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Some Results on Non-Coercive Variational Problems and Applications

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Abstract. We give a necessary and sufficient condition to ensure the existence of solutions of three problems of the calculus of variations with non-coercive integrands. The solutions u we consider are lipschitz functions, i.e. $u \in W^{1,\infty}(0,1)$. The three problems depends on the same functionals but are different in the constraints. We consider respectively a problem without constraint, a problem with $u' \geq 0$ and finally a problem with $u \geq 0$. These problems can be related to optimal foraging models in behavioural ecology.

Keywords: *Calculus of variations, convexity, non-coercive and constraint problems*

AMS subject classification: 49

1. Introduction

In this paper we consider the following three variational problems: Letting $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition

(H) $f(x,\cdot): \mathbb{R} \to \mathbb{R}$ is convex for every $x \in [0,1]$

we investigate the existence of a minimizer u in the cases

$$(\mathbf{P}_S) \quad \inf \left\{ I(u) = \int_0^1 f\big(x,u'(x)\big)\,dx \;\middle|\; u \in W^{1,\infty}(0,1),\ u(0)=0,\ u(1)=S \right\}$$

$$(\mathbf{P}_S^+) \quad \inf \left\{ I(u) = \int_0^1 f\big(x,u'(x)\big)\,dx \;\middle|\; 0 \leq u \in W^{1,\infty}(0,1),\ u(0)=0,\ u(1)=S \right\}$$

$$(\mathbf{P}_S^{\prime +}) \quad \inf \left\{ I(u) = \int_0^1 f\big(x,u'(x)\big)\,dx \;\middle|\; u \in W^{1,\infty}(0,1),\ u(0)=0,\ u(1)=S \right\}.$$

Hereby S is a given number and $W^{1,\infty}(0,1)$ stands for the set of Lipschitz functions (i.e. continuous functions with almost everywhere uniformly bounded derivatives) defined

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on the interval $(0,1)$. It is important to observe that we do not impose any coercivity condition or any differentiability on the function f.

For these three problems we will show the following statements.

(1) In the problems (P_S) and (P_S^+) two possibilities can happen: either there is no solution for every number $S \in \mathbb{R}$ or there exists numbers $S_{\inf}$ and $S_{\sup}$ respectively $S_{\inf}^+$ and $S_{\sup}^+$ such that the following implications

$$\text{problem } (\mathrm{P}_S) \text{ has a solution} \implies S \in \overline{J} = [S_{\inf}, S_{\sup}]$$

$$S \in J = (S_{\inf}, S_{\sup}) \implies \text{problem } (\mathrm{P}_S) \text{ has a solution}$$

respectively

$$\text{problem } (\mathrm{P}_S^+) \text{ has a solution} \implies S \in \overline{J^+} = [S_{\inf}^+, S_{\sup}^+]$$

$$S \in J^+ = (S_{\inf}^+, S_{\sup}^+) \implies \text{problem } (\mathrm{P}_S^+) \text{ has a solution}$$

are true.

(2) For problem $(\mathrm{P}_S'^+)$ there exists always numbers $S_{\inf}'^+$ and $S_{\sup}'^+$ such that the implications

$$\text{problem } (\mathrm{P}_S'^+) \text{ has a solution} \implies S \in \overline{J'^+} = [S_{\inf}'^+ = 0, S_{\sup}'^+]$$

$$S \in J'^+ = [S_{\inf}'^+ = 0, S_{\sup}'^+) \implies \text{problem } (\mathrm{P}_S'^+) \text{ has a solution}$$

are true.

(3) If numbers $S_{\sup}$ and $S_{\sup}^+$ exist, then always the inequalities

$$S_{\sup} \le S_{\sup}^+ \le S_{\sup}'^+$$

are true.

(4) In general (contrary to $S_{\sup}$, $S_{\sup}^+$ and $S_{\sup}'^+$) we cannot establish an order relation between the numbers $S_{\inf}$ and $S_{\inf}^+$.

Remarks: (i) We will characterize explicitely the numbers $S_{\inf}$ and $S_{\sup}$ (the numbers $S_{\inf}^+$, $S_{\sup}^+$ and $S_{\sup}'^+$, respectively).

(ii) In general if $S = S_{\inf}$ or $S = S_{\sup}$ (if $S = S_{\sup}^+$, $S = S_{\inf}^+$ or $S = S_{\sup}'^+$, respectively), we cannot conclude to the existence or non-existence of a solution. In fact it is possible to exhibit cases where we can show the existence of a solution and other cases where we can show the non-existence of a solution if $S = S_{\inf}$ or $S = S_{\sup}$ (if $S = S_{\sup}^+$, $S = S_{\inf}^+$ or $S = S_{\sup}'^+$, respectively).

(iii) If $S_{\inf}$ and $S_{\sup}$ ($S_{\inf}^+$ and $S_{\sup}^+$, respectively) do not exist, then $J = \emptyset$ ($J^+ = \emptyset$, respectively). Finally, if $S_{\sup}'^+ = 0$ (if $S_{\inf} = S_{\sup}$ or $S_{\inf}^+ = S_{\sup}^+$, respectively), then $\overline{J'^+} = [0]$ ($\overline{J} = [S_{\inf}]$ and $\overline{J^+} = [S_{\inf}^+]$, respectively).

(iv) We will show in particular that if $f = f(x,\xi) \in C^0$, $f_\xi \in C^0$ and $f_\xi(x,0) \equiv$ *constant* for all $x \in [0,1]$, then the three problems (P_S), (P_S^+) and $(\mathrm{P}_S^{'+})$ are equivalent for $S \geq 0$. In particular we have the relations

$$S_{\inf} \leq 0, \qquad S_{\inf}^+ = S_{\inf}^{'+} = 0, \qquad S_{\sup} = S_{\sup}^+ = S_{\sup}^{'+}.$$

Although some of these conclusions are already known, the main interest of this paper is to relate all these results. Note that they are optimal. Another important point is the fact that we can compare the existence of solutions between these three variational problems. Indeed this question is the same as to compare the maximal value and the minimal value of S for each one of the three problems.

In the coercive case, i.e. when the function f satisfies the Tonnelli growth condition

$$\lim_{|\xi|\to\infty} \frac{f(x,\xi)}{|\xi|} = +\infty \quad \text{for every } x \in [0,1],$$

it is well known that there exists a solution to the problem (P_S) for any $S \in \mathbb{R}$ (see Cellina and Colombo [8], Marcellini [16] or Olech [17]). Note that for problems (P_S^+) and $(\mathrm{P}_S^{'+})$ existence holds only for every $S \geq 0$ since otherwise the problem is ill posed. In the general case we show that the existence of a solution of problem (P_S) (of problems (P_S^+) and $(\mathrm{P}_S^{'+})$, respectively) depends on the value S: S has to belong to an interval $\overline{J}$ (interval $\overline{J^+}$ and $\overline{J'^+}$, respectively).

The problem (P_S) can be illustrated by the example of Weierstrass (see Cesari [9]): $f(x,\xi) = \frac{1}{2}x\xi^2$. With our notation we will obtain for this problem that $S_{\inf} = S_{\sup} = 0$. In this example we know that there exist a solution $u \in W^{1,\infty}$ if and only if $S = 0$, i.e. if and only if $S = S_{\inf} = S_{\sup} = 0$. The problem of finding geodesics on a cylinder (see Troutman [19]) can also be put into the formalism of the problem (P_S).

The variational problem $(\mathrm{P}_S^{'+})$ appears in models related to behavioural ecology. Arditi and Dacorogna [2, 3] considered models of optimal foraging theory. One of the problems is related to the minimization of a functional F of type

$$F(u) = \int_0^1 \Big(\rho(x)e^{-u'(x)} + K(x)u'(x)\Big)\,dx$$

in the class of functions $u \in W^{1,\infty}(0,1)$ such that $u(0) = 0$, $u(1) = S$ and $u'(x) \geq 0$ a.e. on $[0,1]$. Observe that in this case the integrand is bounded, and therefore non-coercive, with respect to $u' \geq 0$.

The problem $(\mathrm{P}_S^{'+})$ has also been studied by Botteron and Dacorogna [5]. Their approach is slightly different: they fix the value of S and obtain a sufficient condition depending on f and S for the existence of a solution. We will see here that this sufficient condition is included in our framework in such a way that it implies directly the relations $S_{\sup}^{'+} > S > S_{\inf}^{'+} = 0$ (i.e. $S \in J'^+$).

Botteron and Marcellini [6] studied similar problems. In their paper the essential difference is that there the functional f depends not only on x and u' but also on u.

The constraint is the same, i.e. $u' \geq 0$ or $u \geq 0$. In their context they obtain a sufficient condition to prove the existence of a minimizer. Here we can show more precise results since f does not depend on u. Indeed we obtain a necessary and sufficient condition for the existence of a solution for each of our problems (P_S), $(P_S^{'+})$ and (P_S^+).

The study of problem (P_S^+) is motivated by all classical examples illustrating problem (P_S) (see Akhiezer [1], Cesari [9] or Gelfand and Fomin [15]) in which we introduce an obstacle. This is the motivation of the introduction of the contraint $u \geq 0$. Note that we can also consider more general constraint such as $u(x) \geq g(x)$ for all $x \in [0,1]$ where g is a given function representing the obstacle. By changing variable $(v = u - g)$ the problem is then equivalent to problem (P_S^+). Observe that this problem is more difficult than the two others since the function u appears explicitely in the constraint: u must be positive. In the two other problems the integrand (and the constraint for problem $(P_S^{'+})$) depends on the derivative u' only.

In the Section 2 of this article we first present some examples that motivate the study of these three problems (P_S), (P_S^+) and $(P_S^{'+})$. Then we consider some mathematical preliminaries in the goal to study these problems independently. In Section 3 we compare the three problems, i.e. we prove for a fixed S if the poblems have a solution. As we will see, this question is the same as to compare the values $S_{\sup}$, $S_{\sup}^{'+}$ and $S_{\sup}^{+}$ (respectively $S_{\inf}$, $S_{\inf}^{+}$ and $S_{\inf}^{'+}$).

2. Examples and mathematical modelling

2.1 Examples

2.1.1 Geodesics on a cylinder. This is a well known problem but we present it only to show that we can put it into our formalism. We consider a right cylinder C containing two points A and B. We suppose that this cylinder is generated by a Jordan curve in the (x,y)-plane. Assume that this curve is described in polar coordinates by a strictly positive function $\rho \in C^1([0,2\pi])$ with $\rho(0) = \rho(2\pi)$. By using cylindrical coordinates (ρ, θ, z) it implies that the cylinder is defined by

$$\begin{aligned} x(\theta,z) &= \rho(\theta)\cos\theta \\ y(\theta,z) &= \rho(\theta)\sin\theta \qquad (z \in \mathbb{R}, \theta \in [0,2\pi]). \\ z(\theta,z) &= z \end{aligned}$$

We want to join the points A and B by an arc lying on C and having shortest possible length. Let $A = (\rho(\theta_1), \theta_1, z_1)$ and $B = (\rho(\theta_1), \theta_2, z_2)$. The curve joining A and B is given by

$$\begin{aligned} x(\theta) &= \rho(\theta)\cos\theta \\ y(\theta) &= \rho(\theta)\sin\theta \qquad (\theta \in [\theta_1, \theta_2]) \\ z(\theta) &= z(\theta) \end{aligned}$$

where $z(\theta_1) = z_1$ and $z(\theta_2) = z_2$. With such a notation the curve has the length

$$L(z) = \int_{\theta_1}^{\theta_2} \sqrt{x'(\theta)^2 + y'(\theta)^2 + z'(\theta)^2}\, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{\rho(\theta)^2 + \rho'(\theta)^2 + z'(\theta)^2}\, d\theta.$$

Without loss of generality we can suppose that $A = (0,0)$ and $B = (1,S)$. Therefore by letting $f(x,\xi) = \sqrt{\rho(x)^2 + \rho'(x)^2 + \xi^2}$ the problem is

$$(\mathbf{P}_S) \quad \inf \left\{ I(u) = \int_0^1 f(x,u'(x))\, dx \,\middle|\, u \in W^{1,\infty}(0,1),\ u(0) = 0,\ u(1) = S \right\}.$$

Such a problem on a right circular cylinder (i.e. $\rho(\theta) \equiv$ constant > 0) can be found in Troutman [19]. Cesari [9] treats this problem of geodesics on a sphere. Note that in this last case the lenght depends neither on z' and θ but also on z.

2.1.2 Problems with obstacles. Let $g \in C^1([0,1])$ and $S \geq 0$. We then consider two points $A = (0, g(0))$ and $B(1, S + g(1))$ in the (x,y)-plane. We want to find a function v joining A and B such that its length is minimal and $v(x) \geq g(x)$ for every $x \in [0,1]$. This situation is represented in Figure 1. The function g represents an obstacle that the curve v must avoid.

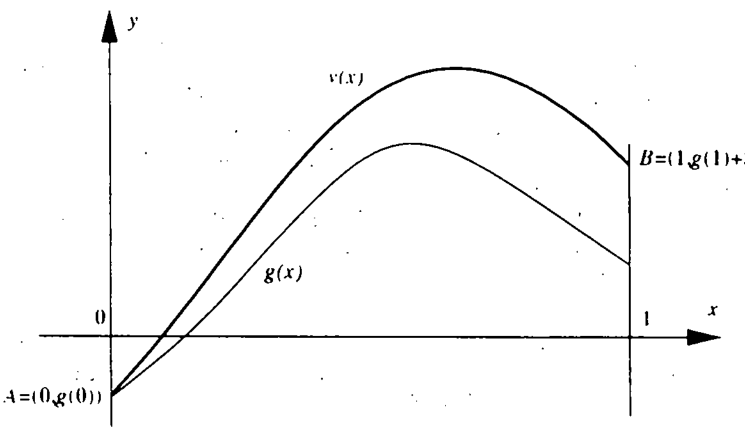

Figure 1: Problem with obstacle

The length of the curve described by v is $L(v) = \int_0^1 \sqrt{1 + v'(x)^2}\, dx$. Hence the problem is to find

$$\inf \left\{ L(v) = \int_0^1 \sqrt{1 + v'(x)^2}\, dx \,\middle|\, \begin{array}{l} v \in W^{1,\infty}(0,1) \text{ with } v \geq g \\ v(0) = g(0),\ v(1) = S + g(1) \end{array} \right\}.$$

By letting $u = v - g$ and $f(x,\xi) = \sqrt{1+(\xi+g'(x))^2}$ this problem is then equivalent to the problem

$$(\mathbf{P}_S^+)\quad \inf\left\{ I(u) = \int_0^1 f(x,u'(x))\,dx \;\middle|\; \begin{array}{l} 0 \le u \in W^{1,\infty}(0,1) \\ u(0)=0,\; u(1)=S \end{array} \right\}.$$

Note that all classical problems of the calculus of variations (see Akhiezer [1], Cesari [9] or Gelfand and Fomin [15]) where the obstacle can be represented by a function such as g and where the integrand depends on x and u' only (i.e. no dependence on u) can be transformed into this formalism.

2.1.3 Biological problems. Some models of behavioural ecology (see Botteron [7]) study the movements of animals while foraging, i.e. while searching food. The problem we want to deal with is the following: an animal is going each day around in its habitat to find food. Suppose that the food resource is renewed each day and with the same distribution. Assuming that the animal has learnt the food distribution, what is the optimal way to exploit the habitat in the goal to maximize the quantity of the food?

In the last few years, optimal foraging theory has developed models to answer such theoretical problems. Charnov [10] attempts to formalize this problem and gives a partial answer in the well-known "patch model": he only considers particular type of food distributions. More recenlty, Arditi, Botteron and Dacorogna [2, 3, 5 - 7] have proposed models in which food distribution is arbitrary.

Their model can be described by the animal's shedule $u = u(x)$ (time in function of position x with $x \in [0,1]$). The interval $[0,1]$ represents a one-dimensional habitat or a closed curve in a plane with $x = 0$ and $x = 1$ corresponding to the start and the end point of this curve. These two points are assumed to be both the nest or the cache of the animal. We suppose that the animal covers its habitat during the "foraging period" S ($u(0) = 0$ and $u(1) = S$). For biological reasons the speed of the animals is upper bounded: after change of variable this is equivalent to $u'(x) \ge 0$ a.e. The food distribution in the habitat is decribed by the food density $\rho = \rho(x)$. We represent the time during which the animal eats the food available at point x by the "foraging presence" $u'(x)$. Finally the dynamics of food acquisition is described by $\phi(u'(x))$ where ϕ is a given function.

