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On the Operator-Valued Nevanlinna-Pick Problem

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Abstract. In this note we construct a specific Schur-Nevanlinna type algorithm that will be used in describing more precisely the solutions of the operator-valued Nevanlinna-Pick problem with the so-called method of Weyl circles. We also present an approach to the Pick criterion in the operator-valued setting.

Keywords: *Cascade transformations, contractions, elementary rotations, operator-valued functions, positive definiteness*

AMS subject classification: Primary 47 A 57, secondary 30 E 05, 42 A 82, 47 A 20, 93 A 20

1. Introduction

The main purpose of this note is to present a variant of a Schur-Nevanlinna type algorithm for the operator-valued Nevanlinna-Pick problem, and we also describe the solution structure using the method of Weyl circles. We obtain the connections between the solvability of the operator-valued Nevanlinna-Pick problem and the classical Pick criterion.

Let us give a short description of the results obtained in this note. In Section 2 we consider an operator-valued setting of the scalar Schur-Nevanlinna algorithm using the Redheffer product [15]. We develop an operator-valued Schur-Nevanlinna type algorithm similar to the scalar one (cf. [10: p. 202]), which yields the solutions ofthe operator-valued Nevanlinna-Pick problem and extends the results in [4]. Then, as in the scalar case, we obtain the correspondence between the operator-valued Nevanlinna-Pick problem data and the associated Schur sequence. In Section 3 we derive the generalized method of Weyl circles, which precisely describes the solution structure in the operator-valued case. In Section 4 we obtain the well known Pick criterion in the operator-valued setting, reflecting the close connection between the Pick matrices of the operator-valued Nevanlinna-Pick problem and the Toeplitz matrices of the operator-valued Schur problem, and generalizing the approach in [4] and [11]. For a different approach to the Pick criterion for the Nevanlinna-Pick problem see also [2: p. 398], [7: p. 197] and [16: p. 25]. Such interpolation problems are of great interest in systems theory (see [2: Chapter 23] and [8: Chapter 6]).

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2. The operatorial Nevanlinna algorithm

We assume that the reader is familiar with the scalar Schur and Nevanlinna-Pick problems and the associated algorithms (see [10: p. 202]). We now introduce some notation and terminology that are needed to formulate the problem in the operator-valued case. Let $\mathbb{N}$ be the set of positive integers, and let us denote by $\mathbb{D}$ the open unit disc and by $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the set of bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and $\mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ the set of analytic and contractive operatorial-valued functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then we consider the following operator-valued Nevanlinna-Pick problem:

(ONPP) Given any set of distinct points $\{z_n\}_{n\geq 1} \subset \mathbb{D}$ and any set of contraction operators $\{W_n\}_{n\geq 1} \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, does there exist a function $F \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ satisfying $F(z_n) = W_n$ for all $n \in \mathbb{N}$?

A sequence $\{z_n, W_n\}_{n\geq 1} \subset (\mathbb{D}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$, where $\{W_n\}_{n\geq 1}$ are contraction operators, is called a set of *initial data*. Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction and denote, as usual (see [13: p. 7]),

$$D_T = (I - T^*T)^{1/2} \qquad \text{and} \qquad \mathcal{D}_T = (D_T\mathcal{H}_1)^-.$$

Let further $z \in \mathbb{D}$. In this context, we can define

$$J[T](z) = \begin{bmatrix} T & zD_{T^*} \\ D_T & -zT^* \end{bmatrix} \tag{1}$$

and then, clearly, $J[T](z)$ is an operator from $\mathcal{H}_1 \oplus \mathcal{D}_{T^*}$ into $\mathcal{H}_2 \oplus \mathcal{D}_T$. It is easy to see that $J[T]$ is inner from both sides and that $J[T](1)$ is the elementary rotation of T (see [13: p. 16]). In the operator-valued setting, it is useful to replace Möbius transformations by the so-called cascade transformations (see [3] and [8: Chapter 6]). Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces and let

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{K}_1, \mathcal{H}_2 \oplus \mathcal{K}_2)$$

be a contraction. For an arbitrary contraction $X \in \mathcal{L}(\mathcal{K}_2; \mathcal{K}_1)$ and $I = I_{\mathcal{K}_2}$, we define the *cascade transformation*

$$C_S(X) = A + BX(I - DX)^{-1}C$$

whenever the inverse of $(I - DX)$ exists. One immediately verifies that $C_S(X)$ is a contraction. In a similar way, we may define the cascade transformation for operator-valued functions, and we have the following result (see [3]).

