a compact closure on  $I_\beta$ . By the corollary of Tychnov's Theorem (see [2: p. 405]), *L* has a fixed point  $\bar{n} \in D_z$ ,  $L\bar{n}=\bar{n}$ . Then

s convergent. Therefore 
$$
L(D_z)
$$
 has a con-  
nov's Theorem (see [2: p. 405]), L has  
 $x(t) = \Gamma\left(\int_{t_0}^t \frac{\overline{n}(s)}{R(s)} ds\right) > 0$   $(t \in I_\beta)$ 

is a non-oscillatory solution of the differential Equation  $(D2)_1$  which contradicts the suppusition.

**Case (ii):** Let the differential equation  $(D2)_1$  be oscillatory at  $\beta$  and x a solution of the differential equation  $(D2)_2$ . By using

Tychnov's Theorem (see [2: p. 405]), *L* has a fixed point 
$$
\overline{n} \in D_z
$$
,  
\n
$$
x(t) = \Gamma \left( \int_{t_0}^t \frac{\overline{n}(s)}{R(s)} ds \right) > 0 \qquad (t \in I_\beta)
$$
\nory solution of the differential Equation (D2)<sub>1</sub> which contradicts the  
\net the differential equation (D2)<sub>1</sub> be oscillatory at  $\beta$  and  $x$  a solution  
\nI equation (D2)<sub>2</sub>. By using  
\n
$$
s = \int_{t_0}^t \frac{d\tau}{r(\tau)w(x(\tau))}, \qquad y(s) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(s) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$
\n
$$
y(t) = x(t(s)) \qquad (t \in I_\beta)
$$

 $\bar{\mu}$  ,  $\bar{\nu}$ 

we transform the differential equation  $(D2)_2$  into the equation

$$
y(s) = x(t(s)) \qquad (t \in I)
$$
  
differential equation (D2)<sub>2</sub> into the equation  

$$
\frac{d^2y}{ds^2} + r(t(s))q(t(s))w(y)f(y) = 0 \qquad (s \in [0, \infty)).
$$

 $\sim$ 

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\nWe set

\n
$$
a(s) = r(t(s))q(t(s)) \qquad \text{and} \qquad F(y(s)) = w(y(s))f(y(s))
$$
\nand obtain the differential equation

and obtain the differential equation

 $\mathbf{r}$ 

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\n
$$
f(t(s))q(t(s))
$$
\nand\n
$$
F(y(s)) = w(y(s))f(y(s))
$$
\nntial equation\n
$$
\frac{d^2y(s)}{ds^2} + a(s)F(y(s)) = 0 \quad (s \in [0, \infty)).
$$
\n(14)

By (13) we get

initial equation

\n
$$
\frac{d^2y(s)}{ds^2} + a(s)F(y(s)) = 0 \quad (s \in [0, \infty)).
$$
\n(14)

\n
$$
A(s) = \int_{s}^{\infty} a(\tau) d\tau \ge 0 \quad (s \in [0, \infty)).
$$

Hence we have the set of the set of

$$
A(s) = \int_{s}^{\infty} a(\tau) d\tau \ge 0 \qquad (s \in [0, \infty)).
$$
  

$$
\int_{T}^{\infty} A(s) ds \ge \frac{1}{S^2} \int_{t(T)}^{\beta} \frac{\overline{Q}(u)}{r(u)} du
$$
  

$$
\int_{T}^{\infty} \left( \int_{s}^{\infty} A^2(\tau) d\tau \right) ds \ge \frac{1}{S^4} \int_{t(T)}^{\beta} \frac{1}{r(u)} \left( \int_{u}^{\beta} \frac{\overline{Q}^2(v)}{r(v)} dv \right) du.
$$
  

$$
= \int_{t_0}^{t} \frac{d\tau}{R(\tau)w(x(\tau))} \qquad \text{and} \qquad y(s) = x(t(s)) \qquad (t \in I_{\beta})
$$
  
