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Degenerate Stieltjes Moment Problem and Associated J-Inner Polynomials

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Abstract. In this paper we consider the truncated Stieltjes matrix moment problem when the so-called information matrices are degenerate and describe the set of all solutions in terms of the linear fractional transformation using the fundamental matrix inequality method.

Keywords: *Moment problems, J-expansive matrix functions, fundamental matrix inequalities*
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1. Introduction

The objective of this paper is to describe the solutions of a degenerate Stieltjes matrix moment problem. We begin with an ordered set of hermitian non-negative matrices $s_0, \dots, s_N \in \mathbb{C}^{m\times m}$

$$\mathbf{H}_N = \{s_0, \dots, s_N\} \tag{1.1}$$

(by $\mathbb{C}^{m\times \ell}$ we denote the space of $m \times \ell$ matrices with complex entries, and, throughout the paper I_m stands for the identity matrix of the order m). Let K and $\widetilde{K}$ denote the associated Hankel block matrices

$$K = (s_{i+j})_{i,j=0}^{[N/2]} \qquad \text{and} \qquad \widetilde{K} = (s_{i+j+1})_{i,j=0}^{[(N-1)/2]}.$$

Definition 1.1: We say that $\mathbf{H}_N$ belongs to $\mathcal{H}$ if the associated matrices K and $\widetilde{K}$ are non-negative. If, moreover, $\mathbf{H}_N$ admits a non-negative extension (i.e. if there exists a matrix $s_{N+1} \in \mathbb{C}^{m\times m}$ such that the extended block matrices $(s_{i+j})_{i,j=0}^{[(N+1)/2]}$ and $(s_{i+j+1})_{i,j=0}^{[N/2]}$ are still non-negative), then $\mathbf{H}_N$ is said to be in $\mathcal{H}^+$.

Let $\mathcal{Z}(\mathbf{H}_N)$ denote the set of all solutions of the associated truncated Stieltjes moment problem, i.e. the set of non-decreasing right-continuous $\mathbb{C}^{m\times m}$-valued functions $\sigma(\lambda)$ such that

$$\int_0^\infty \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \dots, N-1) \tag{1.2}$$

$$\int_0^\infty \lambda^N d\sigma(\lambda) \le s_N. \tag{1.3}$$

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As in the scalar case (see, e.g., [16: §5.1]) $\mathcal{Z}(\mathbf{H}_N)$ is non-empty if and only if $\mathbf{H}_N \in \mathcal{H}$. Moreover, there is a one-to-one correspondence between $\mathcal{Z}(\mathbf{H}_N)$ and the class $\mathcal{S}(\mathbf{H}_N)$ of $\mathbb{C}^{m\times m}$-valued functions $s = s(z)$ analytic in $\mathbb{C}\backslash\mathbb{R}_+$ (where $\mathbb{R}_+ = [0, +\infty)$) and such that

$$\Im\, s(z) \geq 0 \quad (\Im\, z \geq 0) \qquad \text{and} \qquad s(x) \geq 0 \quad (x < 0) \tag{1.4}$$

and, uniformly in the sector $S_\varepsilon = \{z = \rho e^{i\theta} \in \mathbb{C} : \ \varepsilon \leq \theta \leq \pi - \varepsilon\}$ $(\varepsilon > 0)$,

$$\lim_{z\to\infty} \left\{ z^{N+1} s(z) + \sum_{k=0}^{N} s_k z^{N-k} \right\} \geq 0.$$

Indeed, if $K, \widetilde{K} \geq 0$ and $\sigma \in \mathcal{Z}(\mathbf{H}_N)$, then the function

$$s(z) = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - z} \tag{1.5}$$

belongs to $\mathcal{S}(\mathbf{H}_N)$ and conversely, every $s \in \mathcal{S}(\mathbf{H}_N)$ admits such a representation, where σ is obtained from s by the Stieltjes inversion formula (see [16: Appendix]). The parametrization of the set $\mathcal{S}(\mathbf{H}_N)$ in terms of the linear fractional transformation for the non-degenerate case (K and $\widetilde{K}$ are both strictly positive) is given in [8].

We recall some necessary definitions.

Definition 1.2: A $\mathbb{C}^{2m\times 2m}$-valued meromorphic function Θ is of the *class* $\mathbf{W}_\pi$ if

$$\Theta(z)J\Theta(z)^* = J \quad (z \in \mathbb{R}) \qquad \text{and} \qquad \Theta(z)J\Theta(z)^* \geq J \quad (z \in \mathbb{C}_+) \tag{1.6}$$

and

$$\Theta(x)J_\pi\Theta(x)^* \geq J_\pi \quad (x < 0)$$

where

$$J = \begin{pmatrix} 0 & iI_m \\ -iI_m & 0 \end{pmatrix} \qquad \text{and} \qquad J_\pi = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}, \tag{1.7}$$

and Θ is of the *class* $\mathbf{W}$ if it satisfies only conditions (1.6).

The following theorem establishes the connection between the classes $\mathbf{W}$ and $\mathbf{W}_\pi$.

Theorem 1.3 (see [8: §4]): *A $\mathbb{C}^{2m\times 2m}$-valued function Θ belongs to the class $\mathbf{W}_\pi$ if and only if*

$$\Theta \in \mathbf{W} \qquad \textit{and} \qquad \tilde{\Theta}(z) = P(z)\Theta(z)P(z)^{-1} \in \mathbf{W} \tag{1.8}$$

where

$$P(z) = \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}. \tag{1.9}$$

Definition 1.4: Let $\{p, q\}$ be a pair of $\mathbb{C}^{m\times m}$-valued functions meromorphic in $\mathbb{C}\backslash\mathbb{R}_+$.

(i) $\{p, q\}$ is called a *Stieltjes pair* if

(α) $\det\Big(p(z)^*p(z) + q(z)^*q(z)\Big) \not\equiv 0$ (non-degeneracy of the pair)

(β) $\dfrac{q(z)^*p(z) - p(z)^*q(z)}{z - \bar{z}} \geq 0$ $(\Im z \neq 0)$

(γ) $\dfrac{zq(z)^*p(z) - \bar{z}p(z)^*q(z)}{z - \bar{z}} \geq 0$ $(\Im z \neq 0)$.

(ii) $\{p, q\}$ is said to be *equivalent* to another pair $\{p_1, q_1\}$ if there exists a $\mathbb{C}^{m\times m}$-valued function Ω with $\det \Omega(z) \not\equiv 0$ and meromorphic in $\mathbb{C}\backslash \mathbb{R}_+$ such that $p_1 = p\Omega$ and $q_1 = q\Omega$.

The set of all Stieltjes pairs will be denoted by $\overline{\mathcal{S}}_m$.

The degenerate scalar Stieltjes problem ($m = 1$) is simple: $\mathcal{S}(\mathbf{H}_N)$ consists of the unique rational function $s = s(z)$. For the degenerate matrix case the description of $\mathcal{S}(\mathbf{H}_N)$ depends on the character of degeneracy of the *information matrices* K and $\tilde{K}$.

The main result of the paper is

Theorem 1.5: *The following statements are true.*

(i) *All functions $s \in \mathcal{S}(\mathbf{H}_N)$ are given by the linear fractional transformation*

$$s(z) = \Big(\theta_{11}(z)p(z) + \theta_{12}(z)q(z)\Big)\Big(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)\Big)^{-1} \tag{1.10}$$

with the resolvent matrix $\Theta = (\theta_{ij})$ of class $\mathbf{W}_\pi$ and parameters $\{p, q\} \in \overline{\mathcal{S}}_m$ of the form

$$p(z) = \begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix} \quad \text{and} \quad \tilde{q}(z) = \begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \tag{1.11}$$

where

$$\mu = \operatorname{rank}\big(I_m, 0, \dots, 0\big)P_{\mathrm{Ker}K} \quad \text{and} \quad \nu = \operatorname{rank}\big(s_0, \dots, s_{[(N-1)/2]}\big)P_{\mathrm{Ker}\widetilde{K}}.$$

Here $\mathbf{P}_{\mathrm{Ker}K}$ and $P_{\mathrm{Ker}\widetilde{K}}$ denote the orthogonal projections onto the kernels of K and $\widetilde{K}$, respectively.

(ii) *Two pairs $\{p, q\}$, $\{p_1, q_1\} \in \overline{\mathcal{S}}_m$ of the form* (1.11) *are equivalent if and only if they lead under the transformation* (1.10) *to the same function s.*

Note that the non-degenerate case corresponds to $\mu = \nu = 0$. Theorem 1.5 will be proved in Section 5 under the assumption $\mathbf{H}_N \in \mathcal{H}^+$. The general case can be reduced to this particular one in view of the following

Lemma 1.6: *Let $\mathbf{H}_N \in \mathcal{H}$. Then the last moment s_N can be perturbated in such a way that the set*

$$\mathbf{H'}_N = \big\{s_0, \dots, s_{N-1}, s'_N\big\}$$

belongs to $\mathcal{H}^+$ and the associated Stieltjes moment problems have the same solutions: $\mathcal{Z}(\mathbf{H}_N) = \mathcal{Z}(\mathbf{H'}_N)$ *(or equivalently,* $\mathcal{S}(\mathbf{H}_N) = \mathcal{S}(\mathbf{H'}_N)$*).*

It will be shown also that the function Θ is the matrix polynomial of $\deg\Theta = [N/2]+1$ which admits a realization (not minimal, in general) $\Theta(z) = \Theta(0) + A(I - zF)^{-1}B$ with the state space $\mathbb{C}^{m[(N+2)/2]}$. To construct the resolvent matrix of the degenerate Stieltjes moment problem (which is a J-inner polynomial of the non-full rank, see, e.g., [5]) we follow the method of V. Dubovoj which was applied in the series [4] to the degenerate Schur problem. Using this method we obtain in Sections 3 and 6 some special decompositions (see (3.9) and (6.10)) of the state space which allow to construct the explicit formula for Θ for the case that K and $\widetilde{K}$ are not strictly positive (formulas (4.5) and (6.11)). In Section 2 we point out some peculiarities of the degenerate Stieltjes problem. For example, the condition $\mathbf{H}_N \in \mathcal{H}$ (as against the non-degenerate case) does not ensure the existence of a measure $\sigma(\lambda)$ in $\mathcal{Z}(\mathbf{H}_N)$ such that in the inequality (1.3) the equality sign prevails.

Lemma 1.7: *A measure $\sigma(\lambda)$ which satisfies* (1.2) *and* $\int_0^\infty \lambda^N d\sigma(\lambda) = s_N$ *exists if and only if* $\mathbf{H}_N \in \mathcal{H}^+$.

This fact (as well as the statement from Lemma 1.6) will be established separately for N odd and even in Sections 2 and 6, respectively.

The Stieltjes moment problem is in fact the interpolation problem in the class of matrix-valued Stieltjes functions (which by definition are analytic in $\mathbb{C}\backslash\mathbb{R}_+$ and satisfy (1.4); see, e.g., [8]) with the interpolating point at infiniy. It can presumably be solved using a number of approaches, e.g. reproducing kernels method (using this method, the moment problem on the whole axis was considered in [7]), methods based on operator theory [14, 15] or on realization of matrix-valued functions (such approach was applied in [10] to the interpolation problem with interpolating points from the upper half-plane). In this paper we follow the Potapov method of the fundamental matrix inequality (see [4, 5, 8 - 13]). The starting point is the following theorem which describes the class $\mathcal{S}(\mathbf{H}_N)$ in terms of a system of matrix inequalities.

