Degenerate Stieltjes Moment Problem and Associated J-Inner Polynomials

V. Bolotnikov

Abstract. In this paper we consider the truncated Stieltjes matrix moment problem when the so-called information matrices are degenerate and describe the set of all solutions in terms of the linear fractional transformation using the fundamental matrix inequality method.

Keywords: Moment problems, J— expansive matrix functions, *fundamental matrix inequalities* AMS subject classification: Primary 47A57, secondary 30E05

1. Introduction

The objective of this paper is to describe the solutions of a degenerate Stielties matrix moment problem. We begin with an ordered set of hermitian non-negative matrices $s_0, \ldots, s_N \in \mathbb{C}^{m \times m}$ So cance interimental matrices are degenerate and describe the set of a
the linear fractional transformation using the fundamental matrix inequences.
 Keywords: Moment problems, J-expansive matrix functions, fundament
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$$
\mathbf{H}_N = \{s_0, \ldots, s_N\} \tag{1.1}
$$

(by $\mathbb{C}^{m \times \ell}$ we denote the space of $m \times \ell$ matrices with complex entries, and, throughout the paper I_m stands for the identity matrix of the order m). Let *K* and \widetilde{K} denote the associated Hankel block matrices *l l(N-1)/2) l(N-1)/2) l(N-1)/2) l(N-2)* *****l(N-1)/2) l(N-1)/2) l(N-1)/2)*

$$
K = (s_{i+j})_{i,j=0}^{\lfloor N/2 \rfloor} \quad \text{and} \quad \widetilde{K} = (s_{i+j+1})_{i,j=0}^{\lfloor (N-1)/2 \rfloor}.
$$

Definition 1.1: We say that H_N belongs to H if the associated matrices K and \tilde{K} are non-negative. If, moreover, H_N admits a non-negative extension (i.e. if there exists a matrix $s_{N+1} \in \mathbb{C}^{m \times m}$ such that the extended block matrices $(s_{i+j})_{i,j=0}^{\left[(N+1)/2\right]}$ and $(s_{i+j+1})_{i,j=0}^{\lfloor N/2 \rfloor}$ are still non-negative), then H_N is said to be in \mathcal{H}^+ . *z* matrix of the order *m*). Let *K* and \tilde{K} denote the

and $\tilde{K} = (s_{i+j+1})_{i,j=0}^{\lfloor (N-1)/2 \rfloor}$
 H_N belongs to *H* if the associated matrices *K* and
 H_N admits a non-negative extension (i.e. if there

the th

Let $Z(H_N)$ denote the set of all solutions of the associated truncated Stieltjes moment problem, i.e. the set of non-decreasing right-continuous $\mathbb{C}^{m \times m}$ -valued functions $\sigma(\lambda)$ such that

$$
\mathbb{C}^{m \times m} \text{ such that the extended block matrices } (s_{i+j})_{i,j=0}^{\lfloor (N+1)/2 \rfloor}
$$

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the set of all solutions of the associated truncated Stieltjes
e set of non-decreasing right-continuous $\mathbb{C}^{m \times m}$ -valued functions

$$
\int_{0}^{\infty} \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, ..., N - 1) \qquad (1.2)
$$

$$
\int_{0}^{\infty} \lambda^N d\sigma(\lambda) \leq s_N. \qquad (1.3)
$$

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$$
\int_{0}^{\infty} \lambda^{N} d\sigma(\lambda) \leq s_{N}.
$$
\n(1.3)

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As in the scalar case (see, e.g., [16: §5.1]) $\mathcal{Z}(\mathbf{H}_N)$ is non-empty if and only if $\mathbf{H}_N \in \mathcal{H}$. Moreover, there is a one-to-one correspondence between $\mathcal{Z}(\mathbf{H}_N)$ and the class $\mathcal{S}(\mathbf{H}_N)$ of $\mathbb{C}^{m \times m}$ -valued functions $s = s(z)$ analytic in $\mathbb{C} \backslash \mathbb{R}_+$ (where $\mathbb{R}_+ = [0, +\infty)$) and such that **z** ase (see, e.g., [16: §5.1]) $Z(\mathbf{H}_N)$ is non-empty if and only if $\mathbf{H}_N \in \mathcal{H}$.
 s a one-to-one correspondence between $Z(\mathbf{H}_N)$ and the class $S(\mathbf{H}_N)$

unctions $s = s(z)$ analytic in $\mathbb{C}\setminus\mathbb{R}_+$ (wh

$$
\Im s(z) \ge 0 \quad (\Im z \ge 0) \qquad \text{and} \qquad s(x) \ge 0 \quad (x < 0) \tag{1.4}
$$

Therefore, there is a one-to-one consequence between
$$
\Sigma(11N)
$$
 and the condition $s = s(z)$ analytic in $\mathbb{C}\setminus\mathbb{R}_+$ (where $\mathbb{R}_+ = [0, +\infty)$ that

\n $\Im s(z) \geq 0$ $(\Im z \geq 0)$ and $s(x) \geq 0$ $(x < 0)$

\nand, uniformly in the sector $S_{\epsilon} = \{z = \rho e^{i\theta} \in \mathbb{C} : \epsilon \leq \theta \leq \pi - \epsilon\}$ $(\epsilon > 0)$,

\n $\lim_{z \to \infty} \left\{ z^{N+1} s(z) + \sum_{k=0}^{N} s_k z^{N-k} \right\} \geq 0.$

\nIndeed if $K \tilde{K} > 0$ and $\sigma \in \mathcal{Z}(H_N)$, then the function

Indeed, if K, $\widetilde{K} \geq 0$ and $\sigma \in \mathcal{Z}(\mathbf{H}_N)$, then the function

$$
s(z) = \int_{0}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z} \tag{1.5}
$$

belongs to $S(H_N)$ and conversely, every $s \in S(H_N)$ admits such a representation, where σ is obtained from s by the Stieltjes inversion formula (see [16: Appendix]). The parametrization of the set $S(H_N)$ in terms of the linear fractional transformation for the non-degenerate case $(K$ and K are both strictly positive) is given in [8]. *e* is obtained from *s* by the Stieltjes inversion formula (see [16: Appendix]). The setrization of the set $S(H_N)$ in terms of the linear fractional transformation for a-degenerate case (*K* and *K* are both strictly pos

We recall some necessary definitions.

Definition 1.2: A $\mathbb{C}^{2m \times 2m}$ -valued meromorphic function Θ is of the *class* \mathbf{W}_{π} if

$$
\Theta(z)J\Theta(z)^{*} = J \quad (z \in \mathbb{R}) \qquad \text{and} \qquad \Theta(z)J\Theta(z)^{*} \ge J \quad (z \in \mathbb{C}_{+}) \tag{1.6}
$$
\n
$$
\Theta(x)J_{\pi}\Theta(x)^{*} \ge J_{\pi} \qquad (x < 0)
$$

and

$$
\Theta(x)J_{\pi}\Theta(x)^{*}\geq J_{\pi} \qquad (x<0)
$$

where

For the set
$$
O(11N)
$$
 in terms of the mean factorian transformation for the case $(K \text{ and } K \text{ are both strictly positive})$ is given in [8].

\n1.2: A $\mathbb{C}^{2m \times 2m}$ -valued meromorphic function Θ is of the class \mathbf{W}_{π} if $f(x) = 0$ and $O(z)J\Theta(z)^{*} \geq J$ and $O(z)J\Theta(z)^{*} \geq J$ (where $z \in \mathbb{C}_{+}$)

\n1.3. $O(x)J_{\pi}\Theta(x)^{*} \geq J_{\pi}$ and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, I is a matrix of I and I and I and $J_{\pi} = \begin{pmatrix} 0 & I_{m} \\ I_{m} & 0 \end{pmatrix}$, $$

and Θ is of the *class* W if it satisfies only conditions (1.6).

The following theorem establishes the connection between the classes W and W_{π} .

Theorem 1.3 (see [8: §4]): *A* $\mathbb{C}^{2m \times 2m}$ -valued function Θ belongs to the class \mathbf{W}_{π} if and only if $\Theta(z)J\Theta(z)^{*} = J \quad (z \in I\!\!R)$ and

and $\Theta(x)J_{\pi}\Theta(x)$

where $J = \begin{pmatrix} 0 & iI_{m} \\ -iI_{m} & 0 \end{pmatrix}$

and Θ is of the *class* **W** if it satisfies on

The following theorem establishes therefore in 1.3 (see [8: §4]): $A \mathbb{C}^{$

$$
\Theta \in \mathbf{W} \qquad \text{and} \qquad \tilde{\Theta}(z) = P(z)\Theta(z)P(z)^{-1} \in \mathbf{W} \tag{1.8}
$$

where

$$
P(z) = \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}.
$$
 (1.9)

Definition 1.4: Let $\{p,q\}$ be a pair of $\mathbb{C}^{m \times m}$ -valued functions meromorphic in $\mathbb{C}\backslash\mathbb{R}_+$.

(1) {p, q} is called a *Stieltjes pair* if

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\n
$$
\text{(a)} \ \ \det\left(p(z)^*p(z) + q(z)^*q(z)\right) \not\equiv 0 \qquad \text{(non-degeneracy of the pair)}
$$

D
\n(a)
$$
\det(p(z)^*p(z) + q(z)^*q(z)) \neq 0
$$
 (no)
\n
$$
\left(\beta\right) \frac{q(z)^*p(z) - p(z)^*q(z)}{z - \overline{z}} \geq 0
$$

$$
\left(\gamma\right) \frac{zq(z)^*p(z) - \overline{z}p(z)^*q(z)}{z - \overline{z}} \geq 0
$$

$$
\left(\Im z \neq 0\right)
$$

\n(ii)
$$
\{p, q\}
$$
 is said to be equivalent to an
\nangled function 0 with $\det Q(x) \neq 0$ and

$$
(\gamma) \frac{zq(z)^*p(z)-\bar{z}p(z)^*q(z)}{z-\bar{z}}\geq 0 \qquad (\Im z\neq 0).
$$

(ii) $\{p,q\}$ is said to be *equivalent* to another pair $\{p_1,q_1\}$ if there exists a $\mathbb{C}^{m \times m}$. valued function Ω with det $\Omega(z) \neq 0$ and meromorphic in $\mathbb{C}\setminus\mathbb{R}_+$ such that $p_1 = p\Omega$ and $q_1 = q\Omega$.

The set of all Stieltjes pairs will be denoted by \overline{S}_m .

The degenerate scalar Stieltjes problem $(m = 1)$ is simple: $S(H_N)$ consists of the unique rational function $s = s(z)$. For the degenerate matrix case the description of $S(H_N)$ depends on the character of degeneracy of the *information matrices* K and K .

The main result of the paper is

Theorem 1.5: *The following statements are true.*

(i) All functions $s \in \mathcal{S}(\mathbf{H}_N)$ are given by the linear fractional transformation

functions
$$
s \in S(H_N)
$$
 are given by the linear fractional transformation
\n
$$
s(z) = \left(\theta_{11}(z)p(z) + \theta_{12}(z)q(z)\right) \left(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)\right)^{-1}
$$
\n(1.10)

form

(i) All functions
$$
s \in S(H_N)
$$
 are given by the linear fractional transformation
\n
$$
s(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z)) (\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1}
$$
\n(1.10)
\nwith the resolvent matrix $\Theta = (\theta_{ij})$ of class W_{π} and parameters $\{p, q\} \in \overline{S}_{m}$ of the form
\n
$$
p(z) = \begin{pmatrix} \hat{p}(z) \\ 0 \\ I_{\nu} \end{pmatrix}
$$
\nand\n
$$
\hat{q}(z) = \begin{pmatrix} \hat{q}(z) \\ I_{\mu} \\ 0_{\nu} \end{pmatrix}
$$
\n(1.11)
\nwhere
\n $\mu = \text{rank}(I_m, 0, \ldots, 0)P_{KerK}$ and $\nu = \text{rank}(s_0, \ldots, s_{[(N-1)/2]})P_{KerK}$.

where

 $\mu = \text{rank}(I_m, 0, \ldots, 0)P_{\text{Ker }K}$ and $\nu = \text{rank}(s_0, \ldots, s_{[(N-1)/2]})P_{\text{Ker }\widetilde{K}}$.
 Here $\text{P}_{\text{Ker }K}$ and $P_{\text{Ker }\widetilde{K}}$ denote the orthogonal projections onto the kernels of K and \widetilde{K} , respectively. *respectively.*

(ii) *Two pairs* $\{p, q\}$, $\{p_1, q_1\} \in \overline{S}_m$ *of the form* (1.11) are equivalent if and only if *they lead under the transformation* (1.10) *to the same function* **s.**

Note that the non-degenerate case corresponds to $\mu = \nu = 0$. Theorem 1.5 will be proved in Section 5 under the assumption $\mathbf{H}_N \in \mathcal{H}^+$. The general case can be reduced to this particular one in view of the following

Lemma 1.6: Let $H_N \in \mathcal{H}$. Then the last moment s_N can be perturbated in such a *way that the set*

$$
\mathbf{H'}_N = \{s_0,\ldots,s_{N-1},s_N'\}
$$

belongs to H^+ and the associated Stieltjes moment problems have the same solutions: $\mathcal{Z}(\mathbf{H}_N) = \mathcal{Z}(\mathbf{H'}_N)$ (or equivalently, $\mathcal{S}(\mathbf{H}_N) = \mathcal{S}(\mathbf{H'}_N)$).