the differential equation (D2)<sub>1</sub> into the equation  

$$
\frac{d^2y(s)}{ds^2} + \overline{a}(s)F(y(s)) = 0 \qquad (s \in [0, \infty)) \qquad (14)'
$$
  

$$
R(t(s))p(t(s)). \qquad \text{The oscillation properties of the differential equations}
$$
  

$$
= \int_{t_0}^{t} R(t(s))p(t(s)) \qquad \text{the oscillation properties of the differential equations}
$$
  

$$
= \int_{t_0}^{t} R(t(s))p(t(s)) \qquad \text{the oscillation properties of the differential equations}
$$
  

$$
= \int_{t_0}^{t} R(t(s))p(t(s)) \qquad \text{the oscillation properties of the differential equations}
$$

Using

 $\gamma=30$ 

 $\bar{\mathcal{L}}$ 

$$
\int_{T} \left( \int_{s}^{T} \frac{1}{s} e^{i\theta} \right) ds = S^{4} \int_{t(T)} r(u) \left( \int_{u}^{T} r(v) \right) du.
$$
\n
$$
s = \int_{t_0}^{t} \frac{d\tau}{R(\tau)w(x(\tau))} \quad \text{and} \quad y(s) = x(t(s)) \quad (t \in I_{\beta})
$$

we transform the differential equation  $(D2)_1$  into the equation

$$
\frac{d^2y(s)}{ds^2} + \overline{a}(s)F(y(s)) = 0 \qquad (s \in [0, \infty))
$$
\n(14)

Using<br>  $s = \int_{t_0}^{t} \frac{d\tau}{R(\tau)w(x(\tau))}$  and  $y(s) = x(t(s))$   $(t \in I_{\beta})$ <br>
we transform the differential equation (D2)<sub>1</sub> into the equation<br>  $\frac{d^2y(s)}{ds^2} + \overline{a}(s)F(y(s)) = 0$   $(s \in [0, \infty))$  (where  $\overline{a}(s) = R(t(s))p(t(s))$ . The oscillation where  $\overline{a}(s) = R(t(s))p(t(s))$ . The oscillation properties of the differential equations  $(D2)_1$  and  $(D2)_2$  are invariant under these transformations. Analogously, we have the  $\sim$   $\sim$ 

$$
= \int_{t_0} \frac{d\tau}{R(\tau)w(x(\tau))} \quad \text{and} \quad y(s) = x(t(s)) \quad (t \in I_\beta)
$$
\nthe differential equation (D2), into the equation\n
$$
\frac{d^2y(s)}{ds^2} + \bar{a}(s)F(y(s)) = 0 \quad (s \in [0, \infty)) \quad (14)'
$$
\n
$$
R(t(s))p(t(s)). \quad \text{The oscillation properties of the differential equations}
$$
\n
$$
P(t(s))p(t(s)) = \int_{t_0}^{\infty} \bar{a}(\tau) d\tau \quad \text{where} \quad \tau \text{ and } \bar{a}(\tau) \text{ and } \bar{a}
$$

 $\mathcal{A}_\text{c}$  ,  $\mathcal{A}_\text{c}$  ,

Concerning the differential equation (14)' it follows from Theorem 5 that

$$
\int^{\infty}\left(|\overline{A}(s)|+\int\limits_{s}^{\infty}(\overline{A}(u))^{2}du\right)ds=\infty.
$$

On the other hand, by relations (15)' and, without loss of generality,  $s_0 \leq 1$  and  $S \geq 1$ , we have

In. Rudek  
\n: 
$$
\int^{\infty} \left( |\overline{A}(s)| + \int_{s}^{\infty} (\overline{A}(u))^{2} du \right) ds = \infty.
$$
\nthe other hand, by relations (15)' and, without loss of generality,  $s_{0} \leq 1$  and  $S \geq \infty$   
\n
$$
\int^{\infty} \left( |\overline{A}(s)| + \int_{s}^{\infty} (\overline{A}(u))^{2} du \right) ds \leq \frac{1}{s_{0}^{4}} \int_{t(T)}^{\beta} \frac{1}{R(u)} \left( \overline{P}(u) + \int_{u}^{\beta} \frac{(\overline{P}(v))^{2}}{R(v)} dv \right) du.
$$
\nrespectively, in view of inequalities  $0 < r(t) \leq R(t)$  and  $|P(t)| \leq Q(t)$  for all  $t \in \mathbb{R}$ .