Theorem 1.8 (see [8]): *Let s be a $\mathbb{C}^{m\times m}$-valued function analytic in the upper half plane $\mathbb{C}_+$. Then s belongs to $\mathcal{S}(\mathbf{H}_N)$ if and only if it satisfies the system of inequalities*

$$\begin{pmatrix} K & (I - zF_{m,n})^{-1}(\mathbf{e}\,s(z) + F_{m,n}K\mathbf{e}) \\ * & \dfrac{s(z) - s(z)^*}{z - \bar{z}} \end{pmatrix} \geq 0 \tag{1.12}$$

$$\begin{pmatrix} \widetilde{K} & B(I - zF_{m,n})^{-1}(z\mathbf{e}\,s(z) + K\mathbf{e}) \\ * & \dfrac{zs(z) - \bar{z}s(z)^*}{z - \bar{z}} \end{pmatrix} \geq 0 \tag{1.13}$$

for $z \in \mathbb{C}_+$ where

$$F_{m,n} = \begin{pmatrix} 0_m & & & 0 \\ I_m & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I_m & 0_m \end{pmatrix} \in \mathbb{C}^{m(n+1)\times m(n+1)} \quad \text{and} \quad \mathbf{e} = \begin{pmatrix} I_m \\ 0_m \\ \vdots \\ 0_m \end{pmatrix} \tag{1.14}$$

and the matrix B is $I_{m(n+1)}$ or $\left(I_{m(n+1)}, 0_{m(n+1)\times m}\right)$ whenever $N = 2n+1$ or $N = 2n$, respectively.

Note that (1.12) itself is the fundamental matrix inequality of the Hamburger moment problem on $\mathbb{R}$ (see, e.g, [13]).

For the non-degenerate case the difference between "even" and "odd" Stieltjes problems is not essential: they are particular cases of the much more general moment problem [2] and can be considered in a unified way. For the degenerate case it is not so: the difficulties which are arised due to different sizes of K and $\widetilde{K}$ for N even are essential. In Sections 2 - 5 we consider the Stieltjes problem with odd number of moments and postpone the "even" problem up to Section 6.

2. Some auxiliary lemmas

For $N = 2n+1$ we have the moment conditions

$$\int_0^\infty \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n) \qquad \text{and} \qquad \int_0^\infty \lambda^{2n+1} d\sigma(\lambda) \le s_{2n+1}$$

for prescribed non-negative matrices $s_i \in \mathbb{C}^{m\times m}$ $(i = 1, \ldots, 2n+1)$. It sems to be more convenient to deal with associated Hankel block matrices instead of the set $\mathbf{H}_{2n+1}$ itself. So, we reformulate Definition 1.1 for this case as

Definition 2.1: A pair $\{K_n, \widetilde{K}_n\}$ is said to be in $\mathcal{K}_n$ if K_n and $\widetilde{K}_n$ are both non-negative and of the form

$$K_n = (s_{i+j})_{i,j=0}^n \qquad \text{and} \qquad \widetilde{K}_n = (s_{i+j+1})_{i,j=0}^n \tag{2.1}$$

for some square matrices s_i of the same size. If, moreover, K_n admits a non-negative Hankel extension (i.e. if there exists a matrix $s_{2n+2} \in \mathbb{C}^{m\times m}$ such that the block matrix $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ is still non-negative), then we say that the pair $\{K_n, \widetilde{K}_n\}$ belongs to $\mathcal{K}_n^+ \subset \mathcal{K}_n$.

We put the index n in (2.1) to shorten some impending computations with the Hankel block matrices of different sizes. The following proposition can be easily checked by a direct calculation.

Lemma 2.2: *Let matrices $A, C \in \mathbb{C}^{m(n+1)\times m(n+1)}$ be non-negative and let $F_{m,n}$ be the matrix of the m-dimensional shift in the space $\mathbb{C}^{m(n+1)}$ given by* (1.14)*. Then the pair $\{A, C\}$ belongs to $\mathcal{K}_n$ if and only if*

$$F_{m,n}\left(F_{m,n}^* A - C\right) = 0.$$

Given a non-negative matrix K let Q be a matrix such that

$$QKQ^* > 0 \qquad \text{and} \qquad \operatorname{rank} QKQ^* = \operatorname{rank} K. \tag{2.2}$$

We introduce the ***pseudoinverse matrix*** $K^{[-1]}$ by

$$K^{[-1]} = Q^*(QKQ^*)^{-1}Q. \tag{2.3}$$

The pseudoinverse matrix defined by (2.2), (2.3) depends on the choice of Q, nevertheless, some its properties are independent of this choice.

Lemma 2.3 (see [3]): *For every choice of the pseudoinverse matrix* $K^{[-1]}$,

$$I - KK^{[-1]} = \left(I - KK^{[-1]}\right) \mathbf{P}_{\operatorname{Ker} K}. \tag{2.4}$$

Lemma 2.4 (see [3]): *The block matrix*

$$\begin{pmatrix} K & B \\ B^* & C \end{pmatrix}$$

is non-negative if and only if

$$K \geq 0, \qquad P_{\operatorname{Ker} K} B = 0, \qquad R := C - B^* K^{[-1]} B \geq 0.$$

Moreover, if

$$\begin{pmatrix} K & B \\ B^* & C \end{pmatrix} \geq 0,$$

then the matrix R *does not depend on the choice of* $K^{[-1]}$.

This last lemma is a reformulation of the well known lemma about the non-negativity of a block matrix (see, e.g., [6: §0] and [12: §4]).

Lemma 2.5: *Let* $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n$ *and let* $\mathcal{L}$ *be the subspace of* $\mathbb{C}^{1\times m}$ *defined as*

$$\mathcal{L} = \left\{ f \in \mathbb{C}^{1\times m} : (f_0, \ldots, f_{n-1}, f) \in \operatorname{Ker} K_n \text{ for some } f_0, \ldots, f_{n-1} \in \mathbb{C}^{1\times m} \right\}. \tag{2.5}$$

Then the following statements are equivalent:

(i) $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$.

(ii) $\operatorname{Ker} K_n \subseteq \operatorname{Ker} \widetilde{K}_n$. (2.6)

(iii) *The block* s_{2n+1} *is of the form*

$$s_{2n+1} = \left(s_{n+1}, \ldots, s_{2n}\right) \widetilde{K}_{n-1}^{[-1]} \left(s_{n+1}, \ldots, s_{2n}\right)^* + R \tag{2.7}$$

for some non-negative matrix $R \in \mathbb{C}^{m\times m}$ *which vanishes on the subspace* $\mathcal{L}$ *and does not depend on the choice of* $\widetilde{K}_{n-1}^{[-1]}$ *(according to (2.1),* $\widetilde{K}_{n-1} = (s_{i+j+1})_{i,j=0}^{n-1}$*).*

(iv) *There exists a measure* $d\sigma(\lambda) \geq 0$ *such that*

$$\int_0^\infty \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \ldots, 2n+1). \tag{2.8}$$

Proof: *The implication* (i) $\Rightarrow$ (ii). Let $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ be a non-negative Hankel extension of K_n. From the non-negativity of K_{n+1} we receive

$$\mathbf{P}_{\operatorname{Ker} K_n} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n+1} \end{pmatrix} = 0 \tag{2.9}$$

which in view of (2.1) is equivalent to (2.6).

The implication (ii) $\Rightarrow$ (i). Using Lemma 2.3 and taking into account (2.6) (or (2.9)) we conclude that $K_{n+1} \geq 0$ if and only if

$$s_{2n+2} - \left(s_{n+1}, \dots, s_{2n+1}\right) K_n^{[-1]} \left(s_{n+1}, \dots, s_{2n+1}\right)^* \geq 0.$$

Thus, every choice of s_{2n+2} satisfying this last inequality leads to $K_{n+1} \geq 0$ and therefore, $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$.

The equivalence (ii) $\Leftrightarrow$ (iii). Since $\widetilde{K}_n \geq 0$, then by Lemma 2.3,

$$s_{2n+1} - \left(s_{n+1}, \dots, s_{2n}\right) \widetilde{K}_{n-1}^{[-1]} \left(s_{n+1}, \dots, s_{2n}\right)^* \geq 0$$

and hence, s_{2n+1} admits a representation (2.7) for some $R \geq 0$. It remains to show that (2.6) holds if and only if R vanishes on the subspace $\mathcal{L}$ defined by (2.5). Let $(f_0, \dots, f_n)$ be an arbitrary vector from $\operatorname{Ker} K_n$. Then in particular

$$\left(f_0, \dots, f_n\right) \begin{pmatrix} s_1 & \dots & s_n \\ \vdots & & \vdots \\ s_n & \dots & s_{2n-1} \\ s_{n+1} & \dots & s_{2n} \end{pmatrix} = 0$$

and therefore,

$$f_n\left(s_{n+1}, \dots, s_{2n}\right) = -\left(f_0, \dots, f_{n-1}\right) \widetilde{K}_{n-1}.$$

Using this last equality both with (2.9) and (2.4) we obtain

$$\begin{aligned} & f_0 s_{n+1} + \dots + f_{n-1} s_{2n} + f_n \left(s_{n+1}, \dots, s_{2n}\right) \widetilde{K}_{n-1}^{[-1]} \left(s_{n+1}, \dots, s_{2n}\right)^* \\ &= \left(f_0, \dots, f_{n-1}\right) \left\{ I - \widetilde{K}_{n-1} \widetilde{K}_{n-1}^{[-1]} \right\} \left(s_{n+1}, \dots, s_{2n}\right)^* \\ &= \left(f_0, \dots, f_{n-1}\right) \left\{ I - \widetilde{K}_{n-1} \widetilde{K}_{n-1}^{[-1]} \right\} \mathbf{P}_{\operatorname{Ker} \widetilde{K}_{n-1}} \left(s_{n+1}, \dots, s_{2n}\right)^* \\ &= 0 \end{aligned}$$

which in view of the representation (2.7) is equivalent to

$$f_0 s_{n+1} + \dots + f_n s_{2n+1} = f_n R. \tag{2.10}$$

The condition (2.9) means that

$$f_0 s_{n+1} + \dots + f_n s_{2n+1} = 0 \qquad \text{for every vector } \left(f_0, \dots, f_n\right) \in \operatorname{Ker} K_n.$$

The last one is equivalent, in view of (2.5) and (2.10), to $f_n R = 0$ for all $f_n \in \mathcal{L}$. By Lemma 2.3, the matrix

$$R = s_{2n+1} - \left(s_{n+1}, \dots, s_{2n}\right) \widetilde{K}_{n-1}^{[-1]} \left(s_{n+1}, \dots, s_{2n}\right)^*$$

does not depend on the choice of $\widetilde{K}_{n-1}^{[-1]}$.

The implication (iv) $\Rightarrow$ (ii). Let $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ and let K_{n+1} be a non-negative Hankel extension of K_n for some $s_{2n+2} \in \mathbb{C}^{m\times m}$. Since $\widetilde{K}_n$ and K_{n+1} are both non-negative, then by the solvability criterion of the Stieltjes moment problem, there exists a measure $d\sigma(\lambda) \geq 0$ such that

$$\int_0^\infty \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n+1) \qquad \text{and} \qquad \int_0^\infty \lambda^{2n+2} d\sigma(\lambda) \leq s_{2n+2}.$$

In particular, this measure satisfies (2.8).

The implication (i) $\Rightarrow$ (iv). Let $d\sigma$ satisfy (2.8) and therefore,

$$K_n = \int_0^\infty (I_m, \ldots, \lambda^n I_m)^* d\sigma(\lambda)(I_m, \ldots, \lambda^n I_m). \tag{2.11}$$

Let $\mathbf{f} = (f_0, \ldots, f_n)$ be a vector from $\operatorname{Ker} K_n$. Then

$$\int_0^\infty f(\lambda)\, d\sigma(\lambda) f(\lambda)^* = 0$$

where

$$f(\lambda) = f_0 + \lambda f_1 + \ldots + \lambda^n f_n = \mathbf{f}(I_m, \ldots, \lambda^n I_m)^*. \tag{2.12}$$

In particular, for every choice of $0 \leq a < b < +\infty$,

$$\int_a^b f(\lambda)\, d\sigma(\lambda) f(\lambda)^* = 0. \tag{2.13}$$

Let $g \in \mathbb{C}^{1\times m}$ be an arbitrary non-zero vector. By the Cauchy inequality,

$$\int_a^b f(\lambda)\, d\sigma(\lambda) \lambda^{n+1} g^* \leq \left(\int_a^b f(\lambda)\, d\sigma(\lambda) f(\lambda)^* \int_a^b \lambda^{2n+2} g\, d\sigma(\lambda) g^* \right)^{1/2}$$

which in view of (2.13) implies

$$\int_a^b f(\lambda)\, d\sigma(\lambda)\, \lambda^{n+1} g^* = 0.$$

Since $a, b \in \mathbb{R}_+$ and $g \in \mathbb{C}^{1\times m}$ are arbitrary, then

$$\int_0^\infty f(\lambda)\, d\sigma(\lambda)\, \lambda^{n+1} I_m = 0.$$

Therefore, we have also

$$\int_0^\infty f(\lambda)\, d\sigma(\lambda)\big(I_m, \lambda I_m, \ldots, \lambda^{n+1} I_m\big) = 0$$

which on account of (2.8), (2.11) and (2.12) can be rewritten as $\mathbf{f}\,\widetilde{K}_n = 0$. Thus, every vector $\mathbf{f} \in \operatorname{Ker} K_n$ belongs also to $\operatorname{Ker} \widetilde{K}_n$ and so, $\operatorname{Ker} K_n \subseteq \operatorname{Ker} \widetilde{K}_n$, which ends the proof of the lemma ∎

In connection with the statement (iv) from Lemma 2.5 we consider the following problem:

To describe all matrices $s \in \mathbb{C}^{m\times m}$ such that

$$s = \int_0^\infty \lambda^{2n+1} d\sigma(\lambda)$$

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$.