It will be shown also that the function Θ is the matrix polynomial of deg $\Theta = [N/2] +$ 1 which admits a realization (not minimal, in general) $\Theta(z) = \Theta(0) + \tilde{A(I - zF)^{-1}B}$ with the state space $\mathbb{C}^{m[(N+2)/2]}$. To construct the resolvent matrix of the degenerate Stieltjes moment problem (which is a J -inner polynomial of the non-full rank, see, e.g., [5]) we follow the method of V. Dubovoj which was applied in the series [4] to the degenerate Schur problem. Using this method we obtain in Sections 3 and 6 some special decompositions (see (3.9) and (6.10)) of the state space which allow to construct the explicit formula for Θ for the case that *K* and \tilde{K} are not strictly positive (formulas (4.5) and (6.11)). In Section 2 we point out some peculiarities of the degenerate Stieltjes problem. For example, the condition $H_N \in \mathcal{H}$ (as against the non-degenerate case) does not ensure the existence of a measure $\sigma(\lambda)$ in $\mathcal{Z}(\mathbf{H}_N)$ such that in the inequality (1.3) the equality sign prevails.

Lemma 1.7: A measure $\sigma(\lambda)$ which satisfies (1.2) and $\int_0^\infty \lambda^N d\sigma(\lambda) = s_N$ exists if *and only if* $H_N \in \mathcal{H}^+$.

This fact *(as* well as the statement from Lemma 1.6) will be established separately for *N* odd and even in Sections 2 and 6, respectively.

The Stieltjes moment problem is in fact the interpolation problem in the class of matrix–valued Stieltjes functions (which by definition are analytic in $\mathbb{C}\backslash\mathbb{R}_+$ and satisfy (1.4); see, e.g., [8]) with the interpolating point at infiniy. It can presumably be solved using a number of approaches, e.g. reproducing kernels method (using this method, the moment problem on the whole axis was considered in [7]), methods based on operator theory [14, 151 or on realization of matrix-valued functions (such approach was applied in [10] to the interpolation problem with interpolating points from the upper half—plane). In this paper we follow the Potapov method of the fundamental matrix inequality (see *[4,* 5, 8 - 13]). The starting point is the following theorem which describes the class $\mathcal{S}(\mathbf{H}_N)$ in terms of a system of matrix inequalities. presumable
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Theorem 1.8 (see [8]): Let s be a $\mathbb{C}^{m \times m}$ *-valued function analytic in the upper half plane* \mathbb{C}_+ . Then *s belongs to* $S(H_N)$ *if and only if it satisfies the system of inequalities*

From problem with interpolating points from the upper man-plane.

\nNow the Potapov method of the fundamental matrix inequality (see starting point is the following theorem which describes the class a system of matrix inequalities.

\n(see [8]): Let
$$
s
$$
 be a $\mathbb{C}^{m \times m}$ -valued function analytic in the upper half elongs to $\mathcal{S}(\mathbf{H}_N)$ if and only if it satisfies the system of inequalities

\n
$$
\begin{pmatrix}\nK & (I - zF_{m,n})^{-1} \left(e \, s(z) + F_{m,n} K e \right) \\
\ast & \frac{s(z) - s(z)^*}{z - \overline{z}}\n\end{pmatrix}\n\geq 0
$$
\n(1.12)

\n
$$
\begin{pmatrix}\n\widetilde{K} & B\left(I - zF_{m,n}\right)^{-1} \left(z e \, s(z) + K e \right) \\
\ast & \frac{z s(z) - \overline{z} s(z)^*}{z - \overline{z}}\n\end{pmatrix}\n\geq 0
$$
\n(1.13)

$$
\begin{pmatrix}\n\widetilde{K} & B(I - zF_{m,n})^{-1}(ze\,s(z) + Ke) \\
* & \frac{zs(z) - \bar{z}s(z)^*}{z - \bar{z}}\n\end{pmatrix} \geq 0
$$
\n(1.13)

for $z \in \mathbb{C}_+$ *where*

Theorem 1.8 (see [8]): Let *s* be a
$$
\mathbb{C}^{m \times m}
$$
-valued function analytic in the upper half
ne C₊. Then *s* belongs to $S(H_N)$ if and only if it satisfies the system of inequalities

$$
\begin{pmatrix} K & (I - zF_{m,n})^{-1} (e s(z) + F_{m,n} K e) \\ * & \frac{s(z) - s(z)^*}{z - \overline{z}} \end{pmatrix} \geq 0
$$
(1.12)
$$
\begin{pmatrix} \widetilde{K} & B(I - zF_{m,n})^{-1} (ze s(z) + K e) \\ * & \frac{zs(z) - \overline{z} s(z)^*}{z - \overline{z}} \end{pmatrix} \geq 0
$$
(1.13)
$$
z \in \mathbb{C}_+
$$
 where

$$
F_{m,n} = \begin{pmatrix} 0_m & 0 \\ I_m & \\ 0 & I_m & 0_m \end{pmatrix} \in \mathbb{C}^{m(n+1) \times m(n+1)}
$$
 and $e = \begin{pmatrix} I_m \\ 0_m \\ \vdots \\ 0_m \end{pmatrix}$ (1.14)

and the matrix B is $I_{m(n+1)}$ *or* $(I_{m(n+1)}, 0_{m(n+1)\times m})$ *whenever* $N = 2n + 1$ *or* $N = 2n$, *respectively.*

Note that (1.12) itself is the fundamental matrix inequality of the Hamburger moment problem on *JR* (see, e.g, [13]).

For the non-degenerate case the difference between "even" and "odd" Stieltjes problems is not essential: they are particular cases of the much more general moment problem [2] and can be considered in a unified way. For the degenerate case it is not so: the difficulties which are arised due to different sizes of K and \tilde{K} for N even are essential. In Sections 2 - 5 we consider the Stieltjes problem with odd number of moments and postpone the "even" problem up to Section 6.

2. Some auxiliary lemmas

For $N = 2n + 1$ we have the moment conditions

The auxiliary lemmas
\n
$$
2n + 1
$$
 we have the moment conditions
\n
$$
\int_{0}^{\infty} \lambda^{k} d\sigma(\lambda) = s_{k} \quad (k = 0, ..., 2n) \quad \text{and} \quad \int_{0}^{\infty} \lambda^{2n+1} d\sigma(\lambda) \leq s_{2n+1}
$$

for prescribed non-negative matrices $s_i \in \mathbb{C}^{m \times m}$ $(i = 1, ..., 2n + 1)$. It sems to be more convenient to deal with associated Hankel block matrices instead of the set H_{2n+1} itself. So, we reformulate Definition 1.1 for this case as **Example 18 and the moment conditions**
 $\int_{0}^{\infty} \lambda^{2n+1} d\sigma(\lambda) \leq s_{2n+1}$
 A-negative matrices $s_i \in \mathbb{C}^{m \times m}$ $(i = 1, ..., 2n + 1)$. It sems to be

be deal with associated Hankel block matrices instead of the set H_{2n

Definition 2.1: A pair $\{K_n, \widetilde{K}_n\}$ is said to be in \mathcal{K}_n if K_n and \widetilde{K}_n are both non-negative and of the form

$$
K_n = (s_{i+j})_{i,j=0}^n \quad \text{and} \quad \widetilde{K}_n = (s_{i+j+1})_{i,j=0}^n \tag{2.1}
$$

for some square matrices s_i of the same size. If, moreover, K_n admits a non-negative Hankel extension (i.e. if there exists a matrix $s_{2n+2} \in \mathbb{C}^{m \times m}$ such that the block matrix $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ is still non-negative), then we say that the pair $\{K_n, \widetilde{K}_n\}$ belongs to $\mathcal{K}_n^+ \subset \mathcal{K}_n$.

We put the index *n* in (2.1) to shorten some impending computations with the Hankel block matrices of different sizes. The following proposition can be easily checked by a direct calculation.

Lemma 2.2: Let matrices $A, C \in \mathbb{C}^{m(n+1)\times m(n+1)}$ be non-negative and let $F_{m,n}$ be *the matrix of the m-dimensional shift in the space* $\mathbb{C}^{m(n+1)}$ given by (1.14). Then the *pair* $\{A, C\}$ *belongs to* K_n *if and only if a* is still non-negative), then we say that the pair $\{K_n, K_n\}$ belongs

lex *n* in (2.1) to shorten some impending computations with the

ces of different sizes. The following proposition can be easily checked

ion.
 K1 K **C** \in **C**^{*m*(*n*+1)×*m*(*n*+1) *be non-neg*
A, $C \in \mathbb{C}^{m(n+1)\times m(n+1)}$ *be non-neg*
al shift in the space $\mathbb{C}^{m(n+1)}$ *given*
d only if
F_{m,n} $(F_{m,n}^*A - C) = 0$.
ix K let Q be a matrix such that}

$$
F_{m,n}\left(F_{m,n}^*A-C\right)=0.
$$

Given a non-negative matrix *K* let *Q* be a matrix such that

We introduce the *pseudoinverse matrix* $K^{[-1]}$ by

$$
K^{[-1]} = Q^*(QKQ^*)^{-1}Q. \tag{2.3}
$$

The pseudoinverse matrix defined by (2.2), (2.3) depends on the choice of *Q,* nevertheless, some its properties are independent of this choice.

Lemma 2.3 (see [3]): For every choice of the pseudoinverse matrix $K^{[-1]}$,

\n- [3]: For every choice of the pseudoinverse matrix
$$
K^{[-1]}
$$
,
\n- $I - KK^{[-1]} = (I - KK^{[-1]}) P_{\text{Ker }K}$.
\n- [2.4]
\n

Lemma 2.4 (see [3]): *The block matrix*

$$
\begin{pmatrix} K & B \\ B^* & C \end{pmatrix}
$$

is non-negative if and only if

$$
\begin{pmatrix} K & B \\ B^* & C \end{pmatrix}
$$

if and only if
 $K \ge 0$, $P_{\text{Ker }K}B = 0$, $R := C - B^*K^{[-1]}B \ge 0$.

Moreover, if

$$
\begin{pmatrix} K & B \\ B^* & C \end{pmatrix} \geq 0,
$$

then the matrix R does not depend on the choice of $K^{[-1]}$.

This last lemma is a reformulation of the well known lemma about the non-negativity of a block matrix (see, e.g., [6: §01 and [*12: §4]).*

Lemma 2.5: Let $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n$ and let $\mathcal L$ be the subspace of $\mathbb C^{1 \times m}$ defined as

$$
(B^* \t C)^{\leq 0}
$$
\nen the matrix R does not depend on the choice of $K^{[-1]}$.

\nThis last lemma is a reformulation of the well known lemma about the non-negativity a block matrix (see, e.g., [6: §0] and [12: §4]).