Correspondingly, in view of inequalities  $0 < r(t) \le R(t)$  and  $|P(t)| \le Q(t)$  for all  $t \in I_\beta$ we obtain

$$
\frac{1}{S^4} \int_{t(T)}^{\beta} \frac{1}{r(u)} \left( S^2 \overline{Q}(u) + \int_u^{\beta} \frac{\overline{Q}^2(v)}{r(v)} dv \right) du = \infty.
$$
 (16)

Hence, by (15) and (16) we obtain<br> $\int_{0}^{\infty} \int_{0}^{x} A(s)$ 

$$
\int\limits_T^{\infty}\left(A(s)+\int\limits_s^{\infty}A^2(u)\,du\right)\,ds=\infty.
$$

Thus, in view of Theorem 4, the differential equation (14) is oscillatory at  $\beta$  and hence also the differential equation  $(D2)_2$  is oscillatory at  $\beta$ .

**Case (iii):** The inequalities  $0 < r(t) \leq R(t)$  ( $t \in I_\beta$ ) and condition (iii) imply the existence of the integral  $\int_{\mathcal{B}}^{\beta} \frac{du}{R(u)}$ . All suppositions of Theorem 6 are fulfilled for the differential equations  $(D2)_1$  and  $(D2)_2$ . Both equations are non-oscillatory. Thus the proof of Theorem 7 is complete. **I**  *x*  $V(t) \leq R(t)$   $\leq R(t)$   $\leq R(t)$   $\leq R(t)$  and condition (ii) imply<br>integral  $\int_{\mathcal{B}}^{\beta} \frac{du}{R(u)}$ . All suppositions of Theorem 6 are fulfilled for the<br>is  $(D2)_1$  and  $(D2)_2$ . Both equations are non-oscillatory. Thus the<br> *x* (D2)<sub>1</sub> and (D2)<sub>2</sub>. Both equations are non-oscillatory. Thus the scomplete.  $\blacksquare$ <br>
sition  $p(t) \ge 0$  for all  $t \in [t_0, \infty)$ , Theorem 7 generalizes a result of 3 in that paper).<br>
a in that paper).<br>
ardifferential equa

Under the supposition  $p(t) \geq 0$  for all  $t \in [t_0, \infty)$ , Theorem 7 generalizes a result of Butler [i] (Theorem 3 in that paper).

### **3. A corollary**

Consider the nonlinear differential equations

$$
x''(t) + A(t)x'(t) + B(t)f(x(t)) = 0 \qquad (t \in I_\beta) \tag{D3}_1
$$

$$
x''(t) + a(t)x'(t) + b(t)f(x(t)) = 0 \qquad (t \in I_\beta).
$$
 (D3)<sub>2</sub>

**Corollary.** Let  $a, A, b, B \in C(I_\beta)$  be functions such that

A corollary  
\nhsider the nonlinear differential equations  
\n
$$
x''(t) + A(t)x'(t) + B(t)f(x(t)) = 0 \qquad (t \in I_{\beta}) \qquad (I
$$
\n
$$
x''(t) + a(t)x'(t) + b(t)f(x(t)) = 0 \qquad (t \in I_{\beta}). \qquad (I
$$
\nCorollary. Let  $a, A, b, B \in C(I_{\beta})$  be functions such that  
\n
$$
a(t) \leq A(t) \qquad and \qquad |B(t)| \leq b(t) \exp\left(\int_{t_0}^t (a(s) - A(s)) ds\right) \qquad (t \in I_{\beta}).
$$

Suppose that there exist

\n
$$
P^*(t) = \int_t^{\beta} |B(s)| \exp\left(\int_t^s A(u) du\right) ds
$$
\n
$$
Q^*(t) = \int_t^{\beta} b(s) \exp\left(\int_t^s a(u) du\right) ds
$$
\n
$$
A \text{sum of } t \in I_{\beta}
$$
\nAssume further that  $f$  is a function on  $R$  satisfying the condition (a) and either the condition (b) or (c) of Theorem 7, where  $w = 1$  is supposed.

