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On Associated and Co-Associated Complex Differential Operators

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Abstract. The paper deals with initial value problems of the form

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \qquad u = u_0 \ \text{ for } \ t = 0$$

in $[0,T] \times G \subset \mathbb{R}_0^+ \times \mathbb{R}^n$ where $\mathcal{L}$ is a linear first order differential operator. The desired solutions will be sought in function spaces defined as kernel of a linear differential operator l being associated to $\mathcal{L}$. Mainly two assumptions are required for such initial value problems to be solvable: Firstly, the operators have to be associated, i.e. $lu = 0$ implies $l(\mathcal{L}u) = 0$. Secondly, an interior estimate $\|\mathcal{L}u\|_{G'} \leq c(G,G')\,\|u\|_G$ (with $G' \subset G$) must be true. Moreover, operators $\mathcal{L}$ are investigated possessing a family of associated operators l_k (which then are said to be co-associated).

The present paper surveys the use of associated and co-associated differential operators for solving initial value problems of the above (Cauchy-Kovalevskaya) type. Discussing interior estimates as starting point for the construction of related scales of Banach spaces, the paper sets up a possible framework for further generalizations. E.g., that way a theorem of Cauchy-Kovalevskaya type with initial functions satisfying a differential equation of an arbitrary order k (with not necessarily analytic coefficients) is obtained.

Keywords: *Initial value problems, interior estimates, Cauchy-Kovalevskaya theorem*

AMS subject classification: 35 F 10, 35 B 45, 35 A 10

1. An application of interior estimates of solutions of associated differential equations

Let G be a bounded domain in $\mathbb{R}^n$ and T a positive real number. Let, further, $\mathcal{L}$ be a linear first order differential operator defined on a set of functions sufficiently often differentiable. The coefficients are defined on $[0,T] \times G$ and continuous, at least. The paper is aimed at solving differential equations of type

$$\frac{\partial u}{\partial t} = \mathcal{L}u. \tag{1}$$

The solution can be understood in the distributional sense, too. For this purpose introduce the spaces $C_0^\infty(G)$ and $C_0^\infty([0,T] \times G)$ of test functions defined in G and $[0,T] \times G$,

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respectively. Suppose $\mathcal{L}$ possesses an *adjoint* operator $\mathcal{L}^*$ in the distributional sense for every fixed $t \in [0,T]$, i.e. one has

$$\int_G \varphi \mathcal{L} u \, dx = - \int_G u \mathcal{L}^* \varphi \, dx \, .$$

provided u is smooth and φ is an arbitrary test function belonging to $C_0^\infty(G)$. Then a (locally) integrable function $u = u(t,x)$ is called a *distributional solution* of the evolution equation (1) if for all $\varphi \in C_0^\infty([0,T] \times G)$ the relation

$$\int_{[0,T]\times G} u \left(\frac{\partial \varphi}{\partial t} - \mathcal{L}^* \varphi \right) dt dx = 0$$

is satisfied.

Suppose the differential operator l (of order k and having coefficients not depending on time) is *associated* to the differential operator $\mathcal{L}$ on the right-hand side of (1), i.e.

$$lu = 0 \qquad \Longrightarrow \qquad l(\mathcal{L}u) = 0.$$

In case that both the operators $\mathcal{L}$ and l have adjoint operators $\mathcal{L}^*$ and l^*, respectively, the condition for associated differential operators can be rewritten in the following distributional sense: for all test functions φ,

$$\int_G u l^* \varphi \, dx = 0 \qquad \Longrightarrow \qquad \int_G u \mathcal{L}^*(l^* \varphi) \, dx = 0.$$