\nLemma 2.5: Let $\{K_n, \tilde{K}_n\} \in \mathcal{K}_n$ and let C be the subspace of $\mathbb{C}^{1 \times m}$ defined as

\n
$$
\mathcal{L} = \left\{ f \in \mathbb{C}^{1 \times m} : (f_0, \ldots, f_{n-1}, f) \in \text{Ker } K_n \text{ for some } f_0, \ldots, f_{n-1} \in \mathbb{C}^{1 \times m} \right\}. \tag{2.5}
$$
\nthen the following statements are equivalent:

\n(i) $\{K_n, \tilde{K}_n\} \in \mathcal{K}_n^+$.

\n(ii) $\text{Ker } K_n \subseteq \text{Ker } \tilde{K}_n$.

\n(iii) The block s_{2n+1} is of the form

\n
$$
s_{2n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^+ + R \qquad (2.7)
$$
\nwhere non-negative matrix $R \in \mathbb{C}^{m \times m}$ which vanishes on the subspace C and does

\ndefined on the choice of $\tilde{K}^{[-1]}$ (according to (2.1) $\tilde{K}_{n-1} = (s_{1}, \ldots)^{n-1}$).

Then the following statements are equivalent:

(i) ${K_n, \widetilde{K}_n} \in \mathcal{K}_n^+$.

(ii) Ker
$$
K_n \subseteq \text{Ker } \widetilde{K}_n
$$
.

(iii) The block s_{2n+1} is of the form

$$
s_{2n+1} = (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^+ + R \qquad (2.7)
$$

for some non-negative matrix $R \in \mathbb{C}^{m \times m}$ which vanishes on the subspace $\mathcal L$ and does Then the following statements are equivalent:

(i) $\{K_n, \tilde{K}_n\} \in \mathcal{K}_n^+$.

(ii) $\text{Ker } K_n \subseteq \text{Ker } \tilde{K}_n$.

(iii) $\text{The block } s_{2n+1}$ is of the form
 $s_{2n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]}(s_{n+1}, \ldots, s_{2n})^+ + R$

for som

(iv) There exists a measure $d\sigma(\lambda) \geq 0$ such that

$$
\widetilde{K}_{n}.
$$
\n
$$
(2.6)
$$
\n
$$
1 \text{ is of the form}
$$
\n
$$
= (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]}(s_{n+1}, \ldots, s_{2n})^* + R \qquad (2.7)
$$
\n
$$
\text{matrix } R \in \mathbb{C}^{m \times m} \text{ which vanishes on the subspace } \mathcal{L} \text{ and does}
$$
\n
$$
\text{we of } \widetilde{K}_{n-1}^{[-1]} \text{ (according to (2.1), } \widetilde{K}_{n-1} = (s_{i+j+1})_{i,j=0}^{n-1}.
$$
\n
$$
\text{measure } d\sigma(\lambda) \ge 0 \text{ such that}
$$
\n
$$
\int_{0}^{\infty} \lambda^k d\sigma(\lambda) = s_k \qquad (k = 0, \ldots, 2n + 1).
$$
\n
$$
\text{ation (i)} \Rightarrow \text{(ii). Let } K_{n+1} = (s_{i+j})_{i,j=0}^{n+1} \text{ be a non-negative Ban-}
$$

Proof: The implication (i) \Rightarrow (ii). Let $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ be a non-negative Hankel extension of K_n . From the non-negativity of K_{n+1} we receive

$$
t \in \mathbb{C}
$$
 which vanishes on the subspace L and does
\n
$$
t = 1
$$
\n
$$
d\sigma(\lambda) \ge 0 \text{ such that}
$$
\n
$$
\lambda) = s_k \qquad (k = 0, ..., 2n + 1).
$$
\n(2.8)\n
$$
\Rightarrow (\text{ii}). \text{ Let } K_{n+1} = (s_{i+j})_{i,j=0}^{n+1} \text{ be a non-negative Han-\nnon-negativity of } K_{n+1} \text{ we receive}
$$
\n
$$
P_{\text{Ker } K_n} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n+1} \end{pmatrix} = 0
$$
\n(2.9)

which in view of (2.1) is equivalent to (2.6) .

The implication (ii) \Rightarrow (i). Using Lemma 2.3 and taking into account (2.6) (or (2.9)) we conclude that $K_{n+1} \geq 0$ if and only if

$$
s_{2n+2}-(s_{n+1},\ldots,s_{2n+1})K_n^{[-1]}(s_{n+1},\ldots,s_{2n+1})^*\geq 0.
$$

Thus, every choice of s_{2n+2} satisfying this last inequality leads to $K_{n+1} \geq 0$ and therefore, $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$.

The equivalence (ii) \Leftrightarrow (iii). Since $\widetilde{K}_n \geq 0$, then by Lemma 2.3,

$$
s_{2n+1}-(s_{n+1},\ldots,s_{2n})\widetilde{K}_{n-1}^{(-1)}(s_{n+1},\ldots,s_{2n})^*\geq 0
$$

and hence, s_{2n+1} admits a representation (2.7) for some $R \geq 0$. It remains to show that (2.6) holds if and only if *R* vanishes on the subspace $\mathcal L$ defined by (2.5). Let (f_0,\ldots,f_n) be an arbitrary vector from Ker K_n . Then in particular $\begin{array}{l} \n\text{A} \text{A} \text{A} \text{A} \text{B} \text{A} \text{B} \text{B} \text{B} \text{B} \text{C} \text{C} \text{C} \text{D} \text{D} \text{A} \$

$$
-(s_{n+1},...,s_{2n})\widetilde{K}_{n-1}^{[-1]}(s_{n+1},...,s_{2n})
$$

\nis a representation (2.7) for some $R \ge 0$.
\nR vanishes on the subspace L defined b
\nrom Ker K_n . Then in particular
\n
$$
(f_0,...,f_n)\begin{pmatrix} s_1 & \cdots & s_n \\ \vdots & & \vdots \\ s_n & \cdots & s_{2n-1} \\ s_{n+1} & \cdots & s_{2n} \end{pmatrix} = 0
$$

and therefore,

$$
f_n(s_{n+1},\ldots,s_{2n})=-(f_0,\ldots,f_{n-1})\widetilde{K}_{n-1}
$$

Using this last equality both with (2.9) and (2.4) we obtain

$$
(f_0, ..., f_n) \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ s_n & \cdots & s_{2n-1} \\ s_{n+1} & \cdots & s_{2n} \end{pmatrix} = 0
$$

ore,

$$
f_n(s_{n+1}, ..., s_{2n}) = -(f_0, ..., f_{n-1})\tilde{K}_{n-1}.
$$

last equality both with (2.9) and (2.4) we obtain

$$
f_0s_{n+1} + ... + f_{n-1}s_{2n} + f_n(s_{n+1}, ..., s_{2n})\tilde{K}_{n-1}^{[-1]}(s_{n+1}, ..., s_{2n})^*
$$

$$
= (f_0, ..., f_{n-1}) \left\{ I - \tilde{K}_{n-1} \tilde{K}_{n-1}^{[-1]} \right\} P_{\text{Ker}\tilde{K}_{n-1}}(s_{n+1}, ..., s_{2n})^*
$$

$$
= 0
$$

view of the representation (2.7) is equivalent to

$$
f_0s_{n+1} + ... + f_ns_{2n+1} = f_nR.
$$
 (2.10)
tion (2.9) means that

which in view of the representation (2.7) is equivalent to

$$
f_0 s_{n+1} + \ldots + f_n s_{2n+1} = f_n R. \tag{2.10}
$$

The condition (2.9) means that

*f*₀*s*_{n+1} + ... + *f*_n*s*_{2n+1} = *f*_n*R*.

dition (2.9) means that
 $f_0 s_{n+1} + ... + f_n s_{2n+1} = 0$ for every vector $(f_0, ..., f_n) \in \text{Ker } K_n$.

The last one is equivalent, in view of (2.5) and (2.10), to $f_n R = 0$ for all $f_n \in \mathcal{L}$. By Lemma 2.3, the matrix

$$
R = s_{2n+1} - (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]}(s_{n+1}, \ldots, s_{2n})^*
$$

does not depend on the choice of $\widetilde{K}_{n-1}^{[-1]}$.

The implication (iv) \Rightarrow (ii). Let $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ and let K_{n+1} be a non-negative Hankel extension of K_n for some $s_{2n+2} \in \mathbb{C}^{m \times m}$. Since \widetilde{K}_n and K_{n+1} are both nonnegative, then by the solvability criterion of the Stieltjes moment problem, there exists a measure $d\sigma(\lambda) \geq 0$ such that **10 0 l o c c c** *f f* $\tilde{K}_{n-1}^{(-1)}$.
 he implication **(iv**) \Rightarrow (**ii**). Let $\{K_n, \tilde{K}_n\} \in K_n^+$ and le
 el extension of K_n for some $s_{2n+2} \in \mathbb{C}^{m \times m}$. Since \tilde{K}_n
 live, then by

$$
\int_{0}^{\infty} \lambda^{k} d\sigma(\lambda) = s_{k} \quad (k = 0, \ldots, 2n + 1) \quad \text{and} \quad \int_{0}^{\infty} \lambda^{2n+2} d\sigma(\lambda) \leq s_{2n+2}.
$$

In particular, this measure satisfies (2.8).

The implication (i) \Rightarrow (iv). Let *do* satisfy (2.8) and therefore,

the solvability criterion of the Stieltjes moment problem, there exists
\n: 0 such that
\n
$$
s_k \quad (k = 0, ..., 2n + 1) \qquad \text{and} \qquad \int_0^\infty \lambda^{2n+2} d\sigma(\lambda) \le s_{2n+2}.
$$
\nmeasure satisfies (2.8).
\n
$$
n \quad (i) \Rightarrow (iv). \text{ Let } d\sigma \text{ satisfy (2.8) and therefore,}
$$
\n
$$
K_n = \int_0^\infty (I_m, ..., \lambda^n I_m)^* d\sigma(\lambda) (I_m, ..., \lambda^n I_m). \qquad (2.11)
$$
\n
$$
s_k \quad \text{where } K_n \text{ is the same as } \text{and } K_n \text{
$$

Let $f = (f_0, \ldots, f_n)$ be a vector from $\text{Ker } K_n$. Then

$$
\int_{0}^{\infty} f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0
$$

where

$$
f(\lambda) = f_0 + \lambda f_1 + \ldots + \lambda^n f_n = \mathbf{f}(I_m, \ldots, \lambda^n I_m)^*.
$$
 (2.12)

In particular, for every choice of $0 \le a < b < +\infty$,

$$
m_1, ..., \lambda^n I_m) d\sigma(\lambda) (I_m, ..., \lambda^n I_m).
$$
 (2.11)
of from Ker K_n . Then

$$
\int_0^\infty f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0
$$

$$
\lambda f_1 + ... + \lambda^n f_n = f(I_m, ..., \lambda^n I_m)^*.
$$
 (2.12)
of $0 \le a < b < +\infty$,

$$
\int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0.
$$
 (2.13)
non-zero vector. By the Cauchy inequality,

Let $g \in \mathbb{C}^{1 \times m}$ be an arbitrary non-zero vector. By the Cauchy inequality,

$$
\int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* \le \left(\int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* \int_a^b \lambda^{2n+2} g d\sigma(\lambda) g^* \right)^{1/2}
$$

which in view of (2.13) implies

$$
\int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* = 0.
$$

Since $a, b \in I\!\!R_+$ and $g \in \mathbb{C}^{1 \times m}$ are arbitrary, then

$$
\int_{0}^{\infty} f(\lambda) d\sigma(\lambda) \lambda^{n+1} I_m = 0.
$$

Therefore, we have also

$$
\int_{0}^{\infty} f(\lambda) d\sigma(\lambda) (I_m, \lambda I_m, \ldots, \lambda^{n+1} I_m) = 0
$$

which on account of (2.8), (2.11) and (2.12) can be rewritten as **f** $\widetilde{K}_n = 0$. Thus, every vector $f \in \text{Ker } K_n$ belongs also to $\text{Ker } \widetilde{K}_n$ and so, $\text{Ker } K_n \subseteq \text{Ker } \widetilde{K}_n$, which ends the proof of the lemma \blacksquare

In connection with the statement (iv) from Lemma 2.5 we consider the following problem:

To describe all matrices $s \in \mathbb{C}^{m \times m}$ *such that*

$$
s=\int\limits_{0}^{\infty}\lambda^{2n+1}d\sigma(\lambda)
$$

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$.

