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On the Decomposition of Unitary Operators into a Product of Finitely Many Positive Operators

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Abstract. We will show that in an infinite-dimensional separable Hilbert space $\mathcal{H}$, there exist constants $N \in I\!N$ and $c, d \in I\!R$ such that every unitary operator can be written as the product of at most N positive invertible operators $\{a_k\} \subseteq \mathrm{B}(\mathcal{H})$ with $\|a_k\| \leq c$ and $\|a_k^{-1}\| \leq d$ for all k. Some consequences of this result in the context of von Neumann algebras are discussed.

Keywords: *Operator theory, von Neumann algebras, non-commutative geometry*
AMS subject classification: 46L10, 46L30, 46L50

1. The results

In this article, we assume the reader is familiar with the theory of von Neumann algebras (see [8] and [12] for standard references), and subsequently we will make use of the standard settings in this context, e.g. *factor, normal, faithful* etc. without further explanation. For a given Hilbert space $\mathcal{H}$, the algebra of all bounded linear operators over $\mathcal{H}$ will be denoted by $\mathrm{B}(\mathcal{H})$. Furthermore the cone of positive operators in $\mathrm{B}(\mathcal{H})$ and the unitary group over $\mathcal{H}$ will be denoted by $\mathrm{B}(\mathcal{H})_+$ and $\mathrm{U}(\mathcal{H})$, respectively.

The main results of this paper are the following theorems.

Theorem 1. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. There exist constants $N \in I\!N$ and $c, d \in I\!R_+$ such that for every $u \in \mathrm{U}(\mathcal{H})$ there are invertible elements $a_1, \ldots, a_N \in \mathrm{B}(\mathcal{H})_+$ with $\|a_k\| \leq c$ and $\|a_k^{-1}\| \leq d$ $(k = 1, \ldots, N)$ and such that $a_N \cdots a_1 = u$.*

A von Neumann algebra M acting on a Hilbert space $\mathcal{H}$ is called *hyperfinite* if there exist factors $\{M_j\}_{j=0}^{\infty}$ satisfying

- $M_j \simeq \mathrm{M}_{n_j}(\boldsymbol{C})$ for all j with $\{n_j\} \subseteq I\!N$ and $2 \leq n_1 < n_2 < \ldots$
- $\mathbb{1}_M \in M_j$ for all j
- $M_j \subseteq M_{j+1}$ for all j

and such that $\overline{\bigcup_{j=1}^{\infty} M_j}^{st} = M$. Here $\mathrm{M}_n(\boldsymbol{C})$ denotes the full algebra of all $n \times n$ matrices. This definition of a hyperfinite von Neumann algebra is taken from [5: p. 150].

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Theorem 2. *Let M be a hyperfinite von Neumann algebra acting on a separable Hilbert space. There exist constants $\tilde{N} \in \mathbb{N}$ and $c, d \in \mathbb{R}_+$ such that for an arbitrary $\varphi \in [0, 2\pi)$ there are elements $a_1, \ldots, a_{\tilde{N}} \in M_+$ with $\|a_k\| \leq c$ and $\|a_k^{-1}\| \leq d$ $(k = 1, \ldots, \tilde{N})$ and such that $a_{\tilde{N}} \cdots a_1 = \mathrm{e}^{\mathrm{i}\varphi}\mathbb{1}$.*

Remark 1. In fact, more than the assertion of Theorem 2 can be proved. For the assertion of Theorem 2 to hold it is even sufficient that M *contains* a hyperfinite von Neumann algebra.

Remark 2. Hyperfiniteness of a von Neumann algebra M is an expression for M being infinite-dimensional as well as for M being "sufficiently" non-commutative. Neither for a finite-dimensional algebra nor for an abelian von Neumann algebra Theorem 2 can be true.

2. Auxiliary results and proofs

Before we can prove the theorems, we have to consider some auxiliary results.

Proposition 1. *Let $\mathrm{M}_2(\mathbb{C})$ denote the full algebra of all 2×2 matrices with complex entries and $\mathrm{M}_2(\mathbb{C})_+$ the cone of the positive ones. Then there exist constants $N' \in \mathbb{N}$ and $c, d \in \mathbb{R}_+$ such that*

$$\mathrm{SU}(2) = \left\{ u \in \mathrm{M}_2(\mathbb{C}) \,\middle|\, \begin{array}{l} u = a_r \cdots a_1 |a_r \cdots a_1|^{-1} \textit{ with invertible } a_k \in \mathrm{M}_2(\mathbb{C})_+ \textit{ such} \\ \textit{that } \|a_k\| \leq c \textit{ and } \|a_k^{-1}\| \leq d \ \ (k = 1, \ldots, r;\ r \leq N' - 1) \end{array} \right\}.$$

Proof. Since in the above equation elements of the right-hand side are evidently in SU(2), it is sufficient to prove only that each matrix in SU(2) has a representation as shown on the right-hand side of the equation above. Let $a \in \mathrm{M}_2(\mathbb{C})_+$ be invertible and $a \neq \lambda\mathbb{1}$ $(\lambda \in \mathbb{R}_+)$. Then there exists an element $v \in \mathrm{SU}(2)$ such that a does not commute with vav^*. Let $u := a(vav^*)|a(vav^*)|^{-1}$. (Here and in the entire article for an operator or matrix b by $|b|$ will be denoted the modulus of b defined by $\sqrt{b^*b}$.) This construction yields $u \neq \mathbb{1}$. Since the element v is normal, it can be written as $v = w \operatorname{diag}[\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}] w^*$ for some $w \in \mathrm{U}(2)$ and $\varphi \in (0, 2\pi)$. Now let for $\varepsilon \in [0, \varphi]$

$$v_\varepsilon = w \operatorname{diag}[\mathrm{e}^{\mathrm{i}\varepsilon}, \mathrm{e}^{-\mathrm{i}\varepsilon}] w^* \qquad \text{and} \qquad u_\varepsilon = a(v_\varepsilon a v_\varepsilon^*)\,|a(v_\varepsilon a v_\varepsilon^*)|^{-1}.$$

We have $u_0 = \mathbb{1}$ and $u_\varphi = u \neq \mathbb{1}$, and the mapping $[0, \varphi] \to \mathrm{SU}(2), \varepsilon \mapsto u_\varepsilon$ is continuous.