Theorem 1: *For any $G \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ the equation $G = C_{J[T]}(F)$ has a unique solution $F \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$ where $T = G(0)$.*

We now present a way by which we may construct the Schur-Nevanlinna algorithm. For each contraction $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, number $z \in \mathbb{D}$ and integer $n \in \mathbb{N}$ one defines the *generalized elementary rotation*

$$J^n[T](z) = \begin{bmatrix} T & b_n(z)D_{T^*} \\ D_T & -b_n(z)T^* \end{bmatrix} \qquad \text{where} \quad b_n = \frac{|z_n|}{\overline{z_n}} \frac{z_n - z}{1 - \overline{z_n}z}.$$

We then consider the following cascade transformation of F in $\mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$:

$$C_{J^n[T]}(F) = T\Big|_{\mathcal{D}_T} + b_n(z)D_{T^*}F(z)\Big[I + b_n(z)T^*F(z)\Big]^{-1} D_T\Big|_{\mathcal{D}_T}.$$

Clearly the operator $C_{J^n[T]}(F)$ makes sense because for $|z| < 1$ we have $|b_n(z)| < 1$, and since T and $F(z)$ are contractions, $[I + b_nT^*F(z)]$ is invertible. A short computation shows that

$$\begin{aligned} I - C^*_{J^n[T]}(F)(z)C_{J^n[T]}(F)(w) &= D_T\Big[I + \overline{b_n(z)}F^*(z)T\Big]^{-1} \\ &\quad \times \Big[I - \overline{b_n(z)}b_n(w)F^*(z)F(w)\Big]\Big[I + b_n(w)T^*F(w)\Big]^{-1} D_T\Big|_{\mathcal{D}_T} \end{aligned}$$

and, therefore, it results that, for each $n \in \mathbb{N}$, $C_{J^n[T]}(F) \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$. Thus, we have obtained the following result.

Corollary 2: *Let $G \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2)$ and $\{z_n\}_{n\geq 1} \subset \mathbb{D}$, and let us denote for each $n \in \mathbb{N}$, $T_n = G(z_n)$. Then the equation $G = C_{J^n[T_n]}(F)$ has a unique solution $F = F_n \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$.*

The main object used for the description of the Schur-Nevanlinna algorithm is the Schur sequence. A set of contractions $\{\Gamma_n\}_{n\geq 1}$ is called a *Schur sequence* if $\Gamma_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and, for each $n \in \mathbb{N}$, we have $\Gamma_n : \mathcal{D}_{\Gamma_{n-1}} \to \mathcal{D}_{\Gamma^*_{n-1}}$ where by definition $\mathcal{D}_{\Gamma_0} = \mathcal{H}_1$ and $\mathcal{D}_{\Gamma^*_0} = \mathcal{H}_2$.

Theorem 3: *The collection of solutions of the Nevanlinna-Pick problem* (ONPP) *with initial data $\{z_n, W_n\}_{n\geq 1} \subset (\mathbb{D}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ is defined by the set of contractive operator-valued functions $F_{n+1} \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma^*_n})$ $(n \in \mathbb{N})$ satisfying*

$$F_n = C_{S_n}(F_{n+1}) \qquad \text{where} \quad S_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$$

such that, for some set of contractions $\Gamma_{n+1} : \mathcal{D}_{\Gamma_n} \to \mathcal{D}_{\Gamma^*_n}$ $(n \in \mathbb{N})$, the operator-valued functions A_n, B_n, C_n and D_n are given in the following way:

$$\left.\begin{aligned} &A_1(z) = \Gamma_1, \quad B_1(z) = b_1(z)D_{\Gamma^*_1}, \quad C_1(z) = D_{\Gamma_1}, \quad D_1(z) = -b_1(z)\Gamma^*_1 \\ &A_{n+1}(z) = A_n(z) + B_n(z)\Gamma_{n+1}\Big[I - D_n(z)\Gamma_{n+1}\Big]^{-1} C_n(z) \\ &B_{n+1}(z) = b_{n+1}(z)B_n(z)\Gamma_{n+1}\Big[I - D_n(z)\Gamma_{n+1}\Big]^{-1} D_nD_{\Gamma^*_{n+1}} \\ &\qquad\qquad + b_{n+1}(z)B_n(z)D_{\Gamma^*_{n+1}} \\ &C_{n+1}(z) = D_{\Gamma_{n+1}}\Big[I - D_n(z)\Gamma_{n+1}\Big]^{-1} C_n(z) \\ &D_{n+1}(z) = b_{n+1}(z)D_{\Gamma_{n+1}}\Big[I - D_n(z)\Gamma_{n+1}\Big]^{-1} D_n(z)D_{\Gamma^*_{n+1}} - b_{n+1}(z)\Gamma^*_{n+1}. \end{aligned}\right\} \quad (2)$$

Proof: Let $F \in \mathcal{S}(\mathcal{D}_T, \mathcal{D}_{T^*})$ be a solution of the Nevanlinna-Pick problem (ONPP) with initial data given by the sequence $\{z_n, W_n\}_{n\geq 1} \subset (\mathbb{D}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Using Corollary 2 one can define an algorithm similar to the scalar one (see [12, 14, 17]). Let $F_1(z) = F(z)$ and $\Gamma_1 = F_1(z_1) = W_1$. Then F_{n+1} and Γ_{n+1} are implicitly defined by the equations

$$F_n = C_{J^n[\Gamma_n]}(F_{n+1}) \qquad \text{and} \qquad \Gamma_{n+1} = F_{n+1}(z_{n+1}) \tag{3}$$

where $\Gamma_{n+1} : \mathcal{D}_{\Gamma_n} \to \mathcal{D}_{\Gamma_n^*}$ is a contraction and $F_{n+1} \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*})$. We readily deduce from (3) that

$$F = C_{J^1[\Gamma_1]}\Big(C_{J^2[\Gamma_2]} \cdots \Big(C_{J^n[\Gamma_n]}(F_{n+1}) \cdots\Big)\Big). \tag{4}$$

It is easy to check that for any two contractions

$$S_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \qquad (i = 1, 2)$$

we have

$$C_{S_1}(C_{S_2}(X)) = C_{S_1 \star S_2}(X)$$

where $\star$ is the Redheffer product (see [15]) and $S_1 \star S_2$ is defined by the matrix

$$\begin{bmatrix} A_1 + B_1 A_2 (I - D_1 A_2)^{-1} C_1 & B_1 A_2 (I - D_1 A_2)^{-1} D_1 B_2 + B_1 B_2 \\ C_2 (I - D_1 A_2)^{-1} C_1 & C_2 (I - D_1 A_2)^{-1} D_1 B_2 + D_2 \end{bmatrix}$$

whenever the inverse of $(I - D_1 A_2)$ exists. Then, (1) and (4) give the relation

$$F = C_{J^1[\Gamma_1] \star J^2[\Gamma_2] \star \ \dots \ \star J^n[\Gamma_n]}(F_{n+1}) \qquad (n \in \mathbb{N}).$$

Considering the matrix functions

$$S_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = J^1[\Gamma_1] \star J^2[\Gamma_2] \star \ \dots \ \star J^n[\Gamma_n]$$

we get

$$F = C_{S_n}(F_{n+1}). \tag{5}$$

By definition,

$$S_0(z) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

and for $n \geq 1$, by induction, it easily follows that the operator-valued functions A_n, B_n, C_n and D_n satisfy the relations (2) ∎

Corollary 4: *For every $n \in \mathbb{N}$ we have the correspondence*

$$W_n = C_{S_{n-1}}(\Gamma_n)$$

between the set of parameters $\{W_n\}_{n\geq 1}$ of the Nevanlinna-Pick problem (ONPP) *and the associated Schur sequence $\{\Gamma_n\}_{n\geq 1}$.*

The set $\{\Gamma_n\}_{n\geq 1}$ is a so called Schur (-Nevanlinna) sequence and, obviously, $\{\Gamma_n\}_{n\geq 1}$ is uniquely determined by F.