Let again $\{K_n, \widetilde{K}_n\}$ be in $\mathcal{K}_n$. Then $\widetilde{K}_n$ is non-negative and its block s_{2n+1} admits a representation

$$s_{2n+1} = \big(s_{n+1}, \ldots, s_{2n}\big)\widetilde{K}_{n-1}^{[-1]}\big(s_{n+1}, \ldots, s_{2n}\big)^* + L \tag{2.14}$$

for some non-negative matrix L which does not depend on the choice of $\widetilde{K}_{n-1}^{[-1]}$.

Lemma 2.6: *Let $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n$, let s_{2n+1} be of the form (2.14), let $\mathcal{L}$ be the subspace defined by (2.5) and let s be an arbitrary $(m \times m)$-matrix. Then*

$$s = \int_0^\infty \lambda^{2n+1} d\sigma(\lambda) \tag{2.15}$$

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$ if and only if s admits a representation

$$s = \big(s_{n+1}, \ldots, s_{2n}\big)\widetilde{K}_{n-1}^{[-1]}\big(s_{n+1}, \ldots, s_{2n}\big)^* + L_0 \tag{2.16}$$

with a non-negative matrix $L_0 \le L$ such that $L_0|_{\mathcal{L}} = 0$ (i.e. such that $\operatorname{Ker} L_0 \supseteq \mathcal{L}$*).*

Proof: Let us consider the Hankel block matrix

$$\widetilde{K}_n^1 = \begin{pmatrix} s_1 & \cdots & s_n & s_{n+1} \\ \vdots & & & \vdots \\ s_n & & & s_{2n} \\ s_{n+1} & \cdots & s_{2n} & s \end{pmatrix}$$

which differs from $\widetilde{K}_n$ only by the block $s^1_{2n+1} = s$. According to Lemma 2.5, s admits a representation (2.15) for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$ if and only if $\{K_n, \widetilde{K}_n^1\} \in \mathcal{K}_n^+$. This in turn is equivalent (by Lemma 2.5) to the representation (2.16) with some non-negative matrix L_0 which vanishes on the subspace $\mathcal{L}$. It follows from (1.3) and (2.15) that $s \le s_{2n+1}$ which in view of (2.14), (2.16) is equivalent to $L_0 \le L$ ∎

Lemma 2.7: *Let $\mathcal{L}$ be the subspace defined by* (2.5), *let s_{2n+1} and s be matrices defined by* (2.14) *and* (2.16), *respectively, and let*

$$L_0|_{\mathcal{L}} = 0 \qquad \text{and} \qquad L_0|_{\mathcal{L}^\perp} = L|_{\mathcal{L}^\perp}. \tag{2.17}$$

Then the Stieltjes moment problems associated with the sets of matrices

$$\mathbf{H}_{2n+1} = \{s_0, \ldots, s_{2n}, s_{2n+1}\} \qquad \text{and} \qquad \mathbf{H}^1_{2n+1} = \{s_0, \ldots, s_{2n}, s\}$$

have the same solutions: $\mathcal{Z}(\mathbf{H}_{2n+1}) = \mathcal{Z}(\mathbf{H}^1_{2n+1})$.

Proof: Let σ belong to $\mathcal{Z}(\mathbf{H}_{2n+1})$. According to Lemma 2.7, the matrix $\hat{s} = \int_0^\infty \lambda^{2n} d\sigma(\lambda)$ is of the form (2.16) with some non-negative matrix $\hat{L}_0 \leq L$ such that $\hat{L}_0|_{\mathcal{L}} = 0$. In view of (2.17) we have also $\hat{L}_0 \leq L_0$. Therefore, $\hat{s} \leq s$ and $\sigma \in \mathcal{Z}(\mathbf{H}^1_{2n+1})$. So, $\mathcal{Z}(\mathbf{H}_{2n+1}) \subset \mathcal{Z}(\mathbf{H}^1_{2n+1})$. The converse inclusion follows from the inequality $s \leq s_{2n+1}$ ∎

Remark 2.8: In view of Lemmas 2.5 - 2.7 we can assume without lost of generality that $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$. In other case we replace the block s_{2n+1} (which is necessarily of the form (2.14)) by the block $s^1_{2n+1} = s$ defined by (2.16) with L_0 satisfying (2.17). By Lemma 2.5, $\{K_n, \widetilde{K}^1_n\} \in \mathcal{K}_n^+$ and we describe the set $\mathcal{Z}(\mathbf{H}^1_{2n+1})$ of solutions of this new moment problem, which coincides, by Lemma 2.7, with $\mathcal{Z}(\mathbf{H}_{2n+1})$.

3. On decompositions of the state space

In this section we show that under assumption $\operatorname{Ker} K_n \subseteq \operatorname{Ker} \widetilde{K}_n$ there exist subspaces $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ which are complements (not orthogonal, in general) to $\operatorname{Ker} K_n$ and $\operatorname{Ker} \widetilde{K}_n$, respectively, in $\mathbb{C}^{m(n+1)}$ and such that $QF_{m,n} \subseteq \widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$. This leads to special decompositions of $\mathbb{C}^{1 \times m(n+1)}$ which in turn allow to construct the resolvent matrix of the degenerate Stieltjes problem. As it was mentioned above, for the Schur problem (i.e. for the degenerate information matrix K of the form $K = I - TT^*$ where T is of the block Toeplitz structure and I is the identity matrix of the appropriate size) it was done in [4: Part 4]. The case of the degenerate information Hankel block matrix (the degenerate Hamburger moment problem on the line, see, e.g., [11, 13]) was considered in [3].

Lemma 3.1: *Let $T_n = (t_{i+j})_{i,j=0}^n$ and $\widetilde{T}_n = (t_{i+j+1})_{i,j=0}^n$ be Hankel block matrices with non-degenerate blocks $t_0, t_1 \in \mathbb{C}^{l \times l}$ and let T^1_{n-1} and $\widetilde{T}^1_{n-1}$ be block matrices defined by*

$$T^1_{n-1} = D^{-1}\{M - \mathcal{T} t_0^{-1} \mathcal{T}^*\} D^{-*} \qquad \text{and} \qquad \widetilde{T}^1_{n-1} = D^{-1}\{\widetilde{M} - \widetilde{\mathcal{T}} t_1^{-1} \widetilde{\mathcal{T}}^*\} D^{-*} \tag{3.1}$$

where

$$M = (t_{i+j})_{i,j=1}^n, \qquad \widetilde{M} = (t_{i+j+1})_{i,j=1}^n \tag{3.2}$$

and

$$D = \begin{pmatrix} t_1 t_0^{-1} & 0 & \ldots & 0 \\ t_2 t_0^{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ t_n t_0^{-1} & \ldots & t_2 t_0^{-1} & t_1 t_0^{-1} \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \qquad \widetilde{\mathcal{T}} = \begin{pmatrix} t_2 \\ \vdots \\ t_{n+1} \end{pmatrix}. \tag{3.3}$$

Under these assumptions the following statements are true:

(i) *If* $\{T_n, \widetilde{T}_n\} \in \mathcal{K}_n$, *then* $\{T^1_{n-1}, \widetilde{T}^1_{n-1}\} \in \mathcal{K}_{n-1}$.

(ii) *If* $\{T_n, \widetilde{T}_n\} \in \mathcal{K}^+_n$, *then* $\{T^1_{n-1}, \widetilde{T}^1_{n-1}\} \in \mathcal{K}^+_{n-1}$.

Proof: From the block decompositions

$$T_n = \begin{pmatrix} t_0 & \mathcal{T}^* \\ \mathcal{T} & M \end{pmatrix} \qquad \text{and} \qquad \widetilde{T}_n = \begin{pmatrix} t_1 & \widetilde{\mathcal{T}}^* \\ \widetilde{\mathcal{T}}^* & \widetilde{M} \end{pmatrix}$$

we obtain on account of (3.1) the factorizations

$$T_n = \begin{pmatrix} I_l & 0 \\ \mathcal{T} t_0^{-1} & D \end{pmatrix} \begin{pmatrix} t_0 & 0 \\ 0 & T^1_{n-1} \end{pmatrix} \begin{pmatrix} I_l & t_0^{-1}\mathcal{T}^* \\ 0 & D^* \end{pmatrix} \tag{3.4}$$

$$\widetilde{T}_n = \begin{pmatrix} I_l & 0 \\ \widetilde{\mathcal{T}} t_1^{-1} & D \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & \widetilde{T}^1_{n-1} \end{pmatrix} \begin{pmatrix} I_l & t_1^{-1}\widetilde{\mathcal{T}}^* \\ 0 & D^* \end{pmatrix} \tag{3.5}$$

which imply that if T_n and $\widetilde{T}_n$ are non-negative, then T^1_{n-1} and $\widetilde{T}^1_{n-1}$ are non-negative as well.

Let $\{T_n, \widetilde{T}_n\} \in \mathcal{K}_n$. We begin with the identities

$$F_{l,n-1} D^{-1} \mathcal{T} = 0, \qquad F_{l,n-1} F^*_{l,n-1} F_{l,n-1} = F_{l,n-1}$$

and

$$M - F_{l,n-1}\widetilde{M} + F_{l,n-1}\widetilde{\mathcal{T}} t_1^{-1} \widetilde{\mathcal{T}}^* = \mathcal{T} t_1^{-1} \widetilde{\mathcal{T}}^* \tag{3.6}$$

which follow immediately from (1.14), (3.2) and (3.3). Using these identities and taking into account (3.1) we get

$$\begin{aligned}
&F_{l,n-1}\left(F^*_{l,n-1} T^1_{n-1} - \widetilde{T}^1_{n-1}\right) \\
&= F_{l,n-1} F^*_{l,n-1} \left(T^1_{n-1} - F_{l,n-1}\widetilde{T}^1_{n-1}\right) \\
&= F_{l,n-1} F^*_{l,n-1} D^{-1}\left(M - \mathcal{T} t_0^{-1}\mathcal{T}^* - F_{l,n-1}\widetilde{M} + F_{l,n-1}\widetilde{\mathcal{T}} t_1^{-1}\widetilde{\mathcal{T}}^*\right) D^{-*} \\
&= F_{l,n-1} F^*_{l,n-1} D^{-1}\mathcal{T}\left(t_1^{-1}\widetilde{\mathcal{T}}^* - t_0^{-1}\mathcal{T}^*\right) D^{-*} \\
&= 0.
\end{aligned}$$

Thus, $F_{l,n-1}(F^*_{l,n-1}T^1_{n-1} - \widetilde{T}^1_{n-1}) = 0$ and by Lemma 2.5, $\{T^1_{n-1}, \widetilde{T}^1_{n-1}\} \in \mathcal{K}_{n-1}$.