In mathematical terms, if we let $f(x,u'(x)) = \rho(x)\phi(u'(x)) + K(x)u'(x)$ where K represents the costs of the foraging, then the problem is to find the minimum of a non-coercive function of the calculus of variations:

$$(\mathbf{P}_S'^+)\quad \inf\left\{ I(u) = \int_0^1 f(x,u'(x))\,dx \;\middle|\; \begin{array}{c} u \in W^{1,\infty}(0,1),\; u' \ge 0 \\ u(0)=0 \text{ and } u(1)=S \end{array} \right\}$$

where $S \ge 0$ is a given number and $W^{1,\infty}(0,1)$ stands for the set of Lipschitz functions defined on $(0,1)$. Note that $f(x,\xi) = \rho(x)e^{-\xi} + K(x)\xi$ is a relevant example in the optimal foraging described above.

2. 2 Mathematical preliminaries

Let the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfying the condition (H). Since the function $f(x,\cdot)$ is convex we can define its left and right derivative

$$f_\xi^+(x,\xi) = \lim_{t \downarrow 0} \frac{f(x,\xi+t) - f(x,\xi)}{t}$$
$$f_\xi^-(x,\xi) = \lim_{t \uparrow 0} \frac{f(x,\xi+t) - f(x,\xi)}{t}.$$

respectively. Note that these two limits exist everywhere and are bounded (not necessarly uniformly) since f is convex and $\mathrm{dom} f = \{(x,\xi) \in [0,1] \times \mathbb{R} : |f(x,\xi)| < +\infty\} = [0,1] \times \mathbb{R}$. Moreover the functions $f_\xi^+(x,\cdot)$ and $f_\xi^-(x,\cdot)$ are non-decreasing.

2.2.1 Behaviour of f at infinity. The results of this subsection are well known and we state them here for the sake of completeness only (for more details, see Ekeland and Temam [14] or Rockafellar [18]).

Since the function $f(x,\cdot)$ is convex, the function

$$g : [0,1] \times \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad \text{defined by } g(x,\xi) = \frac{f(x,\xi) - f(x,0)}{\xi}$$

is non-decreasing. Therefore we can define the functions

$$d : [0,1] \to \mathbb{R} \cup \{+\infty\} \quad \text{by } d(x) = \lim_{\xi \to +\infty} g(x,\xi)$$
$$c : [0,1] \to \mathbb{R} \cup \{-\infty\} \quad \text{by } c(x) = \lim_{\xi \to -\infty} g(x,\xi).$$

These definitions are clearly equivalent to

$$d(x) = \lim_{\xi \to +\infty} \frac{f(x,\xi)}{\xi} = \lim_{\xi \to +\infty} f_\xi^+(x,\xi) = \lim_{\xi \to +\infty} f_\xi^-(x,\xi)$$
$$c(x) = \lim_{\xi \to -\infty} \frac{f(x,\xi)}{\xi} = \lim_{\xi \to -\infty} f_\xi^+(x,\xi) = \lim_{\xi \to -\infty} f_\xi^-(x,\xi),$$

respectively. We have $c(x) \le d(x)$ for every $x \in [0,1]$. If we suppose the functions c and d to be bounded, then we cannot conclude at their continuity, and this even if $f(\cdot,\xi)$ is of Lipschitz type.

Example 2.1: Let $\varepsilon > 0$ and $f(x,\xi) = \xi \arctan(|\xi|^\varepsilon x)$. It is easy to verify that this function satisfies condition (H). Let $\xi \in \mathbb{R}$. Then for any $x, y \in [0,1]$ we have $|f(x,\xi) - f(y,\xi)| \le |\xi|^{1+\varepsilon}|x-y|$. Therefore $f(\cdot,\xi)$ is of $K(\xi)$-Lipschitz type where $K(\xi) = |\xi|^{1+\varepsilon}$. Note that this constant $K(\xi)$ is the smallest one satisfying the Lipschitz condition. However $d(x) = \pi/2$ for $0 < x \le 1$ and $d(0) = 0$. It means that d is not continuous at the point $x = 0$.

The next lemma gives a possible sufficient condition for the functions c and d to be continuous. We state it as a matter of curiosity but we will not require any extra hypothesis than (H).

Lemma 2.1: *Assume that for all $\xi \in \mathbb{R}$ there exists a constant $K(\xi) \geq 0$ such that*

(i) $|f(x,\xi) - f(y,\xi)| \leq K(\xi)|x-y|$ *for all* $x, y \in [0,1]$

(ii) $\lim_{\xi \to +\infty} \frac{K(\xi)}{\xi} < \infty \quad \left(\lim_{\xi \to -\infty} \frac{K(\xi)}{\xi} > -\infty, \textit{respectively}\right).$

Then the function d (the function c, respectively) is of Lipschitz type.

The proof is elementary. Note that this lemma is in some sense also necessary since we cannot suppress any condition on f. The function f given in the preceding example satisfies the condition (ii) only for $\varepsilon = 0$ but does not satisfy this condition for any $\varepsilon > 0$.

2.2.2 The conjugate function f^*. For the function f the *conjugate* function $f^* : [0,1] \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$f^*(x,\alpha) = \sup_{\xi \in \mathbb{R}} \{\alpha\xi - f(x,\xi)\} \qquad \big((x,\alpha) \in [0,1] \times \mathbb{R}\big).$$

Since $f^*(x,\cdot)$ is convex we can define the left and right derivative of $f^*(x,\cdot)$ by

$$f_\alpha^{*+}(x,\alpha) = \lim_{t \downarrow 0} \frac{f^*(x,\alpha+t) - f^*(x,\alpha)}{t}$$
$$f_\alpha^{*-}(x,\alpha) = \lim_{t \uparrow 0} \frac{f^*(x,\alpha+t) - f^*(x,\alpha)}{t},$$

respectively.

Remarks. (i) Let $x \in [0,1]$ be fixed. When $c(x) < d(x)$, $f^*(x,\alpha)$ is bounded if $\alpha \in \big(c(x), d(x)\big)$, and in the case where $d(x) < +\infty$ ($c(x) > -\infty$, respectively) we have $f^*(x,\alpha) = +\infty$ if $\alpha > d(x)$ (if $\alpha < c(x)$, respectively). When $c(x) = d(x)$ we have clearly $f^*(x,\alpha) = 0$ if $\alpha = d(x)$ and $f^*(x,\alpha) = +\infty$ if $\alpha \neq d(x)$.

(ii) The derivatives f_α^{*-} and f_α^{*+} are bounded in the interior of $\mathrm{dom} f^* = \{(x,\alpha) \in [0,1] \times \mathbb{R} : f^*(x,\alpha) < +\infty\}$.

(iii) By convention (as in Rockafellar [18]), when d (when c, respectively) is bounded, we let $f_\alpha^{*-}(x,\alpha) = f_\alpha^{*+}(x,\alpha) = +\infty$ if $\alpha > d(x)$ ($f_\alpha^{*-}(x,\alpha) = f_\alpha^{*+}(x,\alpha) = -\infty$ if $\alpha < c(x)$). This convention implies that $f_\alpha^{*-}(x,\cdot)$ and $f_\alpha^{*+}(x,\cdot)$ are non-decreasing functions.

These left and right derivative leads us to the definition of the sets

$$\partial f(x,\xi) = \left\{\alpha \in \mathbb{R} \,\middle|\, f_\xi^-(x,\xi) \leq \alpha \leq f_\xi^+(x,\xi)\right\} \qquad (\xi \in \mathbb{R})$$
$$\partial f^*(x,\alpha) = \left\{\xi \in \mathbb{R} \,\middle|\, f_\alpha^{*-}(x,\alpha) \leq \xi \leq f_\alpha^{*+}(x,\alpha)\right\} \qquad (\alpha \in \mathbb{R}).$$

The set $\partial f(x,\xi)$ (the set $\partial f^*(x,\alpha)$, respectively) is called the *subdifferential* of f at point (x,ξ) (of f^* at point (x,α), respectively). Observe that if $f_\alpha^{*+}(x,\alpha) = f_\alpha^{*-}(x,\alpha) = \infty$, then the set $\partial f^*(x,\alpha)$ is empty. In the interior of $\mathrm{dom} f^*$, this set is never empty.

The next lemma follows from the above definitions.

Lemma 2.2: *Let $x \in [0,1]$ be fixed. Then the propositions* (a) $\alpha \in \partial f(x,\xi)$ *and* (b) $\xi \in \partial f^*(x,\alpha)$ *are equivalent.*

The proof of this lemma is given in Rockafellar [18]. It means that $\partial f^*(x,\cdot)$ is a generalized inverse of $\partial f(x,\cdot)$ and conversely.

2.2.3 The Kuhn-Tucker Theory. Let $f:\ [0,1]\times \mathbb{R} \to \mathbb{R}$ and $g:\ [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous functions, $X = W^{1,\infty}(0,1)$, $A_S = \{u \in X|\ u(0)=0 \text{ and } u(1)=S\}$ and $Y = L^2(0,1)$. We define

$$G:\ X \to Y \quad \text{by} \quad G(u)(x) = g\big(x,u(x),u'(x)\big)$$

$$F:\ X \to \mathbb{R} \quad \text{by} \quad F(u) = \int_0^1 f\big(x,u'(x)\big)\,dx$$

and $A_Y = \{v \in Y|\ v(x) \le 0 \ \text{ a.e.}\}$. Let

$$\textbf{(P)}\ \inf\left\{ F(u) = \int_0^1 f\big(x,u'(x)\big)\,dx \,\middle|\, u \in A_S \ \text{ and } \ G(u) \in A_Y \right\}.$$

Note that the problem (P) can be formulated in the form

$$\textbf{(P)}\ \inf\left\{ F(u) = \int_0^1 f\big(x,u'(x)\big)\,dx \,\middle|\, \begin{array}{l} u \in W^{1,\infty}(0,1),\ \ u(0)=0 \\ u(1)=S,\ g\big(x,u(x),u'(x)\big) \le 0 \text{ a.e.} \end{array} \right\}.$$

Definition 2.1: Let

$$\begin{aligned} A_Y^0 &= \left\{\lambda \in L^2(0,1) \,\middle|\, \lambda \ge 0 \ \text{ a.e.}\right\} \\ &= \left\{\lambda \in L^2(0,1) \,\middle|\, \int_0^1 \lambda(x)v(x)\,dx \le 0 \ \text{ for every } \ v \in A_Y \right\}. \end{aligned}$$

(i) We define the *Langrange function* $L:\ A_S \times A_Y^0 \to \mathbb{R}$ *associated to the problem* (P) by

$$L(u,\lambda) = \int_0^1 \Big(f\big(x,u'(x)\big) + \lambda(x) g\big(x,u(x),u'(x)\big)\Big)\,dx.$$

(ii) Let $(\bar u,\bar\lambda) \in A_S \times A_Y^0$. We say that $(\bar u,\bar\lambda)$ is a *saddle point* of the function L if

$$L(\bar u,\lambda) \le L(\bar u,\bar\lambda) \le L(u,\bar\lambda)$$

for every $(u,\lambda) \in A_S \times A_Y^0$.

The proofs of the following two lemmas can be found in Barbu and Precupanu [4].

Lemma 2.3: *The element* $(\bar{u},\bar{\lambda}) \in A_S \times A_Y^0$ *is a saddle point of the function* L *if and only if*

(a) $L(\bar{u},\bar{\lambda}) = \min_{u\in A_S} L(u,\bar{\lambda})$

(b) $g(x,\bar{u}(x),\bar{u}'(x)) \le 0$ *a.e. in* $(0,1)$

(c) $\bar{\lambda}(x)g(x,\bar{u}(x),\bar{u}'(x)) = 0$ *a.e. in* $(0,1)$.

Lemma 2.4: *If* $(\bar{u},\bar{\lambda}) \in A_S \times A_Y^0$ *is a saddle point of the function* L*, then* $\bar{u}$ *is a solution of problem* (P)*. Conversely, if problem* (P) *admits a solution* $\bar{u}$*, then there exists an element* $\bar{\lambda} \in A_Y^0$ *such that* $(\bar{u},\bar{\lambda})$ *is a saddle point of the function* L.

Ekeland and Temam [14] give also a proof of the two lemmas above based on duality theorems. Note that in the general theory the function f can also depend on u, but we avoid deliberately this dependence since the integrand in the problems we want to deal with does not depend on u.

Now we want to apply these results to our problem.

Theorem 2.1: *Let the function* $f: [0,1]\times \mathbb{R} \to \mathbb{R}$ *be continuous and satisfying condition* (H)*. Let further the function* $g: [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *be differentiable and* $g(x,\cdot,\cdot)$ *be convex.*

(i) *If* $\bar{u} \in A_S$ *is a solution of the problem* (P)*, then there exist* $\bar{\lambda} \in A_Y^0$ *and* $\alpha \in \mathbb{R}$ *such that*

(ã) $0 \in \partial L(\bar{u},\bar{\lambda})$, *i.e.* $\alpha \in$

$$\partial f(x,\bar{u}'(x)) + \bar{\lambda}(x)g_\xi(x,\bar{u}(x),\bar{u}'(x)) - \int_0^x \bar{\lambda}(s)g_s(s,\bar{u}(s),\bar{u}'(s))\,ds \text{ a.e. in } (0,1)$$

(b̃) $g(x,\bar{u}(x),\bar{u}'(x)) \le 0$ *a.e. in* $(0,1)$

(c̃) $\bar{\lambda}(x)g(x,\bar{u}(x),\bar{u}'(x)) = 0$ *a.e. in* $(0,1)$.

(ii) *Conversely, if there exist* $\bar{u} \in A_S$, $\bar{\lambda} \in A_Y^0$ *and* $\alpha \in \mathbb{R}$ *such that conditions* $(\tilde{a})$, $(\tilde{b})$ *and* $(\tilde{c})$ *are satisfied, then* $\bar{u}$ *is a solution of the problem* (P).

This theorem is equivalent to Lemma 2.4. The only difficulty comes from condition $(\tilde{a})$. However, since in this case $L(u,\lambda)$ is a convex function of u and a linear function of λ, the minimum of L is attained at a point $(\bar{u},\bar{\lambda})$ if and only if $0 \in \partial L(\bar{u},\bar{\lambda})$ where ∂ represents the subgradient of L as function of u.

2.3 Existence of a solution for the problem (P_S)

In this subsection we will deal with problem (P_S). We begin with the following

Theorem 2.2: *Let*

$$d_{\inf} = \inf_{x\in[0,1]} d(x) = \inf_{x\in[0,1]} \lim_{\xi\to+\infty} \frac{f(x,\xi)}{\xi}$$
$$c_{\sup} = \sup_{x\in[0,1]} c(x) = \sup_{x\in[0,1]} \lim_{\xi\to-\infty} \frac{f(x,\xi)}{\xi}.$$

Then the following statements are true.

(i) *If $d_{\inf} < c_{\sup}$, then problem (P_S) has no solution for every $S \in \mathbb{R}$.*

(ii) *If $d_{\inf} > c_{\sup}$ we define*

$$S_{\sup} := \lim_{\alpha\uparrow d_{\inf}} \int_0^1 f_\alpha^{*+}(x,\alpha)\,dx \qquad \textit{and} \qquad S_{\inf} := \lim_{\alpha\downarrow c_{\sup}} \int_0^1 f_\alpha^{*-}(x,\alpha)\,dx,$$

then problem (P_S) has a solution for every $S \in (S_{\inf}, S_{\sup})$ and has no solution if $S < S_{\inf}$ or if $S > S_{\sup}$.

Remarks: (i) In general if $S = S_{\sup}$ or $S = S_{\inf}$, then one can not conclude at the existence or non-existence of solutions of problem (P_S).

(ii) If $d_{\inf} = c_{\sup}$, then one can not determine in general the values of $S_{\sup}$ and $S_{\inf}$. In some cases (e.g. $f = \text{const}$) we can proof the existence of a solution of problem (P_S) for any $S \in \mathbb{R}$ and in other cases (e.g. $f(x,\xi) = (\sin(2\pi x)+2)(\arctan\xi + \pi)$) we have no solution of problem (P_S) for any $S \in \mathbb{R}$. However the next lemma gives a necessary and sufficient condition to obtain the existence of a solution of problem (P_S) in this particular case.

To establish Theorem 2.2 it is sufficient, by Kuhn-Tucker theory, to observe that $u \in W^{1,\infty}(0,1)$ with $u(0) = 0$ and $u(1) = S$ is a solution of problem (P_S) if and only if there exists an $\alpha \in \mathbb{R}$ such that $\alpha \in \partial f(x, u'(x))$ a.e.