In order to study the converse, let us consider for each $n \in \mathbb{N}$ the decomposition

$$\begin{aligned} F(z) - A_n(z) &= C_{S_{n-1}}(F_n)(z) - C_{S_{n-1}}(\Gamma_n)(z) \\ &= B_{n-1}(z)\Big\{F_n(z)\Big[I - D_{n-1}(z)F_n(z)\Big]^{-1} \\ &\quad - \Gamma_n\Big[I - D_{n-1}(z)\Gamma_n\Big]^{-1}\Big\}C_{n-1}(z). \end{aligned}$$

Then, from (2), we obtain

$$B_n(z) = \prod_{k=1}^{n} b_k(z)\widehat{B}_n(z) \qquad \text{for} \quad \widehat{B}_n \in \mathcal{S}(\mathcal{D}_{\Gamma_n^*}, \mathcal{H}_2)$$

and it follows that

$$F(z) - A_n(z) = \prod_{k=1}^{n} b_k(z)\widehat{F}_n(z) \tag{6}$$

where $\widehat{F}_n$ is a bounded operator-valued function. As a consequence of (6), the sequence $\{A_n\}_{n\geq 1}$ converges uniformly to F on compact subsets of $\mathbb{D}$ if and only if the product $\prod_{k=1}^{n} b_k(z)$ converges uniformly to 0 on compact subsets of $\mathbb{D}$, and this happens if and only if $\sum_{n\geq 1}(1 - |z_n|) = \infty$. In this case the function F is uniquely determined by the set $\{\Gamma_n\}_{n\geq 1}$ and, moreover, $\{\Gamma_n\}_{n\geq 1}$ uniquely determines the sequence $\{A_n\}_{n\geq 1}$.

3. The method of Weyl circles

In order to present the *Method of Weyl circles* in the operator-valued case we shall restrict our study to the non-degenerate case, where we have at least one solution of the problem (ONPP). The matrix case is intensively analysed in many papers (see, e.g., [4, 5, 9]). For the sake of simplicity we consider $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{S}(\mathcal{H}) = \mathcal{S}(\mathcal{H}, \mathcal{H})$. Let us consider a set of initial data $\{z_k, W_k\}_{k=1}^{n}$ for the problem (ONPP) and define, for each $z \in \mathbb{D}$,

$$\mathcal{W}_n(z) = \Big\{W \in \mathcal{L}(\mathcal{H}) \Big| \; W = F(z), \; F \in \mathcal{S}(\mathcal{H}), \; F(z_k) = W_k \;\; (k = 1, \ldots, n)\Big\}.$$

Then, $\mathcal{W}_n(z)$ is characterized by the following result.

Theorem 5: *For the elements $W \in \mathcal{W}_n(z)$ we have the representation*

$$\begin{aligned} W = &\Big[A_n(z) + B_n(z)D_n^*(z)D_{D_n^*(z)}^{-2}C_n(z)\Big] \\ &+ \Big[B_n(z)D_{D_n(z)}^{-2}B_n^*(z)\Big]^{1/2} Z \Big[C_n^*(z)D_{D_n^*(z)}^{-2}C_n(z)\Big]^{1/2} \end{aligned}$$

where $Z \in \mathcal{L}(\mathcal{H})$, and the operator-valued functions A_n, B_n, C_n and D_n are given by (2).

Proof: As a consequence of (5) we get

$$\mathcal{W}_n(z) = \left\{ W \in \mathcal{L}(\mathcal{H}) \middle| \; W = C_{S_n(z)}(\Gamma),\; \Gamma \in \mathcal{S}(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*}) \right\}$$

and thus, for each $W \in \mathcal{W}_n(z)$ we have the decomposition

$$W = A_n(z) + B_n(z)\Gamma\Big[I - D_n(z)\Gamma\Big]^{-1} C_n(z). \tag{7}$$

Let us consider the relations

$$Y = \Gamma\Big[I - D(z)\Gamma\Big]^{-1} \qquad \text{and} \qquad Y = \Big[I + YD(z)\Big]\Gamma$$

which give

$$YY^* = \Big[I + YD(z)\Big]\Gamma\Gamma^*\Big[I + D^*(z)Y^*\Big] \leq \Big[I + YD(z)\Big]\Big[I + YD^*(z)Y^*\Big]$$