Let now $\{T_n, \widetilde{T}_n\} \in \mathcal{K}^+_n$ and let $\mathbf{f} \in \mathbb{C}^{1\times ln}$ be an arbitrary vector from $\operatorname{Ker} T^1_{n-1}$. In view of (3.5), the vector $(-\mathbf{f}D^{-1}\mathcal{T}t_1^{-1}, \mathbf{f}D^{-1})$ belongs to $\operatorname{Ker}\widetilde{T}_n$. By Lemma 2.5, $\operatorname{Ker} T_n \subseteq \operatorname{Ker}\widetilde{T}_n$ and thus, $(-\mathbf{f}D^{-1}\mathcal{T}t_1^{-1}, \mathbf{f}D^{-1})\widetilde{T}_n = 0$. Substituting (3.5) into this last equality we obtain, in particular, $\mathbf{f}\widetilde{T}^1_{n-1} = 0$. So, every vector $\mathbf{f} \in \operatorname{Ker} T^1_{n-1}$ belongs also to $\operatorname{Ker}\widetilde{T}^1_{n-1}$ and so, $\operatorname{Ker} T^1_{n-1} \subseteq \operatorname{Ker}\widetilde{T}^1_{n-1}$. By Lemma 2.5, $\{T^1_{n-1}, \widetilde{T}^1_{n-1}\} \in \mathcal{K}^+_{n-1}$ ∎

Lemma 3.2: *Let* $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ *and let* $\operatorname{rank} K_n = r$ *and* $\operatorname{rank} \widetilde{K}_n = \tilde{r}$. *Then there exist matrices* $Q \in \mathbb{C}^{r\times(n+1)m}$ *and* $\widetilde{Q} \in \mathbb{C}^{\tilde{r}\times(n+1)m}$ *such that*

$$QK_nQ^* > 0 \qquad \text{and} \qquad \widetilde{Q}\widetilde{K}_n\widetilde{Q}^* > 0 \tag{3.7}$$

$$QF_{m,n} = N\widetilde{Q} \qquad \text{and} \qquad \widetilde{Q} = \widetilde{N}Q \tag{3.8}$$

for the shift $F_{m,n}$ *defined via* (1.14) *and some matrices* $N \in \mathbb{C}^{r\times\tilde{r}}$ *and* $\widetilde{N} \in \mathbb{C}^{\tilde{r}\times r}$. *In other words, there exist subspaces*

$$\mathcal{Q} = \operatorname{Ran} Q := \left\{y \in \mathbb{C}^{1\times m(n+1)} : \ y = fQ \ \text{ for some } \ f \in \mathbb{C}^{1\times r}\right\}$$
$$\widetilde{\mathcal{Q}} = \operatorname{Ran} \widetilde{Q}$$

such that $\mathcal{Q}F_{m,n} \subseteq \widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ *and*

$$\mathbb{C}^{m(n+1)} = \operatorname{Ker} K_n \dot{+} \mathcal{Q} = \operatorname{Ker} \widetilde{K}_n \dot{+} \widetilde{\mathcal{Q}}. \tag{3.9}$$

Proof: We prove this lemma by induction. Let, for $n = 0$, $\operatorname{rank} s_0 = l$, and $\operatorname{rank} s_1 = \tilde{l}$ $(l, \tilde{l} \le m)$. By Lemma 2.5, $\operatorname{Ker} s_0 \subseteq \operatorname{Ker} s_1$ and therefore, $l \le \tilde{l}$. Moreover, there exists a unitary matrix $U \in \mathbb{C}^{m\times m}$ such that

$$Us_0U^* = \begin{pmatrix} \hat{s}_0 & 0 \\ 0 & 0_{m-l} \end{pmatrix} \quad \text{and} \quad Us_1U^* = \begin{pmatrix} t_1 & 0 \\ 0 & 0_{m-\tilde{l}} \end{pmatrix} \qquad (\hat{s}_0, t_1 > 0) \tag{3.10}$$

and hence, matrices

$$Q = (I_l, 0)U, \quad \tilde{Q} = (I_{\tilde{l}}, 0)U \qquad \text{and} \qquad N = 0, \quad \tilde{N} = (I_{\tilde{l}}, 0_{\tilde{l}\times(l-\tilde{l})})$$

satisfy (3.7) and (3.8).

Let us suppose the assertion of the lemma hold for all integers up to $n-1$. Let as above $\operatorname{rank} s_0 = l \ge \tilde{l} = \operatorname{rank} s_1$ and s_0, s_1 be of the form (3.10). By Lemma 2.5, $\operatorname{Ker} K_n \subseteq \operatorname{Ker} \widetilde{K}_n$ which implies, in particular,

$$\operatorname{Ker} s_0 \subseteq \operatorname{Ker} s_1 \subseteq \ldots \subseteq \operatorname{Ker} s_{2n+1}.$$

Therefore,

$$Us_iU^* = \begin{pmatrix} t_i & 0 \\ 0 & 0_{m-\tilde{l}} \end{pmatrix} \qquad (i = 1, \ldots, 2n+1) \tag{3.11}$$

for some matrices $t_i \in \mathbb{C}^{\tilde{l}\times\tilde{l}}$. Furthermore, let

$$\hat{s}_0 = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \qquad \left(\alpha \in \mathbb{C}^{\tilde{l}\times\tilde{l}}, \ \beta \in \mathbb{C}^{\tilde{l}\times(l-\tilde{l})}, \ \gamma \in \mathbb{C}^{(l-\tilde{l})\times(l-\tilde{l})}\right) \tag{3.12}$$

be the block decomposition of the matrix $\hat{s}_0$ from the representation (3.10). Introducing the matrices

$$t_0 = \alpha - \beta\gamma^{-1}\beta^* = (I_{\tilde{l}}, -\beta\gamma^{-1})\hat{s}_0 \begin{pmatrix} I_{\tilde{l}} \\ -\gamma^{-1}\beta^* \end{pmatrix} > 0 \tag{3.13}$$

and

$$g = \left(I_{\tilde{l}}, -\beta\gamma^{-1}, 0_{\tilde{l}\times(m-l)}\right) U \tag{3.14}$$

we obtain immediately from (3.10) - (3.12) that

$$g s_i g^* = t_i \qquad (i = 0, \dots, 2n+1). \tag{3.15}$$

Let T_n and $\widetilde{T}_n$ be $\tilde{l}(n+1) \times \tilde{l}(n+1)$ Hankel block matrices defined by

$$T_n = (t_{i+j}) = \Gamma_n K_n \Gamma_n^* \qquad \text{and} \qquad \widetilde{T}_n = (t_{i+j+1}) = \Gamma_n \widetilde{K}_n \Gamma_n^* \tag{3.16}$$

where

$$\Gamma_n = \begin{pmatrix} g & & 0 \\ & \ddots & \\ 0 & & g \end{pmatrix} \in \mathbb{C}^{\tilde{l}(n+1)\times m(n+1)}. \tag{3.17}$$

It follows from (3.10) - (3.17) that

$$\operatorname{rank} T_n = \operatorname{rank} K_n - (l - \tilde{l}) = r + \tilde{l} - l \quad \text{and} \quad \operatorname{rank} \widetilde{T}_n = \operatorname{rank} \widetilde{K}_n = \tilde{r}. \tag{3.18}$$

Since $\operatorname{Ker} K_n \subseteq \operatorname{Ker} \widetilde{K}_n$, then in view of (3.10) - (3.16), $\operatorname{Ker} T_n \subseteq \operatorname{Ker} \widetilde{T}_n$ and by Lemma 2.5, $\{T_n, \widetilde{T}_n\} \in \mathcal{K}_n^+$. Let T_{n-1}^1, $\widetilde{T}_{n-1}^1$ and D be matrices defined by (3.1) and (3.3), respectively. According to Lemma 3.1, $\{T_{n-1}^1, \widetilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}^+$. Moreover, we obtain from (3.4), (3.5) and (3.18) that $\operatorname{rank} T_{n-1}^1 = r - l$ and $\operatorname{rank} \widetilde{T}_{n-1}^1 = \tilde{r} - \tilde{l}$. Hence, by the induction hypothesis there exist matrices

$$Q_1 \in \mathbb{C}^{(r-l)\times \tilde{l}n}, \quad \tilde{Q}_1 \in \mathbb{C}^{(\tilde{r}-\tilde{l})\times \tilde{l}n} \quad \text{and} \quad N_1 \in \mathbb{C}^{(r-l)\times(\tilde{r}-\tilde{l})}, \quad \widetilde{N}_1 \in \mathbb{C}^{(\tilde{r}-\tilde{l})\times(r-l)}$$

such that

$$Q_1 T_{n-1}^1 Q_1^* > 0 \qquad \text{and} \qquad \tilde{Q}_1 \widetilde{T}_{n-1}^1 \tilde{Q}_1^* > 0 \tag{3.19}$$

$$Q_1 F_{\tilde{l},n-1} = N_1 \widetilde{Q}_1 \qquad \text{and} \qquad \tilde{Q}_1 = \widetilde{N}_1 Q_1. \tag{3.20}$$

Furthermore, let $\tilde{e} \in \mathbb{C}^{\tilde{l}n \times \tilde{l}}$ be the matrix defined by

$$\tilde{e} = (I_{\tilde{l}}, 0_{\tilde{l}}, \dots, 0_{\tilde{l}})^*. \tag{3.21}$$

We show that the matrices $Q, \widetilde{Q}$ and $N, \widetilde{N}$ given by

$$Q = \begin{pmatrix} \begin{pmatrix} I_{\tilde{l}} & -\beta\gamma^{-1} & 0 \\ 0 & I_{l-\tilde{l}} & 0 \end{pmatrix} U & 0 \\ -Q_1 D^{-1} T t_0^{-1} g & Q_1 D^{-1} \Gamma_{n-1} \end{pmatrix}$$

$$\widetilde{Q} = \begin{pmatrix} I_{\tilde{l}} & 0 \\ -\widetilde{Q}_1 D^{-1} \widetilde{T} t_1^{-1} & \widetilde{Q}_1 D^{-1} \end{pmatrix} \Gamma_n$$

and

$$N = \begin{pmatrix} 0_{l\times\tilde{l}} & 0 \\ Q_1\tilde{e}t_0t_1^{-1}g & N_1 \end{pmatrix}$$
$$\widetilde{N} = \begin{pmatrix} I_{\tilde{l}} & 0_{\tilde{l}\times(l-\tilde{l})} & 0 \\ \widetilde{N}_1 Q_1 D^{-1}(\mathcal{T}t_0^{-1} - \widetilde{\mathcal{T}}t_1^{-1}) & 0 & \widetilde{N}_1 \end{pmatrix} \tag{3.22}$$

(where $\mathcal{T}, \widetilde{\mathcal{T}}, U$ and $\tilde{e}$ are matrices defined via (3.3), (3.11) and (3.21)) satisfy conditions (3.7) and (3.8). Indeed, in view of (3.4), (3.5), (3.10) - (3.17) and (3.19) we have

$$QK_nQ^* = \begin{pmatrix} t_0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & Q_1T_{n-1}^1Q_1^* \end{pmatrix} > 0$$
$$\widetilde{Q}\widetilde{K}_n\widetilde{Q}^* = \begin{pmatrix} t_1 & 0 \\ 0 & \widetilde{Q}_1\widetilde{T}_{n-1}^1\widetilde{Q}_1 \end{pmatrix} > 0.$$

It follows from (3.23) and the block decompositions

$$\Gamma_n = \begin{pmatrix} g & 0 \\ 0 & \Gamma_{n-1} \end{pmatrix} \qquad \text{and} \qquad F_{\tilde{l},n} = \begin{pmatrix} 0 & 0 \\ \tilde{e} & F_{\tilde{l},n-1} \end{pmatrix}$$

that

$$QF_{m,n} - N\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix} \tag{3.23}$$

where

$$B_1 = Q_1D^{-1}\tilde{e}g - Q_1\tilde{e}t_0t_1^{-1}g + N_1\widetilde{Q}_1D^{-1}\widetilde{\mathcal{T}}t_1^{-1}g$$
$$B_2 = Q_1D^{-1}\Gamma_{n-1}F_{m,n-1} - N_1\widetilde{Q}_1D^{-1}\Gamma_{n-1}.$$