We recall that for $\alpha \in \mathbb{R}$ and $v \in L^\infty(0,1)$, the inclusion $\alpha \in \partial f(x, v(x))$ a.e. means that

$$f_\xi^-(x, v(x)) \le \alpha \le f_\xi^+(x, v(x)) \qquad \text{for almost every } x \in (0,1). \tag{1}$$

The proof of Theorem 2.2 is therefore an immediate consequence of the lemma which follows.

Lemma 2.5: *Let $d_{\inf} = \inf_{x\in[0,1]} d(x)$, $c_{\sup} = \sup_{x\in[0,1]} c(x)$ and $\alpha \in \mathbb{R}$.*

(i) *If $c_{\sup} > d_{\inf}$, then for every $\alpha \in \mathbb{R}$ there is no $v \in L^\infty(0,1)$ for which the inequalities (1) are satisfied.*

(ii) *If $c_{\sup} = d_{\inf}$, then for at most one $\alpha \in \mathbb{R}$ there exists $v \in L^\infty(0,1)$ such that the inequalities (1) are satisfied.*

(iii) *If $c_{\sup} < d_{\inf}$, then for every $\alpha \in (c_{\sup}, d_{\inf})$ there exists $v \in L^\infty(0,1)$ such that the inequalities* (1) *are satisfied. Conversely, if there exists $\alpha \in I\!R$ and $v \in L^\infty(0,1)$ such that inequalities* (1) *are satisfied, then $\alpha \in [c_{\sup}, d_{\inf}]$.*

Proof: First observe that inequalities (1) are equivalent to the inclusion $v(x) \in \partial f^*(x,\alpha)$ almost everywhere. This is equivalent to showing that $f_\alpha^{*-}(x,\alpha)$ and $f_\alpha^{*-}(x,\alpha)$ are bounded in $L^\infty(0,1)$. For $\varepsilon > 0$ let

$$I_d(\varepsilon) = \left\{x \in [0,1] \,\Big|\, |d(x) - d_{\inf}| < \varepsilon\right\}$$
$$I_c(\varepsilon) = \left\{x \in [0,1] \,\Big|\, |c(x) - c_{\sup}| < \varepsilon\right\}.$$

By definition of $d_{\inf}$ and $c_{\sup}$, for every $\varepsilon > 0$, $I_d(\varepsilon)$ and $I_c(\varepsilon)$ are not empty.

Let $\alpha > d_{\inf}$ and $\varepsilon \in (0, \alpha - d_{\inf})$. This implies that $f^*(x,\alpha) = +\infty$ for every $x \in I_d(\varepsilon)$. We then distinguish two cases:

(a) meas $I_d(\varepsilon) > 0$: Then $f_\alpha^{*-}(x,\alpha) = f_\alpha^{*+}(x,\alpha) = +\infty$ for every $x \in I_d(\varepsilon)$ which implies that $v \notin L^\infty(0,1)$ since meas $I_d(\varepsilon) > 0$.

(b) meas $I_d(\varepsilon) = 0$: Let $x \in I_d(\varepsilon)$. Then $f^*(x,\alpha) = +\infty$. Hence for every $k \geq 0$, there exists $\xi_k \in I\!R$ such that $\alpha\xi_k - f(x,\xi_k) \geq k$. Let $x_n \in I_d(\varepsilon)^c$ (i.e. $x_n \in [0,1]\backslash I_d(\varepsilon)$) a sequence such that $\lim_{n\to\infty} x_n = x$. Then we have

$$\alpha\xi_k - f(x_n,\xi_k) + f(x_n,\xi_k) - f(x,\xi_k) \geq k.$$

Since $f^*(x_n,\alpha) \geq \alpha\xi_k - f(x_n,\xi_k)$, we obtain

$$f^*(x_n,\alpha) + f(x_n,\xi_k) - f(x,\xi_k) \geq k \quad \text{and} \quad \liminf_{n\to\infty} f^*(x_n,\alpha) \geq k$$

since $\lim_{n\to\infty} f(x_n,\xi_k) = f(x,\xi_k)$. Therefore for all $k \geq 0$ there exists a neighbourhood V of x such that $f^*(\tilde{x},\alpha) \geq k$ for every $\tilde{x} \in V$. Since this result is true for k as large as we want, it implies that $v \notin L^\infty(0,1)$.

With the same argument we can show that if $\alpha < c_{\sup}$, then $v \notin L^\infty(0,1)$. This implies that inequalities (1) hold for $v \in L^\infty(0,1)$ only if $c_{\sup} \leq \alpha \leq d_{\inf}$. The cases (i), (ii) and (iii) follow easily from this result ∎.

We now give an example which illustrates Theorem 2.2.

Example 2.2: Let $\rho_1, \rho_2 : [0,1] \to I\!R$ be continuous functions, $\rho_1, \rho_2 \geq 0$, and let the function $f : [0,1] \times I\!R \to I\!R$ be defined by

$$f(x,\xi) = \rho_1(x)\phi_1(\xi) + \rho_2(x)\phi_2(\xi) \quad \text{where } \phi_1(\xi) = |\xi| \text{ and } \phi_2(\xi) = \frac{1}{2}\xi^2.$$

We consider the following problem:

$$(\mathbf{P}_S) \quad \inf\left\{ I(u) = \int_0^1 f(x,u'(x))\,dx \,\Big|\, u \in W^{1,\infty}(0,1),\ u(0) = 0,\ u(1) = S \right\}.$$

In this context the classical problem of Weierstrass (i.e. $f(x,\xi) = \frac{1}{2}x\xi^2$) corresponds to $\rho_1 = 0$ and $\rho_2(x) = x$. We will see here how to apply our theorem to the example of Weierstrass.

Let $K = \{x \in [0,1] : \rho_2(x) = 0\}$. This set is compact. Assume that it is not empty. Then we have

$$d(x) = \lim_{\xi\to+\infty} \frac{f(x,\xi)}{\xi} = \begin{cases} \rho_1(x) & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}$$

$$c(x) = \lim_{\xi\to-\infty} \frac{f(x,\xi)}{\xi} = \begin{cases} -\rho_1(x) & \text{if } x \in K \\ -\infty & \text{if } x \notin K. \end{cases}$$

Let $\rho_K = \min_{x\in K}\rho_1(x) \geq 0$. Then $c_{\sup} = -\rho_K$ and $d_{\inf} = \rho_K$. If $\rho_K = 0$ and $\operatorname{meas}\{x \in K : \rho_1(x) = \rho_K = 0\} = 0$, then problem (P_S) has a solution only if $S = 0$. If $\rho_K = 0$ and $\operatorname{meas}\{x \in K : \rho_1(x) = \rho_K = 0\} > 0$, then problem (P_S) has a solution for every $S \in \mathbb{R}$.

Now let us suppose that $\rho_K > 0$. For the left and right derivatives of f we have

$$f_\xi^+(x,\xi) = \begin{cases} \rho_1(x) + \rho_2(x)\xi & \text{if } \xi \geq 0 \\ -\rho_1(x) + \rho_2(x)\xi & \text{if } \xi < 0 \end{cases}$$

$$f_\xi^-(x,\xi) = \begin{cases} \rho_1(x) + \rho_2(x)\xi & \text{if } \xi > 0 \\ -\rho_1(x) + \rho_2(x)\xi & \text{if } \xi \leq 0. \end{cases}$$

The conjugate f^* is defined by

$$f^*(x,\alpha) = \begin{cases} 0 & \text{if } |\alpha| \leq \rho_1(x) \\ +\infty & \text{if } x \in K \text{ and } |\alpha| > \rho_1(x) \\ \dfrac{1}{2}\dfrac{(\alpha - \rho_1(x))^2}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha > \rho_1(x) \\ \dfrac{1}{2}\dfrac{(\alpha + \rho_1(x))^2}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha < -\rho_1(x) \end{cases}$$

and its derivatives are given by

$$f_\alpha^{*+}(x,\alpha) = \begin{cases} 0 & \text{if } -\rho_1(x) \leq \alpha < \rho_1(x) \\ +\infty & \text{if } x \in K \text{ and } \alpha \geq \rho_1(x) \\ -\infty & \text{if } x \in K \text{ and } \alpha < -\rho_1(x) \\ \dfrac{\alpha - \rho_1(x)}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha \geq \rho_1(x) \\ \dfrac{\alpha + \rho_1(x)}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha < -\rho_1(x) \end{cases}$$

$$f_\alpha^{*-}(x,\alpha) = \begin{cases} 0 & \text{if } -\rho_1(x) < \alpha \leq \rho_1(x) \\ +\infty & \text{if } x \in K \text{ and } \alpha > \rho_1(x) \\ -\infty & \text{if } x \in K \text{ and } \alpha \leq -\rho_1(x) \\ \dfrac{\alpha - \rho_1(x)}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha > \rho_1(x) \\ \dfrac{\alpha + \rho_1(x)}{\rho_2(x)} & \text{if } x \notin K \text{ and } \alpha \leq -\rho_1(x). \end{cases}$$

With the help of Theorem 2.2 we have for this problem

$$S_{\sup} = \lim_{\alpha \uparrow \rho_K} \int_{K_1} \frac{\alpha - \rho_1(x)}{\rho_2(x)}\, dx \qquad \text{and} \qquad S_{\inf} = \lim_{\alpha \downarrow -\rho_K} \int_{K_1} \frac{\alpha + \rho_1(x)}{\rho_2(x)}\, dx$$

where $K_1 = \{x \notin K : \rho_1(x) < \rho_K\}$.

If we apply these results to the example of Weierstrass (i.e. for $f(x,\xi) = \frac{1}{2}x\xi^2$), we conclude that $\rho_K = 0$ and that $S_{\sup} = S_{\inf} = 0$. Therefore the problem (P_S) admits a solution only if $S = 0$: the solution is given by $u = 0$. In all other cases (i.e. for $S \neq 0$), the problem of Weierstrass does not admit any solution.

Example 2.3: In geometrical optics, using Fermat's principle (see, for example, Dacorogna [13]), we are led to find a solution of problem (P_S) with

$$f(x,\xi) = a(x)\sqrt{1+\xi^2}$$

where $a : [0,1] \to \mathbb{R}$ is a continuous function representing the refractive index. Let $a_{\min} = \min_{x\in[0,1]} a(x)$. Then we have $c(x) \equiv -a_{\min}$ and $d(x) \equiv a_{\min}$. Therefore, applying Theorem 2.2, we obtain three cases:

(i) $a_{\min} < 0$: There is no solution for any $S \in \mathbb{R}$.

(ii) $a_{\min} = 0$: Let $J = \{x \in [0,1] : a(x) = a_{\min} = 0\}$. If $\mathrm{meas}J = 0$, then problem (P_S) admits a solution only if $S = 0$. If $\mathrm{meas}J > 0$, then problem (P_S) admits a solution for any $S \in \mathbb{R}$.

(iii) $a_{\min} > 0$: We can calculate the minimal and maximal value of S:

$$S_{\sup} = \int_0^1 \frac{a_{\min}}{\sqrt{a^2(x) - a_{\min}}}\, dx \qquad \text{and} \qquad S_{\inf} = -S_{\sup}.$$

2.4 Existence of a solution for the problem $(\mathrm{P}_S'^{+})$

Now we will consider the second of our problems, namely problem $(\mathrm{P}_S'^{+})$ in which the function f is continuous and satisfies condition (H).

Theorem 2.3: *Let $d_{\inf} = \inf_{x\in[0,1]} d(x) = \inf_{x\in[0,1]} \lim_{\xi\to+\infty} \frac{f(x,\xi)}{\xi}$ and $K = \{x \in [0,1] : f_\xi^-(x,0) \le d_{\inf}\}$. Then let*

$$S_{\inf}'^{+} = 0 \qquad \text{and} \qquad S_{\sup}'^{+} = \lim_{\alpha\uparrow d_{\inf}} \int_K f_\alpha^{*+}(x,\alpha)\, dx.$$

If $S_{\inf}'^{+} \le S < S_{\sup}'^{+}$, then problem $(\mathrm{P}_S'^{+})$ has a solution. Conversely, if $S < S_{\inf}'^{+}$ or $S > S_{\sup}'^{+}$, then problem $(\mathrm{P}_S'^{+})$ has no solution.

Remarks: **(i)** If $S = S'^{+}_{\sup} > 0$, then we cannot conclude in general that problem (P'^{+}_S) admits a solution. **(ii)** If $S < S'^{+}_{\inf} = 0$, then the problem (P'^{+}_S) is ill posed. **(iii)** If meas $K = 0$, then $S'^{+}_{\sup} = S'^{+}_{\inf} = 0$.

Proof of Theorem 2.3: Let $S \geq 0$ and $A_S = \{u \in W^{1,\infty}(0,1) : u(0) = 0 \text{ and } u(1) = S\}$. With the help of the Kuhn-Tucker theory, there exists a solution for the problem (P'^{+}_S) if and only if there exist $u \in A_S$, $\alpha \in \mathbb{R}$ and $\lambda \in L^2(0,1)$ such that a.e.

$$u'(x) \geq 0, \qquad \lambda(x) \geq 0, \qquad \alpha + \lambda(x) \in \partial f\big(x, u'(x)\big), \qquad \lambda(x)u'(x) = 0.$$

For $\alpha \in \mathbb{R}$ we define

$$K(\alpha) = \big\{x \in [0,1] : \ f^+_\xi(x,0) < \alpha\big\} \qquad \text{and} \qquad K^c(\alpha) = [0,1] \setminus K(\alpha).$$

By letting $\lambda(x) = 0$ for a.e. $x \in K(\alpha)$ and $\lambda(x) \in \partial f\big(x, u'(x)\big) - \alpha$ for a.e. $x \in K^c(\alpha)$ we conclude that there exists a solution for the problem (P'^{+}_S) if and only if there exist $u \in A_S$ and $\alpha \in \mathbb{R}$ such that

$$\alpha \in \partial f\big(x, u'(x)\big) \quad \text{for a.e. } x \in K(\alpha) \tag{2}$$

$$u'(x) = 0 \quad \text{for a.e. } x \in K^c(\alpha). \tag{3}$$

Let us consider $u \in A_S$ and $\alpha \in \mathbb{R}$ such that conditions (2) and (3) are satirfied. Then $u'(x) \geq 0$ for a.e. $x \in (0,1)$. More precisely we have $u'(x) > 0$ for a.e. $x \in K(\alpha)$ and $u'(x) = 0$ for a.e. $x \in K^c(\alpha)$.

Moreover, let $\alpha_1 < \alpha_2$ and assume the existence of elements $u_1, u_2 \in W^{1,\infty}(0,1)$ such that

$$\alpha_1 \in \partial f\big(x, u_1'(x)\big) \quad \text{for a.e. } x \in K(\alpha_1)$$
$$\alpha_2 \in \partial f\big(x, u_2'(x)\big) \quad \text{for a.e. } x \in K(\alpha_2).$$

Then $K(\alpha_1) \subset K(\alpha_2)$ and $u_1'(x) \leq u_2'(x)$ for a.e. $x \in K(\alpha_1)$. This implies that the maximal value of S will be obtained by taking the maximal value of α for which the conditions (2) and (3) are satisfied. With the same arguments as in Theorem 2.2 on the existence of a solution to the problem (P_S) we know that $\alpha \in \partial f\big(x, u'(x)\big)$ has a meaning only if $\alpha \leq d_{\inf}$ since u is of Lipschitz type. This leads us to consider the maximal set

$$K = \Big\{x \in (0,1) : \ f^-_\xi(x,0) \leq d_{\inf}\Big\}.$$

With the set K we finally obtain

$$S'^{+}_{\sup} = \lim_{\alpha \uparrow d_{\inf}} \int_K f^{*+}_\alpha(x,\alpha)\, dx$$

and the statement is proved ∎

Botteron and Dacorogna [5] study the same problem (P'^{+}_{S}) in a different way: they fix the value of S and obtain a sufficient condition depending on f and S to conclude at the existence of a solution. This condition is given by

$$\gamma(S) < d_{\inf}$$

where $\gamma(S) = \sup_{x\in[0,1]} f^{+}_{\xi}(x,S)$.