Assume, finally, that the solutions of the associated differential equation $lu = 0$ satisfy an interior estimate of type

$$\|\mathcal{L}u\|_{G'} \le \frac{\text{const}}{\text{dist}(G', \partial G)} \|u\|_G$$

where G' is a subdomain of G having positive distance from the boundary ∂G of the given domain G and $\| \cdot \|$ denotes the norm of a suitably chosen function space (with respect to G' and G). Note that this interior estimate must not be true for all first order derivatives separately (e.g., in the case of holomorphic functions the interior estimate is needed for the complex derivative only but not for the real derivatives of real and imaginary part). Exhaust the domain G by a family $\{G_s\}_{0<s<s_0}$ of subdomains G_s satisfying the condition

$$\text{dist}(G_{s'}, \partial G_s) \ge \text{const} \cdot (s - s')$$

for any pair s, s' with $0 < s' < s < s_0$ where the constant does not depend on the choice of s and s'. Introduce a corresponding family $\{\mathcal{W}_s\}_{0<s<s_0}$ of Banach spaces $\mathcal{W}_s$ (with norm $\| \cdot \|_s$) embedded in each other by restriction, i.e. the restriction of a function $u \in \mathcal{W}_s$ to the subdomain $G_{s'}$ belongs to the space $\mathcal{W}_{s'}$ where the norm of the restriction can be estimated by $\|u\|_{s'} \le \|u\|_s$. Then the above interior estimate yields

$$\|\mathcal{L}u\|_{s'} \le \frac{\text{const}}{s - s'} \|u\|_s$$

and, consequently, the abstract Cauchy-Kovalevskaya theorem is applicable [1].

That way the following theorem has been proved.

Theorem. *Suppose the linear first order differential operator $\mathcal{L}$ has an associated differential operator l (of order k) where the solutions of the equation $lu = 0$ permit an interior estimate. Suppose, moreover, that u_0 is a solution of that equation having finite norm $\|u_0\|_G$. Then the initial value problem*

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \qquad u(0) = u_0$$

has a solution u in the distributional sense existing in G_s provided $0 \le t < a(s-s_0)$ where a is a suitably chosen positive number. The solution depends continuously differentiable on t and is a solution of the associated differential equation $lu = 0$ for every fixed t.

2. Interior estimates depending on the order of the associated differential operator and the choice of the norm

Next we are going to show that both well-known classical statements and new results as well can be subordinated under the above general theorem.

Example 1. Let G be a domain in the complex plane, $z = x + iy$ $(x, y \in \mathbb{R})$. As function space choose the space $\mathcal{H}(G)$ of all holomorphic and bounded functions in G. This space equipped with the supremum norm turns out to be a Banach space. Define the operator $\mathcal{L}$ by

$$(\mathcal{L}u)(t,z) = C(t,z)\,\frac{du}{dz}(t,z) + A(t,z)\,u(t,z)$$

where du/dz means the ordinary complex derivative, A and C are elements of the space $\mathcal{C}([0,T],\mathcal{H}(G))$, the set of all continuous mappings of $[0,T]$ into $\mathcal{H}(G)$. Introduce the partial complex differentiations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then the so-called *Cauchy-Riemann operator*

$$l = \frac{\partial}{\partial \overline{z}}$$

is associated to $\mathcal{L}$. Notice that for holomorphic functions the ordinary complex differentiation coincides with the partial complex differentiation with respect to z, i.e.

$$\frac{du}{dz} = \frac{\partial u}{\partial z} \qquad \text{while} \quad lu = \frac{\partial u}{\partial \overline{z}} = 0.$$

[1] Concerning linear versions of the abstract Cauchy-Kovalevskaya theorem cf. T. Yamanaka [17], L. V. Ovsyannikov [13, 14], and F. Treves [15], while non-linear versions can be found in L. Nirenberg [10, 11] and T. Nishida [12]. The functional-analytic approach to the Cauchy-Kovalevskaya problem was initiated by M. Nagumo [9]. Concerning further historical remarks see the appendix of the Russian translation of L. Nirenberg's book [11]; see also [16].

The necessary interior estimate follows from Cauchy's integral formula and the above theorem yields the simplest form of the (linear) classical Cauchy-Kovalevskaya theorem.