Using (3.20) and identities

$$D^{-1}F_{\tilde{l},n-1} = F_{\tilde{l},n-1}D^{-1}, \quad \Gamma_{n-1}F_{m,n-1} = F_{\tilde{l},n-1}\Gamma_{n-1}, \quad \tilde{e}t_1 + F_{m,n-1}\widetilde{\mathcal{T}} = \mathcal{T}$$

which follow immediately from (2.2), (3.3), (3.17) and (3.21), we get

$$\begin{aligned} B_1 &= Q_1D^{-1}\left(\tilde{e}g - D\tilde{e}t_0t_1^{-1}g + F_{\tilde{l},n-1}\widetilde{\mathcal{T}}t_1^{-1}g\right) \\ &= Q_1D^{-1}\left(\tilde{e}t_1 - \mathcal{T} + F_{\tilde{l},n-1}\widetilde{\mathcal{T}}\right)t_1^{-1}g \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} B_2 &= Q_1D^{-1}\Gamma_{n-1}F_{m,n-1} - Q_1F_{\tilde{l},n-1}D^{-1}\Gamma_{n-1} \\ &= Q_1D^{-1}\left(\Gamma_{n-1}F_{m,n-1} - F_{\tilde{l},n-1}\Gamma_{n-1}\right) \\ &= 0. \end{aligned}$$

So, $B_1 = B_2 = 0$ and (3.23) implies the first equality from (3.8). Similarly, in view of (3.20) and (3.23),

$$\widetilde{Q} - \widetilde{N}Q = \begin{pmatrix} g & 0 \\ -\widetilde{Q}_1 D^{-1}\widetilde{T}t_1^{-1}g & \widetilde{Q}_1 D^{-1}\Gamma_{n-1} \end{pmatrix} - \begin{pmatrix} g & 0 \\ -\widetilde{N}_1 Q_1 D^{-1}\widetilde{T}t_1^{-1}g & \widetilde{N}_1 Q_1 D^{-1}\Gamma_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ (N_1Q_1 - \widetilde{Q}_1)D^{-1}\widetilde{T}t_1^{-1}g & (\widetilde{Q}_1 - N_1Q_1)D^{-1}\Gamma_{n-1} \end{pmatrix}$$
$$= 0.$$

Thus, $\widetilde{Q} = \widetilde{N}Q$ which ends the proof of the lemma ∎

Corollary 3.3: *Let Q and $\widetilde{Q}$ be matrices satisfying* (3.8). *Then for all $l \in \mathbb{N}$*

$$QF^l_{m,n} = (N\widetilde{N})^l N\widetilde{Q} \qquad \text{and} \qquad \widetilde{Q}F^l_{m,n} = (\widetilde{N}N)^l\widetilde{N}Q = (\widetilde{N}N)^{l+1}\widetilde{Q}. \tag{3.24}$$

4. J-inner polynomials associated with $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$

In this section we associate to every pair $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ a function Θ of the class $\mathbf{W}_\pi$. It will be shown in Section 5 that this function is the resolvent matrix of the corresponding Stieltjes moment problem. To simplify the further computations, the index n will be omitted, and up to Section 6, by K, $\widetilde{K}$ and F we mean matrices K_n, $\widetilde{K}_n$ and $F_{m,n}$ given by (2.1) and (1.14).

Lemma 4.1: *Let $\{K, \widetilde{K}\} \in \mathcal{K}^+$, let Q and $\widetilde{Q}$ be matrices satisfying* (3.7) *and* (3.8), *and let $K^{[-1]}$ and $\widetilde{K}^{[-1]}$ be pseudoinverse matrices defined as*

$$K^{[-1]} = Q^*(QKQ^*)^{-1}Q \qquad \text{and} \qquad \widetilde{K}^{[-1]} = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}\widetilde{Q}^*)^{-1}\widetilde{Q}. \tag{4.1}$$

Then, for all $l \in \mathbb{N}_0$,

$$K^{[-1]}F^l\left(I - \widetilde{K}\widetilde{K}^{[-1]}\right) = \widetilde{K}^{[-1]}F^l\left(I - KK^{[-1]}\right) = 0 \tag{4.2}$$

$$\widetilde{K}^{[-1]}F^l\left(I - \widetilde{K}\widetilde{K}^{[-1]}\right) = K^{[-1]}F^{l+1}\left(I - KK^{[-1]}\right) = 0. \tag{4.3}$$

Proof: It follows from (4.1) that

$$Q\left(I - KK^{[-1]}\right) = 0 \qquad \text{and} \qquad \widetilde{Q}\left(I - \widetilde{K}\widetilde{K}^{[-1]}\right) = 0. \tag{4.4}$$

Using again (4.1) together with (3.24) we get

$$K^{[-1]}F^l = Q^*(QK_nQ^*)^{-1}(N\widetilde{N})^lQ = Q^*(QK_nQ^*)^{-1}(N\widetilde{N})^lN\widetilde{Q}$$

$$\widetilde{K}^{[-1]}F^{l+1} = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}\widetilde{Q}^*)^{-1}(\widetilde{N}N)^l\widetilde{N}Q = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}\widetilde{Q}^*)^{-1}(\widetilde{N}N)^{l+1}\widetilde{Q}.$$

Substituting these last equalities (for $l = 0, 1, \ldots$) into (4.2) and (4.3) and taking into account (4.4) we obtain the required equalities ∎

Lemma 4.2: *Under the assumptions of Lemma 4.1 let Θ and $\widetilde{\Theta}$ be the $\mathbb{C}^{2m\times 2m}$-valued functions defined by*

$$\Theta(z)=I_{2m}+\begin{pmatrix}\mathbf{e}^*K & 0\\ 0 & \mathbf{e}^*\end{pmatrix}\Omega(z)\begin{pmatrix}K^{[-1]} & 0\\ 0 & \widetilde{K}^{[-1]}\end{pmatrix}\begin{pmatrix}\mathbf{e} & 0\\ 0 & K\mathbf{e}\end{pmatrix} \tag{4.5}$$

and

$$\widetilde{\Theta}(z)=P(z)\Theta(z)P^{-1}(z) \tag{4.6}$$

where $P(z)$ and $\mathbf{e}$ are given by (1.9) and (1.14), respectively, and

$$\Omega(z)=\begin{pmatrix} zF^*(I-zF^*)^{-1} & (I-zF^*)^{-1}\\ -z(I-zF^*)^{-1} & -z(I-zF^*)^{-1}\end{pmatrix}.$$

Then Θ is of the class W_π (see Definition 1.2) and moreover, for all $z,\omega\in\mathbb{C}\setminus\mathbb{R}$,

$$\frac{\Theta(z)J\Theta(\omega)^*-J}{i(\overline{\omega}-z)}=\begin{pmatrix}\mathbf{e}^*KF^*\\ -\mathbf{e}^*\end{pmatrix}(I-zF^*)^{-1}K^{[-1]}(I-\overline{\omega}F)^{-1}(FK\mathbf{e},-\mathbf{e}) \tag{4.7}$$

$$\frac{\widetilde{\Theta}(z)J\widetilde{\Theta}(\omega)^*-J}{i(\overline{\omega}-z)}=\begin{pmatrix}\mathbf{e}^*K\\ -\mathbf{e}^*\end{pmatrix}(I-zF^*)^{-1}\widetilde{K}^{[-1]}(I-\overline{\omega}F)^{-1}(K\mathbf{e},-\mathbf{e}). \tag{4.8}$$

Proof: Since $\{K,\widetilde{K}\}\in\mathcal{K}^+$, then by Lemma 2.2, $F(F^*K-\widetilde{K})=0$, or equivalently,

$$(I-\mathbf{e}\mathbf{e}^*)K=F\widetilde{K}. \tag{4.9}$$

This last identity implies that

$$FK\mathbf{e}\mathbf{e}^*-\mathbf{e}\mathbf{e}^*KF^*=FK-KF^* \quad\text{and}\quad K\mathbf{e}\mathbf{e}^*-\mathbf{e}\mathbf{e}^*K=F\widetilde{K}-\widetilde{K}F^*. \tag{4.10}$$

Using (4.9) and (4.10) both with (4.2) and (4.3), it is easy to check that the functions Θ and $\widetilde{\Theta}$ admit the factorizations

$$\Theta(z)=\left\{I_{2m}+z\begin{pmatrix}\mathbf{e}^*KF^*\\ -\mathbf{e}^*\end{pmatrix}(I-zF^*)^{-1}K^{[-1]}(\mathbf{e},FK\mathbf{e})\right\}\Psi \tag{4.11}$$

$$\widetilde{\Theta}(z)=\left\{I_{2m}+z\begin{pmatrix}\mathbf{e}^*K\\ -\mathbf{e}^*\end{pmatrix}(I-zF^*)^{-1}\widetilde{K}^{[-1]}(\mathbf{e},K\mathbf{e})\right\}\widetilde{\Psi} \tag{4.12}$$

where

$$\Psi=\begin{pmatrix}I_m & \mathbf{e}^*K\widetilde{K}^{[-1]}K\mathbf{e}\\ 0 & I_m\end{pmatrix}\quad\text{and}\quad\widetilde{\Psi}=\begin{pmatrix}I_m & 0\\ \mathbf{e}^*K^{[-1]}\mathbf{e} & I_m\end{pmatrix}. \tag{4.13}$$

Since the matrix Ψ is J-unitary, we obtain from (4.11)

$$\Theta(z)J\Theta(\omega)^*-J=-i\begin{pmatrix}\mathbf{e}^*KF^*\\ -\mathbf{e}^*\end{pmatrix}(I-zF^*)^{-1}\Phi(z,\omega)(I-\overline{\omega}F)^{-1}(FK\mathbf{e},-\mathbf{e}) \tag{4.14}$$

where

$$\begin{aligned}\Phi(z,\omega)&=zK^{[-1]}(I-\overline{\omega}F)-\overline{\omega}(I-zF^*)K^{[-1]}\\ &\quad+z\overline{\omega}K^{[-1]}\{FK\mathbf{e}\mathbf{e}^*-\mathbf{e}\mathbf{e}^*KF^*\}K^{[-1]}.\end{aligned} \tag{4.15}$$

Substituting the first equality from (4.10) into (4.15) and using (4.2) we get

$$\begin{aligned}\Phi(z,\omega) &= (z-\overline{\omega})K^{[-1]} + z\overline{\omega}\Big\{F^*K^{[-1]} - K^{[-1]}F + K^{[-1]}(FK - KF^*)K^{[-1]}\Big\}\\ &= (z-\overline{\omega})K^{[-1]} + z\overline{\omega}\Big\{(I - K^{[-1]}K)F^*K^{[-1]} - K^{[-1]}F(I - KK^{[-1]})\Big\}\\ &= (z-\overline{\omega})K^{[-1]}\end{aligned}$$

which both with (4.14) implies (4.7). Similarly, taking into account the J-unitarity of $\widetilde{\Psi}$ we obtain from (4.8)

$$\widetilde{\Theta}(z)J\widetilde{\Theta}(\omega)^* - J = \begin{pmatrix} \mathbf{e}^*K \\ -\mathbf{e}^* \end{pmatrix} (I - zF^*)^{-1}\widetilde{\Phi}(z,\omega)(I - \overline{\omega}F)^{-1}(K\mathbf{e}, -\mathbf{e}) \qquad (4.16)$$

where

$$\begin{aligned}\widetilde{\Phi}(z,\omega) = z\widetilde{K}^{[-1]}(I - \overline{\omega}F) - \overline{\omega}(I - zF^*)\widetilde{K}^{[-1]}&\\ + z\overline{\omega}\widetilde{K}^{[-1]}\big\{K\mathbf{e}\mathbf{e}^* - \mathbf{e}\mathbf{e}^*K\big\}\widetilde{K}^{[-1]}.&\end{aligned} \qquad (4.17)$$