In our framework, $\gamma(S) < d_{\inf}$ implies $S'^{+}_{\sup} > S \geq 0$: since $c_{\sup} \leq \gamma(S)$ we have clearly $c_{\sup} < d_{\inf}$; let us consider $\alpha = \gamma(S)$ and $K(\alpha) = \{x \in [0,1] : f^{*+}_{\alpha}(x,\alpha) > 0\}$. The function u defined by

$$u(0) = 0 \qquad \text{and} \qquad u'(x) = \begin{cases} f^{*+}_{\alpha}(x,\alpha) & \text{if } x \in K(\alpha) \\ 0 & \text{if } x \notin K(\alpha) \end{cases}$$

is then a solution of problem $(\mathrm{P}'^{+}_{S_\alpha})$ for $S_\alpha = \int_{K(\alpha)} f^{*+}_{\alpha}(x,\alpha)\,dx$. Finally, we have

$$S'^{+}_{\sup} > S_\alpha \geq \int_0^1 f^{*+}_{\alpha}(x,\alpha)\,dx \geq \int_0^1 f^{*+}_{\alpha}\left(x, f^{+}_{\xi}(x,S)\right)dx = S \geq 0.$$

It means that in general the sufficiency condition $\gamma(S) < d_{\inf}$ is not necessary. We give below such an example.

Example 2.4: We consider the problem

$$(\mathbf{P}'^{+}_{S}) \ \inf\left\{ I(u) = \int_0^1 f(x,u'(x))\,dx \;\middle|\; \begin{array}{l} u \in W^{1,\infty}(0,1),\ u' \geq 0 \\ u(0) = 0,\ u(1) = S \end{array} \right\}$$

where $f(x,\xi) = e^{x}\sqrt{1+\xi^2}$ and we show that the result of Botteron and Dacorogna cannot be applied if S is sufficiently large. Indeed, we have

$$d(x) = e^{x}, \qquad d_{\inf} = 1, \qquad \gamma(S) = \frac{eS}{\sqrt{1+S^2}}.$$

Therefore $\gamma(S) < d_{\inf}$ if and only if $S < 1/\sqrt{e^2-1}$. Moreover

$$S'^{+}_{\sup} = \lim_{\alpha\uparrow 1} \int_0^1 \frac{\alpha}{\sqrt{e^{2x}-\alpha}}\,dx = \arctan\sqrt{e^2-1}.$$

As $\arctan\sqrt{e^2-1} > 1/\sqrt{e^2-1}$ we conclude that there exists a solution of problem (P'^{+}_{S}) if $S'^{+}_{\sup} > S \geq 0$ but $\gamma(S) > d_{\inf}$ if $\arctan\sqrt{e^2-1} > S > 1/\sqrt{e^2-1}$. Therefore the condition $\gamma(S) < d_{\inf}$ is not necessary in general.

2. 5 Existence of a solution for the problem (P_S^+)

The third problem we want to deal with is (P_S^+). This problem is more difficult than the two first one since the function u appears explicitely in the constraint: u must be positive. In the two other problems the integrand and the constraint for problem $(\mathrm{P}_S'^+)$ depend only on the derivative u'.

For the sake of clarity, instead of giving immediately the most general theorem (see Theorem 2.7), we will start by an abstract existence theorem (see Theorem 2.4) which gives the existence of solutions provided one can find an appropriate function l. We then give two results (see Theorems 2.5 and 2.6) where such a function can be constructed easily. Finally we show how l can be found in general (see Theorem 2.7).

Theorem 2.4: *The problem (P_S^+) admits a solution if and only if there exist functions $l \in W^{1,2}(0,1)$ and $u \in W^{1,\infty}(0,1)$ such that*

$$\begin{aligned}
&u(0)=0, \quad u(1)=S, \quad u \geq 0\\
&l \text{ is non-increasing}\\
&u'(x) \in \partial f^*\big(x, l(x)\big) \text{ a.e. in } (0,1)\\
&l'(x)u(x)=0 \text{ a.e. in } (0,1).
\end{aligned}$$

Remarks: (i) A function $u \in W^{1,\infty}(0,1)$ which satisfies all the conditions of this theorem is of course a solution of problem (P_S^+).

(ii) The set $W^{1,2}(0,1)$ stands for the set of functions defined on [0,1] and whose first derivative in a weak sense belongs to $L^2(0,1)$.

(iii) Theorem 2.4 is a direct application of the Kuhn-Tucker theory. We will see that in the case where f is differentiable and $f_\xi(\cdot,0)$ is monotone, this theorem implies that problem (P_S^+) can be easily compared to problems (P_S) or $(\mathrm{P}_S'^+)$.

Proof of Theorem 2.4: The Kuhn-Tucker theory leads us to the following result: problem (P_S^+) has a solution $u \in W^{1,\infty}(0,1)$ if and only if there exist $\lambda \in L^2(0,1)$ and $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}
&u(0)=0, \quad u(1)=S, \quad u \geq 0\\
&\lambda \geq 0 \text{ a.e. in } (0,1)\\
&\alpha \in \partial f\big(x, u'(x)\big) + \int_0^x \lambda(\sigma)\, d\sigma \text{ a.e. in } (0,1)\\
&\lambda(x)u(x)=0 \text{ a.e. in } (0,1).
\end{aligned}$$

Let $l(x) = \alpha - \int_0^x \lambda(\sigma)\, d\sigma$. Since $0 \leq \lambda \in L^2(0,1)$, l is non-increasing and $l \in W^{1,2}(0,1)$. With the help of the conjugate function f^* we can conclude that the problem (P_S^+) has a solution $u \in W^{1,\infty}(0,1)$ if and only if there exists a function $l \in W^{1,2}(0,1)$ such that

$$\begin{aligned}
&u(0)=0, \quad u(1)=S, \quad u \geq 0\\
&l \text{ is non-increasing}\\
&u'(x) \in \partial f^*\big(x, l(x)\big) \text{ a.e. in } (0,1)\\
&l'(x)u(x)=0 \text{ a.e. in } (0,1).
\end{aligned}$$

These last relations represent the result we are looking for ∎

We now present two cases where the function l can be found easily.

Theorem 2.5: *Let $f, f_\xi \in C^0([0,1] \times I\!R)$. Assume that the function $f(x,\cdot)$ is convex and the derivative $f_\xi(\cdot,0) : [0,1] \to I\!R$ is non-decreasing. Then for a given number $S \geq 0$ the problems (P_S^+) and (P_S) are equivalent. In particular either the two problems have no solution for every $S \geq 0$ or*

$$S_{\sup}^{+} = S_{\sup} = \lim_{\alpha \uparrow d_{\inf}} \int_0^1 f_\alpha^{*+}(x,\alpha)\,dx$$

and

$$S_{\inf}^{+} = \max(S_{\inf}, 0) = \max\left(\lim_{\alpha \downarrow c_{\sup}} \int_0^1 f_\alpha^{*-}(x,\alpha)\,dx, 0\right).$$

Proof: Let $S \geq 0$.

(a) Suppose that problem (P_S) admits a solution $u \in W^{1,\infty}(0,1)$. Let $d_{\inf} = \inf_{x\in[0,1]} d(x)$ and $c_{\sup} = \sup_{x\in[0,1]} c(x)$. We know that problem (P_S) admits a solution u only if

(i) $d_{\inf} \geq c_{\sup}$

(ii) $\alpha = f_\xi(x, u'(x))$ a.e. in $(0,1)$ for some $\alpha \in [c_{\sup}, d_{\inf}]$.

By hypothesis the derivative $f_\xi(\cdot,0)$ is non-decreasing and since the function $f(x,\cdot)$ is convex the derivative $f_\xi(x,\cdot)$ is also non-decreasing. Suppose that there exist $\bar{x}$ and $u'(\bar{x}) \leq 0$ such that $\alpha = f_\xi(\bar{x}, u'(\bar{x}))$. Then since $f_\xi(x,\cdot)$ and $f_\xi(\cdot,0)$ are non-decreasing we have

$$\alpha = f_\xi(\bar{x}, u'(\bar{x})) \leq f_\xi(\bar{x},0) \leq f_\xi(x,0) \qquad \text{for all } x \in [\bar{x},1].$$

Since $\alpha = f_\xi(x, u'(x))$ a.e. in $(0,1)$ and $\alpha \leq f_\xi(x,0)$ for all $x \in [\bar{x},1]$ we have $u'(x) \leq 0$ for a.e. $x \in (\bar{x},1)$. It means that if $u'(\bar{x}) \leq 0$, then $u'(x) \leq 0$ for a.e. $x \in [\bar{x},1]$. Therefore there exists $0 < a \leq 1$ such that the equation $\alpha = f_\xi(x,u'(x))$ has a solution which satisfies $u'(x) \geq 0$ for a.e. $x \in (0,a)$ and $u'(x) \leq 0$ for a.e. $x \in (a,1)$. Since $u(1) = S \geq 0$ we conclude that $u(x) \geq 0$ for every $x \in [0,1]$. By letting $l(x) \equiv \alpha$ the functions l and u satisfy the conditions of Theorem 2.4. This implies that for $S \geq 0$ problem (P_S) admits a solution u which is also solution of problem (P_S^+).

(b) Suppose that problem (P_S^+) admits a solution $u \in W^{1,\infty}(0,1)$. By Theorem 2.4 we know that there exists a function $l \in W^{1,2}(0,1)$ such that

$$\begin{aligned} &l \text{ is non-increasing} \\ &u'(x) \in \partial f^*(x, l(x)) \text{ a.e. in } (0,1) \\ &l'(x)u(x) = 0 \text{ a.e. in } (0,1). \end{aligned}$$

It implies that $l(x) = f_\xi(x, u'(x))$ a.e. in (0,1). Moreover if there exists an open set I such that $u(x) = 0$ for all $x \in I$, then $l(x) = f_\xi(x,0)$ for all $x \in I$. Since the function $f_\xi(\cdot,0)$ is non-decreasing l must be constant on I. Finally if there exists an open set J such that $u(x) > 0$ for all $x \in J$, then l must be clearly constant on J. Therefore since l must be continuous there exists $\alpha \in I\!R$ such that $l(x) = \alpha$ and $\alpha = f_\xi(x, u'(x))$ a.e. in (0,1). Moreover since $u' \in L^\infty(0,1)$ we must have $\alpha \in [c_{\sup}, d_{\inf}]$. It means that u is also solution of problem (P_S) ∎

Theorem 2.6: *Let $f, f_\xi \in C^0([0,1] \times I\!R)$. Assume that $f(x,\cdot): I\!R \to I\!R$ is convex and $f_\xi(\cdot,0): [0,1] \to I\!R$ is non-increasing. Then the problems* (P_S^+) and $(\mathrm{P}_S^{'+})$ are equivalent. In particular, let $K = \{x \in (0,1): d_{\inf} \geq f_\xi(x,0)\}$. Then

$$S_{\sup}^+ = S_{\sup}^{'+} = \lim_{\alpha \uparrow d_{\inf}} \int_K f_\alpha^{*+}(x,\alpha)\,dx \qquad \text{and} \qquad S_{\inf}^+ = S_{\inf}^{'+} = 0.$$

Proof: Let $S \geq 0$.

(a) Suppose problem (P_S^+) admits a solution. By Theorem 2.4 there exist $l \in W^{1,2}(0,1)$ and $u \in W^{1,\infty}(0,1)$ such that

$$\begin{aligned} &u(0) = 0, \quad u(1) = S, \quad u \geq 0 \\ &l \ \text{ is non-increasing} \\ &u'(x) \in \partial f^*(x, l(x)) \ \text{ a.e. in } (0,1) \\ &l'(x)u(x) = 0 \ \text{ a.e. in } (0,1). \end{aligned}$$

It implies that $l(x) = f_\xi(x, u'(x))$ a.e. in $(0,1)$. Moreover if there exists an open set I such that $u(x) = 0$ for all $x \in I$, then $l(x) = f_\xi(x,0)$ for all $x \in I$. Conversely, if there exists an open set J such that $u(x) > 0$ for all $x \in J$, then l must be clearly constant on J. Therefore on a sufficient small interval $l(x)$ is either constant or equal to $f_\xi(x,0)$. Since $f_\xi(\cdot,0)$ and l are non-increasing and continuous there are only two ways to define the function l:

(i) There exists $\alpha \geq f_\xi(0,0)$ such that $l(x) \equiv \alpha$.

(ii) There exists $\bar{x} \in [0,1]$ such that $l(x) = \begin{cases} f_\xi(x,0) & \text{if } 0 \leq x \leq \bar{x} \\ f_\xi(\bar{x},0) & \text{if } \bar{x} < x \leq 1. \end{cases}$

In the second case $u'(x) = 0$ for a.e. $x \in (0,\bar{x})$ and $l'(x) = 0$ for every $x \in (\bar{x},1)$.

Let us show that this function u is also a solution of problem $(\mathrm{P}_S^{'+})$. By letting $\alpha = l(1)$ we can say that in the two cases above u satisfies

$$u'(x) \begin{cases} \in \partial f^*(x,\alpha) & \text{for a.e. } x \in K(\alpha) \\ = 0 & \text{for a.e. } x \in K^c(\alpha) \end{cases}$$

where

$$K(\alpha) = \Big\{x \in [0,1]: \ f_\xi(x,0) < \alpha\Big\} \qquad \text{and} \qquad K^c(\alpha) = [0,1] \setminus K(\alpha).$$

Moreover since $f_\xi(\cdot,0)$ is non-increasing we have either $K(\alpha) = [0,1]$ in the case where $l(x) \equiv \alpha$ or $K(\alpha) = [\bar{x},1]$ in the case where l is defined as in the point (ii). In both of these cases $u'(x) \geq 0$ for a.e. $x \in K(\alpha)$. This last result implies that u is also a solution of problem $(\mathrm{P}_S^{'+})$.

Note that $l(x) \equiv \alpha$ with $\alpha < f_\xi(0,0)$ is not possible since in this case $u'(x) < 0$ for a.e. $x \in (0,\tilde{x})$ for some $\tilde{x} > 0$ and hence $u(x) < 0$ for $x \in (0,\tilde{x})$, which contradicts the fact that u is a solution of problem (P_S^+).

(b) Suppose problem $(\mathrm{P}_S'^{+})$ admits a solution. By proceeding as in the proof of Theorem 2.3 (see relations (2) and (3)) there exist $\alpha \in I\!R$ and $u \in A_S$ (recall that $A_S = \{u \in W^{1,\infty}(0,1) : u(0) = 0 \text{ and } u(1) = S\}$) such that

$$\alpha = f_\xi(x, u'(x)) \text{ for a.e. } x \in K(\alpha) \qquad \text{and} \qquad u'(x) = 0 \text{ for a.e. } x \in K^c(\alpha).$$

Since $f_\xi(\cdot, 0)$ is non-increasing we deduce that there exists $\bar{x} \in [0,1]$ such that $K = (\bar{x}, 1]$. Therefore by letting

$$l(x) \equiv \alpha \quad \text{if } \bar{x} = 0 \qquad \text{or} \qquad l(x) = \begin{cases} f_\xi(x,0) & \text{if } 0 \leq x \leq \bar{x} \\ f_\xi(\bar{x},0) & \text{if } \bar{x} < x \leq 1 \end{cases} \quad \text{if } \bar{x} > 0$$

we conclude that u is also solution of problem (P_S^+) ∎

Corollary 2.1: *Let $f, f_\xi \in C^0([0,1] \times I\!R)$. Assume that $f(x,\cdot) : I\!R \to I\!R$ is convex and $f_\xi(\cdot, 0) : [0,1] \to I\!R$ is constant. Then in view of Theorems* 2.5 *and* 2.6 *the problems* (P_S), (P_S^+) *and* $(\mathrm{P}_S'^{+})$ *are equivalent for all* $S \geq 0$.

Before describing the interval to which S must belong to conclude at the existence of a solution for the problem (P_S^+) in the general case, we introduce a set E of functions which will describe the solutions of the problem. The definition of this set is motivated by the results of Theorem 2.4. We want to exhibit all the functions l which can satisfy the conditions of this theorem.

Definition 2.2: We define E as the set of functions $l \in W^{1,2}(0,1)$ such that the following conditions are fulfilled:

$(\mathbf{E}_a)$ l is non-increasing.

$(\mathbf{E}_b)$ $c(x) \leq l(x) \leq d(x)$ for every $x \in [0,1]$.

$(\mathbf{E}_c)$ For every $x_0 \in (0,1)$ at least one of the two following conditions is satisfied:

$(\mathbf{E}_c)_1$ $l(x_0) \in \partial f(x_0, 0)$

$(\mathbf{E}_c)_2$ $l(x) = l(x_0)$ $\big(x \in (x_0 - \delta_0, x_0 + \delta_0)\big)$ for some $\delta_0 > 0$.