The spectrum of a matrix is a continuous function of its entries, and so are the real parts of the spectrum. Since all matrices u_ε are elements of SU(2), their spectra are of the kind $\{\mathrm{e}^{\mathrm{i}\vartheta}, \mathrm{e}^{-\mathrm{i}\vartheta}\}$ which means that the real parts of this two numbers coincide, and therefore we will speak from now on of the real part of the spectrum. Let the real part of the spectrum of u be x (note that it has to be $x < 1$). When ε goes from 0 to φ, the set of real parts $\Re(\operatorname{spec} u_\varepsilon)$ covers at least the interval $[x, 1]$. Let $\psi := \operatorname{Arccos} x$. Then for each $\vartheta \in [0, \psi]$ there is an $\varepsilon(\vartheta) \in [0, \varphi]$ such that $\operatorname{spec} u_{\varepsilon(\vartheta)} = \{\mathrm{e}^{\mathrm{i}\vartheta}, \mathrm{e}^{-\mathrm{i}\vartheta}\}$. From this and since SU(2) is a normal subgroup of U(2) it now follows that each matrix in SU(2) the spectrum of which is $\{\mathrm{e}^{\mathrm{i}\vartheta}, \mathrm{e}^{-\mathrm{i}\vartheta}\}$ $(\vartheta \in [0, \psi])$ can be written as $ab|ab|^{-1}$ for some matrices $a, b \in \mathrm{M}_2(\mathbb{C})_+$.

Let $k = \left[\frac{2\pi}{\psi}\right] + 1$ (the brackets stand for the integer part). Now we get the universal constant N' by setting $N' = 3k$ because each matrix in SU(2) can be generated as a product of at most $3k$ positive matrices as follows: Let $u' \in \mathrm{SU}(2)$ be given arbitrarily,

$$u' = w' \operatorname{diag}[\mathrm{e}^{\mathrm{i}\varrho}, \mathrm{e}^{-\mathrm{i}\varrho}]\, w'^{*} \qquad \text{for some } w' \in \mathrm{U}(2) \text{ and } \varrho \in [0, 2\pi).$$

We have $\frac{\varrho}{k} \in [0, \psi]$, hence there are elements $a, b \geq 0$ with

$$ab|ab|^{-1} = \operatorname{diag}[\mathrm{e}^{\mathrm{i}\frac{\varrho}{k}}, \mathrm{e}^{-\mathrm{i}\frac{\varrho}{k}}].$$

But then

$$(ab|ab|^{-1})^{k} = \operatorname{diag}[\mathrm{e}^{\mathrm{i}\varrho}, \mathrm{e}^{-\mathrm{i}\varrho}]$$

and, finally,

$$u' = (w'aw'^{*}\, w'bw'^{*}\, w'|ab|^{-1}w'^{*})^{k}.$$

Since the norm of a matrix and that of its inverse are continuous functions of the entries, too, it is easy to see that there are upper bounds for both of them for all $v_\varepsilon a v_\varepsilon^{*}$ as well as for all $|a\,(v_\varepsilon a v_\varepsilon^{*})|^{-1}$. This completes the proof ∎

Now let $\mathcal{H}$ be a separable Hilbert space of finite or infinite dimension. Let $\mathrm{U_d}(\mathcal{H})$ be the subset of $\mathrm{U}(\mathcal{H})$ which consists of those operators which have a complete orthonormal system of eigenvectors. Then each element of $\mathrm{U_d}(\mathcal{H})$ admits a representation as a diagonal matrix. Now we can state the following.

Proposition 2. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. There exist constants $N'' \in \mathbb{N}$ and $c, d \in \mathbb{R}_+$ such that the following is true: For an arbitrary $u \in \mathrm{U_d}(\mathcal{H})$ there exist invertible elements $a_1(u), \ldots, a_{N''}(u) \in \mathrm{B}(\mathcal{H})_+$ with $\|a_i(u)\| \leq c$ and $\|a_i(u)^{-1}\| \leq d$ $(i = 1, 2, \ldots, N'')$ and such that $a_{N''}(u) \cdots a_1(u) = u$.*

Proof. Let $u = \sum_{k=1}^{\infty}{}^{\oplus}\, \mathrm{e}^{\mathrm{i}\varphi_k} p_k$ be given with $\{p_k\}$ being a set of mutually orthogonal one-dimensional orthoprojections which sum up to $\mathbb{1}$. Since u is an element of $\mathrm{U_d}(\mathcal{H})$, this is always possible. Now let us define two specific operators $v(u), w(u) \in \mathrm{U_d}(\mathcal{H})$ as follows: There are

$$\vec{\chi} = (\chi_1, \chi_2, \ldots) \quad \text{and} \quad \vec{\psi} = (\psi_1, \psi_2, \ldots) \qquad \text{with } \chi_k, \psi_k \in [0, 2\pi) \text{ for all } k.$$

such that

$$\begin{aligned} \chi_{2k-1} + \chi_{2k} \bmod 2\pi &= 0 \qquad \text{for all } k \in \mathbb{N} \\ \psi_1 = 0,\ \psi_{2k} + \psi_{2k+1} \bmod 2\pi &= 0 \qquad \text{for all } k \in \mathbb{N} \end{aligned}$$

and with the property that $\chi_k + \psi_k \bmod 2\pi = \varphi_k$ for all $k \in \mathbb{N}$. To this end, we have to set

$$\chi_1 = \varphi_1, \qquad \psi_2 = \varphi_1 + \varphi_2 \bmod 2\pi, \qquad \chi_3 = \varphi_1 + \varphi_2 + \varphi_3 \bmod 2\pi$$

and so on. Now we can set

$$v(u) = \sum_{k=1}^{\infty}{}^{\oplus}\, \mathrm{e}^{\mathrm{i}\chi_k} p_k \qquad \text{and} \qquad w(u) = \sum_{k=1}^{\infty}{}^{\oplus}\, \mathrm{e}^{\mathrm{i}\psi_k} p_k.$$

The special choice of $\vec{\chi}$ and $\vec{\psi}$ brings about

$$v(u)w(u) = \sum_{k=1}^{\infty}{}^{\oplus}\, e^{i\varphi_k} p_k = u.$$

This means if we are able to represent both $v(u)$ and $w(u)$ as products of positive invertible operators with the required properties we will have proved the proposition.