Substituting the second equality from (4.10) into (4.17) and using (4.2) we receive

$$\begin{aligned}\widetilde{\Phi}(z,\omega) &= (z-\overline{\omega})\widetilde{K}^{[-1]} + z\overline{\omega}\Big\{F^*\widetilde{K}^{[-1]} - \widetilde{K}^{[-1]}F + \widetilde{K}^{[-1]}(F\widetilde{K} - \widetilde{K}F^*)\widetilde{K}^{[-1]}\Big\}\\ &= (z-\overline{\omega})\widetilde{K}^{[-1]} + z\overline{\omega}\Big\{(I - \widetilde{K}^{[-1]}\widetilde{K})F^*\widetilde{K}^{[-1]} - \widetilde{K}^{[-1]}F(I - \widetilde{K}\widetilde{K}^{[-1]})\Big\}\\ &= (z-\overline{\omega})\widetilde{K}^{[-1]}\end{aligned}$$

which both with (4.17) leads to (4.8). From (4.7) and (4.8) we conclude that Θ and $\widetilde{\Theta}$ belong to the class $\mathbf{W}$, and hence, Θ belongs to $\mathbf{W}_\pi$ by Theorem 1.3 ∎

Remark 4.3: Since the functions Θ and $\widetilde{\Theta}$ are both J-unitary on the real axis, then by the symmetry principle, $\Theta^{-1}(z) = J\Theta(\overline{z})^*J$ and $\widetilde{\Theta}^{-1}(z) = J\widetilde{\Theta}(\overline{z})^*J$, and on account of (4.7) and (4.8), the relations

$$\begin{aligned}&J - \Theta(\omega)^{-*}J\Theta^{-1}(z)\\ &\quad = J\big(J - \Theta(\overline{\omega})J\Theta(\overline{z})^*\big)J\\ &\quad = i(\overline{\omega} - z)\begin{pmatrix} \mathbf{e}^* \\ \mathbf{e}^*KF^* \end{pmatrix}(I - \overline{\omega}F^*)^{-1}K^{[-1]}(I - zF)^{-1}(\mathbf{e}, FK\mathbf{e})\end{aligned} \qquad (4.18)$$

and

$$\begin{aligned}&P(\omega)^*JP(z) - \Theta(\omega)^{-*}P(\omega)^*JP(z)\Theta^{-1}(z)\\ &\quad = i(\overline{\omega} - z)P(\omega)^*\begin{pmatrix} \mathbf{e}^* \\ \mathbf{e}^*K \end{pmatrix}(I - \overline{\omega}F^*)^{-1}\widetilde{K}^{[-1]}(I - zF)^{-1}(\mathbf{e}, K\mathbf{e})P(z)\end{aligned} \qquad (4.19)$$

are true.

For the further purposes we need those J-forms of Θ and $\widetilde{\Theta}$ which are dual to (4.7) and (4.8).

Lemma 4.4: *Let Θ and $\widetilde{\Theta}$ be the functions defined by* (4.5) *and* (4.6), *respectively. Then*

$$\Theta(\omega)^* J\Theta(z) - J = i(\overline{\omega} - z)R^* K^{[-1]}(I - \overline{\omega}F)^{-1} K(I - zF^*)^{-1} K^{[-1]} R \quad (4.20)$$

$$\widetilde{\Theta}(\omega)^* J\widetilde{\Theta}(z) - J = i(\overline{\omega} - z)\widetilde{R}^* \widetilde{K}^{[-1]}(I - \overline{\omega}F)^{-1} \widetilde{K}(I - zF^*)^{-1} \widetilde{K}^{[-1]} \widetilde{R} \quad (4.21)$$

where

$$R = (\mathbf{e}, FK\mathbf{e})\Psi \qquad \text{and} \qquad \widetilde{R} = (\mathbf{e}, K\mathbf{e})\widetilde{\Psi}\begin{pmatrix} I & 0 \\ -\mathbf{e}^* K^{[-1]}\mathbf{e} & I \end{pmatrix}. \quad (4.22)$$

Proof: Using the representation (4.7) of Θ and taking into account (4.10) and (4.21), we obtain

$$\begin{aligned} &\Theta(\omega)^* J\Theta(z) - J \\ &= -iR^*\Big\{z(I - zF^*)^{-1}K^{[-1]} - \overline{\omega}K^{[-1]}(I - \overline{\omega}F)^{-1} \\ &\quad + z\overline{\omega}K^{[-1]}(I - \overline{\omega}F)^{-1}\{FK - KF^*\}(I - zF^*)^{-1}K^{[-1]}\Big\}R \\ &= i(\overline{\omega} - z)R^* K^{[-1]}(I - \overline{\omega}F)^{-1}K(I - zF^*)^{-1}K^{[-1]}R \\ &\quad - izR^*(I - K^{[-1]}K)(I - zF^*)^{-1}K^{[-1]}R \\ &\quad + i\overline{\omega}R^* K^{[-1]}(I - \overline{\omega}F)^{-1}(I - KK^{[-1]})R. \end{aligned} \quad (4.23)$$

Since

$$(I - zF^*)^{-1} = \sum_{j=0}^{n} z^j F^{*j}, \quad (4.24)$$

then by (4.3)

$$(I - K^{[-1]}K)(I - zF^*)^{-1}K^{[-1]} = 0 \qquad \text{for all } z \in \mathbb{C}.$$

Substituting this last equality into (4.23) we obtain (4.20). The equality (4.21) can be checked quite similarly ∎

Remark 4.5: Using (4.22), (4.13) and the equalities

$$\mathbf{e}\mathbf{e}^* K\widetilde{K}^{[-1]}K\mathbf{e} + FK\mathbf{e} = K\widetilde{K}^{[-1]}K\mathbf{e} + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}$$

$$\mathbf{e} - K\mathbf{e}\mathbf{e}^* K^{[-1]}\mathbf{e} = (I - KK^{[-1]})\mathbf{e} + \widetilde{K}F^* K^{[-1]}\mathbf{e}$$

which follow from (4.9), we obtain

$$R = \left(\mathbf{e},\, K\widetilde{K}^{[-1]}K\mathbf{e} + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}\right) \quad (4.25)$$

$$\widetilde{R} = \left((I - KK^{[-1]})\mathbf{e} + \widetilde{K}F^* K^{[-1]}\mathbf{e},\, K\mathbf{e}\right). \quad (4.26)$$

Note that in view of (4.5), (4.6) and (4.24), Θ and $\widetilde{\Theta}$ are matrix polynomials of degree $n + 1$.

5. Description of all solutions

In this section we parametrize the set $\mathcal{Z}(\mathbf{H}_{2n+1})$ of all solutions to the degenerate Stieltjes moment problem in terms of a linear fractional transformation. We begin with the following auxiliary results.

Theorem 5.1 (see [10: §3]): *Let*

$$\Theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}$$

be the block decomposition of a $\mathbb{C}^{2m\times 2m}$-valued function $\Theta \in \mathbf{W}_\pi$ into four $\mathbb{C}^{m\times m}$-valued blocks. Then the following statements are true.

(i) *All $\mathbb{C}^{m\times m}$-valued meromorphic in $\mathbb{C}\backslash\mathbb{R}_+$ solutions s of the system of inequalities*

$$\big(s(z)^*, I_m\big)\frac{\Theta(z)^{-*}J\Theta^{-1}(z)}{i(\overline{z}-z)}\begin{pmatrix} s(z) \\ I_m \end{pmatrix} \geq 0 \tag{5.1}$$

$$\big(s(z)^*, I_m\big)\frac{\Theta(z)^{-*}P(z)^*JP(z)\Theta^{-1}(z)}{i(\overline{z}-z)}\begin{pmatrix} s(z) \\ I_m \end{pmatrix} \geq 0 \tag{5.2}$$

are given by the linear fractional transformation (1.10) *when the parameter $\{p,q\}$ varies in the set $\overline{\mathcal{S}}_m$ of all Stieltjes pairs and satisfies the condition*

$$\det\Big(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)\Big) \not\equiv 0. \tag{5.3}$$

(ii) *Two pairs $\{p,q\}$ and $\{p_1,q_1\}$ lead by* (1.10) *to the same function s if and only if these pairs are equivalent.*

Lemma 5.2 *Let $\mathcal{F}$ and $\mathcal{G}$ be two orthogonal subspaces in $\mathbb{C}^{1\times m}$ (i.e. $fg^* = 0$ for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$). Let*

$$\dim \mathcal{F} = \mu \qquad \text{and} \qquad \dim \mathcal{G} = \nu \tag{5.4}$$

and let $P_\mathcal{F}$ and $P_\mathcal{G}$ be the orthogonal projections onto $\mathcal{F}$ and $\mathcal{G}$, respectively. Then every pair $\{p,q\} \in \overline{\mathcal{S}}_m$ such that

$$P_\mathcal{F}\, p(z) \equiv P_\mathcal{G}\, q(z) \equiv 0 \tag{5.5}$$

is equivalent to a pair $\{p_1,q_1\}$ of the form

$$p_1(z) = V\begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix} \qquad \text{and} \qquad q_1(z) = V\begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \tag{5.6}$$

for some unitary $(m\times m)$-matrix V which depends only on $\mathcal{F}$ and $\mathcal{G}$.

Proof: Since the subspaces $\mathcal{F}$ and $\mathcal{G}$ are orthogonal, there exists a unitary matrix $V \in \mathbb{C}^{m\times m}$ such that

$$V^*P_\mathcal{F}V = \begin{pmatrix} 0_{m-\mu-\nu} & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \qquad \text{and} \qquad V^*P_\mathcal{G}V = \begin{pmatrix} 0_{m-\mu-\nu} & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix}.$$

with μ and ν given by (5.4). Therefore, the Stieltjes pair $\{V^*p, V^*q\}$ satisfies

$$\begin{pmatrix} 0 & I_\mu & 0_{\mu\times\nu} \end{pmatrix} V^*p(z) \equiv 0 \qquad \text{and} \qquad \begin{pmatrix} 0 & 0_{\nu\times\mu} & I_\nu \end{pmatrix} V^*q(z) \equiv 0$$

and hence (see [3: Lemma 4.3]), it is equivalent to some Stieltjes pair of the form

$$\left\{ \begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix}, \begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \right\}.$$

This in turn means that the initial pair $\{p, q\}$ is equivalent to a pair $\{p_1, q_1\}$ of the form (5.6) ∎

The following theorem describes the set $\mathcal{S}(\mathbf{H}_{2n+1})$ under the assumption $\{K, \widetilde{K}\} \in \mathcal{K}^+$ (or equivalently, $\operatorname{Ker} K \subseteq \operatorname{Ker} \widetilde{K}$).