\nThen if the differential equation (D3)<sub>1</sub> is oscillatory at  $\beta$ , so also the differentiation (D3)<sub>2</sub> is oscillatory at  $\beta$ .

\nProof. By setting

Assume further that f is a function on IR satisfying the condition (a) and either the *condition* (b) *or* (c) *of Theorem* 7, where  $w = 1$  is supposed.

Then if the differential equation  $(D3)_1$  is oscillatory at  $\beta$ , so also the differential  $\ddot{\phantom{0}}$ 

Proof. By *setting* 

$$
Q^*(t) = \int_{t} b(s) \exp\left(\int_{t_0} a(u) du\right) ds
$$
  
\nAssume further that  $f$  is a function on  $\mathbb{R}$  satisfying the condition (a) and either  
\ncondition (b) or (c) of Theorem 7, where  $w = 1$  is supposed.  
\nThen if the differential equation (D3)<sub>1</sub> is oscillatory at  $\beta$ , so also the difference  
\nequation (D3)<sub>2</sub> is oscillatory at  $\beta$ .  
\nProof. By setting  
\n
$$
r(t) = \exp\left(\int_{t_0}^t a(s) ds\right) \quad \text{and} \quad \mathbb{R}(t) = \exp\left(\int_{t_0}^t A(s) ds\right) \quad (t \in I_{\beta})
$$
\nthe corollary immediately follows from Theorem 7.

#### 4. An example

Let  $\beta < \infty$  and  $t_0 > 0$ . In view of Theorem 2 in [4] we can see that the differential *equation*

example

\n
$$
\infty \text{ and } t_0 > 0. \text{ In view of Theorem 2 in [4] we can see that the diffi-\n
$$
\left( (\beta - t)^5 \left( \frac{a}{e^{x(t)} + e^{-x(t)}} + b \right) x'(t) \right)' + \frac{1}{t} \left( |x(t)|^\alpha \operatorname{sign} x(t) + cx(t) \right) = 0
$$
\n
$$
\beta \text{ with}
$$
\n
$$
c > 0, \qquad 0 < a \le b, \qquad 1 < \alpha = \frac{2\beta + 1}{2\gamma + 1} \quad (\beta, \gamma \in \mathbb{N})
$$
\natory at  $\beta$ . Hence, by Theorem 7 it follows that the differential equation
$$

for  $t \in I_\beta$  with

$$
c>0, \qquad 0
$$

is oscillatory at  $\beta$ . Hence, by Theorem 7 it follows that the differential equation

$$
\left(r(t)\left(\frac{a}{e^{x(t)}+e^{-x(t)}}+b\right)x'(t)\right)' + q(t)\left(|x(t)|^{\alpha}\operatorname{sign} x(t)+cx(t)\right) = 0
$$

is oscillatory at  $\beta$ , if the functions  $r$  and  $q$  satisfy the conditions

scillatory at 
$$
\beta
$$
. Hence, by Theorem 7 it follows that the differential equation  
\n
$$
\left(r(t)\left(\frac{a}{e^{x(t)}+e^{-x(t)}}+b\right)x'(t)\right)' + q(t)\left(|x(t)|^{\alpha}\operatorname{sign} x(t) + cx(t)\right) = 0
$$
\nscillatory at  $\beta$ , if the functions  $r$  and  $q$  satisfy the conditions  
\n $r, q \in C(I_{\beta}), \qquad 0 < r(t) \leq (\beta - t)^5, \qquad \ln \frac{\beta}{t} \leq \int_{\epsilon}^{\beta} q(s) ds < \infty \qquad (t \in I_{\beta}).$ 

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Received 25.04.1994