But since both $v(u)$ and $w(u)$ are direct sums of SU(2) and SU(1) matrices (note the special construction of these operators), Proposition 1 states that this is the case here. Simply take the operators provided by Proposition 1 and form the direct sum of them. There are no convergence problems or ambiguities since the application to vectors of $\mathcal{H}$ is well-defined. (The case of SU(1) is trivial since it contains only $\mathbb{1}$.) As for the constants, the number N'' of Proposition 2 can be chosen twice the number N' of Proposition 1 – should there be less matrices one has to add a suitable number of unit matrices –; c and d can be the same as in Proposition 1 since the direct sum of operators has a norm which does not exceed the maximum of the initial norms ∎

Remark 3. The argumentation used in the preceding proof can be easily adjusted to the case of an n-dimensional Hilbert space $\mathcal{H}$. There we can find for each operator (matrix) $u \in \mathrm{SU}(n)$ at most N'' positive invertible operators the product of which is u. This is the case since in a finite-dimensional Hilbert space, the matrix of every unitary operator is unitarily diagonalizable. The restriction to $\mathrm{SU}(n)$ is necessary because the application of $v(u)$ and $w(u)$ to Proposition 1 requires that they have determinant 1. In fact, only operators with a positive determinant can be the product of positive invertible operators, and therefore only elements of $\mathrm{SU}(n)$ can occur.

Finally we have to consider whether *every* unitary operator can be the product of positive invertible operators. To this end, we use the following result.

Proposition 3. *Every unitary operator on an infinite-dimensional Hilbert space is the product of at most four symmetries, i.e. unitary Hermitian operators.*

This result can be found, e.g., in [6: Problem 112], and for reasons of convenience the idea of the proof is restated here.

Proof of Proposition 3. Begin by representing $\mathcal{H}$ as the direct sum of a sequence $\{\mathcal{H}_n\}$ of equidimensional subspaces each of which reduces the given unitary operator u. It is convenient to let the index n run through all integers.

Relative to the fixed direct sum decomposition $\mathcal{H} = \sum_n{}^{\oplus} \mathcal{H}_n$, define a right shift as a unitary operator s such that $s\mathcal{H}_n = \mathcal{H}_{n+1}$, and define a left shift as a unitary operator t such that $t\mathcal{H}_n = \mathcal{H}_{n-1}$ $(n = 0, \pm 1, \pm 2, \ldots)$. The equi-dimensionality of all the $\mathcal{H}_n$'s guarantees the existence of shifts. If s is an arbitrary right shift, write $t = s^* u$. Since $t\mathcal{H}_n = s^* u \mathcal{H}_n = s^* \mathcal{H}_n = \mathcal{H}_{n-1}$ for all n, it follows that t is a left shift. Since $u = st$, it follows that every unitary operator is the product of two shifts; the proof will be completed by showing that every shift is the product of two symmetries.

Since the inverse (equivalently, the adjoint) of a left shift is a right shift, it is sufficient to consider right shifts. Suppose then that s is a right shift; let p be the operator that is equal to s^{1-2n} on $\mathcal{H}_n$ and let q be the operator that is equal to s^{-2n} on $\mathcal{H}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$. It is easy to see that p and q are symmetries. If $\varphi \in \mathcal{H}_n$, then

$q\varphi = s^{-2n}\varphi \in s^{-2n}\mathcal{H}_n = \mathcal{H}_{-n}$, so that $pq\varphi = ps^{-2n}\varphi = s^{1-2(-n)}s^{-2n}\varphi = s\varphi$. This completes the proof ∎

Now we have the tools to prove Theorems 1 and 2.

Proof of Theorem 1. Every symmetry is an element of $\mathrm{U_d}(\mathcal{H})$ since every symmetry is the difference of two orthoprojections where the one is the orthogonal complement of the other. Hence we can apply Proposition 2 to each of the symmetries of Proposition 3 and get the desired result. It is clear that c and d can be chosen the same as in Proposition 2 and $N = 4N''$ ∎

Proof of Theorem 2. It is known that $n_j | n_{j+1}$ holds for all j. Let $k_1 = n_1$ and $k_j = n_j/(n_{j-1})$ for $j \geq 2$. Without loss of generality, we can assume that $k_j \geq 3$ for all j. (If necessary, we can achieve it by thinning out the type sequence $\{n_j\}$.)

In M there exist systems of mutually orthogonal and mutually equivalent (in von Neumann sense) orthoprojections

$$\begin{array}{rll} \{p_i\}_{i=1}^{k_1} \subseteq M_1 & \text{with} & p_1 + p_2 + \dots p_{k_1} = \mathbb{1}_M \\ \{p_{ij}\}_{i=1}^{k_1}{}_{j=1}^{k_2} \subseteq M_2 & \text{with} & p_{i1} + p_{i2} + \dots p_{ik_2} = p_i \\ \{p_{ijk}\}_{i=1}^{k_1}{}_{j=1}^{k_2}{}_{k=1}^{k_3} \subseteq M_3 & \text{with} & p_{ij1} + p_{ij2} + \dots + p_{ijk_3} = p_{ij} \end{array}$$

etc. Now we define orthoprojections P_i and P_i' as follows:

$$\begin{array}{lcl} P_1 = p_1 + p_2 + \dots + p_{k_1-2} & & P_1' = p_{k_1-1} \\ P_2 = p_{k_1 1} + p_{k_1 2} + \dots + p_{k_1 k_2 - 2} & \text{and} & P_2' = p_{k_1 k_2 - 1} \\ P_3 = p_{k_1 k_2 1} + p_{k_1 k_2 2} + \dots + p_{k_1 k_2 k_3 - 2} & & P_3' = p_{k_1 k_2 k_3 - 1} \end{array}$$

etc. Finally, let

$$\begin{array}{lcl} Q_1 = P_1 + P_1' & & Q_1' = P_1' + P_2 \\ Q_2 = P_2 + P_2' & \text{and} & Q_2' = P_2' + P_3 \\ Q_3 = P_3 + P_3' & & Q_3' = P_3' + P_4 \end{array}$$

etc. Let for the moment $\varphi, \varphi_2, \varphi_4, \ldots \in [0, 2\pi)$ and in addition $\psi_1, \psi_3, \psi_5, \ldots \in [0, 2\pi)$ be arbitrarily given. Let $\vec{\varphi} := (\varphi, \varphi_2, \varphi_4, \ldots)$ and $\vec{\psi} := (\psi_1, \psi_3, \psi_5, \ldots)$.