Remarks: (i) The condition $(\mathrm{E}_c)_1$ is equivalent to $f_\xi^-(x_0,0) \leq l(x_0) \leq f_\xi^+(x_0,0)$. Note that the functions $f_\xi^+(\cdot,0) : [0,1] \to I\!R$ and $f_\xi^-(\cdot,0) : [0,1] \to I\!R$ are continuous.

(ii) The set E can be empty. However if, for example, $\inf_{x\in[0,1]} d(x) = d_{\inf} \geq c_{\sup} = \sup_{x\in[0,1]} c(x)$, then E is not empty since $l \equiv d_{\inf}$ or $l \equiv c_{\sup}$ belongs to E.

Definition 2.3: In the case where E is not empty we define the functions

$$l_{\sup} : [0,1] \to I\!R \cup \{+\infty\} \quad \text{by } l_{\sup}(x) = \sup_{l \in E} l(x)$$
$$l_{\inf} : [0,1] \to I\!R \cup \{-\infty\} \quad \text{by } l_{\inf}(x) = \inf_{l \in E} l(x).$$

We give below two examples how to obtain these function for a given function f.

Example 2.5: Let us consider the function $f : [0,1] \times I\!R \to I\!R$ defined by $f(x,\xi) = \rho(x)\phi(\xi)$. Assume that f is continuous and satifies condition (H) (this is equivalent to saying that ρ is continuous and ϕ is convex).

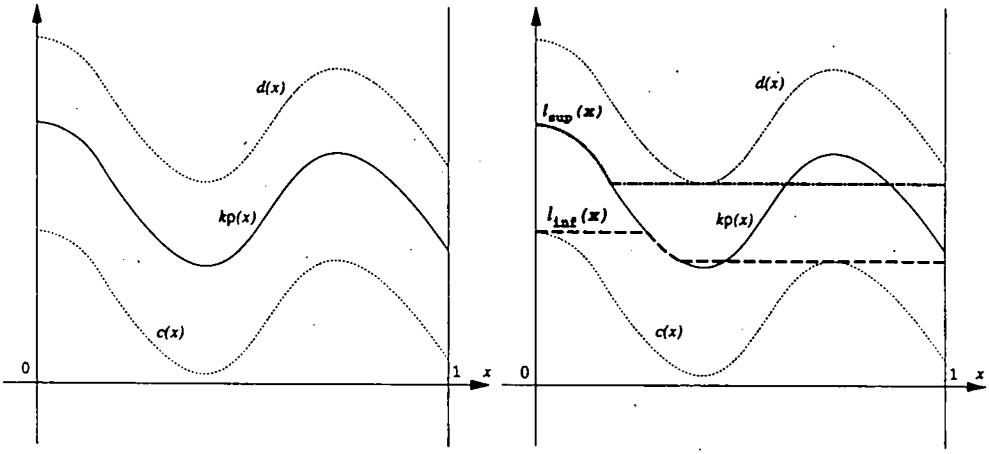

Figure 2: Construction of $l_{\sup}$ and $l_{\inf}$ when $f(x,\xi) = \rho(x)\phi(\xi)$

We suppose that ϕ satisfies the conditions

$$\lim_{\xi \to -\infty} \frac{\phi(\xi)}{\xi} = \phi_- > -\infty \qquad \text{and} \qquad \lim_{\xi \to +\infty} \frac{\phi(\xi)}{\xi} = \phi_+ < +\infty.$$

Therefore we have $c(x) = \phi_-\rho(x)$, $d(x) = \phi_+\rho(x)$ and $f_\xi^+(x,0) = k\rho(x)$, where $k \in \partial\phi(0)$. Finally suppose that the set E associated to f is not empty. We then can construct the functions $l_{\sup}$ and $l_{\inf}$. In Figure 2 we give a representation of all these functions in a general example.

Example 2.6: We take the same example with

$$\rho(x) = \sin(2\pi x) + \frac{7}{4} \qquad \text{and} \qquad \phi(\xi) = \int_0^\xi (\arctan s + \pi)\, ds.$$

Then we have

$$f_\xi^+(x,\xi) = f_\xi^-(x,\xi) = f_\xi(x,\xi) = \left(\sin(2\pi x) + \frac{7}{4}\right)(\arctan \xi + \pi)$$

$$d(x) = \frac{3\pi}{2}\left(\sin(2\pi x) + \frac{7}{4}\right) \qquad \text{and} \qquad c(x) = \frac{\pi}{2}\left(\sin(2\pi x) + \frac{7}{4}\right).$$

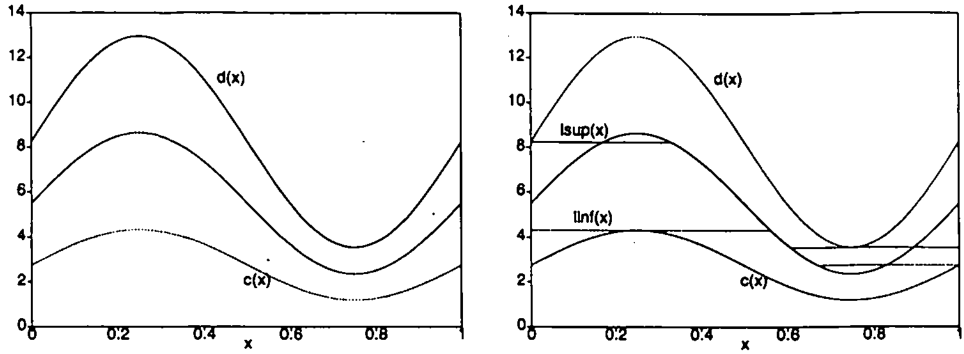

Figure 3: Construction of $l_{\sup}$ and $l_{\inf}$, $f(x,\xi) = \left(\sin(2\pi x) + \frac{7}{4}\right)\left(\int_0^\xi (\arctan s + \pi)\, ds\right)$

The functions $l_{\sup}$ and $l_{\inf}$ are given by

$$l_{\sup}(x) = \begin{cases} d(0) & \text{if } 0 \le x \le d_1 \\ f_\xi^+(x,0) & \text{if } d_1 < x < d_2 \\ d(\frac{3}{4}) & \text{if } d_2 \le x \le 1 \end{cases} \qquad l_{\inf}(x) = \begin{cases} c(\frac{1}{4}) & \text{if } 0 \le x \le c_1 \\ f_\xi^+(x,0) & \text{if } c_1 < x < c_2 \\ c(1) & \text{if } c_2 \le x \le 1 \end{cases}$$

where

$$\frac{1}{4} < d_1 < \frac{1}{2}, \quad \frac{1}{2} < d_2 < \frac{3}{4}, \quad \frac{1}{4} < c_1 < \frac{1}{2}, \quad \frac{1}{2} < c_2 < \frac{3}{4}$$

satisfy the equations

$$d(0) = f_\xi(d_1,0), \quad d\left(\frac{3}{4}\right) = f_\xi(d_2,0), \quad c\left(\frac{1}{4}\right) = f_\xi(c_1,0), \quad c(1) = f_\xi(c_2,0).$$

In Figure 3 above the representation of all these functions is given.

With the definition of the set E we can study the problem (P_S^+) in the general case. We only restrict our work by introducing the following two weak assumptions:

(A1) $f \in C^1([0,1] \times \mathbb{R})$

(A2) There exist disjoint open intervals $J_1, \ldots, J_n$ such that $\bigcup_{k=1}^n \bar{J}_k = [0,1]$ and $f_\xi(\cdot,0)$ is monotone on the intervals J_k $(k = 1, \ldots, n)$.

Observe that we can show the existence of a solution in the general case (i.e. f only satisfies condition (H)) but we avoid deliberately this general proof, the notations being too heavy in this case. Moreover the results are qualitatively not better.

The assumptions (A1) and (A2) imply that, for every function $l \in E$, there exist disjoint open intervals $I_1, \ldots, I_n$ such that $\bigcup_{k=1}^n \bar{I}_k = [0,1]$ and l is either constant or equal to $f_\xi(x,0)$ on the intervals I_k $(k = 1, \ldots, n)$. However these assumptions are not sufficient to prove that E is not empty. We give below a necessary and sufficient condition to determine if E has an element.

Lemma 2.6: *Assume that the function f satisfies conditions* (H), (A1) *and* (A2). *Then E is empty if and only if there exist elements x_1 and x_2 such that $x_1 < x_2$ and $d(x_1) < c(x_2)$.*

Proof: Let $x_1 < x_2$ and $d(x_1) < c(x_2)$. Suppose that E is not empty and $l \in E$. This implies that $l(x_1) \geq l(x_2)$ and $d(x_1) \geq l(x_1) \geq l(x_2) \geq c(x_2)$ which is absurd since $d(x_1) < c(x_2)$. Therefore E is empty.

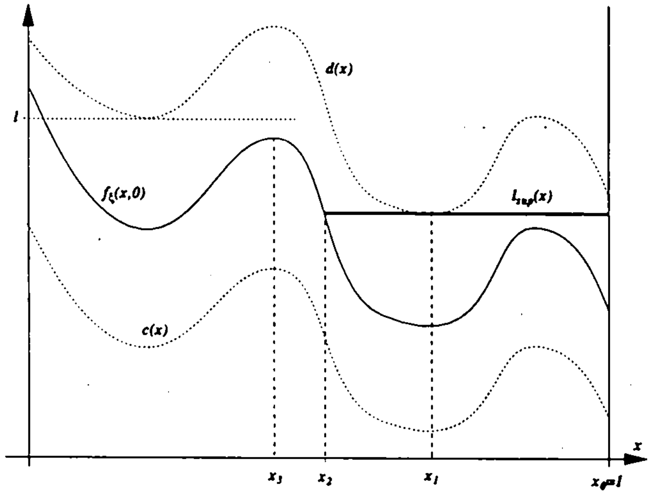

Figure 4: Beginning of the construction of l_{sup}

Conversely, if for every $x_1 \in [0,1]$ we have $d(x_1) \geq c(x_2)$ for every $x_2 \in [x_1, 1]$, then we can exhibit a function which belongs to E. Indeed two cases can happen:

(a) If $d(x) = +\infty$ for every $x \in [0,1]$, then the function $l(x) \equiv \max_{y\in[0,1]} f_\xi(y,0)$ belongs to E since for every $x \in [0,1]$ we have

$$c(x) \leq f_\xi(x,0) \leq \max_{y\in[0,1]} f_\xi(y,0) < d(x) \equiv +\infty.$$

(b) If $\inf_{x\in[0,1]} d(x) < +\infty$, then we can construct a function $l_{\text{sup}} \in W^{1,2}(0,1)$ which belongs to E. We will see that this function is actually the supremum of all the functions which belongs to E.

We construct l_{sup} from right to left, i.e. from $x = 1$ to $x = 0$. Let $x_0 = 1$.

(i) Let $x_1 = \min\{x \in [0, x_0] : d(x) = d_{\text{inf}}\}$. We let $l_{\text{sup}}(x) = d_{\text{inf}}$ for all $x \in [x_1, x_0]$.

(ii) If $f_\xi(x,0) \leq d_{\text{inf}}$ for all $x \in [0, x_1]$, then we let $l_{\text{sup}}(x) = d_{\text{inf}}$ for all $x \in [0, x_1]$ and the construction is finished.

(iii) Assume that (ii) does not hold. We then let

$$x_2 = \max\Big\{x \in [0, x_1] : \ f_\xi(x,0) = d_{\text{inf}} \ \text{ and } \ f_\xi(\cdot,0) \ \text{ is decreasing at } \ x\Big\}$$

and set $l_{\sup}(x) = d_{\inf}$ for all $x \in [x_2, x_1]$. We finally consider

$$x_3 = \min\Big\{x \in [0, x_2] : \ f_\xi(\cdot, 0) \quad \text{is non-increasing on the interval } [x, x_2]\Big\}$$

and $l = \inf_{x\in[0,x_3]} d(x)$. See Figure 4 on the previous page to observe what x_1, x_2, x_3 and l represent in a concrete example.

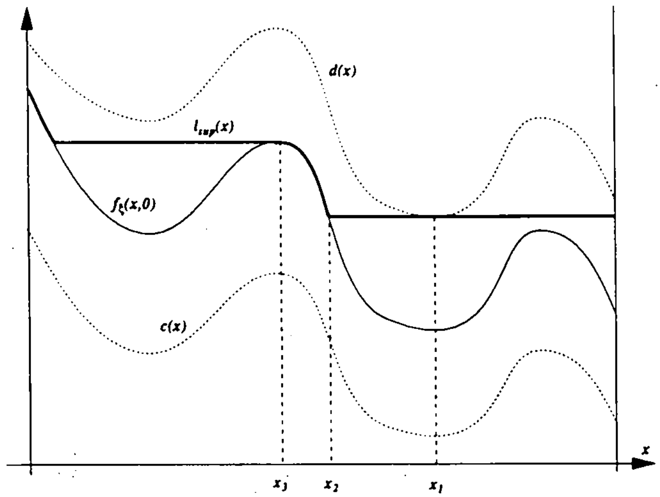

Figure 5: Construction of $l_{\sup}$

At this point of the construction two cases can happen:

(iii)$_a$ $l \geq f_\xi(x_3, 0)$. Then set $l_{\sup}(x) = f_\xi(x, 0)$ for all $x \in [x_3, x_2]$.

(iii)$_b$ $l < f_\xi(x_3, 0)$. Since $l > f_\xi(x_2, 0)$ there exists $x_4 \in (x_3, x_2)$ such that $f_\xi(x_4, 0) = l$. We then set $l_{\sup}(x) = f_\xi(x, 0)$ for all $x \in [x_4, x_2]$.

The points (i), (ii) and (iii) represent the first step of the construction of $l_{\sup}$. To go further it is sufficient to take again in account the following two cases:

(iii)$_a$ We replace x_1 by x_3 and $d_{\inf}$ by $f_\xi(x_3, 0)$. We then return to point (ii).

(iii)$_b$ We replace $x_0 = 1$ by x_4 and $d_{\inf}$ by $f_\xi(x_4, 0)$. We then return to point (i).

In Figure 5 above the representation of such a construction is given. Since $f_\xi(\cdot, 0)$ is piecewise monotone $l_{\sup}$ is constructed in a finite number of steps. Moreover we have $c(x) \leq l_{\sup}(x) \leq d(x)$ since $l_{\sup}(x)$ is either equal to $f_\xi(x, 0)$ or equal to a constant ∎

Lemma 2.7: *Assume that the function f satisfies conditions* (H), (A1) *and* (A2). *Assume that E is not empty.*

(a) *If $d(x) = +\infty$ (if $c(x) = -\infty$, respectively) for every $x \in [0, 1]$, then $l_{\sup}(x) \equiv +\infty$ ($l_{\inf}(x) \equiv -\infty$, respectively).*

(b) *If there exists $\bar{x} \in [0, 1]$ such that $d(\bar{x}) < +\infty$ ($c(\bar{x}) > -\infty$, respectively), then $l_{\sup} \in E$ ($l_{\inf} \in E$, respectively).*

Proof: We prove the statements (a) and (b) for the function $l_{\sup}$ only. For $l_{\inf}$ proceed analogously.

(a) Suppose that $d(x) \equiv +\infty$. Let $l(x) \equiv k$ with $k \geq \max_{x\in[0,1]} f_\xi(x,0)$. Then $l \in E$ since l satisfies clearly the conditions (E_a) - (E_c) of Definition 2.2. Therefore $\sup_{x\in E} l(x) = +\infty$ for every $x \in [0,1]$.

(b) Let there exists $\bar{x} \in [0,1]$ such that $d(\bar{x}) < +\infty$. Then for any $l \in E$ we have $l(x) \leq d(\bar{x})$ for all $x \geq \bar{x}$. Moreover l can be strictly decreasing at a point $\tilde{x}$ only if $l(\tilde{x}) = f_\xi(\tilde{x},0)$. Therefore

$$l_{\sup}(x) \leq \max\left(\max_{x\in[0,1]} f_\xi(x,0),\ d_{\inf}\right) \qquad \text{for every } x \in [0,1]$$

where $d_{\inf} = \inf_{x\in[0,1]} d(x)$. Hence $l_{\sup}$ satisfies condition (E_b) since $d(x) \geq f_\xi(x,0) \geq c(x)$ for all $x \in [0,1]$.