Theorem 5.3: *Let $\{K, \widetilde{K}\} \in \mathcal{K}^+$, let Q and $\widetilde{Q}$ be matrices satisfying* (3.7) *and* (3.8), *and let $K^{[-1]}$ and $\widetilde{K}^{[-1]}$ be the pseudoinverse matrices defined by* (4.1). *Then the linear fractional transformation* (1.10) *with the resolvent matrix Θ defined by* (4.5) *gives a parametrization of the set $\mathcal{S}(\mathbf{H}_{2n+1})$ when the parameter $\{p, q\}$ varies in $\overline{\mathcal{S}}_m$ (the set of $\mathbb{C}^{m\times m}$-valued Stieltjes pairs) and is of the form*

$$p(z) = V \begin{pmatrix} \hat{p}(z) & & \\ & 0_\mu & \\ & & I_\nu \end{pmatrix} \qquad \text{and} \qquad q(z) = V \begin{pmatrix} \hat{q}(z) & & \\ & I_\mu & \\ & & 0_\nu \end{pmatrix} \tag{5.7}$$

with a unitary matrix V (which depends only on the initial data $\mathbf{H}_{2n+1}$) and a Stieltjes pair $\{\hat{p}, \hat{q}\} \in \overline{\mathcal{S}}_{m-\mu-\nu}$ where

$$\mu = \operatorname{rank} P_{\operatorname{Ker} K}\, \mathbf{e} \qquad \text{and} \qquad \nu = \operatorname{rank} P_{\operatorname{Ker} \widetilde{K}} K\mathbf{e}. \tag{5.8}$$

More precisely: every function $s \in \mathcal{S}(\mathbf{H}_{2n+1})$ is of the form (1.10) *for some pair $\{p, q\} \in \overline{\mathcal{S}}_m$ of the form* (5.7). *Conversely, for every pair $\{p, q\} \in \overline{\mathcal{S}}_m$ of the form* (5.7) *the transformation* (1.10) *is well-defined* ($\det(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \not\equiv 0$) *and leads to some $s \in \mathcal{S}(\mathbf{H}_{2n+1})$. Two pairs lead by* (1.10) *to the same function s if and only if these pairs are equivalent.*

Proof: According to Theorem 1.2 the set $\mathcal{S}(\mathbf{H}_{2n+1})$ coincides with the set of all solutions of the system of inequalities (1.12) and (1.13) which is equivalent, by Lemma 2.4, to the following system:

$$\begin{aligned} &\frac{s(z) - s(z)^*}{z - \overline{z}} \\ &\qquad - \left(\mathbf{e}s(z) + FK\mathbf{e}\right)^* (I - zF)^{-*} K^{[-1]} (I - zF)^{-1} \left(\mathbf{e}s(z) + FK\mathbf{e}\right) \geq 0 \\ &\frac{zs(z) - \overline{z}s(z)^*}{z - \overline{z}} \\ &\qquad - \left(z\mathbf{e}s(z) + K\mathbf{e}\right)^* (I - zF)^{-*} \widetilde{K}^{[-1]} (I - zF)^{-1} \left(\overline{z}\mathbf{e}s(z) + K\mathbf{e}\right) \geq 0 \end{aligned} \tag{5.9}$$

and

$$P_{\mathrm{Ker}\,K}(I-zF)^{-1}(\mathbf{e},FK\mathbf{e})\begin{pmatrix} s(z) \\ I \end{pmatrix} \equiv 0 \tag{5.10}$$

$$P_{\mathrm{Ker}\,\widetilde{K}}(I-zF)^{-1}(\mathbf{e},K\mathbf{e})\begin{pmatrix} zs(z) \\ I \end{pmatrix} \equiv 0. \tag{5.11}$$

It is easy to see that inequalities (5.9) can be written as

$$\left(s(z)^*, I\right)\left\{\frac{J}{i(\bar{z}-z)} - \begin{pmatrix} \mathbf{e}^* \\ \mathbf{e}^*KF^* \end{pmatrix}\right.$$
$$\left.(I-zF^*)^{-1}K^{[-1]}(I-zF)^{-1}(\mathbf{e},FK\mathbf{e})\right\}\begin{pmatrix} s(z) \\ I \end{pmatrix} \geq 0$$

$$\left(\bar{z}s(z)^*, I\right)\left\{\frac{J}{i(\bar{z}-z)} - \begin{pmatrix} \mathbf{e}^* \\ \mathbf{e}^*K \end{pmatrix}\right.$$
$$\left.(I-zF^*)^{-1}\widetilde{K}^{[-1]}(I-zF)^{-1}(\mathbf{e},K\mathbf{e})\right\}\begin{pmatrix} zs(z) \\ I \end{pmatrix} \geq 0$$

which in turn, on account of (4.17) and (4.18), can be represented in the form (5.1) and (5.2) with the function Θ defined by (4.5) which is of the class $\mathbf{W}_\pi$ by Lemma 4.1. According to Theorem 5.1, all solutions s of the system (5.9) are parametrized by the linear fractional transformation (1.10) when the parameter $\{p,q\}$ varies in the set $\overline{\mathcal{S}}_m$ of all Stieltjes pairs and satisfies (5.3). It remains to choose among these solutions all functions s which satisfy also identities (5.10) and (5.11). The further proof is divided into three steps which we now detail.

Step 1: *The function s of the form* (1.10) *satisfies the identities* (5.10) *and* (5.11) *if and only if the corresponding parameter $\{p,q\}$ satisfies*

$$P_{\mathrm{Ker}\,K}\,\mathbf{e}p(z) \equiv P_{\mathrm{Ker}\,\widetilde{K}}K\mathbf{e}q(z) \equiv 0. \tag{5.12}$$

Step 2: *If a pair $\{p,q\} \in \overline{\mathcal{S}}_m$ satisfies* (5.12), *then it also satisfies* (5.3).

Step 3: *There exists a unitary matrix $V \in \mathbb{C}^{m\times m}$ such that every pair $\{p,q\} \in \overline{\mathcal{S}}_m$ satisfying* (5.12) *is equivalent to some pair $\{p_1,q_1\}$ of the form* (5.6) *with μ and ν defined by* (5.8).

Proof of Step 1: Let s be of the form (1.10) for some pair $\{p,q\} \in \overline{\mathcal{S}}_m$ which satisfies the condition (5.3). Then

$$\begin{pmatrix} s(z) \\ I \end{pmatrix} = \Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}\left(\theta_{21}(z)p(z)+\theta_{22}(z)q(z)\right)^{-1}$$

and, in view of (4.6), the identities (5.10) and (5.11) are equivalent to

$$P_{\mathrm{Ker}\,K}(I-zF)^{-1}(\mathbf{e},FK\mathbf{e})\Theta(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0 \tag{5.13}$$

$$P_{\mathrm{Ker}\,\widetilde{K}}(I-zF)^{-1}(\mathbf{e},K\mathbf{e})\widetilde{\Theta}(z)P(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0, \tag{5.14}$$

respectively. To simplify (5.13) and (5.14) we begin with identities

$$\begin{aligned} z(I-zF)^{-1}\{\mathbf{e}\mathbf{e}^*KF^*-FK\mathbf{e}\mathbf{e}^*\}(I-zF^*)^{-1} &\\ = K(I-zF^*)^{-1}-(I-zF)^{-1}K &\\ z(I-zF)^{-1}\{\mathbf{e}\mathbf{e}^*K-K\mathbf{e}\mathbf{e}^*\}(I-zF^*)^{-1} &\\ = \widetilde{K}(I-zF^*)^{-1}-(I-zF)^{-1}\widetilde{K} & \end{aligned} \tag{5.15}$$

which follow from (4.10). Substituting (4.11) and (4.12) into (5.13) and (5.14), respectively, and using (5.15), we rewrite (5.13) and (5.14) as

$$P_{\operatorname{Ker} K}(I-zF)^{-1}(I-KK^{[-1]})R\begin{pmatrix} p(z)\\ q(z)\end{pmatrix}\equiv 0 \tag{5.16}$$

$$P_{\operatorname{Ker}\widetilde{K}}(I-zF)^{-1}(I-\widetilde{K}\widetilde{K}^{[-1]})\widetilde{R}P(z)\begin{pmatrix} p(z)\\ q(z)\end{pmatrix}\equiv 0, \tag{5.17}$$

respectively, where R and $\widetilde{R}$ are matrices given by (4.22). Using (2.4) one can rewrite (5.16) as

$$\Big(I+zP_{\operatorname{Ker} K}F(I-zF)^{-1}(I-KK^{[-1]})\Big)P_{\operatorname{Ker} K}R\begin{pmatrix} p(z)\\ q(z)\end{pmatrix}\equiv 0$$

and since

$$\det\Big(I+zP_{\operatorname{Ker} K}F(I-zF)^{-1}(I-KK^{[-1]})\Big)\not\equiv 0,$$

then (5.16) is equivalent to

$$P_{\operatorname{Ker} K}R\begin{pmatrix} p(z)\\ q(z)\end{pmatrix}\equiv 0. \tag{5.18}$$

Similarly, the identity (5.17) is equivalent to

$$P_{\operatorname{Ker}\widetilde{K}}\widetilde{R}P(z)\begin{pmatrix} p(z)\\ q(z)\end{pmatrix}\equiv 0. \tag{5.19}$$

Substituting (4.25) into (5.18) and (4.26), (1.9) into (5.19) we get

$$P_{\operatorname{Ker} K}\Big(\mathbf{e}p(z)+F(I-\widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}q(z)\Big)\equiv 0 \tag{5.20}$$

$$P_{\operatorname{Ker}\widetilde{K}}\Big(z(I-KK^{[-1]})\mathbf{e}p(z)+K\mathbf{e}q(z)\Big)\equiv 0. \tag{5.21}$$

Multiplying (5.20) by $zP_{\operatorname{Ker}\widetilde{K}}(I-KK^{[-1]})$ from the left, subtracting the obtained identity from (5.21) and using the assumption $\operatorname{Ker} K\subseteq\operatorname{Ker}\widetilde{K}$, we obtain

$$\Big\{I-zP_{\operatorname{Ker} K}F(I-\widetilde{K}\widetilde{K}^{[-1]})\Big\}P_{\operatorname{Ker}\widetilde{K}}K\mathbf{e}q(z)\equiv 0$$

which is equivalent to

$$P_{\operatorname{Ker}\widetilde{K}}K\mathbf{e}q(z)\equiv 0. \tag{5.22}$$

Substituting (5.22) into (5.20) we conclude that $P_{\mathrm{Ker}\,K}\mathbf{e}p(z) \equiv 0$, which ends the proof of Step 1.

Proof of Step 2: Let a pair $\{p, q\} \in \overline{S}_m$ satisfy conditions (5.12). We introduce a pair $\{x, y\}$ by

$$\begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \tag{5.23}$$

and show that $\det y(z) \neq 0$. Indeed, suppose that the point $\lambda \in \mathbb{C}_+$ and the non-zero vector $h \in \mathbb{C}^{m\times 1}$ are such that $\det \Theta(\lambda) \neq 0$ and

$$y(\lambda)h = 0. \tag{5.24}$$

Since

$$h^*(p(\lambda)^*, q(\lambda)^*)\Theta(\lambda)^* J\Theta(\lambda) \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h = h^*(x(\lambda)^*, 0) J \begin{pmatrix} x(\lambda) \\ 0 \end{pmatrix} h = 0,$$

then

$$\begin{aligned} 0 &\leq h^*(p(\lambda)^*, q(\lambda)^*) J \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h \\ &= h^*(p(\lambda)^*, q(\lambda)^*)\{J - \Theta(\lambda)^* J \Theta(\lambda)\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h \\ &\leq 0. \end{aligned}$$

Substituting (4.20) (with $z = \omega = \lambda$) into the last inequality we conclude that

$$K(I - \lambda F^*)^{-1} K^{[-1]} R \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h = 0 \tag{5.25}$$

(where R is the matrix given by (4.22)). It follows from (4.11), (4.22) and (5.24) that

$$\begin{aligned} x(\lambda)h &= (I_m, O_m)\Theta(\lambda) \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h \\ &= \left\{(I, \mathbf{e}^* K \tilde{K}^{[-1]} K\mathbf{e}) + \lambda \mathbf{e}^* K F^* (I - \lambda F^*)^{-1} K^{[-1]} R\right\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h. \end{aligned}$$

Since

$$\lambda F^*(I - \lambda F^*)^{-1} = (I - \lambda F^*)^{-1} - I,$$

then on account of (5.25)

$$x(\lambda)h = \left\{(I, \mathbf{e}^* K \tilde{K}^{[-1]} K\mathbf{e}) - \mathbf{e}^* K K^{[-1]} R\right\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h. \tag{5.26}$$

The equalities

$$K^{[-1]} F (I - \tilde{K}\tilde{K}^{[-1]}) = 0 \qquad \text{and} \qquad K^{[-1]} K \tilde{K}^{[-1]} = \tilde{K}^{[-1]}$$

(see (4.5)) both with (5.23) lead to

$$KK^{[-1]}R = \left(KK^{[-1]}\mathbf{e}, K\widetilde{K}^{[-1]}K\mathbf{e}\right).$$

Substituting this last equality into (5.26) and using (2.4) we obtain

$$\begin{aligned} x(\lambda)h &= \left\{\mathbf{e}^*(I - KK^{[-1]})\mathbf{e}, 0\right\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h \\ &= \mathbf{e}^*(I - KK^{[-1]})P_{\mathrm{Ker}\,K}\mathbf{e}p(\lambda)h \\ &= 0. \end{aligned}$$

Since $\det\Theta(\lambda) \neq 0$, the equality $x(\lambda)h = 0$ both with (5.24) and (5.23) implies

$$\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h = \Theta^{-1}(\lambda) \begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix} h = 0$$

which contradicts (since λ is an arbitrary point in $\mathbb{C}_+$) the non-degeneracy of the Stieltjes pair $\{p, q\}$.