Example 2. Again let G be a bounded domain in the complex plane. Let $\mathcal{C}^\lambda(\overline{G})$ be the Banach space of Hölder continuous functions (with Hölder exponent λ, $0 < \lambda < 1$) equipped with the norm

$$\|u\|_G = \max\left\{\sup_{z\in\overline{G}} |u(z)|\ ,\ \sup_{z_1\neq z_2} \frac{|u(z_1)-u(z_2)|}{|z_1-z_2|^\lambda}\right\}.$$

Define the first order differential operator $\mathcal{L}$ by

$$\mathcal{L}u = C_1\left(\frac{\partial u}{\partial z}\right) + C_2\left(\frac{\partial u}{\partial \overline{z}}\right) + C_3\overline{\left(\frac{\partial u}{\partial z}\right)} + C_4\overline{\left(\frac{\partial u}{\partial \overline{z}}\right)} + Au + B\overline{u}$$

where the coefficients may depend on (t,z) and, e.g., are supposed to belong to the space $\mathcal{C}([0,T];\mathcal{C}^\lambda(\overline{G}))$. Then for large classes of operators $\mathcal{L}$ of the above form there exist operators l of both first order and of second order as well being associated to $\mathcal{L}$. This can be shown by (elliptic) operators of type

$$lu = \delta\frac{\partial^2 u}{\partial z\partial\overline{z}} + \gamma_1\frac{\partial u}{\partial z} + \gamma_2\frac{\partial u}{\partial \overline{z}} + \gamma_3\overline{\left(\frac{\partial u}{\partial z}\right)} + \gamma_4\overline{\left(\frac{\partial u}{\partial \overline{z}}\right)} + \alpha u + \beta\overline{u}$$

having coefficients belonging to $\mathcal{C}^\lambda(\overline{G})$, for instance, where $\delta = 0$ or $\delta = 1$.

In the case $\delta = 0$ sufficient conditions for associated pairs and interior estimates for solutions of the associated equation can be found in [16]. As an example of associated pairs with non-analytic coefficients (even not in the sense of real analyticity) in this book, moreover, associated pairs with piecewise constant coefficients are constructed. Using mapping properties of both weakly and strongly singular integral operators (with the singularities $1/(z-\zeta)$ and $1/(z-\zeta)^2$, respectively) the interior estimates may be given not only in the Hölder norm but also in the L_p-norm provided $p > 2$. Using B. Bojarski's theory of generalized analytic vectors [1], in his paper [2] A. Crodel solves initial value problems with generalized vectors as initial elements. On the other hand, for the case $\delta = 1$ in the paper [5] a system of 10 (sufficient) non-linear second order equations (joined with each other) for the 12 coefficients $C_1, C_2, C_3, C_4,\ A, B,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4,\ \alpha, \beta$ is deduced from the above definition of associated differential operators. Special solutions of that system are contained in the same paper [5] and, in addition, in the paper [4]. A further solution of the system for associated pairs not contained in the quoted papers [4, 5] and not only consisting of (real-) analytic coefficients will be given in the next section.

In the case $\delta = 1$ the associated differential equation $lu = 0$ is a second order equation with principal part Δ for a complex-valued function u. Splitting up this equation into real and imaginary part, one gets a system of form

$$\Delta u_1 + \cdots = 0$$
$$\Delta u_2 + \cdots = 0$$

where $u = u_1 + iu_2$. In A. Douglis' and L. Nirenberg's paper [3] interior estimates are given for general elliptic systems of arbitrary order. These estimates show, especially, that the product of the absolute value of the first order derivatives of a solution and the distance from the boundary is uniformly bounded by the supremum norm of the solution multiplied with a constant not depending on the special choice of the solution. This estimate implies that the above mentioned scale method is applicable.

It remains to prove that the above system satisfies the assumptions being sufficient for applying the results of the paper [3]:

The characteristic determinant of the above system is

$$\begin{vmatrix} \xi_1^2 + \xi_2^2 & 0 \\ 0 & \xi_1^2 + \xi_2^2 \end{vmatrix} = (\xi_1^2 + \xi_2^2)^2$$

and, therefore, the system is elliptic, as expected[2].

Concerning the coefficients $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and α, β suppose that their Hölder norms are uniformly bounded (it would be sufficient, however, that a weighted Hölder norm containing the distance from the boundary is finite).