The function $l_{\sup}$ is non-increasing. Let $0 \leq y \leq x \leq 1$. For every $\varepsilon > 0$ there exists $l_{\varepsilon,x} \in E$ such that $l_{\sup}(x) \leq l_{\varepsilon,x}(x) + \varepsilon$. Since $l_{\varepsilon,x}$ is non-increasing, we have

$$l_{\sup}(x) \leq l_{\varepsilon,x}(x) + \varepsilon \leq l_{\varepsilon,x}(y) + \varepsilon \leq l_{\sup}(y) + \varepsilon.$$

Since this result is true for every $\varepsilon > 0$ we conclude that $l_{\sup}$ satisfies condition (E_a).

The function $l_{\sup}$ is continuous from the right. Indeed, let $x_0 \in [0,1)$. For every $\varepsilon > 0$ there exists $l_{\varepsilon,x_0} \in E$ such that $l_{\sup}(x_0) \leq l_{\varepsilon,x_0}(x_0) + \varepsilon/2$. Therefore for every $x \in [x_0, 1]$ we have

$$0 \leq l_{\sup}(x_0) - l_{\sup}(x) \leq \frac{\varepsilon}{2} + l_{\varepsilon,x_0}(x_0) - l_{\varepsilon,x_0}(x).$$

Since l_{ε,x_0} is continuous, there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that $|x - x_0| < \delta$ implies $|l_{\varepsilon,x_0}(x_0) - l_{\varepsilon,x_0}(x)| < \varepsilon/2$ and finally

$$0 \leq x - x_0 \leq \delta \Rightarrow 0 \leq l_{\sup}(x_0) - l_{\sup}(x) \leq \varepsilon.$$

The function $l_{\sup}$ is continuous from the left. Indeed, let $x_0 \in (0,1]$. For every $\varepsilon > 0$ there exists $\delta = \delta(x_0,\varepsilon) > 0$ such that $|x - x_0| < \delta$ implies $|f_\xi(x,0) - f_\xi(x_0,0)| < \varepsilon/2$.

With the definition of the set E, these results implies that, for every $l \in E$, $|x - x_0| < \delta$ yields $|l(x_0) - l(x)| < \varepsilon/2$. Let $x \in [x_0 - \delta, x_0]$. Then there exists $l_{x,\varepsilon} \in E$ and $l_{x_0,\varepsilon} \in E$ such that

$$l_{x,\varepsilon}(x) \leq l_{\sup}(x) \leq l_{x,\varepsilon}(x) + \frac{\varepsilon}{4} \quad \text{and} \quad l_{x_0,\varepsilon}(x_0) \leq l_{\sup}(x_0) \leq l_{x_0,\varepsilon}(x_0) + \frac{\varepsilon}{4}.$$

This implies

$$\begin{aligned} 0 &\leq l_{\sup}(x) - l_{\sup}(x_0) \\ &\leq l_{x,\varepsilon}(x) + \frac{\varepsilon}{4} - l_{x_0,\varepsilon}(x_0) \\ &= \frac{\varepsilon}{2} + l_{x,\varepsilon}(x) - l_{x,\varepsilon}(x_0) + l_{x,\varepsilon}(x_0) - l_{x_0,\varepsilon}(x_0) - \frac{\varepsilon}{4}. \end{aligned}$$

Since

$$l_{x,\varepsilon}(x_0) \le l_{\sup}(x_0) \le l_{x_0,\varepsilon}(x_0) + \frac{\varepsilon}{4}$$

we have finally

$$0 \le l_{\sup}(x) - l_{\sup}(x_0) \le \frac{\varepsilon}{2} + l_{x,\varepsilon}(x) - l_{x,\varepsilon}(x_0).$$

Moreover since $l_{x,\varepsilon} \in E$ we know that $0 \le x_0 - x < \delta$ implies $0 \le |l_{x,\varepsilon}(x) - l_{x,\varepsilon}(x_0)| < \frac{\varepsilon}{2}$. Therefore $l_{\sup}$ is continuous. We can then show in an elementary way that $l_{\sup}$ satisfies condition (E_c). This implies that $l_{\sup} \in E$ ∎

Now we are able to present the general theorem on the problem (P_S^+).

Theorem 2.7: *Assume that there exists l belonging to the interior of E and assume that the function f satisfies conditions* (H), (A1) *and* (A2). *Then*

(i) *either $l_{\inf}(x) \equiv c_{\sup}$ or there exist $b_0 = 0 \le a_1 < b_1 < \ldots < a_k < b_k < \ldots < a_n < b_n \le 1 = a_{n+1}$ such that*

$$l_{\inf}(x) = \begin{cases} f_\xi(x,0) & \text{if } x \in I_k = (a_k, b_k) \quad (k = 1, \ldots, n) \\ f_\xi(b_k,0) & \text{if } x \in [b_k, a_{k+1}] \quad (k = 0, 1, \ldots, n). \end{cases}$$

(ii) *either $l_{\sup}(x) \equiv d_{\inf}$ or there exist $\tilde b_0 = 0 \le \tilde a_1 < \tilde b_1 < \ldots < \tilde a_k < \tilde b_k < \ldots < \tilde a_n < \tilde b_n \le 1 = \tilde a_{n+1}$ such that*

$$l_{\sup}(x) = \begin{cases} f_\xi(x,0) & \text{if } x \in \tilde I_k = (\tilde a_k, \tilde b_k) \quad (k = 1, \ldots, n) \\ f_\xi(\tilde b_k,0) & \text{if } x \in [\tilde b_k, \tilde a_{k+1}] \quad (k = 0, 1, \ldots, n). \end{cases}$$

Let

$$K = \left(\bigcup_{k=1}^{n} I_k\right) \bigcap \left(\bigcup_{k=1}^{n} \tilde I_k\right)$$

$$v_{\sup}(x) = \lim_{\varepsilon \downarrow 0} \int_0^x f_\alpha^{*+}(\sigma, l_{\sup}(\sigma) - \varepsilon)\, d\sigma, \qquad v_{\inf}(x) = \lim_{\varepsilon \downarrow 0} \int_0^x f_\alpha^{*-}(\sigma, l_{\inf}(\sigma) + \varepsilon)\, d\sigma.$$

Then problem (P_S^+) *admits a solution for a certain $S \ge 0$ if and only if the two conditions*

(a1) *$v_{\sup}(x) \ge 0$ for every $x \in [0,1]$*

(a2) *$v_{\inf}(x) \le 0$ for every $x \in K$*

are fulfilled. Moreover, if conditions (a1) *and* (a2) *are satisfied then, let*

$$x_{\sup} = \min\left\{x \in [0,1] : \ l_{\sup}(x) = l_{\sup}(1)\right\}$$
$$x_{\inf} = \min\left\{x \in [0,1] : \ l_{\inf}(x) = l_{\inf}(1)\right\}.$$

Then

$$S_{\sup}^+ = \lim_{\varepsilon \downarrow 0} \int_{x_{\sup}}^1 f_\alpha^{*+}(\sigma, d_{\inf} - \varepsilon)\, d\sigma, \qquad S_{\inf}^+ = \max\left(0, \lim_{\varepsilon \downarrow 0} \int_{x_{\inf}}^1 f_\alpha^{*-}(\sigma, c_{\sup} + \varepsilon)\, d\sigma\right).$$

Remarks: (i) If E is empty, then problem (P_S^+) does not admit any solution for every $S \geq 0$.

(ii) K is the open set where $l_{\inf}$ and l_{sup} are equal and strictly decreasing. Moreover, if $x \in K$, then $u'_{\sup}(x) = u'_{\inf}(x) = 0$.

Proof of Theorem 2.7: *Step 1: Necessary and sufficient condition for the existence of a solution.* Theorem 2.4 says that problem (P_S^+) has a solution if and only if there exist $l \in W^{1,2}(0,1)$ and $u \in W^{1,\infty}(0,1)$ such that

$$u(0) = 0, \quad u(1) = S, \quad u \geq 0 \tag{4}$$

$$l \text{ is non-increasing} \tag{5}$$

$$u'(x) \in \partial f^*\big(x, l(x)\big) \quad \text{a.e. in } (0,1) \tag{6}$$

$$l'(x)u(x) = 0 \quad \text{a.e. in } (0,1). \tag{7}$$

These relations imply that $l \in E$. Since E is not empty we can define the functions

$$l_{\sup}(x) = \sup_{l \in E} l(x) \qquad \text{and} \qquad l_{\inf}(x) = \inf_{l \in E} l(x).$$

They are respectively the smallest and the greatest function for which the relations (5) and (6) are well defined. Moreover $l_{\sup}(x) \geq d_{\inf}$ and $l_{\inf}(x) \leq c_{\sup}$ for every $x \in [0,1]$.

Step 2: Condition (a1) *is necessary.* Let

$$v_{\sup}(x) = \lim_{\varepsilon \downarrow 0} \int_0^x f_\alpha^{*+}\big(\sigma, l_{\sup}(\sigma) - \varepsilon\big)\, d\sigma, \qquad v_{\inf}(x) = \lim_{\varepsilon \downarrow 0} \int_0^x f_\alpha^{*-}\big(\sigma, l_{\inf}(\sigma) + \varepsilon\big)\, d\sigma.$$

Letting $l \in E$ we define $u \in W^{1,\infty}$ by $u(0) = 0$ and $u'(x) \in \partial f^*\big(x, l(x)\big)$. Then $v_{\inf}(x) \leq u(x) \leq v_{\sup}(x)$. Therefore $v_{\sup}(x) \geq 0$ for every $x \in [0,1]$ is a necessary condition for the existence of a solution of problem (P_S^+).

Step 3: Condition (a2) *is necessary.* Letting $l \in E$ we define $u \in W^{1,\infty}$ by $u(0) = 0$ and $u'(x) \in \partial f^*\big(x, l(x)\big)$. Such a function u is a solution of problem (P_S^+) if and only if $S = \int_0^1 u'(x)\, dx$, $u \geq 0$ and $l'(x)u(x) = 0$ for a.e. $x \in [0,1]$. Therefore we define

$$g_- : \; E \to L^2(0,1) \quad \text{by} \quad g_-(l)(x) = l'(x) \int_0^x f_\alpha^{*-}\big(\sigma, l(\sigma)\big)\, d\sigma$$

$$g_+ : \; E \to L^2(0,1) \quad \text{by} \quad g_+(l)(x) = l'(x) \int_0^x f_\alpha^{*+}\big(\sigma, l(\sigma)\big)\, d\sigma.$$

To explain the intoduction of the functionals g_- and g_+ we consider $l \in E$ and u defined as above, i.e. $u \in W^{1,\infty}(0,1)$, $u(0) = 0$ and $u'(x) \in \partial f^*\big(x, l(x)\big)$. We then obtain

$$g_+(l)(x) \leq l'(x)u(x) \leq g_-(l)(x) \qquad \text{for a.e. } x \in [0,1]$$

since, for any $l \in E$, $l'(x) \leq 0$. With this definition, $l \in E$ and u satisfy all the conditions of Theorem 2.4 if and only if

$$g_+(l)(x) \leq 0 \leq g_-(l)(x) \qquad \text{a.e. in } [0,1]. \tag{8}$$

Let

$$K = \left(\bigcup_{k=1}^{n} I_k\right) \bigcap \left(\bigcup_{k=1}^{n} \tilde{I}_k\right)$$

This is clearly an open set. For every $x \in K$ and for every $l \in E$ we have $l(x) = l_{\inf}(x) = l_{\sup}(x) = f_\xi(x,0)$ and $l'(x) < 0$.

Suppose that for a certain $\bar{x} \in K$ we have $v_{\inf}(\bar{x}) > 0$. Then $v_{\sup}(\bar{x}) > 0$. Moreover for every $l \in E$ we have

$$g_+(l_{\sup})(\bar{x}) \leq g_+(l)(\bar{x}) \leq g_-(l)(\bar{x}) \leq g_-(l_{\inf})(\bar{x}) < 0.$$

Therefore the relation (7) cannot be satisfied for any $l \in E$ and problem (P_S^+) does not admit any solution for any $S \geq 0$. Condition (a2) is then necessary.

Step 4: Conditions (a1) *and* (a2) *are sufficient.* Observe first that $g_+(l_{\sup})(x) \leq 0$ for all $x \in [0,1]$, and for every $l \in E$ and for every $x \in [0,1]$ we have

$$g_+(l_{\sup})(x) \leq g_+(l)(x) \leq g_-(l)(x) \leq g_-(l_{\inf})(x). \tag{9}$$

(a) Let $x \in K$. Since conditions (a1) and (a2) are valid we have $l'(x) < 0$ and

$$g_+(l_{\sup})(x) \leq 0 \qquad \text{and} \qquad g_-(l_{\inf})(x) \geq 0 \qquad \text{for all } x \in K.$$

(b) Let $x \notin K$. We then have $l'_{\sup}(x) = 0$ or $l'_{\inf}(x) = 0$. Therefore we have

$$g_+(l_{\sup})(x) = 0 \qquad \text{or} \qquad g_-(l_{\inf})(x) = 0 \qquad \text{for all } x \notin K.$$

The cases (a) and (b) composed with relation (9) imply finally that

$$g_+(l_{\sup})(x) \leq 0 \leq g_-(l_{\inf})(x) \qquad \text{for all } x \in [0,1].$$

Since the functionals g_- and g_+ are continuous and E is connected there exists $\bar{l} \in E$ such that

$$g_+(\bar{l})(x) \leq 0 \leq g_-(\bar{l})(x) \qquad \text{for a.e. } x \in [0,1].$$

It means that the function $\bar{u}$ defined by $\bar{u}(0) = 0$ and $\bar{u}'(x) \in \partial f^*(x, \bar{l}(x))$ is a solution of the problem (P_S^+) for $S = \int_0^1 \bar{u}'(x)\,dx$.

Step 5: Characterization of $S_{\sup}^+$ and $S_{\inf}^+$. Suppose that conditions (a1) and (a2) are satisfied. Then there exists a solution of problem (P_S^+) for a certain $S \geq 0$. It means that there exists a function $\bar{l} \in E$ such that

$$g_+(\bar{l})(x) \leq 0 \leq g_-(\bar{l})(x) \qquad \text{for a.e. } x \in [0,1].$$

We then consider the following two cases:

(i) $\bar{l}(0) \leq d_{\inf}$. Then since $l_{\sup}(x) \geq d_{\inf}$ we conclude that $\tilde{l} \equiv d_{\inf}$ satisfies the conditions of Theorem 2.4 and we obtain

$$S^{+}_{\sup} = S_{\sup} = \lim_{\alpha \uparrow d_{\inf}} \int_0^1 f^{**+}_{\alpha}(x, \alpha)\, dx.$$

(ii) $\bar{l}(0) > d_{\inf}$. Then since $\bar{l}(1) \leq d_{\inf}$ we let $\bar{x} = \min\{x \in [0,1] : \bar{l}(x) = d_{\inf}\}$, $\tilde{l}(x) = \bar{l}(x)$ if $x < \bar{x}$ and $\tilde{l}(x) = d_{\inf}$ if $x \geq \bar{x}$. Then $\tilde{l}$ satisfies the conditions of Theorem 2.4. To obtain the maximal value of S it is sufficient to replace $\bar{x}$ by $x_{\sup} = \min\{x \in [0,1] : l_{\sup}(x) = l_{\sup}(1) = d_{\inf}\}$ and we have

$$S^{+}_{\sup} = \lim_{\varepsilon \downarrow 0} \int_{x_{\sup}}^1 f^{**+}_{\alpha}(\sigma, d_{\inf} - \varepsilon)\, d\sigma.$$

For the minimal value of S we consider $x_{\inf} = \min\{x \in [0,1] : l_{\inf}(x) = l_{\inf}(1)\}$. Taking into account that S must be positive we obtain

$$S^{+}_{\inf} = \max\left(0, \lim_{\varepsilon \downarrow 0} \int_{x_{\inf}}^1 f^{**-}_{\alpha}(\sigma, c_{\sup} + \varepsilon)\, d\sigma\right)$$

and the statement is proved ■

3. Comparison between the three problems

In this section we want to compare the minimal and the maximal values of S for the three problems (P_S), (P'^{+}_S) and (P^{+}_S). We suppose that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies conditions (H), (A1) and (A2). Note that the assumptions (A1) and (A2) can be relaxed but we avoid deliberately to consider the general case, the notations being too heavy. Observe that these two assumptions are not very strong.

3.1 Comparison between the maximal values of S

In Section 2 we characterized the maximal values $S_{\sup}$, $S^{+}_{\sup}$ and $S'^{+}_{\sup}$ for the three problems. Here we give relations between these values.

Theorem 3.1: *Assume that there exist numbers $S_1 \in \mathbb{R}$ and $S_2 \geq 0$ such that problems (P_{S_1}) and $(P^{+}_{S_2})$ have a solution. Then $S_{\sup} \leq S^{+}_{\sup} \leq S'^{+}_{\sup}$.*

Remark: $S_{\sup}$, $S^{+}_{\sup}$ and $S'^{+}_{\sup}$ are allowed to take the value $+\infty$.