Let again $\{K_n, \widetilde{K}_n\}$ be in \mathcal{K}_n . Then \widetilde{K}_n is non-negative and its block s_{2n+1} admits a representation he statement (iv) from Lemma 2.5 we consider the following
 $ces \ s \in \mathbb{C}^{m \times m} \ such \ that$
 $s = \int_{0}^{\infty} \lambda^{2n+1} d\sigma(\lambda)$

be in \mathcal{K}_n . Then \widetilde{K}_n is non-negative and its block s_{2n+1} admits
 $= (s_{n+1},...,s_{2n}) \widetilde{K}_{n-1$

$$
s_{2n+1} = (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L \qquad (2.14)
$$

for some non-negative matrix L which does not depend on the choice of $\widetilde{K}_{n-1}^{[-1]}$.

Lemma 2.6: Let $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n$, let s_{2n+1} be of the form (2.14), let \mathcal{L} be the subspace defined by (2.5) and let s be an arbitrary $(m \times m)$ -matrix. Then

$$
\tilde{X}_n
$$
 be in \mathcal{K}_n . Then \tilde{K}_n is non-negative and its block s_{2n+1} admits
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L
$$
\n
$$
s_{2n+1} = \sum_{n=0}^{\infty} \lambda^{2n+1} d\sigma(\lambda)
$$
\n
$$
s_{n+1} = \sum_{n=0}^{\infty} \lambda^{2n+1} d\sigma(\lambda)
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n
$$
s_{n+1} = (s_{n+1}, \ldots, s_{2n}) \tilde{K}_{n-1}^{[-1]} (s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\n

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$ *if and only if s admits a representation*

for some
$$
\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})
$$
 if and only if s admits a representation
\n
$$
s = (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]}(s_{n+1}, \ldots, s_{2n})^* + L_0
$$
\nwith a non-negative matrix $L_0 \leq L$ such that $L_0|_{\mathcal{L}} = 0$ (i.e. such that Ker $L_0 \supseteq \mathcal{L}$).

Proof: Let us consider the Hankel block matrix

$$
s = \int_{0}^{3} \lambda^{2n+1} d\sigma(\lambda)
$$
 (2.15)
for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$ if and only if s admits a representation

$$
s = (s_{n+1}, \ldots, s_{2n}) \widetilde{K}_{n-1}^{[-1]}(s_{n+1}, \ldots, s_{2n})^* + L_0
$$
 (2.16)
with a non-negative matrix $L_0 \le L$ such that $L_0|_{\mathcal{L}} = 0$ (i.e. such that Ker $L_0 \supseteq \mathcal{L}$).
Proof: Let us consider the Hankel block matrix

$$
\widetilde{K}_n^1 = \begin{pmatrix} s_1 & \cdots & s_n & s_{n+1} \\ \vdots & & \vdots & \vdots \\ s_n & & s_{2n} \\ s_{n+1} & \cdots & s_{2n} & s \end{pmatrix}
$$

which differs from \widetilde{K}_n only by the block $s_{2n+1}^1 = s$. According to Lemma 2.5, s admits

a representation (2.15) for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1})$ if and only if $\{K_n, \widetilde{K}_n^1\} \in \mathcal{K}_n^+$. This in turn is equivalent (by Lemma 2.5) to the representation (2.16) with some non-negative matrix L_0 which vanishes on the subspace \mathcal{L} . It follows from (1.3) and (2.15) that $s \leq s_{2n+1}$ which in view of (2.14), (2.16) is equivalent to $L_0 \leq L$

Lemma 2.7: *Let <i>L* be the subspace defined by (2.5), let s_{2n+1} and s be matrices
 and $L_0|_{\mathcal{L}} = 0$ and $L_0|_{\mathcal{L}^{\perp}} = L|_{\mathcal{L}^{\perp}}$. (2.17)
 n the Stieltjes moment problems associated with the sets of *defined by (2.14) and (2.16), respectively, and let L* be the subspace defined b
(2.16), respectively, and let
 $L_0|_{\mathcal{L}} = 0$ and
ment problems associated wit *y* (2.5), *let* s_{2n+1} and *s* be matrices
 $L_0|_{\mathcal{L}^{\perp}} = L|_{\mathcal{L}^{\perp}}.$ (2.17)
 ih the sets of matrices
 $H_{2n+1}^1 = \{s_0, \ldots, s_{2n}, s\}$ **Lemma 2.7:** Let \mathcal{L} be the subspace defined by (2.14) and (2.16), respectively, an $L_0|_{\mathcal{L}} = 0$ and
 T hen the Stieltjes moment problems associa:
 $\mathbf{H}_{2n+1} = \{s_0, \ldots, s_{2n}, s_{2n+1}\}$
have the same solutions: $\$

$$
L_0|_{\mathcal{L}}=0 \qquad \text{and} \qquad L_0|_{\mathcal{L}^\perp}=L|_{\mathcal{L}^\perp}. \qquad (2.17)
$$

Then the Stieltjes moment problems associated with the sets of matrices

$$
\mathbf{H}_{2n+1} = \{s_0, \ldots, s_{2n}, s_{2n+1}\} \quad \text{and} \quad \mathbf{H}_{2n+1}^1 = \{s_0, \ldots, s_{2n}, s\}
$$

Proof: Let σ belong to $Z(\mathbf{H}_{2n+1})$. According to Lemma 2.7, the matrix \hat{s} = **Proof:** Let σ belong to $Z(\mathbf{H}_{2n+1})$. According to Lemma 2.7, the matrix $\hat{s} = \int_0^{\infty} \lambda^{2n} d\sigma(\lambda)$ is of the form (2.16) with some non-negative matrix $\hat{L}_0 \leq L$ such that $\hat{L}_0|_{\mathcal{L}} = 0$. In view of (2.17) $\hat{L}_0|\mathcal{L}=0$. In view of (2.17) we have also $\hat{L}_0 \leq L_0$. Therefore, $\hat{s} \leq s$ and $\sigma \in \mathcal{Z}(\mathbf{H}_{2n+1}^1)$. So, $\mathcal{Z}(\mathbf{H}_{2n+1}) \subset \mathcal{Z}(\mathbf{H}_{2n+1}^1)$. The converse inclusion follows from the inequality $s \leq$ s_{2n+1}

Remark 2.8: In view of Lemmas *2.5 - 2.7* we can assume without lost of generality that $\{K_n, K_n\} \in \mathcal{K}_n^+$. In other case we replace the block s_{2n+1} (which is necessarily of the form (2.14)) by the block $s_{2n+1}^1 = s$ defined by (2.16) with L_0 satisfying (2.17). By Lemma 2.5, $\{K_n, \tilde{K}_n^1\} \in \mathcal{K}_n^+$ and we describe the set $\mathcal{Z}(\mathbf{H}_{2n+1}^1)$ of solutions of this new moment problem, which coincides, by Lemma 2.7, with $\mathcal{Z}(\mathbf{H}_{2n+1})$.

3. On decompositions of the state space

In this section we show that under assumption Ker $K_n \subseteq \text{Ker } \widetilde{K}_n$ there exist subspaces G and \widetilde{G} which are complements (not orthogonal, in general) to Ker K_n , and Ker \widetilde{K}_n , respectively, in $\mathbb{C}^{m(n+1)}$ and such that $\mathcal{Q}F_{m,n} \subseteq \tilde{\mathcal{Q}} \subseteq \mathcal{Q}$. This leads to special decompositions of $\mathbb{C}^{1 \times m(n+1)}$ which in turn allow to construct the resolvent matrix of the degenerate Stieltjes problem. As it was mentioned above, for the Schur problem (i.e. for the degenerate information matrix *K* of the form $K = I - TT^*$ where *T* is of the block Toeplitz structure and *I* is the identity matrix of the appropriate size) it was done in *[4:* Part *41.* The case of the degenerate information Hankel block matrix (the degenerate Hamburger moment problem on the line, see, e.g., [11, 13]) was considered in [3].

Lemma 3.1: Let $T_n = (t_{i+j})_{i,j=0}^n$ and $\widetilde{T}_n = (t_{i+j+1})_{i,j=0}^n$ be Hankel block matrices *with non-degenerate blocks* $t_0, t_1 \in \mathbb{C}^{l \times l}$ and let T_{n-1}^1 and \widetilde{T}_{n-1}^1 be block matrices defined *by* $\begin{aligned} & \text{appropriate size)}\ \text{like} \ \text{block matrix}\ & \text{1, 13]} \ \text{was consid} \ & \text{final } \hat{T}^1_{n-1} \ \text{be block}\ & \text{and}\ \hat{T}^1_{n-1} \ \text{be block}\ & \text{in} \ \text{of } \hat{T}^1_{n-1} \ \text{in} \ \text{f}^1_{n-1} \ \text{if} \ \text{f}^1_{n+1} \ & \text{if} \$

$$
T_{n-1}^1 = D^{-1}\left\{M - T t_0^{-1} T^*\right\} D^{-*} \qquad \text{and} \qquad \widetilde{T}_{n-1}^1 = D^{-1}\left\{\widetilde{M} - \widetilde{T} t_1^{-1} \widetilde{T}^*\right\} D^{-*} \tag{3.1}
$$

where

$$
M = (t_{i+j})_{i,j=1}^n, \qquad \widetilde{M} = (t_{i+j+1})_{i,j=1}^n \tag{3.2}
$$

and

t 4]. The case of the degenerate information Hankel block matrix (the degenerate
aburger moment problem on the line, see, e.g., [11, 13]) was considered in [3].
Lemma 3.1: Let
$$
T_n = (t_{i+j})_{i,j=0}^n
$$
 and $\widetilde{T}_n = (t_{i+j+1})_{i,j=0}^n$ be Hankel block matrices
i non-degenerate blocks $t_0, t_1 \in \mathbb{C}^{l \times l}$ and let T_{n-1}^1 and \widetilde{T}_{n-1}^1 be block matrices defined

$$
\widetilde{T}_{n-1}^1 = D^{-1} \{M - T t_0^{-1} T^*\} D^{-*}
$$
 and $\widetilde{T}_{n-1}^1 = D^{-1} \{ \widetilde{M} - \widetilde{T} t_1^{-1} \widetilde{T}^*\} D^{-*}$ (3.1)

$$
T = \begin{pmatrix} t_1 t_0^{-1} & 0 & \dots & 0 \\ t_2 t_0^{-1} & \dots & \vdots \\ \vdots & \ddots & \vdots \\ t_n t_0^{-1} & \dots & t_2 t_0^{-1} & t_1 t_0^{-1} \end{pmatrix}, \qquad T = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \qquad \widetilde{T} = \begin{pmatrix} t_2 \\ \vdots \\ t_{n+1} \end{pmatrix}.
$$
 (3.3)

Under these assumptions the following statements are true:

-
- (i) *If* $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n$, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}$.