and we obtain

$$Y\Big[I - D(z)D^*(z)\Big]Y^* - YD(z) - D^*(z)Y^* - I \leq 0.$$

A short computation yields

$$\begin{aligned} &\left\{Y\Big[I - D(z)D^*(z)\Big]^{1/2} - D^*(z)\Big[I - D(z)D^*(z)\Big]^{-1/2}\right\} \\ &\times \left\{Y\Big[I - D(z)D^*(z)\Big]^{1/2} - D^*(z)\Big[I - D(z)D^*(z)\Big]^{-1/2}\right\}^* \leq \Big[I - D^*(z)D(z)\Big]^{-1} \end{aligned}$$

and we also have the factorization

$$Y\Big[I - D(z)D^*(z)\Big]^{1/2} - D^*(z)\Big[I - D(z)D^*(z)\Big]^{-1/2} = \Big[I - D^*(z)D(z)\Big]^{-1/2}X$$

for $X \in \mathcal{L}(\mathcal{H})$. It results that $Y = D^*(z)D^{-2}_{D(z)} + D^{-1}_{D(z)}XD^{-1}_{D^*(z)}$ and by (7) we infer that, for some $X \in \mathcal{L}(\mathcal{H})$,

$$W = A_n(z) + B_n(z)D_n^*(z)D^{-2}_{D_n(z)}C_n(z) + B_n(z)D^{-1}_{D_n(z)}XD^{-1}_{D_n^*(z)}C_n(z).$$

Another factorization leads to

$$\begin{aligned} W = &\Big[A_n(z) + B_n(z)D_n^*(z)D^{-2}_{D_n(z)}C_n(z)\Big] \\ &+ \Big[B_n(z)D^{-2}_{D_n(z)}B_n^*(z)\Big]^{1/2} Z \Big[C_n^*(z)D^{-2}_{D_n^*(z)}C_n(z)\Big]^{1/2} \end{aligned}$$

where $Z = E\,X\,F$ for some $E, F \in \mathcal{L}(\mathcal{H})$ ∎

Corollary 6: *For any $z \in \mathbb{D}$ and for every $n \in \mathbb{N}$ the set $\mathcal{W}_n(z)$ may be thought of as a closed sphere with "center"*

$$O(z) = A_n(z) + B_n(z)D_n^*(z)D^{-2}_{D_n(z)}C_n(z)$$

and right and left "radii" given by

$$R_n^{\mathrm{r}}(z) = C_n^*(z)D^{-2}_{D_n(z)}C_n(z) \qquad \text{and} \qquad R_n^{\mathrm{l}}(z) = B_n(z)D^{-2}_{D_n^*(z)}B_n^*(z),$$

respectively.

4. The Pick criterion

In this section, following the method in [11] (note that the same method is used in the matricial case by the authors of [4]), we consider the connection between the problem (ONPP) and the classical Pick criterion.

Let $\mathcal{C}(\mathcal{H})$ be the Carathéodory class of positive operator-valued analytic functions on $\mathbb{D}$. We then define for each $G \in \mathcal{C}(\mathcal{H})$ the function $F \in \mathcal{S}(\mathcal{H})$ given by

$$F(z) = [I - G(z)]\,[I + G(z)]^{-1} \qquad (z \in \mathbb{D}). \tag{8}$$

This also yields a one-to-one correspondence between the functions $F \in \mathcal{S}(\mathcal{H})$ with $F(0) = 0$ and the functions $G \in \mathcal{C}(\mathcal{H})$ with $G(0) = I$. For any set $\{z_n\}_{n\geq 0} \subset \mathbb{D}$ and for any sequence of contractions $\{W_n\}_{n\geq 0} \subset \mathcal{L}(\mathcal{H})$ (by definition $z_0 = 0$ and $W_0 = 0_{\mathcal{H}}$) one can define the sequence of matrices

$$P_n = \left[\frac{I - W_k^* W_l}{1 - \overline{z_k} z_l}\right]_{k,l=0}^{n} \qquad (n \geq 0).$$

Let us assume that the inverse of $(I + W_n)$ exists for $n \geq 0$, and consider the correspondence given by

$$Y_n = (I + W_n)^{-1}(I - W_n) \qquad \text{and} \qquad W_n = (I - Y_n)(I + Y_n)^{-1}. \tag{9}$$