We have $M_1 \simeq \mathrm{M}_{n_1}(\boldsymbol{C})$ and $Q_1 \in M_1$, hence we can embed $Q_1 M_1 Q_1$ into $\mathrm{M}_{n_1}(\boldsymbol{C})$; thus we have $Q_1 M_1 Q_1 \simeq \mathrm{M}_{k_1-1}(\boldsymbol{C})$. Using Proposition 2/Remark 3, this means there exist invertible $a_1^1(\varphi), \ldots, a_{N''}^1(\varphi) \in \mathrm{M}_{k_1-1}(\boldsymbol{C})_+$ with the boundedness properties

$$\|a_i^1(\varphi)\| \leq c \qquad \text{and} \qquad \|a_i^1(\varphi)^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$a_{N''}^1(\varphi) \cdots a_1^1(\varphi) = \operatorname{diag}\left[e^{i\varphi}, e^{i\varphi}, \ldots, e^{i\varphi}, e^{i\varphi_1}\right]$$
$$\varphi_1 = -(k_1 - 2)\varphi \bmod 2\pi.$$

Since $Q_1 M_1 Q_1 \simeq \mathrm{M}_{k_1-1}(\boldsymbol{C})$ holds we can regard $a_1^1(\varphi), \ldots, a_{N''}^1(\varphi)$ as elements of $(Q_1 M_1 Q_1)_+$ and in this view we can write $a_{N''}^1(\varphi) \cdots a_1^1(\varphi) = e^{i\varphi} P_1 \oplus e^{i\varphi_1} P_1'$.

Similarly, we have $M_2 \simeq \mathrm{M}_{n_2}(\boldsymbol{C})$ and $Q_2 \in M_2$, hence we can embed $Q_2 M_2 Q_2$ into $\mathrm{M}_{n_2}(\boldsymbol{C})$, too; thus we have $Q_2 M_2 Q_2 \simeq \mathrm{M}_{k_2-1}(\boldsymbol{C})$. Again there exist invertible $a_1^2(\varphi_2), \ldots, a_{N''}^2(\varphi_2) \in \mathrm{M}_{k_2-1}(\boldsymbol{C})_+$ with the boundedness properties

$$\|a_i^2(\varphi_2)\| \leq c \qquad \text{and} \qquad \|a_i^2(\varphi_2)^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$a_{N''}^2(\varphi_2) \cdots a_1^2(\varphi_2) = \operatorname{diag}\left[e^{i\varphi_2}, e^{i\varphi_2}, \ldots, e^{i\varphi_2}, e^{i\varphi_3}\right]$$
$$\varphi_3 = -(k_2 - 2)\varphi_2 \bmod 2\pi.$$

Since $Q_2 M_2 Q_2 \simeq \mathrm{M}_{k_2-1}(\boldsymbol{C})$ holds we can again regard $a_1^2(\varphi_2), \ldots, a_{N''}^2(\varphi_2)$ as elements of $(Q_2 M_2 Q_2)_+$ and write $a_{N''}^2(\varphi_2) \cdots a_1^2(\varphi_2) = e^{i\varphi_2} P_2 \oplus e^{i\varphi_3} P_2'$.

Proceeding this way, we finally obtain $a_i^j(\varphi_{2(j-1)}) \in Q_j M_j Q_j \subseteq Q_j M Q_j$ for all i, j. All these operators are bounded by c and act on mutually orthogonal subspaces of $\mathcal{H}$ for different j. Hence for each i the operator

$$a_i(\vec{\varphi}) = a_i^1(\varphi) \oplus \sum_{j=2}^{\infty}{}^{\oplus} a_i^j(\varphi_{2(j-1)})$$

is well defined. For these operators we have the inclusion $a_i(\vec{\varphi}) \in M_+$, the boundedness properties

$$\|a_i(\vec{\varphi})\| \leq c \qquad \text{and} \qquad \|a_i(\vec{\varphi})^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$a_{N''}(\vec{\varphi}) \cdots a_1(\vec{\varphi}) = e^{i\varphi} P_1 \oplus e^{i\varphi_1} P_1' \oplus e^{i\varphi_2} P_2 \oplus e^{i\varphi_3} P_2' \oplus \cdots.$$

Analogously we have $M_2 \simeq \mathrm{M}_{n_2}(\boldsymbol{C})$ and $Q_1' \in M_2$, therefore we can embed $Q_1' M_2 Q_1'$ into $\mathrm{M}_{n_2}(\boldsymbol{C})$; thus we have $Q_1' M_2 Q_1' \simeq \mathrm{M}_{2k_2-2}(\boldsymbol{C})$. This means that there exist invertible $b_1^1(\psi_1), \ldots, b_{N''}^1(\psi_1) \in \mathrm{M}_{2k_2-2}(\boldsymbol{C})_+$ with the boundedness properties

$$\|b_i^1(\psi_1)\| \leq c \qquad \text{and} \qquad \|b_i^1(\psi_1)^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$b_{N''}^1(\psi_1) \cdots b_1^1(\psi_1) = \operatorname{diag}\Big[\underbrace{e^{i\psi_1}, \ldots, e^{i\psi_1}}_{k_2 \text{ times}}, \underbrace{e^{i\psi_2}, \ldots, e^{i\psi_2}}_{(k_2-2) \text{ times}}\Big]$$
$$\psi_2 = -\frac{k_2}{k_2 - 2}\psi_1 \bmod 2\pi.$$

Since $Q_1' M_2 Q_1' \simeq \mathrm{M}_{2k_2-2}(\boldsymbol{C})$ holds we can regard $b_1^1(\psi_1), \ldots, b_{N''}^1(\psi_1)$ as elements of $(Q_1' M_2 Q_1')_+$, and under this agreement we can write $b_{N''}^1(\psi_1) \cdots b_1^1(\psi_1) = e^{i\psi_1} P_1' \oplus e^{i\psi_2} P_2$.