(ii) *If* $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n^+$, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}^+$. *Complementagler* these assumptions the following stateme

(i) If $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n$, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in$

(ii) If $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n^+$, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in$
 Proof: From the block

Proof: From the block decompositions

Degenerate Stieltjes Momen

\nptions the following statements are true:

\n
$$
\in \mathcal{K}_n, \text{ then } \{T_{n-1}^1, \widetilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}.
$$
\n
$$
\in \mathcal{K}_n^+, \text{ then } \{T_{n-1}^1, \widetilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}^+.
$$
\nthe block decompositions

\n
$$
T_n = \begin{pmatrix} t_0 & T^* \\ T & M \end{pmatrix} \qquad \text{and} \qquad \widetilde{T}_n = \begin{pmatrix} t_1 & \widetilde{T}^* \\ \widetilde{T}^* & \widetilde{M} \end{pmatrix}
$$
\nunt of (3.1) the factorizations

we obtain on account of (3.1) the factorizations

Under these assumptions the following statements are true:
\n(i) If
$$
\{T_n, \tilde{T}_n\} \in \mathcal{K}_n
$$
, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}$.
\n(ii) If $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n^+$, then $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}^+$.
\nProof: From the block decompositions
\n
$$
T_n = \begin{pmatrix} t_0 & T^* \\ T & M \end{pmatrix} \text{ and } \tilde{T}_n = \begin{pmatrix} t_1 & \tilde{T}^* \\ \tilde{T}^* & \tilde{M} \end{pmatrix}
$$
\nwe obtain on account of (3.1) the factorizations
\n
$$
T_n = \begin{pmatrix} I_l & 0 \\ T t_0^{-1} & D \end{pmatrix} \begin{pmatrix} t_0 & 0 \\ 0 & T_{n-1}^1 \end{pmatrix} \begin{pmatrix} I_l & t_0^{-1} T^* \\ 0 & D^* \end{pmatrix}
$$
\n
$$
\tilde{T}_n = \begin{pmatrix} I_l & 0 \\ \tilde{T} t_1^{-1} & D \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & \tilde{T}_{n-1}^1 \end{pmatrix} \begin{pmatrix} I_l & t_1^{-1} \tilde{T}^* \\ 0 & D^* \end{pmatrix}
$$
\nwhich imply that if T_n and \tilde{T}_n are non-negative, then T_{n-1}^1 and \tilde{T}_{n-1}^1 are non-negative as well.
\nLet $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n$. We begin with the identities
\n
$$
F_{l,n-1}D^{-1}T = 0, \qquad F_{l,n-1}F_{l,n-1}^*F_{l,n-1} = F_{l,n-1}
$$

$$
T_{n} = \begin{pmatrix} T_{t_{0}}^{1} & D \end{pmatrix} \begin{pmatrix} 0 & T_{n-1}^{1} \end{pmatrix} \begin{pmatrix} 0 & D^{*} \end{pmatrix}
$$
\n
$$
\widetilde{T}_{n} = \begin{pmatrix} I_{l} & 0 \\ \widetilde{T}t_{1}^{-1} & D \end{pmatrix} \begin{pmatrix} t_{1} & 0 \\ 0 & \widetilde{T}_{n-1}^{1} \end{pmatrix} \begin{pmatrix} I_{l} & t_{1}^{-1}\widetilde{T}^{*} \\ 0 & D^{*} \end{pmatrix}
$$
\n
$$
(3.5)
$$
\n
$$
T_{n} \text{ and } \widetilde{T}_{n} \text{ are non-negative, then } T_{n-1}^{1} \text{ and } \widetilde{T}_{n-1}^{1} \text{ are non-negative}
$$
\n
$$
K_{n}. \text{ We begin with the identities}
$$
\n
$$
T_{n-1}D^{-1}T = 0, \qquad F_{l,n-1}F_{l,n-1}^{*}F_{l,n-1} = F_{l,n-1}
$$
\n
$$
M - F_{l,n-1}\widetilde{M} + F_{l,n-1}\widetilde{T}t_{1}^{-1}\widetilde{T}^{*} = Tt_{1}^{-1}\widetilde{T}^{*} \qquad (3.6)
$$
\n
$$
\text{intely from (1.14), (3.2) and (3.3). Using these identities and taking}
$$
\n
$$
T_{n} = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0 \end{pmatrix}
$$

as well.

Let $\{T_n, \widetilde{T}_n\} \in \mathcal{K}_n$. We begin with the identities

$$
F_{l,n-1}D^{-1}T=0, \qquad F_{l,n-1}F_{l,n-1}^*F_{l,n-1}=F_{l,n-1}
$$

and

$$
M - F_{l,n-1}\widetilde{M} + F_{l,n-1}\widetilde{T}t_1^{-1}\widetilde{T}^* = Tt_1^{-1}\widetilde{T}^* \qquad (3.6)
$$

which follow immediately from (1.14), (3.2) and (3.3). Using these identities and taking into account (3.1) we get

$$
F_{l,n-1}\left(F_{l,n-1}^*T_{n-1}^1 - \tilde{T}_{n-1}^1\right)
$$

= $F_{l,n-1}F_{l,n-1}^*\left(T_{n-1}^1 - F_{l,n-1}\tilde{T}_{n-1}^1\right)$
= $F_{l,n-1}F_{l,n-1}^*D^{-1}\left(M - Tt_0^{-1}T^* - F_{l,n-1}\widetilde{M} + F_{l,n-1}\tilde{T}t_1^{-1}\tilde{T}^*\right)D^{-*}$
= $F_{l,n-1}F_{l,n-1}^*D^{-1}T\left(t_1^{-1}\tilde{T}^* - t_0^{-1}T^*\right)D^{-*}$
= 0.

Thus, $F_{l,n-1}(F_{l,n-1}^*T_{n-1}^1 - \tilde{T}_{n-1}^1) = 0$ and by Lemma 2.5, $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}$.

Let now $\{T_n, \tilde{T}_n\} \in \mathcal{K}_n^+$ and let $f \in \mathbb{C}^{1 \times l_n}$ be an arbitrary vector from $\text{Ker } T_{n-1}^1$.

In view of (3.5), the vector $(-fD^{-1}Tt_1^{-1}, fD^{-1})$ belongs to $\text{Ker } \tilde{T}_n$. By Lemma 2.5,
 $\text{Ker } T_n \subseteq \text{Ker } \tilde$ In view of (3.5), the vector $(-fD^{-1}Tt_1^{-1},fD^{-1})$ belongs to Ker \widetilde{T}_n . By Lemma 2.5, $Ker T_n \subseteq Ker \widetilde{T}_n$ and thus, $(-fD^{-1}Tt_1^{-1}, fD^{-1})\widetilde{T}_n = 0$. Substituting (3.5) into this last equality we obtain, in particular, $\mathbf{f} \widetilde{T}_{n-1}^1 = 0$. So, every vector $\mathbf{f} \in \text{Ker } T_{n-1}^1$ belongs also to $\text{Ker } \widetilde{T}_{n-1}^1$ and so, $\text{Ker } T_{n-1}^1 \subseteq \text{Ker } \widetilde{T}_{n-1}^1$. By Lemma 2.5, $\{T_{n-1}^1, \widetilde{T}_{n-1}$

Lemma 3.2: *Let* $\{K_n, \tilde{K}_n\} \in K_n^+$ *and let* rank $K_n = r$ *and* rank $\tilde{K}_n = \tilde{r}$. *Then e exist matrices* $Q \in \mathbb{C}^{r \times (n+1)m}$ *and* $\tilde{Q} \in \mathbb{C}^{\tilde{r} \times (n+1)m}$ *such that* **452** V. Bolotnikov
 Lemma 3.2: Let $\{K_n, \widetilde{K}_n\}$

there exist matrices $Q \in \mathbb{C}^{r \times (n+1)}$
 QK_nQ^* :
 $QF_{m,n} = I$ $and \ \widetilde{Q} \in \mathbb{C}^{\widetilde{r} \times (n+1)m}$ such that $\{X_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ and let $\text{rank } K_n = r$ and $\text{rank } \widetilde{K}_n = \tilde{r}$. Then
 $\mathbb{C}^{r \times (n+1)m}$ and $\widetilde{Q} \in \mathbb{C}^{\tilde{r} \times (n+1)m}$ such that
 $QK_nQ^* > 0$ and $\widetilde{Q}\widetilde{K}_n\widetilde{Q}^* > 0$ (3.7)
 $F_{m,n} = N\widetilde{Q}$ an $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ and let rank $K_n = r$ and rank $\widetilde{K}_n = \tilde{r}$. Then
 $\mathbb{C}^{r \times (n+1)m}$ and $\widetilde{Q} \in \mathbb{C}^{r \times (n+1)m}$ such that
 $QK_nQ^* > 0$ and $\widetilde{Q} \widetilde{K}_n\widetilde{Q}^* > 0$ (3.7)
 $QF_{m,n} = N\widetilde{Q}$ and

$$
QK_nQ^* > 0 \qquad \text{and} \qquad \widetilde{Q}\widetilde{K}_n\widetilde{Q}^* > 0 \tag{3.7}
$$

$$
QF_{m,n} = N\widetilde{Q} \qquad and \qquad \widetilde{Q} = \widetilde{N}Q \qquad (3.8)
$$

for the shift $F_{m,n}$ defined via (1.14) and some matrices $N \in \mathbb{C}^{r \times \bar{r}}$ and $\widetilde{N} \in \mathbb{C}^{\bar{r} \times r}$. In *other words, there exist subspaces*

$$
Q = \text{Ran } Q := \left\{ y \in \mathbb{C}^{1 \times m(n+1)} : y = fQ \text{ for some } f \in \mathbb{C}^{1 \times r} \right\}
$$

$$
\widetilde{Q} = \text{Ran } \widetilde{Q}
$$

such that $QF_{m,n} \subseteq \widetilde{Q} \subseteq Q$ and

$$
\mathbb{C}^{m(n+1)} = \text{Ker } K_n + \mathcal{Q} = \text{Ker } \widetilde{K}_n + \widetilde{\mathcal{Q}}.
$$
 (3.9)

Proof: We prove this lemma by induction. Let, for $n = 0$, rank $s_0 = l$, and rank $s_1 = \tilde{l}$ ($l, \tilde{l} \le m$). By Lemma 2.5, Ker. $s_0 \subseteq$ Ker. s_1 and therefore, $l \le \tilde{l}$. Moreover, there exists a unitary matrix $U \in \mathbb$ Froot: we prove this lemma by induction.

rank $s_1 = \tilde{l}$ ($l, \tilde{l} \le m$). By Lemma 2.5, Ker $s_0 \subseteq K$

there exists a unitary matrix $U \in \mathbb{C}^{m \times m}$ such that
 $Us_0U^* = \begin{pmatrix} \hat{s}_0 & 0 \\ 0 & 0_{m-l} \end{pmatrix}$ and $Us_1U^* = \begin{pmatrix}$ \widetilde{Q}
 $\subseteq Q$ and
 $\mathbb{C}^{m(n+1)}$
 \vdots this lem

matrix U
 0
 0
 0 _{m- l} = Ker $\widetilde{K}_n + \widetilde{Q}$.
 n. Let, for *n* =
 \widetilde{E} Ker *s*₁ and there

hat
 $\begin{pmatrix} t_1 & 0 \\ 0 & 0_{m-1} \end{pmatrix}$

exists a unitary matrix
$$
U \in \mathbb{C}^{m \times m}
$$
 such that
\n
$$
U s_0 U^* = \begin{pmatrix} \hat{s}_0 & 0 \\ 0 & 0_{m-l} \end{pmatrix} \text{ and } U s_1 U^* = \begin{pmatrix} t_1 & 0 \\ 0 & 0_{m-l} \end{pmatrix} \qquad (\hat{s}_0, t_1 > 0) \qquad (3.10)
$$
\n
$$
\text{nence, matrices}
$$
\n
$$
Q = (I_l, 0)U, \quad \tilde{Q} = (I_{\tilde{l}}, 0)U \qquad \text{and} \qquad N = 0, \quad \tilde{N} = (I_{\tilde{l}}, 0_{\tilde{l} \times (l - \tilde{l})})
$$

and hence, matrices

$$
Q = (I_l, 0)U
$$
, $\tilde{Q} = (I_{\tilde{l}}, 0)U$ and $N = 0$, $\tilde{N} = (I_{\tilde{l}}, 0_{\tilde{l} \times (l - \tilde{l})})$

satisfy (3.7) and (3.8).