Proof of Step 3: Let us consider the subspaces $\mathcal{F}$ and $\mathcal{G}$ defined by

$$\mathcal{F} = \mathrm{Ran}\,(P_{\mathrm{Ker}\,K}\,\mathbf{e}) \qquad \text{and} \qquad \mathcal{G} = \mathrm{Ran}\left(P_{\mathrm{Ker}\,\widetilde{K}}K\mathbf{e}\right). \tag{5.27}$$

For such a choice of $\mathcal{F}$ and $\mathcal{G}$ the equalities (5.4) and (5.5) are equivalent to (5.8) and (5.12), respectively. Meaning to apply Lemma 5.2 we show that the subspaces $\mathcal{F}$ and $\mathcal{G}$ in (5.27) are orthogonal. Indeed, let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then $f = \tilde{f}\mathbf{e}$ and $g = \tilde{g}K\mathbf{e}$ for some vectors $\tilde{f} \in \mathrm{Ker}\,K$ and $\tilde{g} \in \mathrm{Ker}\,\widetilde{K}$. Using (4.9) we obtain

$$fg^* = \tilde{f}\mathbf{e}\mathbf{e}^*K\tilde{g}^* = \tilde{f}K\tilde{g}^* - \tilde{f}(I - \mathbf{e}\mathbf{e}^*)K\tilde{g}^* = \tilde{f}K\tilde{g}^* - \tilde{f}F\widetilde{K}\tilde{g}^* = 0.$$

The application of Lemma 5.2 to subspaces (5.27) finishes the proof of Step 3.

By Theorem 5.1, different pairs lead under the transformation (1.10) to the same s if and only if they are equivalent. Hence, instead of all Stieltjes pairs satisfying (5.12) we can substitute in (1.10) all pairs of the form (5.7). This ends the proof of Theorem 5.3 ∎

In view of Remark 2.8, the condition $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ is not restrictive and hence, the description obtained in Theorem 5.3 is applicable to the general situation $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n$.

6. Even Stieltjes moment problem

As it was mentioned above the Stieltjes moment problem for $N = 2n$ is solvable if and only if the *information* matrices K_n and $\widetilde{K}_{n-1}$ defined by

$$K_n = (s_{i+j})_{i,j=0}^{n} \qquad \text{and} \qquad \widetilde{K}_{n-1} = (s_{i+j+1})_{i,j=0}^{n-1} \tag{6.1}$$

are non-negative. Similarly to the odd case we denote by $\widetilde{\mathcal{K}}_n$ the set of all such pairs $\{K_n, \widetilde{K}_{n-1}\}$. If, moreover, K_{n-1} admits a non-negative Hankel extension (i.e. if there exists a matrix $s_{2n+1} \in \mathbb{C}^{m\times m}$ such that the block matrix $\widetilde{K}_n = (s_{i+j+1})_{i,j=0}^{n}$ is still non-negative), then we say that the pair $\{K_n, \widetilde{K}_{n-1}\}$ belongs to $\widetilde{\mathcal{K}}_n^+ \subset \widetilde{\mathcal{K}}_n$.

It can be easily checked that matrices K_n and $\widetilde{K}_{n-1}$ are of structure (6.1) if and only if

$$(I - \mathbf{e}\mathbf{e}^*)K_n B^* - C\widetilde{K}_{n-1} = 0 \tag{6.2}$$

where

$$\mathbf{e} = \begin{pmatrix} I_m \\ 0_{mn\times m} \end{pmatrix}, \qquad B = \begin{pmatrix} 0_{m\times mn} \\ I_{mn} \end{pmatrix}, \qquad C = (I_{mn}, 0_{m\times mn}) \tag{6.3}$$

(an "even" analogue of Lemma 2.1). Note that $F_{m,n} = CB$, where $F_{m,n}$ is a shift given by (1.14).

We ommit proofs of the following lemmas which are obtained closely to the corresponding results from Sections 2 - 5.

Lemma 6.2: *Let $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n$ and let $\widetilde{\mathcal{L}}$ be the subspace of $\mathbb{C}^{1\times m}$ defined as*

$$\widetilde{\mathcal{L}} = \left\{ f \in \mathbb{C}^{1\times m} \,\middle|\, \begin{array}{l} (f_1, \ldots, f_{n-1}, f) \in \operatorname{Ker} \widetilde{K}_{n-1} \\ \text{for some } f_1, \ldots, f_{n-1} \in \mathbb{C}^{1\times m} \end{array} \right\}. \tag{6.4}$$

Then the following statements are equivalent:

(i) $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n^+$.

(ii) $\mathbf{P}_{\operatorname{Ker} \widetilde{K}_{n-1}} (s_{n+1}, \ldots, s_{2n+1})^{\top} = 0$ (6.5)

(iii) *The block s_{2n} is of the form*

$$s_{2n} = (s_n, \ldots, s_{2n-1})K_{n-1}^{[-1]}(s_n, \ldots, s_{2n-1})^* + \widetilde{L} \tag{6.6}$$

for some non-negative matrix $\widetilde{L} \in \mathbb{C}^{m\times m}$ which vanishes on the subspace $\mathcal{L}$ and does not depend on the choice of $K_{n-1}^{[-1]}$.

(iv) *There exists a measure $d\sigma(\lambda) \geq 0$ such that*

$$\int_0^\infty \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \ldots, 2n).$$

Lemma 6.3: *Let $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n$, let s_{2n} be of the form* (6.6), *let $\widetilde{\mathcal{L}}$ be the subspace defined by* (6.4) *and let s be an arbitrary $(m \times m)$-matrix. Then*

$$s = \int_0^\infty \lambda^{2n} d\sigma(\lambda)$$

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n})$ if and only if s admits a representation

$$s = \left(s_n, \ldots, s_{2n-1}\right) K_{n-1}^{[-1]} \left(s_n, \ldots, s_{2n-1}\right)^* + \widetilde{L}_0 \tag{6.7}$$

for some non-negative matrix $\widetilde{L}_0 \in \mathbb{C}^{m\times m}$ such that $\widetilde{L}_0 \leq \widetilde{L}$ and $\widetilde{L}_0|_{\widetilde{\mathcal{L}}} = 0$.

If, moreover, $\widetilde{L}_0|_{\widetilde{\mathcal{L}}^\perp} = \widetilde{L}|_{\widetilde{\mathcal{L}}^\perp}$, then the Stieltjes moment problems associated with the sets

$$\mathbf{H}_{2n} = \left\{s_0, \ldots, s_{2n-1}, s_{2n}\right\} \qquad \text{and} \qquad \mathbf{H}_{2n}^1 = \left\{s_0, \ldots, s_{2n-1}, s\right\}$$

have the same solutions: $\mathcal{Z}(\mathbf{H}_{2n}) = \mathcal{Z}(\mathbf{H}_{2n}^1)$.

Lemma 6.4: *Let $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n^+$, let* rank $K_n = r$ *and* rank $\widetilde{K}_{n-1} = \widetilde{r}$, *and let B and C be matrices given by* (6.3). *Then there exist matrices $Q \in \mathbb{C}^{r\times m(n+1)}$ and $\widetilde{Q} \in \mathbb{C}^{\widetilde{r}\times mn}$ such that*

$$QK_nQ^* > 0 \qquad \text{and} \qquad \widetilde{Q}\widetilde{K}_{n-1}\widetilde{Q}^* > 0 \tag{6.8}$$

$$QC = N\widetilde{Q} \qquad \text{and} \qquad \widetilde{Q}B = \widetilde{N}Q \tag{6.9}$$

for some matrices $N \in \mathbb{C}^{r\times\widetilde{r}}$ and $\widetilde{N} \in \mathbb{C}^{\widetilde{r}\times r}$. In other words, there exist subspaces $\mathcal{Q} = \operatorname{Ran} Q$ and $\widetilde{\mathcal{Q}} = \operatorname{Ran}\widetilde{Q}$ such that $\mathcal{Q}C \subseteq \widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{Q}}B \subseteq \mathcal{Q}$, and

$$\mathbb{C}^{m(n+1)} = \operatorname{Ker} K_n \dot{+} \mathcal{Q} \qquad \text{and} \qquad \mathbb{C}^{mn} = \operatorname{Ker}\widetilde{K}_{n-1} \dot{+} \widetilde{\mathcal{Q}}. \tag{6.10}$$

Lemma 6.5 : *Let $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n^+$, let Q and $\widetilde{Q}$ be matrices satisfying conditions* (6.8) *and* (6.9), *let $K_n^{[-1]}$ and $\widetilde{K}_{n-1}^{[-1]}$ be pseudoinverse matrices defined as*

$$K_n^{[-1]} = Q^*(QK_nQ^*)^{-1}Q \qquad \text{and} \qquad \widetilde{K}_{n-1}^{[-1]} = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}_{n-1}\widetilde{Q}^*)^{-1}\widetilde{Q}$$

and let $\Psi \in \mathbb{C}^{2m\times 2m}$ be the J-unitary matrix given by

$$\begin{aligned}\Psi &= \begin{pmatrix} I_m & \mathbf{e}^* K_n B^* \widetilde{K}_{n-1}^{[-1]} B K_n \mathbf{e} \\ 0 & I_m \end{pmatrix} \\ &= \begin{pmatrix} I_m & \left(s_0, \ldots, s_{n-1}\right)\widetilde{K}_{n-1}^{[-1]}\left(s_0, \ldots, s_{n-1}\right)^\top \\ 0 & I_m \end{pmatrix}.\end{aligned}$$

Then the $\mathbb{C}^{2m\times 2m}$-valued function

$$\Theta(z) = \left\{ I_{2m} + z \begin{pmatrix} \mathbf{e}^* K_n F_{m,n}^* \\ -\mathbf{e}^* \end{pmatrix} (I - zF_{m,n}^*)^{-1} K_n^{[-1]} (\mathbf{e}, F_{m,n}K_n\mathbf{e}) \right\} \Psi \tag{6.11}$$

is of class $\mathbf{W}_\pi$.

Theorem 6.6: *Let* $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n^+$ *(or equivalently, let* $\mathbf{H}_{2n} \in \mathcal{H}^+$*). Then the linear fractional transformation* (1.10) *with the resolvent matrix* Θ *defined by* (6.10) *gives a parametrization of the set* $\mathcal{S}(\mathbf{H}_{2n})$ *when the parameter* $\{p,q\}$ *varies in* $\overline{\mathcal{S}}_m$ *(the set of* $\mathbb{C}^{m\times m}$*-valued Stieltjes pairs) and is of the form* (5.6) *with a unitary matrix* V *(which depends only on the initial data* $\mathbf{H}_{2n}$*) and a Stieltjes pair* $\{\hat{p},\hat{q}\} \in \overline{\mathcal{S}}_{m-\mu-\nu}$ *where*

$$\mu = \operatorname{rank} P_{\operatorname{Ker} K_n} \mathrm{e} = \operatorname{rank} (I_m, 0, \ldots, 0) P_{\operatorname{Ker} K} \tag{6.12}$$

$$\nu = \operatorname{rank} P_{\operatorname{Ker} \widetilde{K}_{n-1}} B K_n \mathrm{e} = \operatorname{rank} (s_0, \ldots, s_{n-1}) P_{\operatorname{Ker} \widetilde{K}}. \tag{6.13}$$

In conclusion we note that Lemmas 1.6 and 1.7 follows immediately from Lemmas 2.7, 6.3 and 2.5, 6.2, respectively. Theorem 1.5 is a consequence of Theorems 5.3 and 6.6. Distinction of parameters in (1.11) and (5.6) is not essential: the linear fractional transformation with the resolvent matrix $\Theta \in \mathbf{W}_\pi$ and parameters $\{p,q\}$ of the form (5.6) is equivalent to the linear fractional transformation with the resolvent matrix

$$\hat{\Theta}(z) = \Theta(z) \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$$

and parameters of the form (1.11). Since the matrix V is unitary, it is easy to see that the function $\hat{\Theta}$ is still of the class $\mathbf{W}_\pi$.

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