Again, we have $M_3 \simeq \mathrm{M}_{n_3}(\boldsymbol{C})$ and $Q_2' \in M_3$, hence we can embed $Q_2'M_3Q_2'$ into $\mathrm{M}_{n_3}(\boldsymbol{C})$; again we have $Q_2'M_3Q_2' \simeq \mathrm{M}_{2k_3-2}(\boldsymbol{C})$. There exist invertible operators $b_1^2(\psi_3), \ldots, b_{N''}^2(\psi_3) \in \mathrm{M}_{2k_3-2}(\boldsymbol{C})_+$ with the boundedness properties

$$\|b_i^2(\psi_3)\| \leq c \qquad \text{and} \qquad \|b_i^2(\psi_3)^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$b_{N''}^2(\psi_3)\cdots b_1^2(\psi_3) = \operatorname{diag}\Big[\underbrace{\mathrm{e}^{\mathrm{i}\psi_3},\ldots,\mathrm{e}^{\mathrm{i}\psi_3}}_{k_3 \text{ times}},\underbrace{\mathrm{e}^{\mathrm{i}\psi_4},\ldots,\mathrm{e}^{\mathrm{i}\psi_4}}_{(k_3-2) \text{ times}}\Big]$$

$$\psi_4 = -\frac{k_3}{k_3-2}\psi_3 \bmod 2\pi.$$

Since $Q_2'M_3Q_2' \simeq \mathrm{M}_{2k_3-2}(\boldsymbol{C})$ holds we can regard $b_1^2(\psi_3), \ldots, b_{N''}^2(\psi_3)$ as elements of $(Q_2'M_3Q_2')_+$ and write $b_{N''}^2(\psi_3)\cdots b_1^2(\psi_3) = \mathrm{e}^{\mathrm{i}\psi_3}P_2' \oplus \mathrm{e}^{\mathrm{i}\psi_4}P_3$.

Proceeding further as indicated, we finally arrive at a family $\{b_i^j(\psi_{2j-1})\}$ of operators obeying $b_i^j(\psi_{2j-1}) \in Q_j'M_{j+1}Q_j' \subseteq Q_j'MQ_j'$ for all i, j. These operators are also bounded by c and act on mutually orthogonal subspaces of $\mathcal{H}$ for different j. Hence, the setting

$$b_i(\vec{\psi}) := P_1 \oplus \sum_{j=1}^{\infty}{}^{\oplus}\, b_i^j(\psi_{2j-1})$$

yields an expression which makes sense as a bounded operator. We have the inclusion $b_i(\vec{\psi}) \in M_+$, the boundedness properties

$$\|b_i(\vec{\psi})\| \leq c \qquad \text{and } \|b_i(\vec{\psi})^{-1}\| \leq d \qquad \text{for all } i$$

and the product representation

$$b_{N''}(\vec{\psi})\cdots b_1(\vec{\psi}) = P_1 \oplus \mathrm{e}^{\mathrm{i}\psi_1}P_1' \oplus \mathrm{e}^{\mathrm{i}\psi_2}P_2 \oplus \mathrm{e}^{\mathrm{i}\psi_3}P_2' \oplus \mathrm{e}^{\mathrm{i}\psi_4}P_3 \oplus \cdots .$$

For given φ, we now define $\{\varphi_j\}$ and $\{\psi_k\}$ successively by requiring

$$\begin{aligned} \psi_1 &= \varphi - \varphi_1 \bmod 2\pi, & \varphi_2 &= \varphi - \psi_2 \bmod 2\pi \\ \psi_3 &= \varphi - \varphi_3 \bmod 2\pi, & \varphi_4 &= \varphi - \psi_4 \bmod 2\pi \end{aligned}$$

etc. Thus we finally get the product representation

$$a_{N''}(\vec{\varphi})\cdots a_1(\vec{\varphi})b_{N''}(\vec{\psi})\cdots b_1(\vec{\psi}) = \mathrm{e}^{\mathrm{i}\varphi}\mathbb{1}_M.$$

The constant $\tilde{N}$ in the theorem can therefore be chosen twice the constant N'' in Proposition 2; c and d apply as well in this theorem ∎

3. Some applications and discussion

The results of the first section are useful especially for the investigation of holonomy groups of states over von Neumann algebras. In the following, the concept of the holonomy group of a normal state, i.e. a normalized to one normal positive linear form, over a von Neumann algebra will be briefly introduced. For more detailed information, the reader is referred to [1] and [2].

Throughout this section we suppose that M is a von Neumann algebra acting on some Hilbert space $\mathcal{H}$. We assume that M admits a cyclic and separating vector Ω. (For definitions, cf. [12: Definition 3.16].) Then it is known that there exists a faithful normal state on M (cf. [5: Proposition 2.5.6]) and that for any normal positive linear form ω over M there is at least one vector $\varphi \in \mathcal{H}$ which implements ω, i.e. $\omega(x) = \langle x\varphi, \varphi\rangle$ for all $x \in M$ (cf. [5: Theorem 2.5.31]). The set of all vectors which implement ω will be denoted by $\mathcal{S}(\omega)$.

A pair $\{\omega, \sigma\}$ of normal positive linear forms over M such that there do not exist normal positive linear forms ω', σ' with $\omega' \leq \omega$, $\sigma' \leq \sigma$ and $\omega' \perp \sigma$, $\sigma' \perp \omega$ apart from the trivial pair $\{0, 0\}$ in [1] has been referred to as a $\ll$-minimal pair. In this case, the situation that for $\psi \in \mathcal{S}(\sigma)$ and $\varphi \in \mathcal{S}(\omega)$ the condition

$$h_{\psi,\varphi} : M' \to \boldsymbol{C}, \quad h_{\psi,\varphi}(\cdot) := \langle(\cdot)\varphi, \psi\rangle \quad \text{positive}$$

is satisfied will be referred to by the notation $\psi \parallel \varphi$. One can easily prove (cf. [1: Remark 6.8(2)]) that for $\{\omega, \sigma\}$ such that each $\varphi \in \mathcal{S}(\omega)$ uniquely determines $\psi \in \mathcal{S}(\sigma)$ with $h_{\psi,\varphi} \geq 0$, and vice versa, the $\ll$-minimality is fulfilled. Note that any pair of faithful normal positive linear forms yields a $\ll$-minimal pair. As it comes out the relation $\parallel$ is reflexive and symmetric, but in general not transitive.