Let us suppose the assertion of the lemma hold for all integers up to $n-1$. Let as above rank $s_0 = l \geq \overline{l} = \text{rank } s_1$ and s_0, s_1 be of the form (3.10). By Lemma 2.5, Ker $K_n \subseteq \text{Ker } K_n$ which implies, in particular,

$$
\operatorname{Ker} s_0 \subseteq \operatorname{Ker} s_1 \subseteq \ldots \subseteq \operatorname{Ker} s_{2n+1}.
$$

Therefore,

Ker
$$
K_n \subseteq
$$
 Ker K_n which implies, in particular,
\n
$$
\text{Ker } s_0 \subseteq \text{Ker } s_1 \subseteq \dots \subseteq \text{Ker } s_{2n+1}.
$$
\nTherefore,
\n
$$
Us_i U^* = \begin{pmatrix} t_i & 0 \\ 0 & 0_{m-i} \end{pmatrix} \qquad (i = 1, \dots, 2n + 1) \qquad (3.11)
$$
\nfor some matrices $t_i \in \mathbb{C}^{\{x\}}.$ Furthermore, let
\n
$$
\hat{s}_0 = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \qquad \left(\alpha \in \mathbb{C}^{\{x\}}; \ \beta \in \mathbb{C}^{\{x(t-\tilde{t})\}}, \ \gamma \in \mathbb{C}^{(t-\tilde{t}) \times (t-\tilde{t})} \right) \qquad (3.12)
$$
\nbe the block decomposition of the matrix \hat{s}_0 from the representation (3.10). Introducing
\nthe matrices
\n
$$
t_0 = \alpha - \beta \gamma^{-1} \beta^* = (I_{\tilde{t}}, -\beta \gamma^{-1}) \hat{s}_0 \begin{pmatrix} I_{\tilde{t}} \\ -\gamma^{-1} \beta^* \end{pmatrix} > 0 \qquad (3.13)
$$

e matrices
$$
t_i \in \mathbb{C}^{\tilde{l} \times \tilde{l}}
$$
. Furthermore, let
\n
$$
\hat{s}_0 = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \qquad \left(\alpha \in \mathbb{C}^{\tilde{l} \times \tilde{l}}, \ \beta \in \mathbb{C}^{\tilde{l} \times (l-\tilde{l})}, \ \gamma \in \mathbb{C}^{(l-\tilde{l}) \times (l-\tilde{l})} \right) \qquad (3.12)
$$

be the block decomposition of the matrix \hat{s}_0 from the representation (3.10). Introducing. the matrices $\left(\begin{array}{cc}I_{\bar{l}}&\\&1\end{array}\right)_{\geq 0}$

$$
t_0 = \alpha - \beta \gamma^{-1} \beta^* = (I_{\tilde{l}}, -\beta \gamma^{-1}) \hat{s}_0 \begin{pmatrix} I_{\tilde{l}} \\ -\gamma^{-1} \beta^* \end{pmatrix} > 0
$$
 (3.13)

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and

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\n
$$
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$$
\n
$$
g = \left(I_{\hat{l}}, -\beta\gamma^{-1}, 0_{\hat{l}\times(m-l)}\right)U
$$
\n
$$
(3.10) - (3.12) that
$$
\n
$$
g^* = t_i \qquad (i = 0, \ldots, 2n + 1).
$$
\n
$$
\tilde{l}(n + 1) \text{ Hankel block matrices defined by}
$$
\n(3.15)

we obtain immediately from (3.10) - (3.12) that

$$
g s_i g^* = t_i \qquad (i = 0, \ldots, 2n + 1). \tag{3.15}
$$

$$
g = (I_{\tilde{l}}, -\beta \gamma^{-1}, 0_{\tilde{l} \times (m-l)}) U
$$
(3.14)
mmmediately from (3.10) - (3.12) that

$$
g s_i g^* = t_i \qquad (i = 0, ..., 2n + 1).
$$
(3.15)

$$
\tilde{T}_n
$$
 be $\tilde{l}(n + 1) \times \tilde{l}(n + 1)$ Hankel block matrices defined by

$$
T_n = (t_{i+j}) = \Gamma_n K_n \Gamma_n^*
$$
 and
$$
\tilde{T}_n = (t_{i+j+1}) = \Gamma_n \tilde{K}_n \Gamma_n^*
$$
(3.16)

where

we obtain immediately from (3.10) - (3.12) that
\n
$$
g_{sj}^* = t_i \qquad (i = 0, ..., 2n + 1).
$$
\n(3.14)
\nLet T_n and \tilde{T}_n be $\tilde{l}(n + 1) \times \tilde{l}(n + 1)$ Hankel block matrices defined by
\n
$$
T_n = (t_{i+j}) = \Gamma_n K_n \Gamma_n^* \qquad \text{and} \qquad \tilde{T}_n = (t_{i+j+1}) = \Gamma_n \tilde{K}_n \Gamma_n^* \qquad (3.16)
$$
\nwhere
\n
$$
\Gamma_n = \begin{pmatrix} g & 0 \\ & \cdot \\ 0 & g \end{pmatrix} \in \mathbb{C}^{\tilde{l}(n+1) \times m(n+1)} \qquad (3.17)
$$
\nIt follows from (3.10) - (3.17) that
\n
$$
\text{rank } T_n = \text{rank } K_n - (l - \tilde{l}) = r + \tilde{l} - l \quad \text{and} \quad \text{rank } \tilde{T}_n = \text{rank } \tilde{K}_n = \tilde{r}. \qquad (3.18)
$$
\nSince $K_n K_n \subset K_n \tilde{K}_n$ then is of (3.10) - (3.11).

It follows from (3.10) - (3.17) that

$$
\operatorname{rank} T_n = \operatorname{rank} K_n - (l - \tilde{l}) = r + \tilde{l} - l \quad \text{and} \quad \operatorname{rank} \widetilde{T}_n = \operatorname{rank} \widetilde{K}_n = \tilde{r}.\tag{3.18}
$$

Since Ker $K_n \subseteq \text{Ker }\widetilde{K}_n$, then in view of (3.10) - (3.16), $\text{Ker }T_n \subseteq \text{Ker }\widetilde{T}_n$ and by Lemma 2.5, $\{T_n, \widetilde{T}_n\} \in \mathcal{K}_n^+$ Let T_{n-1}^1 , \widetilde{T}_{n-1}^1 and *D* be matrices defined by (3.1) and (3.3), respectively. According to Lemma 3.1, $\{T_{n-1}^1, \widetilde{T}_{n-1}^1\} \in \mathcal{K}_{n-1}^+$. Moreover, we obtain from (3.4), (3.5) and (3.18) that rank $T_{n-1}^1 = r - l$ and rank $\widetilde{T}_{n-1}^1 = \widetilde{r} - \widetilde{l}$. Hence, by the induction hypothesis there exist matrices $\begin{aligned} \text{e Ker } K_n &\subseteq \text{Ker } \widetilde{K}_n, \ \{T_n, \widetilde{T}_n\} &\in \mathcal{K}_n^+. \ \text{Let} \ \text{ectively.} \ \ \text{According to } \text{f} \ (3.4), \ (3.5) \ \text{and } \ (3.1) \ \text{induction hypothesis} \ Q_1 &\in \mathbb{C}^{(r-l) \times ln}, \ \ \widetilde{Q}_1 \end{aligned}$ $n = \binom{n}{0}$
 $\binom{n}{0} = r + \tilde{l} - l$ a

hen in view of (3.10) -
 T_{n-1}^1 , \tilde{T}_{n-1}^1 and *D* b

to Lemma 3.1, $\{T_{n-1}^1 = r\}$

8) that rank $T_{n-1}^1 = r$

there exist matrices
 $\in \mathbb{C}^{(\tilde{r}-\tilde{l}) \times \tilde{l}n}$ and N_1 $= r + \tilde{l} - l$ and rank $\tilde{T}_n = \text{rank }\widetilde{K}_n = \tilde{r}$. (3.18)

ew of (3.10) - (3.16), Ker $T_n \subseteq \text{Ker }\widetilde{T}_n$ and by Lemma
 \tilde{r}_{n-1} and *D* be matrices defined by (3.1) and (3.3),

a 3.1, $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in \mathcal{K$ in view of (3.10) - (3.16), Ker $T_n \subseteq$ Ker T_n and by Lemma
 -1 , \tilde{T}_{n-1}^1 and *D* be matrices defined by (3.1) and (3.3),

emma 3.1, $\{T_{n-1}^1, \tilde{T}_{n-1}^1\} \in K_{n-1}^+$. Moreover, we obtain

that rank $T_{n-1}^1 = r$ rank $T_{n-1}^1 = r$
ist matrices
 x^{in} and N_1
 \cdots
 $N_1 \widetilde{Q}_1$ and
 \cdots
 \cdots
 \widetilde{Q}_1 and
 \cdots matrix defined
 $\widetilde{e} = (I_{\overline{t}}, 0_{\overline{t}}, \ldots, 0)$
and N, \widetilde{N} given

$$
Q_1 \in \mathbb{C}^{(r-l)\times \bar{l}n}, \quad \widetilde{Q}_1 \in \mathbb{C}^{(\bar{r}-\bar{l})\times \bar{l}n} \quad \text{and} \quad N_1 \in \mathbb{C}^{(r-l)\times (\bar{r}-\bar{l})}, \quad \widetilde{N}_1 \in \mathbb{C}^{(\bar{r}-\bar{l})\times (r-l)}
$$

such that

$$
Q_1 T_{n-1}^1 Q_1^* > 0 \quad \text{and} \quad \tilde{Q}_1 \tilde{T}_{n-1}^1 \tilde{Q}_1^* > 0 \tag{3.19}
$$

$$
Q_1 F_{\tilde{\ell}, n-1} = N_1 \tilde{Q}_1 \quad \text{and} \quad \tilde{Q}_1 = \tilde{N}_1 Q_1. \tag{3.20}
$$

Furthermore, let $\tilde{\textbf{e}} \in \mathbb{C}^{ln \times l}$ be the matrix defined by

$$
\tilde{\mathbf{e}} = (I_{\tilde{l}}, 0_{\tilde{l}}, \dots, 0_{\tilde{l}})^* \tag{3.21}
$$

We show that the matrices Q,\tilde{Q} and N,\tilde{N} given by

$$
Q_1 T_{n-1}^1 Q_1^* > 0 \quad \text{and} \quad \tilde{Q}_1 \tilde{T}_{n-1}^1 \tilde{Q}_1^* >
$$

\n
$$
Q_1 F_{\tilde{\ell}, n-1} = N_1 \tilde{Q}_1 \quad \text{and} \quad \tilde{Q}_1 = \tilde{N}_1 Q_1.
$$

\n
$$
\in \mathbb{C}^{\tilde{l} n \times \tilde{l}} \text{ be the matrix defined by}
$$

\n
$$
\tilde{e} = (I_{\tilde{l}}, 0_{\tilde{l}}, \dots, 0_{\tilde{l}})^*
$$

\nmatrices Q, \tilde{Q} and N, \tilde{N} given by
\n
$$
Q = \begin{pmatrix} \begin{pmatrix} I_{\tilde{l}} & -\beta \gamma^{-1} & 0 \\ 0 & I_{l-\tilde{l}} & 0 \end{pmatrix} U & 0 \\ -Q_1 D^{-1} T t_0^{-1} g & Q_1 D^{-1} \Gamma_{n-1} \end{pmatrix}
$$

\n
$$
\tilde{Q} = \begin{pmatrix} I_{\tilde{l}} & 0 \\ -\tilde{Q}_1 D^{-1} \tilde{T} t_1^{-1} & \tilde{Q}_1 D^{-1} \end{pmatrix} \Gamma_n
$$

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and

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\nand
\n
$$
N = \begin{pmatrix} 0_{l \times \tilde{l}} & 0 \\ Q_1 \tilde{e} t_0 t_1^{-1} g & N_1 \end{pmatrix}
$$
\n
$$
\tilde{N} = \begin{pmatrix} I_{\tilde{l}} & 0_{\tilde{l} \times (l - \tilde{l})} & 0 \\ \tilde{N}_1 Q_1 D^{-1} (T t_0^{-1} - \tilde{T} t_1^{-1}) & 0 & \tilde{N}_1 \end{pmatrix}
$$
\n(where T, \tilde{T} , U and \tilde{e} are matrices defined via (3.3), (3.11) and (3.21)) satisfy conditions
\n(3.7) and (3.8). Indeed, in view of (3.4), (3.5), (3.10) - (3.17) and (3.19) we have
\n
$$
Q K_n Q^* = \begin{pmatrix} t_0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} > 0
$$
\n(3.22)

(3.7) and (3.8). Indeed, in view of (3.4), (3.5), (3.10) - (3.17) and (3.19) we have

$$
N = \begin{pmatrix} 0_{l \times \tilde{l}} & 0 \\ Q_{1} \tilde{e} t_{0} t_{1}^{-1} g & N_{1} \end{pmatrix}
$$