Note that the Hölder continuity of solutions to equation $lu = 0$ follows from general regularity theorems also contained in A. Douglis' and L. Nirenberg's paper [3]. In the above case the Hölder continuity can be proved immediately using techniques of Complex Analysis: The differential equation $lu = 0$ in the domain Ω implies the representation

$$u = \overline{T}_\Omega T_\Omega \left(-\gamma_1 \frac{\partial u}{\partial z} - \cdots - \alpha u - \beta \overline{u} \right) + \Phi_1 + \overline{\Phi}_2$$

where T_Ω is the (weakly singular) operator with the Cauchy singularity $1/(z - \zeta)$ over Ω and Φ_1, Φ_2 are holomorphic functions defined in Ω. This representation shows, for instance, that u is Hölder continuously differentiable provided u is supposed to be continuously differentiable. Knowing this, the same representation shows that u is even *twice* Hölder continuously differentiable.

Example 3. The characteristic determinant of the (plane) bipotential equation

$$\Delta^2 u = 0$$

is equal to $\left(\xi_1^2 + \xi_2^2\right)^2$ and, therefore, A. Douglis' and L. Nirenberg's result [3] is applicable to this equation, too. Consequently, (real- and complex-valued) solutions to the bipotential equation satisfy an interior estimate of the desired type (in the supremum norm). An easy example of a first order differential operator $\mathcal{L}$ to which the bipotential operator is associated is given by

$$\mathcal{L}u = C_1(t,z) \frac{\partial u}{\partial z} + C_2(t,z) \frac{\partial u}{\partial \overline{z}}$$

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2 In the quoted paper [3] the system is supposed to have the form $\sum_{j=1}^{N} l_{ij} u_j = f_i$ $(i = 1, \dots, N)$, where l_{ij} is an operator of order not greater than $s_i + t_j$. Theorem 1 of [3], for instance, can be applied with $s_i = 1$ and $t_j = 1$ or with $s_i = 0$ and $t_j = 2$ (for all i and j).

provided C_1 is a linear holomorphic function and C_2 is a linear anti-holomorphic function (for every fixed t). Then the corresponding initial value problem can be solved (in case the initial function is a solution to the bipotential equation). At the same time, the above theorem states that the solution turns out to be a solution of the bipotential equation for every t.

3. Examples of associated operators with non-analytic coefficients

While the classical Cauchy-Kovalevskaya theory is applicable to differential equations with analytic coefficients only, the above concept is always applicable if only there exists an associated differential equation whose solutions satisfy an interior estimate.

In the case $\delta = 1$ in Example 2 the pair $l, \mathcal{L}$ is associated if the 12 coefficients $C_1, C_2, C_3, C_4, A, B, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and α, β satisfy 10 non-linear second order partial differential equations joined with each other (see [5]). In that paper and in [4], too, one can find many special solutions of the system of (sufficient) conditions for associated operators. A further example not quoted in these two papers and having non-analytic coefficients is the following:

Let A and k be any real numbers and $b = b(x)$ any real-valued twice continuously differentiable function defined in G and depending on x only. Then the operators $\mathcal{L}$ and l defined by

$$\mathcal{L}u = i\frac{\partial u}{\partial \overline{z}} + Au - b(x)\overline{u}$$

$$lu = \frac{\partial^2 u}{\partial z \partial \overline{z}} + \left(k - b^2(x)\right)u + \frac{i}{2}b'(x)\overline{u}$$

are associated to each other.

Analogous constructions leading to permissible non-analytic initial functions can be carried out for differential operators of arbitrary order and for differential operators with respect to real differentiations as well. To be short, regard, for instance, in the (x, y)-plane the differential operator

$$\mathcal{L}u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + A(x)u.$$

The desired function u and the coefficient A (depending on x only) are supposed to be complex-valued, in general. We look for an associated fourth order differential operator

$$lu = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + a_1(x)\frac{\partial^3 u}{\partial x^3} + a_2(x)\frac{\partial^2 u}{\partial x^2} + a_3(x)\frac{\partial u}{\partial x} + a_4(x)u$$