The proof of Theorem 3.1 is given by the following two lemmas.

Lemma 3.1: *Assume that there exist numbers $S_1 \in I\!R$ and $S_2 \geq 0$ such that problems (P_{S_1}) and $(\mathrm{P}^+_{S_2})$ have a solution. Then the following statements are true.*

(i) $S_{\sup} \leq S^+_{\sup}$.

(ii) *If $S_{\sup}$ is bounded, then $S_{\sup} = u_{\sup}(1)$ and $S^+_{\sup} = S_{\sup} - \min_{x\in[0,1]} u_{\sup}(x)$ for $u_{\sup} : [0,1] \to I\!R \cup \{+\infty\}$ defined by $u_{\sup}(x) = \lim_{\alpha\uparrow d_{\inf}} \int_0^x f^{*+}_\alpha(\sigma,\alpha)\,d\sigma$ with $d_{\inf} = \inf_{x\in[0,1]} d(x)$.*

Proof: **(i)** Since problem (P_S) admits a solution for some $S \in I\!R$ we can apply Theorem 2.2, i.e.

$$S_{\sup} = \lim_{\alpha\uparrow d_{\inf}} \int_0^1 f^{*+}_\alpha(\sigma,\alpha)\,d\sigma.$$

Since problem (P^+_S) admits a solution for some $S \in I\!R$ we can apply Theorem 2.7, i.e. there exists $l \in E$ such that $l(x) \geq d_{\inf}$ for all $x \in [0,1]$ and

$$S^+_{\sup} = \lim_{\varepsilon\downarrow 0} \int_0^1 f^{*+}_\alpha\big(\sigma, l(\sigma) - \varepsilon\big)\,d\sigma.$$

Since the function $f^{*+}_\alpha(x,\cdot)$ is non-decreasing and $l(x) \geq d_{\inf}$ for all $x \in [0,1]$ we conclude that $S_{\sup} \leq S^+_{\sup}$.

(ii) If $\lim_{\alpha\uparrow d_{\inf}} \int_0^1 f^{*+}_\alpha(\sigma,\alpha)\,d\sigma < +\infty$, then we can define $u_{\sup} : [0,1] \to I\!R$ by

$$u_{\sup}(x) = \lim_{\alpha\uparrow d_{\inf}} \int_0^x f^{*+}_\alpha(\sigma,\alpha)\,d\sigma.$$

Let x_0 be the smallest $x \in [0,1]$ such that $d_{\inf} = \lim_{\xi\to+\infty} f^+_\xi(x_0,\xi)$. This implies that $f^{*+}_\alpha(x_0,d_{\inf}) = +\infty$. Let $\bar{x}$ be the smallest $x \in [0,1]$ sucht that $u_{\sup}(\bar{x}) = \min_{x\in[0,1]} u_{\sup}(x)$. This means that if $\bar{x} > 0$, then $u_{\sup}(\bar{x}) < u_{\sup}(x)$ for every $0 \leq x < \bar{x}$. We then consider the following two cases:

(a) $u_{\sup}(\bar{x}) = 0$ (i.e. $\bar{x} = 0$). Then $u_{\sup}(x) \geq 0$ for every $x \in [0,1]$ and $u_{\sup}$ is a solution of the problem (P^+_S) with $S = S_{\sup}$. Consequently $S^+_{\sup} \geq S_{\sup}$. It is therefore sufficient to show that $S^+_{\sup} \leq S_{\sup}$.

Let $u : [0,1] \to I\!R$ be a solution of problem (P^+_S) for a certain number S. Then there exists $l \in E$ (see Subsection 2.5 for the definition of the set E) such that

$$u'(x) \in \partial f^*(x, l(x)) \qquad \text{and} \qquad l'(x)u(x) = 0 \qquad \text{a.e. in } [0,1].$$

Since $l \in E$ we have $l(x) \leq d_{\inf}$ for all $x \in [x_0, 1]$. Then two possibilities can happen:

(a)$_1$ There exists $\tilde{x} \geq x_0$ such that $u(\tilde{x}) = 0$. Since $u_{\text{sup}}(\tilde{x}) \geq u_{\text{sup}}(\bar{x}) = 0$ we have

$$\begin{aligned} S &= u(1) - u(\tilde{x}) \\ &= \int_{\tilde{x}}^{1} f_\alpha^{**+}(\sigma, l(\sigma))\, d\sigma \leq \int_{\tilde{x}}^{1} f_\alpha^{**+}(\sigma, d_{\text{inf}})\, d\sigma \\ &= u_{\text{sup}}(1) - u_{\text{sup}}(\tilde{x}) = S_{\text{sup}} - u_{\text{sup}}(\tilde{x}) \\ &\leq S_{\text{sup}}. \end{aligned}$$

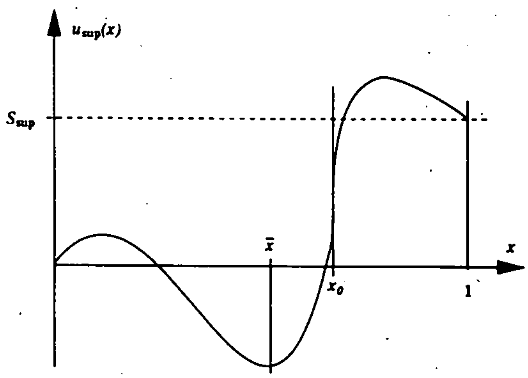

Figure 6: Representation of u_{sup} when $x_0 > \bar{x}$

(a)$_2$ $u(x) > 0$ for every $x \geq x_0$. Let $\tilde{x} < x_0$ such that $u(\tilde{x}) = 0$ and $u(x) > 0$ for all $x \in (\tilde{x}, 1]$. Such a point $\tilde{x}$ exists since $u(0) = 0$. Since $u(x) > 0$ for all $x \in (\tilde{x}, 1]$ there exist $k \in \mathbb{R}$ such that $l(x) = k$ for all $x \in (\tilde{x}, 1]$ (i.e. $l'(x) = 0$ for all $x \in (\tilde{x}, 1]$). Hence for $x > \tilde{x}$ we have

$$u(x) = \int_{\tilde{x}}^{x} f_\alpha^{**+}(\sigma, l(\sigma))\, d\sigma = \int_{\tilde{x}}^{x} f_\alpha^{**+}(\sigma, k)\, d\sigma.$$

Moreover, we know that $k \leq d_{\text{inf}}$. Therefore we obtain

$$S = u(1) - u(\tilde{x}) = \int_{\tilde{x}}^{1} f_\alpha^{**+}(\sigma, k)\, d\sigma \leq \int_{\tilde{x}}^{1} f_\alpha^{**+}(\sigma, d_{\text{inf}})\, d\sigma = u_{\text{sup}}(1) - u_{\text{sup}}(\tilde{x}) \leq S_{\text{sup}}.$$

Considering that $S^+_{\text{sup}} \geq S_{\text{sup}}$ this last inequality implies that $S^+_{\text{sup}} = S_{\text{sup}}$.

(b) $u_{\text{sup}}(\bar{x}) < 0$.

(b)$_1$ $x_0 > \bar{x}$ (see Figure 6). We then consider the following problem: Find $u \in W^{1,\infty}(0, \bar{x})$ and $l \in E$ such that

$(\mathbf{P}_{\bar{x}})$ $u(0) = u(\bar{x}) = 0, \quad u(x) \geq 0$ for every $x \in (0, \bar{x})$

$l(\bar{x}) = d_{\inf}$
$u'(x) \in \partial f^*(x, l(x))$ and $l'(x)u(x) = 0$ a.e. in $[0, \bar{x}]$.

(b)$_{11}$ Problem $(\mathrm{P}_{\bar{x}})$ admits a solution $\bar{u}: [0, \bar{x}] \to \mathbb{R}$. Then we can extend this function on $[0, 1]$ by letting

$$\bar{u}(x) = \int_{\bar{x}}^{x} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma \qquad \text{if } x \in (\bar{x}, 1].$$

With such a definition $\bar{u}$ is a solution of problem (P_S^+) where $S = S_{\sup} - u_{\sup}(\bar{x})$. This implies that $S_{\sup}^+ \geq S_{\sup} - u_{\sup}(\bar{x})$.

The reverse inequality can be shown as follows. Suppose that there exists a solution u of problem (P_S^+) for a certain $S > 0$. Let $\tilde{x} \in [0, 1]$ such that $u(\tilde{x}) = 0$ and $u(x) > 0$ for all $x \in (\tilde{x}, 1]$. Such a $\tilde{x}$ exists since u is continuous, $u(0) = 0$ and $u(1) = S > 0$. Then there exists $k \leq d_{\inf}$ such that

$$S = \int_{\tilde{x}}^{1} f_\alpha^{*+}(\sigma, k)\, d\sigma \leq \int_{\tilde{x}}^{1} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma = S_{\sup} - u_{\sup}(\tilde{x}) \leq S_{\sup} - u_{\sup}(\bar{x}).$$

(b)$_{12}$ Problem $(\mathrm{P}_{\bar{x}})$ does not admit any solution. Then we show that this case is impossible. Suppose that there exists a function u which is solution of problem (P_S^+) for a certain number $S \geq 0$. Then $u(\bar{x}) > 0$ since otherwise problem $(\mathrm{P}_{\bar{x}})$ admits a solution. Since u is a solution of problem (P_S^+) there exists $l \in E$ such that

$$u'(x) \in \partial f^*(x, l(x)) \qquad \text{and} \qquad l'(x)u(x) = 0 \qquad \text{a.e. in } [0, 1].$$

Let $\tilde{x} < \bar{x}$ such that $u(\tilde{x}) = 0$ and $u(x) > 0$ for every $x \in (\tilde{x}, \bar{x}]$. This implies that

$$l(x) = k \qquad \text{and} \qquad u(x) = \int_{\tilde{x}}^{x} f_\alpha^{*+}(\sigma, k)\, d\sigma \qquad \text{for all } x \in (\tilde{x}, \bar{x}]$$

where $k = f_\xi^+(\tilde{x}, 0)$. Since $u(\bar{x}) > 0$ and $u(\tilde{x}) = 0$ we have

$$0 < u(\bar{x}) - u(\tilde{x}) = \int_{\tilde{x}}^{\bar{x}} f_\alpha^{*+}(\sigma, k)\, d\sigma.$$

Moreover

$$0 \geq u_{\sup}(\bar{x}) - u_{\sup}(\tilde{x}) = \int_{\tilde{x}}^{\bar{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma$$

which implies that $k > d_{\inf}$. Since $l(x_0) \leq d_{\inf}$, $l(\bar{x}) = k > d_{\inf}$ and $l'(x)u(x) = 0$ a.e. there exists $\hat{x} \in (\bar{x}, x_0)$ such that $u(\hat{x}) = 0$. We finally obtain

$$u_{\sup}(\hat{x}) - u_{\sup}(\bar{x}) = \int_{\bar{x}}^{\hat{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma < \int_{\bar{x}}^{\hat{x}} f_\alpha^{*+}(\sigma, k)\, d\sigma = u(\hat{x}) - u(\bar{x}) < 0$$

which is absurd since $u_{\sup}(x) - u_{\sup}(\bar{x}) \geq 0$ for every $x \in [0,1]$. This implies that, for every $S \geq 0$, problem (P_S^+) does not admit any solution. This is in contradiction with the hypothesis of the lemma.

(b.2) $x_0 \leq \bar{x}$ (see Figure 7). This last case is impossible since we can show that if $x_0 \leq \bar{x}$, then problem (P_S^+) does not admit any solution.

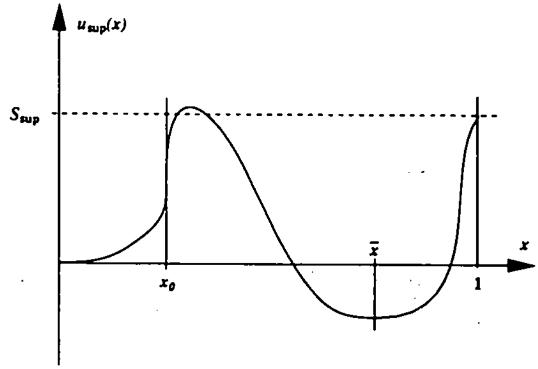

Figure 7: Representation of $u_{\sup}$ when $x_0 < \bar{x}$

Suppose that there exists a function $u: [0,1] \to I\!R$ that is solution of problem (P_S^+) for a certain $S \geq 0$. It means that there exists a function $l \in E$ (see Subsection 2.5) with $l(x) \leq d_{\inf}$ for all $x \in [x_0, 1]$ such that $u(x) = \int_0^x f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma$. Hence for $x \geq x_0$ we obtain

$$\begin{aligned} u(x) &= \int_0^{x_0} f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma + \int_{x_0}^{x} f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma \\ &= u(x_0) + \int_{x_0}^{x} f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma \\ &\leq u(x_0) + \int_{x_0}^{x} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma. \end{aligned}$$

Since $\bar{x} \geq x_0$ we have

$$u(\bar{x}) \leq u(x_0) + \int_{x_0}^{\bar{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma = u(x_0) + u_{\sup}(\bar{x}) - u_{\sup}(x_0).$$

Since $u(\bar{x}) \geq 0$ and $u_{\sup}(\bar{x}) - u_{\sup}(x_0) < 0$ we conclude that $u(x_0) > 0$.

Let $\tilde{x} < x_0$ such that $u(\tilde{x}) = 0$ and $u(x) > 0$ for every $x \in (\tilde{x}, x_0]$. Such an $\tilde{x}$ exists since $u(x_0) > 0$ and $u(0) = 0$. Moreover since $u(x) > 0$ for all $x \in (\tilde{x}, x_0)$ there exists a constant $k \le d_{\inf}$ such that $l(x) = k$ for all $x \in (\tilde{x}, x_0)$ and

$$u(x) = \int_{\tilde{x}}^{x} f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma \qquad \text{for all } \ x \in [\tilde{x}, 1].$$

Finally since $u(\bar{x}) \ge 0$ and $u(\tilde{x}) = 0$, we have

$$0 \le u(\bar{x}) - u(\tilde{x}) = \int_{\tilde{x}}^{\bar{x}} f_\alpha^{*+}(\sigma, l(\sigma))\, d\sigma \le \int_{\tilde{x}}^{\bar{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma = u_{\sup}(\bar{x}) - u_{\sup}(\tilde{x}).$$

This last result is in contradiction with the fact that $\bar{x}$ is the smallest element of $[0, 1]$ which realizes the minimum of the function $u_{\sup}$. It implies that problem (P_S^+) does not admit any solution in this case ∎

Lemma 3.2: *Assume that there exists a number $S \ge 0$ such that problem (P_S^+) has a solution. Then $S_{\sup}^+ \le S_{\sup}^{\prime +}$.*

Proof: In Subsection 2.4 we have shown that

$$S_{\sup}^{\prime +} = \int_K f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma$$

where $K = \{x \in [0, 1] : \ f_\xi^-(x, 0) \le d_{\inf}\}$. From the preceding lemma we know that

$$S_{\sup}^+ = S_{\sup} - \min_{x \in [0,1]} \left(\int_0^x f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma \right).$$

Let $\bar{x} \in [0, 1]$ be a point which realizes the minimum of $\int_0^x f_\alpha^{*+}(\sigma, d_{\inf})$. Therefore we have

$$\begin{aligned} S_{\sup}^+ - S_{\sup}^{\prime +} &= \int_0^1 f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma - \int_0^{\bar{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma - \int_K f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma \\ &= \int_{K^c} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma - \int_0^{\bar{x}} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma \\ &= \int_{K^c \cap (\bar{x}, 1)} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma - \int_{K \cap (0, \bar{x})} f_\alpha^{*+}(\sigma, d_{\inf})\, d\sigma \\ &\le 0 \end{aligned}$$

since $f_\alpha^{*+}(x, d_{\inf}) \le 0$ if $x \in K^c$ and $f_\alpha^{*+}(x, d_{\inf}) > 0$ if $x \in K$ ∎

3.2 Comparison between the minimal values of S

We first recall that $S'^{+}_{\inf}$ is always zero. Concerning $S_{\inf}$ and $S^{+}_{\inf}$, it is not possible in general to say if one is greather than the other. As we will see in the following examples, $S_{\inf}$ and $S^{+}_{\inf}$ are in general independent. Recall that we suppose the function f to satisfy conditions (H), (A1) and (A2). In particular it implies that $f^{+}_{\xi}(x,\xi) = f^{-}_{\xi}(x,\xi) = f_{\xi}(x,\xi)$.