For a given linear form ω on M and $a \in M$, the notation ω^a is an abbreviation for the linear form $\omega(a^*(\cdot)a)$. Then we have $(\omega^a)^b = \omega^{ba}$. Assume $\{\omega, \sigma\}$ is $\ll$-minimal. In the special case $\sigma = \omega^a$ $(a \in M_+)$ for given $\varphi \in \mathcal{S}(\omega)$ the vector $\psi = a\varphi$ is the uniquely defined vector $\psi \in \mathcal{S}(\sigma)$ with $\psi \parallel \varphi$ (cf. [1: Example 6.2(1)]).

One might consider a "path" $\gamma \subseteq M_{*+}$, which is a finite sequence $\{\omega_j\}_{0\leq j\leq n}$ $(n \in \mathbb{N}_0)$ of normal positive linear forms such that $\{\omega_k, \omega_{k+1}\}$ is $\ll$-minimal for $0 \leq k \leq n-1$. According to the above considerations, for such a path to given $\varphi \in \mathcal{S}(\omega_0)$ there exist in a unique manner vectors $\varphi_j \in \mathcal{S}(\omega_j)$ $(j = 0, 1, \ldots, n)$ such that $\varphi_0 = \varphi$ and $\varphi_k \parallel \varphi_{k+1}$ for $0 \leq k \leq n-1$. Call $\varphi(\gamma) := \varphi_n$. For $\omega = \omega_0 = \omega_n$ the path is closed and is referred to as an *ω-loop*. Then, the map $\mathcal{S}(\omega) \ni \varphi \longmapsto \varphi(\gamma) \in \mathcal{S}(\omega)$ is an automorphism (a homeomorphism onto itself) of the complete metric space $\mathcal{S}(\omega)$. The effect of the non-transitivity of the relation $\parallel$ then is that in general the group of all these automorphisms $G_0(\omega)$, the ***holonomy group*** of ω, is non-trivial provided the algebra M is non-commutative. This means that the transport of implementing vectors along paths accomplished in accordance with $\parallel$ in general leads to an effect of anholonomy which manifests in the fibre $\mathcal{S}(\omega)$ over ω.

It turns out that in case γ is an ω-loop, for a fixed vector $\varphi \in \mathcal{S}(\omega)$, each homeomorphism in the holonomy group can be written as the action of a certain unitary operator coming from the *commutant* M' of M. For ω being a faithful state, this operator is uniquely determined (cf. [2: Lemma 8.2]). The group of all such operators is called the ***ω-phase group at φ***. It is anti-isomorphic to the holonomy group at ω. Thus, we can discuss the holonomy group in terms of the ω-phase group at some $\varphi \in \mathcal{S}(\omega)$, which is more convenient.

It is known (cf. [2: Theorem 11.5]) that in case of $M \simeq \mathrm{M}_N(\boldsymbol{C})$, $G_0(\omega) \simeq \mathrm{SU}(N)$ holds for a faithful ω, and $G_0(\omega) \simeq \mathrm{U}(k)$ whenever $\dim s(\omega) = k < N$ occurs. In the case $M \simeq \mathrm{B}(\mathcal{H}_0)$, with a separable Hilbert space $\mathcal{H}_0$, the holonomy group of a normal positive linear form with finite-dimensional support is isomorphic with $\mathrm{U}(\dim s(\omega))$, whereas for those the linear forms of which have infinite-dimensional supports one easily sees that the holonomy groups are mutually isomorphic (thus one needs only to calculate the holonomy group of a faithful normal positive linear form).

Now consider the following construction: Assume $M = \{\mathrm{L}_x \mid x \in \mathrm{B}(\mathcal{H}_0)\}$ is the von Neumann algebra of left multiplications on elements of the Hilbert space $\mathcal{H}$ of all Hilbert-Schmidt operators over the separable Hilbert space $\mathcal{H}_0$. The scalar product in this Hilbert space is given as $\langle x, y\rangle_{\mathrm{HS}} = \operatorname{tr} xy^*$. Then, $M' = \{\mathrm{R}_x \mid x \in \mathrm{B}(\mathcal{H}_0)\}$ is the von Neumann algebra of right multiplications. It is known that $M \simeq \mathrm{B}(\mathcal{H}_0)$ holds, i.e. we are in the standard situation of a factor of type I_n, $n \in \mathbb{N}$ or $n = \aleph_0$.

By means of Theorem 1, we can state the following

Theorem 3. *Let M be the type-I_∞ factor of left multiplications as in the above situation. Let a faithful normal state ω be given. Let $\sqrt{\omega}$ be the square root of the density matrix of ω. Then $\varphi := \sqrt{\omega} \in \mathcal{S}(\omega)$, and the ω-phase group of ω at φ contains every unitary operator that commutes with $\sqrt{\omega}$ as a Hilbert-Schmidt operator, i.e. every $\mathrm{R}_u \in \mathrm{B}(\mathcal{H})$ with $u\sqrt{\omega} = \sqrt{\omega}u$ on $\mathcal{H}_0$.*

Proof. Let $u \in \mathrm{B}(\mathcal{H}_0)$ be a unitary operator that commutes with $\sqrt{\omega}$. In accordance with Theorem 1, there are positive invertible operators $a_1, \ldots, a_n \in \mathrm{B}(\mathcal{H}_0)$ so that $a_n \cdots a_1 = u$. Consider the path

$$\gamma : \omega \longrightarrow \omega^{a_1} \longrightarrow \omega^{a_2 a_1} \longrightarrow \cdots \longrightarrow \omega^{a_n \cdots a_1} = \omega^u.$$

All pairs of neighbouring linear forms are $\ll$-minimal due to [1: Example 6.2(1)]. Since u commutes with $\sqrt{\omega}$, it commutes with ω as well, and we have

$$\omega^u = \omega(u^*(\cdot)u) = \operatorname{tr} \omega u^*(\cdot)u = \operatorname{tr} u\omega u^*(\cdot) = \operatorname{tr} uu^*\omega(\cdot) = \operatorname{tr} \omega(\cdot) = \omega,$$

i.e. γ is a loop. Furthermore, we have

$$\varphi(\gamma) = a_n \cdots a_1 \varphi = u\sqrt{\omega} = \sqrt{\omega}u = \mathrm{R}_u \varphi.$$

(Note the above remark on the $\|$ relation for vectors that implement neighbouring linear forms.) Since $\mathrm{R}_u \in M'$, this must be the uniquely determined element of the ω-phase group for γ ∎

The above mentioned operator $\omega^{1/2}$ is a Hilbert-Schmidt operator over $\mathcal{H}_0$. Since it belongs to a faithful normal state ω its kernel is $\{0\}$, and therefore it is invertible. The inverse is an unbounded, closed, selfadjoint operator over $\mathcal{H}_0$ and will be denoted by $\omega^{-1/2}$. Its domain of definition will be denoted by $\mathcal{D}(\omega^{-1/2})$.