For the problem (ONPP) with initial data $\{z_n, W_n\}_{n\geq 0}$ and a solution F such that $F(z_n) = W_n$ we clearly have $G(z_n) = Y_n$ $(n \geq 0)$, where the Y_n are given by (9). For $n \geq 0$ and $k = 0, 1, \ldots, n$ let

$$\zeta_k = \prod_{\substack{0 \leq l \leq n \\ l \neq k}} (z_k - z_l)^{-1} \qquad \text{and} \qquad U_n = \left[\zeta_k (I + Y_k)(z_k)^l\right]_{k,l=0}^{n}.$$

Thus, we conclude that $M_n = \frac{1}{2} U_n^* P_n U_n$ is an $(n+1) \times (n+1)$ Hermitian Toeplitz matrix for $n \geq 0$. It is easy to see from the definitions that each M_n is a compression of each M_N with $N \geq n$. Therefore, the sequence $\{M_n\}_{n\geq 0}$ can be used to define a single Toeplitz form, indexed by Z, which we denote by $\mathcal{T}$. The positivity of $\mathcal{T}$ is equivalent to the positivity of each M_n for $n \geq 0$. In this case, we obtain that $\mathcal{T}$ on Z is a realization of a semispectral measure E on $\boldsymbol{T}$ (a semispectral measure on $\boldsymbol{T}$ is a linear positive map $E : C(\boldsymbol{T}) \to \mathcal{L}(\mathcal{H})$ where $C(\boldsymbol{T})$ denotes the set of continuous functions on $\boldsymbol{T}$). For every $m \in Z$ we define $S_m = E(e_m)$ and $e_m(e^{it}) = e^{imt}$. Obviously, we have $S_{-m} = S_m^*$. Let us assume that P_n are non-negative definite for all $n \geq 0$. Taking into account (8) we may define $G_n \in \mathcal{C}(\mathcal{H})$ that satisfies problem (ONPP) with data $\{z_k, Y_k\}_{k=0}^{n}$. First we consider the semispectral measure $E_n : C(\boldsymbol{T}) \to \mathcal{L}(\mathcal{H})$ defined by

$$E_n(f) = \frac{1}{2} E\left(|m_n|^2 f\right) \qquad \text{where} \quad m_n(e^{i\theta}) = \prod_{k=0}^{n} \left(e^{i\theta} - z_k\right)$$

and the family of continuous functions

$$g_z : \boldsymbol{T} \to \mathcal{L}(\mathcal{H}) \qquad \text{given by} \quad g_z\left(e^{i\theta}\right) = \frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

We then consider

$$G_n \in \mathcal{C}(\mathcal{H}) \qquad \text{given by} \quad G_n(z) = E_n(\dot{g}_z) \;\; (z \in \mathbb{D}).$$

A short computation shows that

$$\frac{G_n^*(z_k) + G_n(z_l)}{1 - \overline{z_k} z_l} = \frac{Y_k^* + Y_l}{1 - \overline{z_k} z_l} \qquad (k, l = 0, 1, \ldots, n)$$

and it is clear that, for a fixed $n \geq 0$ and for any $k = 0, 1, \ldots, n$, $G_n(z_k) = Y_k$. On the other hand, for the problem (ONPP) with initial data $\{z_n, W_n\}_{n \geq 0}$ let us consider the sequences

$$\begin{aligned} &\{G_n\}_{n\geq 0} \subset \mathcal{C}(\mathcal{H}) \qquad \text{satisfying} \quad G_n(z_k) = Y_k \quad (0 \leq k \leq n) \\ &\{F_n\}_{n\geq 0} \subset \mathcal{S}(\mathcal{H}) \qquad \text{satisfying} \quad F_n(z_k) = W_k \quad (0 \leq k \leq n) \end{aligned}$$

(see (5) and (8)). The sequence $\{F_n\}_{n \geq 0}$ has a subsequence converging (weakly) to an operator-valued function $F \in \mathcal{S}(\mathcal{H})$ which clearly satisfies the problem (ONPP) with initial data $\{z_n, W_n\}_{n \geq 0}$.

Theorem 7: *A necessary and sufficient condition for the solvability of the problem* (ONPP) *is that, for each* $n \geq 0$, P_n *is non-negative definite.*

Proof: Sufficiency was already proved, and necessity is a consequence of the Riesz-Herglotz theorem for functions in the class $\mathcal{C}(\mathcal{H})$ ∎

Remark: This paper is part of the author's Ph.D. thesis, written at Texas A&M University under the direction of Professor Carl Pearcy.

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