\n
$$
\tilde{N} = \begin{pmatrix} I_{\tilde{l}} & 0_{\tilde{l} \times (l - \tilde{l})} & 0 \\ \tilde{N}_{1} Q_{1} D^{-1} (T t_{0}^{-1} - \tilde{T} t_{1}^{-1}) & 0 & \tilde{N}_{1} \end{pmatrix}
$$

\nand \tilde{e} are matrices defined via (3.3), (3.11) and (3.21)
\nIndeed, in view of (3.4), (3.5), (3.10) - (3.17) and (3
\n
$$
Q K_{n} Q^{*} = \begin{pmatrix} t_{0} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & Q_{1} T_{n-1}^{1} Q_{1}^{*} \end{pmatrix} > 0
$$

\n $\tilde{Q} \tilde{K}_{n} \tilde{Q}^{*} = \begin{pmatrix} t_{1} & 0 \\ 0 & \tilde{Q}_{1} \tilde{T}_{n-1}^{1} \tilde{Q}_{1} \end{pmatrix} > 0.$
\n3.23) and the block decompositions
\n
$$
\Gamma_{n} = \begin{pmatrix} g & 0 \\ 0 & \Gamma_{n-1} \end{pmatrix} \text{ and } F_{\tilde{l}, n} = \begin{pmatrix} 0 & 0 \\ \tilde{e} & F_{\tilde{l}, n-1} \end{pmatrix}
$$

\n
$$
Q F_{m,n} - N \tilde{Q} = \begin{pmatrix} 0 & 0 \\ B_{1} & B_{2} \end{pmatrix}
$$

\n
$$
B_{1} = Q_{1} D^{-1} \tilde{e} g - Q_{1} \tilde{e} t_{0} t_{1}^{-1} g + N_{1} \tilde{Q}_{1} D^{-1} \tilde{T} t_{1}^{-1} g
$$

It follows from (3.23) and the block decompositions

$$
\widetilde{Q}\widetilde{K}_n\widetilde{Q}^* = \begin{pmatrix} t_1 & 0 \\ 0 & \widetilde{Q}_1\widetilde{T}_{n-1}^1\widetilde{Q}_1 \end{pmatrix} > 0.
$$
\n(3.23) and the block decompositions\n
$$
\Gamma_n = \begin{pmatrix} g & 0 \\ 0 & \Gamma_{n-1} \end{pmatrix} \quad \text{and} \quad F_{\widetilde{l},n} = \begin{pmatrix} 0 & 0 \\ \widetilde{e} & F_{\widetilde{l},n-1} \end{pmatrix}
$$
\n
$$
QF_{m,n} - N\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}
$$

that

$$
QF_{m,n} - N\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}
$$
 (3.23)

where

$$
I_{n} = \begin{pmatrix} 0 & \Gamma_{n-1} \end{pmatrix} \text{ and } F_{\tilde{l},n} = \begin{pmatrix} \tilde{e} & F_{\tilde{l},n-1} \end{pmatrix}
$$

$$
QF_{m,n} - N\tilde{Q} = \begin{pmatrix} 0 & 0 \\ B_{1} & B_{2} \end{pmatrix} \qquad (i
$$

$$
B_{1} = Q_{1}D^{-1}\tilde{e}g - Q_{1}\tilde{e}t_{0}t_{1}^{-1}g + N_{1}\tilde{Q}_{1}D^{-1}\tilde{T}t_{1}^{-1}g
$$

$$
B_{2} = Q_{1}D^{-1}\Gamma_{n-1}F_{m,n-1} - N_{1}\tilde{Q}_{1}D^{-1}\Gamma_{n-1}.
$$

l identities

$$
= F_{\tilde{l},n-1}D^{-1}, \quad \Gamma_{n-1}F_{m,n-1} = F_{\tilde{l},n-1}\Gamma_{n-1}, \quad \tilde{e}t_{1} + F_{m,n-1}\tilde{T} = T
$$

mediately from (2.2), (3.3), (3.17) and (3.21), we get

Using (3.20) and identities

$$
D^{-1}F_{\bar{l},n-1} = F_{\bar{l},n-1}D^{-1}, \quad \Gamma_{n-1}F_{m,n-1} = F_{\bar{l},n-1}\Gamma_{n-1}, \quad \tilde{e}t_1 + F_{m,n-1}\tilde{T} = T
$$

which follow immediately from (2.2) , (3.3) , (3.17) and (3.21) , we get

$$
B_1 = Q_1 D^{-1} \left(\tilde{e}g - D \tilde{e}t_0 t_1^{-1} g + F_{\tilde{t}, n-1} \tilde{\mathcal{T}} t_1^{-1} g \right)
$$

= $Q_1 D^{-1} \left(\tilde{e}t_1 - \mathcal{T} + F_{\tilde{t}, n-1} \tilde{\mathcal{T}} \right) t_1^{-1} g$
= 0

and

$$
B_2 = Q_1 D^{-1} \Gamma_{n-1} F_{m,n-1} - Q_1 F_{\tilde{l},n-1} D^{-1} \Gamma_{n-1}
$$

= $Q_1 D^{-1} \left(\Gamma_{n-1} F_{m,n-1} - F_{\tilde{l},n-1} \Gamma_{n-1} \right)$
= 0.

So, $B_1 = B_2 = 0$ and (3.23) implies the first equality from (3.8). Similarly, in view of (3.20) and (3.23),

\n Degenerate Stieltjes Moment Problem 455
\n
$$
1 = B_2 = 0 \text{ and } (3.23) \text{ implies the first equality from } (3.8). \text{ Similarly, in view of and } (3.23),
$$
\n
$$
\widetilde{Q} - \widetilde{N}Q = \begin{pmatrix} g & 0 \\ -\widetilde{Q}_1 D^{-1} \widetilde{T} t_1^{-1} g & \widetilde{Q}_1 D^{-1} \Gamma_{n-1} \end{pmatrix}
$$
\n
$$
-\begin{pmatrix} g & 0 \\ -\widetilde{N}_1 Q_1 D^{-1} \widetilde{T} t_1^{-1} g & \widetilde{N}_1 Q_1 D^{-1} \Gamma_{n-1} \end{pmatrix}
$$
\n
$$
= \begin{pmatrix} 0 & 0 \\ (N_1 Q_1 - \widetilde{Q}_1) D^{-1} \widetilde{T} t_1^{-1} g & (\widetilde{Q}_1 - N_1 Q_1) D^{-1} \Gamma_{n-1} \end{pmatrix}
$$
\n
$$
= 0.
$$
\n
$$
\widetilde{Q} = \widetilde{N}Q \text{ which ends the proof of the lemma}
$$
\n\n orollary 3.3: Let Q and \widetilde{Q} be matrices satisfying (3.8). Then for all $l \in \mathbb{N}$

\n
$$
QF_{m,n}^l = (N\widetilde{N})^l N\widetilde{Q} \qquad \text{and} \qquad \widetilde{Q}F_{m,n}^l = (\widetilde{N}N)^l \widetilde{N}Q = (\widetilde{N}N)^{l+1}\widetilde{Q}. \tag{3.24}
$$
\n

Thus, $\widetilde{Q} = \widetilde{N}Q$ which ends the proof of the lemma \blacksquare

Corollary 3.3: Let Q and \widetilde{Q} be matrices satisfying (3.8). Then for all $l \in I\!\!N$

$$
QF_{m,n}^l = (N\widetilde{N})^l N\widetilde{Q} \qquad and \qquad \widetilde{Q}F_{m,n}^l = (\widetilde{N}N)^l \widetilde{N}Q = (\widetilde{N}N)^{l+1}\widetilde{Q}.
$$
 (3.24)

4. J-inner polynomials associated with ${K_n, \widetilde{K}_n} \in \mathcal{K}_n^+$

In this section we associate to every pair $\{K_n, \tilde{K}_n\} \in \mathcal{K}_n^+$ a function Θ of the class W_{π} . It will be shown in Section 5 that this function is the resolvent matrix of the corresponding Stieltjes moment problem. To simplify the further computations, the index *n* will be omitted, and up to Section 6, by *K*, \widetilde{K} and *F* we mean matrices K_n , \widetilde{K}_n and $F_{m,n}$ given by (2.1) and (1.14). ion we associate to every pair $\{K_n, \tilde{K}_n\} \in \mathcal{K}_n^+$ a function Θ of the
il be shown in Section 5 that this function is the resolvent matrix of
ng Stieltjes moment problem. To simplify the further computations,
 polynomials associated with { K_n , K_n } $\in K_n^+$
we associate to every pair { K_n , \tilde{K}_n } $\in K_n^+$ a function Θ of the class
r shown in Section 5 that this function is the resolvent matrix of the
stilletly momen *k* is shown in Section 5 that this function is the resolvent matrix of the omitted, and up to Section 6, by K, \tilde{K} and F we mean matrices K_n , wen by (2.1) and (1.14).
 i. Let $\{K, \tilde{K}\} \in K^+$, let Q and $\tilde{Q$

Lemma 4.1: Let $\{K,\widetilde{K}\} \in \mathcal{K}^+$, let Q and \widetilde{Q} be matrices satisfying (3.7) and (3.8), and let $K^{[-1]}$ and $\widetilde{K}^{[-1]}$ be pseudoinverse matrices defined as *Q* (2.1) and (1.14).
 Let $\{K, \tilde{K}\} \in \mathcal{K}^+$, let Q and \tilde{Q} be matrices satisfying (3.7) and (3.8),
 nd $\tilde{K}^{[-1]}$ be pseudoinverse matrices defined as
 $\tilde{K}^{[-1]} = Q^*(QKQ^*)^{-1}Q$ and $\tilde{K}^{[-1]} = \tilde{Q}^$

$$
K^{[-1]} = Q^*(QKQ^*)^{-1}Q \qquad \text{and} \qquad \widetilde{K}^{[-1]} = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}\widetilde{Q}^*)^{-1}\widetilde{Q}. \tag{4.1}
$$

Then, for all $l \in \mathbb{N}_0$,

$$
K^{[-1]}F^{l}\left(I-\widetilde{K}\widetilde{K}^{[-1]}\right)=\widetilde{K}^{[-1]}F^{l}\left(I-KK^{[-1]}\right)=0
$$
\n(4.2)

$$
\widetilde{K}^{\{-1\}}F^{l}\left(I-\widetilde{K}\widetilde{K}^{\{-1\}}\right)=K^{\{-1\}}F^{l+1}\left(I-KK^{\{-1\}}\right)=0.\tag{4.3}
$$

Proof: It follows from (4.1) that

$$
Q\left(I - KK^{[-1]}\right) = 0 \quad \text{and} \quad \widetilde{Q}\left(I - \widetilde{K}\widetilde{K}^{[-1]}\right) = 0. \quad (4.4)
$$

Using again (4.1) together with (3.24) we get

$$
K^{[-1]}F^l = Q^{\bullet}(QK_nQ^{\bullet})^{-1}(N\widetilde{N})^lQ = Q^{\bullet}(QK_nQ^{\bullet})^{-1}(N\widetilde{N})^lN\widetilde{Q}
$$

$$
\widetilde{K}^{[-1]}F^{l+1} = \widetilde{Q}^{\bullet}(\widetilde{Q}\widetilde{K}\widetilde{Q}^{\bullet})^{-1}(\widetilde{N}N)^l\widetilde{N}Q = \widetilde{Q}^{\bullet}(\widetilde{Q}\widetilde{K}\widetilde{Q}^{\bullet})^{-1}(\widetilde{N}N)^{l+1}\widetilde{Q}.
$$

Substituting these last equalities (for $l = 0, 1, \ldots$) into (4.2) and (4.3) and taking into account (4.4) we obtain the required equalities \blacksquare

Lemma 4.2: Under the assumptions of Lemma 4.1 let Θ and $\widetilde{\Theta}$ be the $\mathbb{C}^{2m \times 2m}$ *valued functions defined by*

Solotnikov

\n1 4.2: Under the assumptions of Lemma 4.1 let
$$
\Theta
$$
 and $\widetilde{\Theta}$ be the $\mathbb{C}^{2m \times 2m}$.