Lemma 1. *Let M and ω be given as in the preceding theorem. The ω-phase group at the implementing vector $\omega^{1/2}$ contains all unitaries R_u for which $\omega^{1/2}u\omega^{-1/2}$ as well as $\omega^{1/2}u^*\omega^{-1/2}$, both defined on $\mathcal{D}(\omega^{-1/2})$, are closable operators over $\mathcal{H}_0$.*

Proof. All we have to do is finding an ω-loop γ with $\omega^{1/2}(\gamma) = \mathrm{R}_u\omega^{1/2}$. Since $\omega^{1/2}u\omega^{-1/2}$ is closable, there exists a polar decomposition of the closure of this operator as follows:

$$\overline{\omega^{1/2}u\omega^{-1/2}} = v|\overline{\omega^{1/2}u\omega^{-1/2}}|$$

with a certain partial isometry $v \in \mathrm{B}(\mathcal{H}_0)$. Since $\omega^{-1/2}$ is closed and $\omega^{1/2}u^*$ is bounded, we have

$$(\omega^{1/2}u^*\omega^{-1/2})^* = \omega^{-1/2}u\omega^{1/2}$$

(cf. [9: §5.10.III.ε)]) and the operator on the right-hand side is densely defined. Hence the range $\mathcal{R}(\omega^{-1/2}u^*\omega^{1/2})$ is dense in $\mathcal{H}_0$. Similarly, since $\omega^{1/2}u^*\omega^{-1/2}$ is densely defined, $\mathcal{R}(\omega^{1/2}u\omega^{-1/2})$ is dense in $\mathcal{H}_0$. Hence v is a unitary operator.

We have

$$|\overline{\omega^{1/2}u\omega^{-1/2}}|\omega^{1/2} = v^*\overline{\omega^{1/2}u\omega^{-1/2}}\omega^{1/2} = v^*\omega^{1/2}u\omega^{-1/2}\omega^{1/2} = v^*\omega^{1/2}u$$

and

$$\omega^{1/2}v^*\omega^{1/2}u = \omega^{1/2}|\overline{\omega^{1/2}u\omega^{-1/2}}|\omega^{1/2} \geq 0.$$

Since $\omega^{1/2}v^*\omega^{1/2}u$ is positive, the linear form

$$h_{\omega^{1/2},v^*\omega^{1/2}u}(\cdot) = \langle v^*\omega^{1/2}u(\cdot),\omega^{1/2}\rangle_{\mathrm{HS}} = \operatorname{tr}\omega^{1/2}v^*\omega^{1/2}u(\cdot)$$

is positive, too, and therefore we have $\omega^{1/2}\|v^*\omega^{1/2}u$.

Let now $a_1,\ldots,a_N \in \mathrm{B}(\mathcal{H}_0)_+$ be given such that $a_N\cdots a_1 = v$ and consider the loop

$$\gamma : \omega \to \omega^{v^*} \to \omega^{a_1v^*} \to \omega^{a_2a_1v^*} \to \cdots \to \omega^{a_N\cdots a_1v^*} = \omega.$$

Since the $a_1,\ldots,a_N$ are positive, for

$$\varphi = \varphi_0 = \omega^{1/2}, \qquad \varphi_1 = v^*\omega^{1/2}u$$
$$\varphi_k = a_{k-1}\cdots a_1v^*\omega^{1/2}u \qquad (k = 2,3,\ldots,N+1)$$

we have

$$\varphi_0 \in \mathcal{S}(\omega), \qquad \varphi_1 \in \mathcal{S}(\omega^{v^*})$$
$$\varphi_k \in \mathcal{S}(\omega^{a_{k-1}\cdots a_1v^*}) \qquad (k = 2,3,\ldots,N+1)$$

and $\varphi_k \,\|\, \varphi_{k+1}$ $(k = 0,1,\ldots,N)$. Hence we have

$$\varphi(\gamma) = a_N\cdots a_1v^*\omega^{1/2}u = vv^*\omega^{1/2}u = \omega^{1/2}u = \mathrm{R}_u\omega^{1/2}$$

or R_u is the ω-phase of the loop γ ∎

Remark 4. By Lemma 1 the problem of finding the holonomy group of ω is transformed to an operator theoretic problem. Note that the condition of Lemma 1 is only sufficient.

In the case of a more general hyperfinite von Neumann algebra, we can say the following.

Theorem 4. *Let M be given as in Theorem 2. Let ω be an arbitrary faithful normal state over M, and $\varphi \in S(\omega)$. The ω-phase group at φ contains all unitaries that are multiples of $\mathbb{1}$.*

Proof. Since for each $\lambda \in \mathbb{R}$ there are $a_1, \ldots, a_{\tilde{N}} \in M_+$ with $a_{\tilde{N}} \cdots a_1 = e^{i\lambda}\mathbb{1}$, we can construct an ω-loop which maps φ to $e^{i\lambda}\varphi$ in a similar way as in the proof of Theorem 3. Since $e^{i\lambda}\mathbb{1}$ is always a unitary element of M' for all $\lambda \in \mathbb{R}$, the proof is complete ∎

Remark 5. The last theorem states that in all hyperfinite von Neumann algebras the full unitary group U(1) is contained in the holonomy group of a faithful normal state. Note that for pure, non-tracial normal states, it is known that the holonomy group is isomorphic to U(1) (cf. [2: Theorem 11.2]). This means that if we restrict ourselves to finite-dimensional and hyperfinite factors the following is true: With the only exception of faithful states over a finite-dimensional factor, the holonomy group of every normal state contains at least U(1).