Examples 3.1: Let us consider the situation described in Figure 8 where x_0 is that point in which c realizes its maximum and $\bar{x}$ is defined as a point where $f_{\xi}(\cdot,0)$ is decreasing and such that $c_{\sup} = f_{\xi}(\bar{x},0)$. Note that here we have $c_{\sup} = c(x_0 = 1)$.

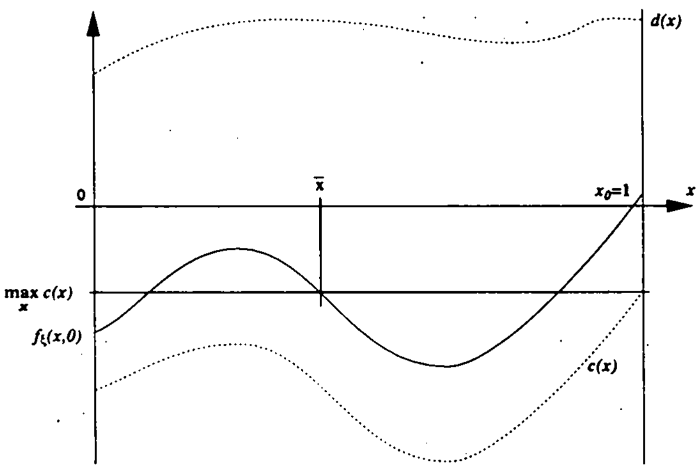

Figure 8: Graph of the functions c, d and $f(\cdot,0)$ in Example 3.1

We suppose that $\inf_x d(x) > \max_x f_{\xi}(x,0)$. This hypothesis implies that for some $S \geq 0$ there exists a solution for each one of the three problems (P_S), (P'^{+}_{S}) and (P^{+}_{S}). With the example presented in Figure 8 we construct in Figure 9 the values of $S_{\inf}$ and $S^{+}_{\inf}$. In Figure 9 we can observe that the point $\bar{x}$ is the global minimum of the function

$$u_{\inf}(x) = \int_0^x f^{*-}_{\alpha}(\sigma, c_{\sup})\,d\sigma.$$

Note that $S_{\inf} = u_{\inf}(1)$. Moreover, by continuity, there exist $x_1 < \bar{x}$ and $\alpha_1 > c_{\sup}$ (see Figure 9) such that

$$\alpha_1 = f_{\xi}(x_1, 0) \qquad \text{and} \qquad \int_0^{x_1} f^{*-}_{\alpha}(\sigma, \alpha_1)\,d\sigma = 0.$$

Therefore by using Theorem 2.4 and by letting

$$l(x) = \begin{cases} \alpha_1 & \text{if } 0 \le x \le x_1 \\ f_\xi(x,0) & \text{if } x_1 < x \le \bar{x} \\ c_{\sup} & \text{if } \bar{x} < x \le 1 \end{cases} \quad \text{and} \quad u(x) = \int_0^x f_\alpha^{*-}(\sigma, l(\sigma))\, d\sigma$$

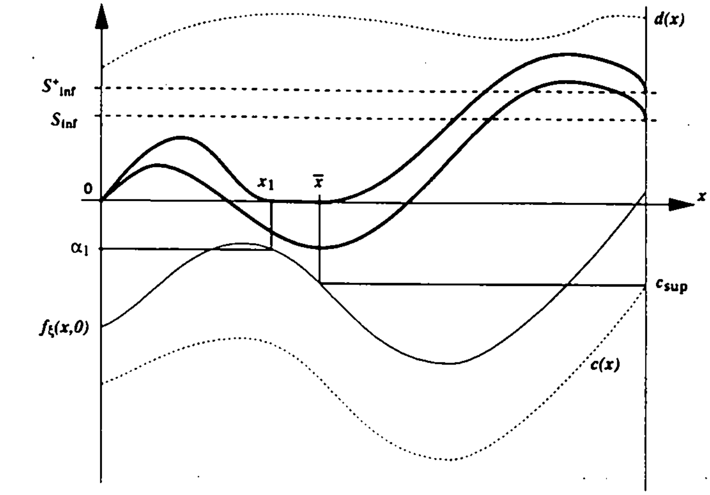

Figure 9: Construction of $S_{\inf}$ and $S^+_{\inf}$ in Example 3.1

we conclude that u is the minimal solution of problem (P^+_S). It means that $S^+_{\inf} = S_{\inf} - u_{\inf}(\bar{x})$, and since $u_{\inf}(\bar{x}) < 0$ we have $S^+_{\inf} > S_{\inf}$.

To show that this last result is not true in general we consider a second example.

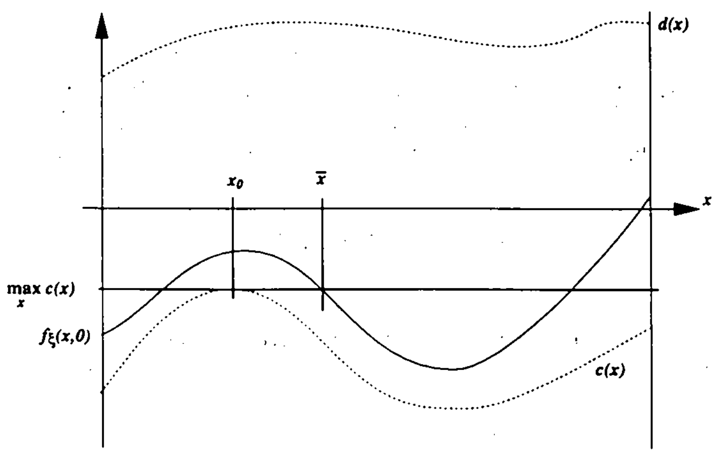

Figure 10: Graph of the functions c, d and $f_\xi(\cdot, o)$ in Example 3.2

Example 3.2: Let the situation described in Figure 10. In this figure, x_0 is always the point where c realizes its maximum, and $\bar{x}$ is defined as a point where $f_\xi(\cdot, 0)$ is decreasing and $f_\xi(\bar{x}, 0) = c(x_0)$. Here again we suppose that the three problems (P_S), (P_S^+) and $(P_S'^+)$ admit a solution for some $S \geq 0$.

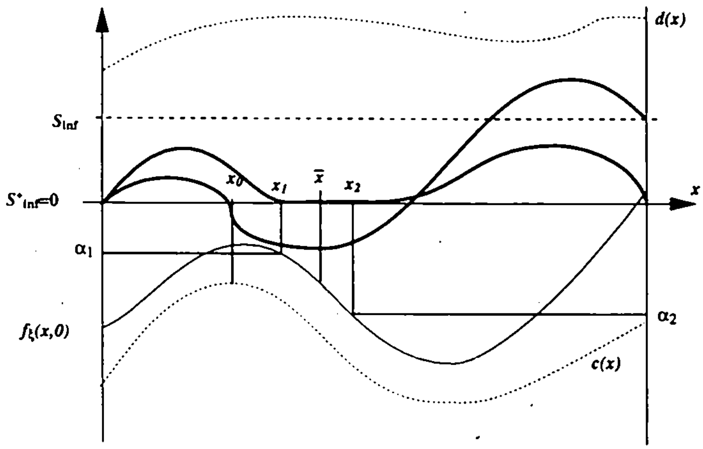

Figure 11: Construction of $S_{\inf}$ and $S^+_{\inf}$ in Example 3.2

In Figure 11 we can observe that the point $\bar{x}$ is the global minimum of the function

$$u_{\inf}(x) = \int_0^x f_\alpha^{*-}(\sigma, c_{\sup})\, d\sigma.$$

Note that $S_{\inf} = u_{\inf}(1) > 0$. Moreover, by continuity, there exist $x_1 < \bar{x}$, $x_2 > \bar{x}$, $\alpha_1 > c_{\sup}$ and $\alpha_2 < c_{\sup}$ (see Figure 11) such that

$$\alpha_1 = f_\xi(x_1, 0), \quad \alpha_2 = f_\xi(x_2, 0), \quad \int_0^{x_1} f_\alpha^{*-}(\sigma, \alpha_1)\, d\sigma = 0, \quad \int_{x_2}^1 f_\alpha^{*-}(\sigma, \alpha_2) d\sigma = 0.$$

Therefore by using Theorem 2.4 and letting

$$l(x) = \begin{cases} \alpha_1 & \text{if } 0 \leq x \leq x_1 \\ f_\xi(x, 0) & \text{if } x_1 < x \leq x_2 \\ \alpha_2 & \text{if } x_2 < x \leq 1 \end{cases} \quad \text{and} \quad u(x) = \int_0^x f_\alpha^{*-}(\sigma, l(\sigma))\, d\sigma$$

we conclude that u is the minimal solution of problem (P_S^+) since $u(x) \geq 0$ for all $x \in [0,1]$ and $u(1) = 0$ (i.e. $S = 0$). It implies in this example that $0 = S^+_{\inf} < S_{\inf}$. This last inequality, composed with the inequality obtained in Example 3.1 (see Figure 9), shows that $S^+_{\inf}$ and $S_{\inf}$ are in general independent.

3. 3 Independence of the problems (P_S) and (P_S^+).

3.3.1 Existence of a solution of problem (P_S) for every $S \in \mathbb{R}$ and non-existence of a solution of problem (P_S^+) for every $S \geq 0$. In this subsection we exhibit a function f such that problem (P_S) has a solution for every $S \in \mathbb{R}$ and, for the same f, problem (P_S^+) does not admit any solution for every $S \geq 0$. In Figure 12 we illustrate such a possibility.

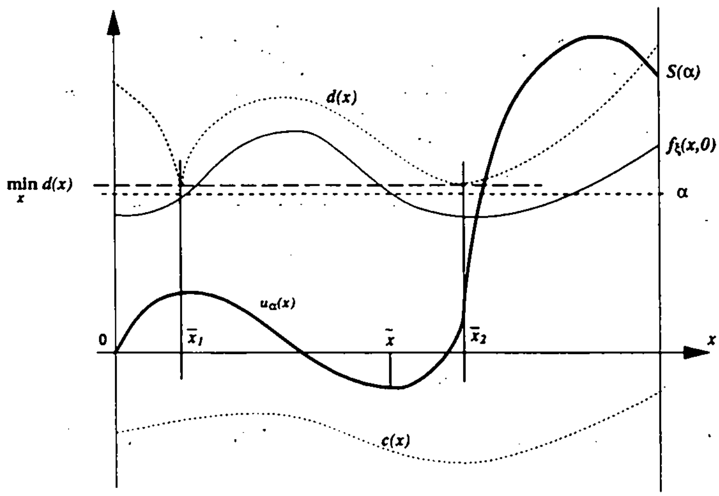

Figure 12: Problem (P_S^+) does not admit any solution

In this example we assume that $\bar{x}_1$ and $\bar{x}_2$ are two points in which d realizes its minimum, i.e. $d(\bar{x}_1) = d(\bar{x}_2) = d_{\inf}$ and $f_\xi(x,0) < d(x)$ for all $x \in [0, \bar{x}_1]$. This means that $l_{\sup}(x) \equiv d_{\inf}$. Moreover we suppose that there exists $\tilde{x} \in (\bar{x}_1, \bar{x}_2)$ such that

$$\lim_{\alpha \uparrow d_{\inf}} \int_0^{\tilde{x}} f_\alpha^{*+}(\sigma, \alpha)\, d\sigma < 0. \tag{10}$$

Let $\alpha \in (c_{\sup}, d_{\inf})$ and define

$$u_\alpha(x) = \int_0^x f_\alpha^{*+}(\sigma, \alpha)\, \sigma \qquad \text{and} \qquad S(\alpha) = u_\alpha(1).$$

We suppose that

$$\lim_{\alpha \uparrow d_{\inf}} S(\alpha) = +\infty \qquad \text{and} \qquad \lim_{\alpha \downarrow c_{\sup}} S(\alpha) = -\infty.$$

This means that problem (P_S) admits a solution for any $S \in \mathbb{R}$. We then consider the function

$$v_{\sup}(x) = \lim_{\varepsilon \downarrow 0} \int_0^x f_\alpha^{**+}\left(\sigma, l_{\sup}(\sigma) - \varepsilon\right) d\sigma.$$

But since relation (10) is valid and since $l_{\sup}(x) \equiv d_{\inf}$ we conclude that v is not always greater than zero since $v_{\sup}(\tilde{x}) < 0$. This implies (see Theorem 2.7) that problem (P_S^+) does not admit any solution for every $S \geq 0$.

3.3.2 Existence of a solution of problem (P_S^+) for every $S \geq 0$ and non-existence of a solution of problem (P_S) for every $S \in \mathbb{R}$. In this subsection we show that we can construct a function f for which problem (P_S^+) has a solution for every $S \geq 0$ and, for the same f, problem (P_S) does not admit any solution for every $S \in \mathbb{R}$. To exhibit such an example, it is sufficient to consider a function $f \in C^1([0,1] \times \mathbb{R})$ such that

(a) $f_\xi(\cdot, 0)$, d and $c: [0,1] \to \mathbb{R}$ are continuous and decreasing,

(b) $c(0) > d(1)$.

The conditions (a) and (b) imply that $c_{\sup} = c(0) > d(1) = d_{\inf}$. Therefore (see Subsection 2.3) problem (P_S) does not admit any solution for every $S \in \mathbb{R}$.

However conditions (a) and (b) imply that there exists $\bar{x} \in (0,1)$ such that $f_\xi(\bar{x}, 0) = d(1)$. We deduce that

$$l_{\sup}(x) = \begin{cases} f_\xi(x,0) & \text{if } 0 \leq x \leq \bar{x} \\ f_\xi(\bar{x},0) & \text{if } \bar{x} < x \leq 1. \end{cases}$$

Therefore $l_{\sup}(x) = d(1)$ for all $x \in [\bar{x}, 1]$. Moreover assume that

$$\lim_{\varepsilon \downarrow 0} \int_{\bar{x}}^1 f_+^{*'}\left(\sigma, d(1) - \varepsilon\right) d\sigma = +\infty. \tag{11}$$

Then problem (P_S^+) has a solution for every $S \geq 0$. Let $\tilde{x} \in (\bar{x}, 1]$. We consider $l \in C^0([0,1])$ and $u \in W^{1,\infty}$ defined by

$$l(x) = \begin{cases} f_\xi(x,0) & \text{if } 0 \leq x \leq \tilde{x} \\ f_\xi(\tilde{x},0) & \text{if } \tilde{x} < x \leq 1. \end{cases} \quad \text{and} \quad u(x) = \int_0^x f_\alpha^{**+}\left(\sigma, l(\sigma)\right) d\sigma,$$

respectively. Then $u(x) = 0$ for all $x \in [0, \tilde{x}]$ and $u'(x) > 0$ for a.e. $x \in (\tilde{x}, 1]$. It implies that both l and u satisfy the conditions of Theorem 2.4. Hence u is a solution of problem (P_S^+) for $S = \int_0^1 u'(x)\, dx = \int_{\tilde{x}}^1 u'(x)\, dx$.

In Figure 13 we can see how to construct such a solution u for problem (P_S^+) by considering the function $l \in E$ associated to the function u (see Subsection 2.5). Finally we obtain that $S_{\text{sup}}^+ = +\infty$ by using relation (11).

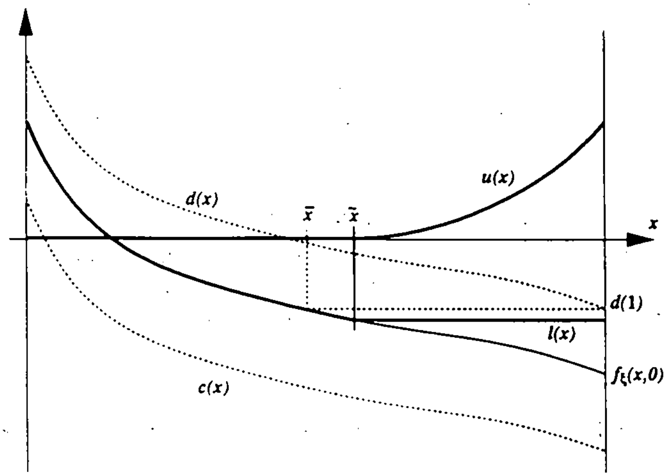

Figure 13: Problem (P_S) does not admit any solution

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