whose (complex-valued) coefficients a_j depend on x only. Let $A = A(x)$ be any four times continuously differentiable function. Then the pair L, l turns out to be associated provided the a_j are (complex-valued) solutions of the system

$$\begin{aligned}\frac{da_1}{dx} &= 4\frac{dA}{dx}\\ \frac{da_2}{dx} &= 3a_1\frac{dA}{dx} + 6\frac{d^2A}{dx^2}\\ \frac{da_3}{dx} &= 2a_2\frac{dA}{dx} + 3a_1\frac{d^2A}{dx^2} + 4\frac{d^3A}{dx^3}\\ \frac{da_4}{dx} &= a_3\frac{dA}{dx} + a_2\frac{d^2A}{dx^2} + a_1\frac{d^3A}{dx^3} + \frac{d^4A}{dx^4}\end{aligned}$$

of ordinary differential equations. Note that the a_j may depend not only on x but also on y. Then the left-hand sides of the above system are to be replaced by

$$\frac{\partial a_j}{\partial x} + \frac{\partial a_j}{\partial y} \qquad (j = 1, ..., 4).$$

Consequently, in this case the permissible coefficients $a_j = a_j(x, y)$ are uniquely determined up to four arbitrary functions only.

4. Solubility of initial value problems using co-associated differential operators

The first example of the previous section shows that for fixedly chosen $A \in \mathbb{R}$ and $b \in C^2(G)$ there exists a whole family of associated operators l because k may be chosen arbitrarily. This phenomenon can be found also for associated first order differential operators such as those given in Example 2 of Section 2 with $\delta = 0$. E.g., the operator

$$\mathcal{L}u = C(t,z)\frac{\partial u}{\partial z} + A(t,z)u$$

with $A, C \in \mathcal{C}([0,T], \mathcal{H}(G))$ and the family of operators l_k defined by

$$l_k u = \frac{\partial u}{\partial \bar{z}} - \bar{z}^k u$$

are associated where k is an arbitrary non-negative integer (cf. [6]).

In such cases the operators of the family l_k are said to be *co-associated* with respect to $\mathcal{L}$ (shortly: $\mathcal{L}$-co-associated).

Provided the operators l_k are $\mathcal{L}$-co-associated, the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \qquad u(0) = u_0$$

where

$$u_0 = \sum_{k=0}^{n} u_0^{(k)} \qquad \text{and} \qquad l_k u_0^{(k)} = 0$$

has the solution

$$u = \sum_{k=0}^{n} u^{(k)}$$

where $u^{(k)}$ is a solution of the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \qquad u(0) = u_0^{(k)}.$$

belonging to the function space defined by the differential equation $l_k u = 0$. In case the initial function u_0 cannot be represented by a finite sum $\sum_{k=0}^{n} u_0^{(k)}$ one can represent the initial function by a series $\sum_{k=0}^{\infty} u_0^{(k)}$ in some cases or one can approximate the initial function by a finite sum, at least. For details we refer to [7].

Note that in the second example of the last section the coefficients $a_j(x, y)$ depend on four arbitrary functions even.

5. Concluding remark

The theory of solving initial value problems in classes of solutions of an associated differential equation makes it possible to solve initial value problems with more general initial functions. The case of (complex-valued) initial functions satisfying a uniformly elliptic first order system in the plane is developed in the book [16] citing A. Crodel's results [2] on generalized analytic initial vectors, too. At present generalizations of these constructions to the case of higher order side conditions are under consideration. The higher the order of the associated differential operator, the larger the set of solutions (e.g., a solution of the Laplace equation satisfies the bipotential equation, too, but not vice versa). Therefore, the conditions for associated pairs are more restrictive in the case of higher order side conditions. On the other hand, the variety of co-associated differential operators is larger for higher order side conditions.

Notice that the famous H. Lewy example [8] shows that there are linear differential equations with C^∞-coefficients not having any solutions. The existence of an associated differential operator satisfying an interior estimate implies not only the solubility of the differential equation but also the possibility to choose arbitrary solutions of the associated differential equation as initial values.

The present paper contains some results obtained recently and characterizes also the directions of generalizations to be considered in the future.

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