Remark 6. Apart from the applications of Propositions 1 and 2 to the theory of holonomy groups, these propositions and Theorem 1 could be of use in other fields of mathematics, too. Therefore it could be interesting to find the best upper bound for the number N. Starting with Proposition 1, N' has to be greater than 3 because $-\mathbb{1}$ can not be the product of three positive invertible operators since $abc = -\mathbb{1}$ implies $ab = -c^{-1}$, and this is impossible since ab has a non-negative spectrum. A similar argument rules out $N' = 4$ such that N' has at least to be 5. On the other hand, $N' = 27$ suffices which can be shown by an example that has $k = 9$, k defined as in the proof of Proposition 1.

Following the line of conclusions, an upper bound for N'' of Proposition 2 is 54, and, finally, an upper bound for N of Theorem 1 is 216. But since the proofs were only intended to yield the *existence* of these constants, these numbers must be far from the respective best upper bounds. Furthermore, N' heavily depends on the choice of the matrices v and a, and therefore on the constants c and d as well. Making N' small means making c and d large and vice versa. Finding the best upper bounds for N', N'' and N in dependence on the choice of c and d could be the subject of separate research.

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4. Note added in proof

While proof-reading the author got aware of some other results on products of special matrices, as positive or Hermitian ones, and which are closely related to this subject. A short discussion of these results follows.

In the context of von Neumann algebras, Størmer in 1965 proved a lemma which deals with the decomposition of $i\mathbb{1}$ into a sum of two products of four self-adjoint operators (cf. [11: Lemma 2.13]). In special cases there is even a decomposition of $i\mathbb{1}$ into a product of three self-adjoint operators.

In 1968 and 1970 Ballantine examined products of positive $(n \times n)$-matrices over the real [3] and the complex field [4]. In the latter work it is shown that every invertible complex matrix is the product of at most five positive invertible matrices. Sourour gave in [10] a shorter proof of this fact using a more general result about factorization of matrices. Finally, Wu [13] extended Ballantine's result to positive *semi*definite matrices.

An extensive survey of solved and unsolved problems concerning products of symmetric, Hermitean, positive and normal matrices is given in [7], where also the case of bounded operators over infinite-dimensional Hilbert spaces is considered.

The difference of these results to the ones of this work lies in the observation that there is a global upper bound for the norms of the positive operators as well as for those of their inverses. This is a vital feature of the constructed matrices for the direct sum of them to yield a *bounded* invertible operator.

Now another remark on the occurring constants. As what has been said above, since our first goal was to say something about holonomy groups, we were only interested in the *existence* of constants N and $\tilde{N}$. Therefore we did not draw our attention to seeking best upper bounds. Nevertheless, something better than the statement in Remark 6 can be said about the constant N' of Proposition 1.

Ballantine's result is that every invertible complex matrix is the product of at most five positive invertible matrices. Since this cannot be true if we require that the latter fulfil the norm inequalities of Proposition 1 and set the constants c and d simultaneously "too close" to 1, one could ask for the dependence on c and d of N'. Conversely, if we allow for "sufficiently" large upper bounds, $N' = 5$ should do for Proposition 1. This can be seen easily by slightly modifying the proof of Proposition 1 as follows.

Consider instead of terms of the kind $a(vav^*)|a(vav^*)|^{-1}$ now terms of the kind

$$(v^3av^{*3})(v^2av^{*2})(vav^*)\,a\,\left|(v^3av^{*3})(v^2av^{*2})(vav^*)a\right|^{-1} =: u.$$

For, say, $a = \operatorname{diag}[25, 1]$ and $v = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$ and t going from 0 to $\frac{\pi}{2}$, u yields matrices with all possible spectra for unitary matrices. This is due to the fact that, for $a = \operatorname{diag}[1, 0]$, the partial isometry in the polar decomposition of $(v^3av^{*3})(v^2av^{*2})(vav^*)a$ can be uniquely extended to the following orthogonal matrix with determinant 1:

$$\begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix} \quad \text{for } t \in \left[0, \frac{\pi}{2}\right),$$

and this is equal to the pointwise with respect to t limit of u for $a = \operatorname{diag}[k, 1]$ as k tends to infinity. The convergence is the better the nearer t is to 0. The number 25 was found numerically. A by-product of these calculations was the following equality which was found by suitably rounding the numbers obtained by numerical calculations:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -9 \\ -9 & 20 \end{pmatrix} \begin{pmatrix} 10 & 12 \\ 12 & 16 \end{pmatrix} \begin{pmatrix} \frac{346}{15} & -5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} \frac{163}{164840} & \frac{30}{4121} \\ \frac{30}{4121} & \frac{467}{8242} \end{pmatrix}$$

Another factorization of $-\mathbb{1}$ into five positive (2×2)-matrices can be found in [7]. The example given there is "more sound" than the above mentioned one in so far as the entries of the positive matrices have smaller denominators, but unfortunately it is given without any word of reasoning.

As for what matrices can be the product of four positive matrices obeying the norm inequalities for given c and d, things seem to be more intricate. In this case, at least the obvious approach $u = (v^2 a v^{*2})(vav^*)a|(v^2 a v^{*2})(vav^*)a|^{-1}$ fails, for, then in case of $a = \operatorname{diag}[1,0]$ the special orthogonal matrix in the polar decomposition of $(v^2 a v^{*2})(vav^*)a$ equals $\left(\begin{smallmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{smallmatrix}\right)$ for $t \in [0, \frac{\pi}{2})$, and hence values of $2t$ near π can only be obtained by letting k for $a = \operatorname{diag}[k,1]$ and therefore the constants c as well as d tend to infinity. The latter is due to the term $|(v^2 a v^{*2})(vav^*)a|^{-1}$.

Without knowing whether or not the following numbers are the best upper bounds, we can at least state that for, say, $c = 25$ and $d = 25^4 = 390625$ we have $N' = 5$ (Proposition 1), $N'' \leq 10$ (Proposition 2/Remark 3), $\widetilde{N} \leq 20$ (Theorem 2) and $N \leq 40$ (Theorem 1).

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