\n $\Theta(z) = I_{2m} + \begin{pmatrix} e^*K & 0 \\ 0 & e^* \end{pmatrix} \Omega(z) \begin{pmatrix} K^{[-1]} & 0 \\ 0 & \widetilde{K}^{[-1]} \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & Ke \end{pmatrix}$ (4.5)

\n $\widetilde{\Theta}(z) = P(z)\Theta(z)P^{-1}(z)$ (4.6)

\nand e are given by (1.9) and (1.14), respectively, and

\n $\begin{pmatrix} xF^*(I - zF^*)^{-1} & (I - zF^*)^{-1} \end{pmatrix}$

and

$$
\widetilde{\Theta}(z) = P(z)\Theta(z)P^{-1}(z) \tag{4.6}
$$

where P(z) and e are given by (1.9) and (1.14), respectively, and

$$
\widetilde{\Theta}(z) = P(z)\Theta(z)P^{-1}(z)
$$
\nre given by (1.9) and (1.14), respectively, and

\n
$$
\Omega(z) = \begin{pmatrix} zF^*(I - zF^*)^{-1} & (I - zF^*)^{-1} \\ -z(I - zF^*)^{-1} & -z(I - zF^*)^{-1} \end{pmatrix}.
$$
\nss W_π (see Definition 1.2) and moreover, for a

\n
$$
\frac{-J}{} = \begin{pmatrix} e^*KF^* \\ -e^* \end{pmatrix} (I - zF^*)^{-1}K^{[-1]}(I - \overline{\omega}F)^{-1}.
$$
\n
$$
\overline{K}, \widetilde{K} \} \in \mathcal{K}^+, \text{ then by Lemma 2.2, } F(F^*K - \widetilde{K})
$$
\n
$$
(I - e e^*)K = F\widetilde{K}.
$$
\nsplies that

Then Θ is of the class W_{π} (see Definition 1.2) and moreover, for all $z,\omega\in\mathbb{C}\setminus{I\!\!R}_{\pi}$

$$
\Omega(z) = \begin{pmatrix} zF^*(I - zF^*)^{-1} & (I - zF^*)^{-1} \\ -z(I - zF^*)^{-1} & -z(I - zF^*)^{-1} \end{pmatrix}.
$$

is of the class W_{π} (see Definition 1.2) and moreover, for all $z, \omega \in \mathbb{C} \setminus \mathbb{R}$,

$$
\frac{\Theta(z)J\Theta(\omega)^* - J}{i(\overline{\omega} - z)} = \begin{pmatrix} e^*KF^* \\ -e^* \end{pmatrix} (I - zF^*)^{-1}K^{[-1]}(I - \overline{\omega}F)^{-1}(FKe, -e) \quad (4.7)
$$

$$
\frac{\widetilde{\Theta}(z)J\widetilde{\Theta}(\omega)^* - J}{i(\overline{\omega} - z)} = \begin{pmatrix} e^*K \\ -e^* \end{pmatrix} (I - zF^*)^{-1}\widetilde{K}^{[-1]}(I - \overline{\omega}F)^{-1}(Ke, -e). \quad (4.8)
$$

$$
\frac{\Theta(z)J\Theta(\omega)^*-J}{i(\overline{\omega}-z)}=\begin{pmatrix}e^*K\\-e^*\end{pmatrix}(I-zF^*)^{-1}\widetilde{K}^{[-1]}(I-\overline{\omega}F)^{-1}(K\mathbf{e},-\mathbf{e}).\tag{4.8}
$$

Proof: Since $\{K, \widetilde{K}\}\in \mathcal{K}^+$, then by Lemma 2.2, $F(F^*K-\widetilde{K})=0$, or equivalently,

$$
(I - ee^*)K = F\widetilde{K}.
$$
 (4.9)

This last identity implies that

$$
FKe\mathbf{e}^* - \mathbf{e}\mathbf{e}^* K F^* = F K - K F^* \quad \text{and} \quad K \mathbf{e}\mathbf{e}^* - \mathbf{e}\mathbf{e}^* K = F\widetilde{K} - \widetilde{K} F^*.
$$
 (4.10)

Using (4.9) and *(4.10)* both with (4.2) and (4.3), it is easy to check that the functions Θ and $\widetilde{\Theta}$ admit the factorizations

$$
(I - ee^*)K = F\tilde{K}.
$$
\n
$$
(4.9)
$$
\ntity implies that

\n
$$
ee^*KF^* = FK - KF^*
$$
\nand

\n
$$
Kee^* - ee^*K = F\tilde{K} - \tilde{K}F^*.
$$
\n
$$
(4.10)
$$
\nand

\n
$$
(4.10)
$$
\nboth with (4.2) and (4.3), it is easy to check that the functions

\n
$$
\Theta(z) = \left\{ I_{2m} + z \begin{pmatrix} e^*KF^* \\ -e^* \end{pmatrix} (I - zF^*)^{-1} K^{[-1]}(e, FKe) \right\} \Psi
$$
\n
$$
\widetilde{\Theta}(z) = \left\{ I_{2m} + z \begin{pmatrix} e^*K \\ 0 \end{pmatrix} (I - zF^*)^{-1} \tilde{K}^{[-1]}(e, Ke) \right\} \widetilde{\Psi}
$$
\n
$$
(4.11)
$$
\n
$$
\widetilde{\Theta}(z) = \left\{ I_{2m} + z \begin{pmatrix} e^*K \\ 0 \end{pmatrix} (I - zF^*)^{-1} \tilde{K}^{[-1]}(e, Ke) \right\} \widetilde{\Psi}
$$
\n
$$
(4.12)
$$

Since
$$
\{K, K\} \in \mathcal{K}^+
$$
, then by Lemma 2.2, $F(F^*K - K) = 0$, or equivalently,
\n $(I - ee^*)K = F\tilde{K}$. (4.9)
\n
\n(4.9)
\n
\n(4.9)
\n
\n(4.10) both with (4.2) and (4.3) , it is easy to check that the functions
\n
\n $\Theta(z) = \begin{cases} I_{2m} + z \begin{pmatrix} e^*KF^* \\ -e^* \end{pmatrix} (I - zF^*)^{-1} K^{[-1]}(e, FKe) \end{cases} \Psi$ (4.11)
\n $\tilde{\Theta}(z) = \begin{cases} I_{2m} + z \begin{pmatrix} e^*K \\ -e^* \end{pmatrix} (I - zF^*)^{-1} \tilde{K}^{[-1]}(e, Ke) \end{cases} \tilde{\Psi}$ (4.12)
\n $\tilde{\Theta}(z) = \begin{cases} I_{2m} + z \begin{pmatrix} e^*K \\ -e^* \end{pmatrix} (I - zF^*)^{-1} \tilde{K}^{[-1]}(e, Ke) \end{cases} \tilde{\Psi}$ (4.12)
\n $\Psi = \begin{pmatrix} I_m & e^*K\tilde{K}^{[-1]}Ke \\ 0 & I_m \end{pmatrix}$ and $\tilde{\Psi} = \begin{pmatrix} I_m & 0 \\ e^*K^{[-1]}e & I_m \end{pmatrix}$. (4.13)
\n
\ntrix Ψ is *J*-unitary, we obtain from (4.11)

where

$$
\Psi = \begin{pmatrix} I_m & e^* K \widetilde{K}^{[-1]} K e \\ 0 & I_m \end{pmatrix} \quad \text{and} \quad \widetilde{\Psi} = \begin{pmatrix} I_m & 0 \\ e^* K^{[-1]} e & I_m \end{pmatrix}.
$$
 (4.13)

$$
\tilde{\Theta}(z) = \left\{ I_{2m} + z \begin{pmatrix} e^* K \\ -e^* \end{pmatrix} (I - zF^*)^{-1} \tilde{K}^{[-1]}(e, Ke) \right\} \tilde{\Psi}
$$
(4.12)
where

$$
\Psi = \begin{pmatrix} I_m & e^* K \tilde{K}^{[-1]} K e \\ 0 & I_m \end{pmatrix} \text{ and } \tilde{\Psi} = \begin{pmatrix} I_m & 0 \\ e^* K^{[-1]} e & I_m \end{pmatrix}.
$$
(4.13)
Since the matrix Ψ is *J*-unitary, we obtain from (4.11)

$$
\Theta(z) J \Theta(\omega)^* - J = -i \begin{pmatrix} e^* K F^* \\ -e^* \end{pmatrix} (I - zF^*)^{-1} \Phi(z, \omega) (I - \overline{\omega} F)^{-1} (F K e, -e)
$$
(4.14)
where

where

$$
\Phi(z,\omega) = zK^{[-1]}(I - \overline{\omega}F) - \overline{\omega}(I - zF^*)K^{[-1]} + z\overline{\omega}K^{[-1]}\{FKee^* - ee^*KF^*\}K^{[-1]}.
$$
\n(4.15)

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Substituting the first equality from (4.10) into (4.15) and using (4.2) we get

$$
\Phi(z,\omega) = (z - \overline{\omega})K^{[-1]} + z\overline{\omega}\Big\{F^*K^{[-1]} - K^{[-1]}F + K^{[-1]}(FK - KF^*)K^{[-1]}\Big\}
$$

\n
$$
= (z - \overline{\omega})K^{[-1]} + z\overline{\omega}\Big\{(I - K^{[-1]}K)F^*K^{[-1]} - K^{[-1]}F(I - KK^{[-1]})\Big\}
$$

\n
$$
= (z - \overline{\omega})K^{[-1]}
$$

\n
$$
\text{h both with (4.14) implies (4.7). Similarly, taking into account the } J\text{-unitari}
$$

\n
$$
\tilde{\Theta}(z)J\tilde{\Theta}(\omega)^* - J = \begin{pmatrix} e^*K \\ -e^* \end{pmatrix}(I - zF^*)^{-1}\tilde{\Phi}(z,\omega)(I - \overline{\omega}F)^{-1}(Ke, -e)
$$

which both with (4.14) implies (4.7) . Similarly, taking into account the J-unitarity of Ψ we obtain from (4.8)

$$
\widetilde{\Theta}(z)J\widetilde{\Theta}(\omega)^{*}-J=\begin{pmatrix} e^{*}K\\-e^{*}\end{pmatrix}(I-zF^{*})^{-1}\widetilde{\Phi}(z,\omega)(I-\overline{\omega}F)^{-1}(K\mathbf{e},-\mathbf{e})\tag{4.16}
$$

$$
\widetilde{\Phi}(z,\omega)=z\widetilde{K}^{[-1]}(I-\overline{\omega}F)-\overline{\omega}(I-zF^{*})\widetilde{K}^{[-1]}\tag{4.17}
$$

$$
+\overline{z\overline{\omega}}\widetilde{K}^{[-1]}K\mathbf{e}\mathbf{e}^{*}-\mathbf{e}\mathbf{e}^{*}K^{[}\widetilde{K}^{[-1]}\tag{4.17}
$$

where

$$
\widetilde{\Phi}^{(z,\omega)} = z \widetilde{K}^{[-1]}(I - \overline{\omega}F) - \overline{\omega}(I - zF^*) \widetilde{K}^{[-1]} + z \overline{\omega} \widetilde{K}^{[-1]} \{Kee^* - ee^*K\} \widetilde{K}^{[-1]}.
$$
\n(4.17)

Substituting the second equality from (4.10) into (4.17) and using (4.2) we receive

$$
\widetilde{\Theta}(z)J\widetilde{\Theta}(\omega)^{*}-J=\begin{pmatrix} e^{*}K\\-e^{*}\end{pmatrix}(I-zF^{*})^{-1}\widetilde{\Phi}(z,\omega)(I-\overline{\omega}F)^{-1}(K\mathbf{e},-\mathbf{e})\qquad(
$$

\n
$$
\widetilde{\Phi}^{(z},\omega)=z\widetilde{K}^{[-1]}(I-\overline{\omega}F)-\overline{\omega}(I-zF^{*})\widetilde{K}^{[-1]}\qquad (
$$

\n
$$
+z\overline{\omega}\widetilde{K}^{[-1]}\{K\mathbf{e}\mathbf{e}^{*}-\mathbf{e}\mathbf{e}^{*}K\}\widetilde{K}^{[-1]}.
$$

\nstituting the second equality from (4.10) into (4.17) and using (4.2) we receive
\n
$$
\widetilde{\Phi}^{(z},\omega)=(z-\overline{\omega})\widetilde{K}^{[-1]}+z\overline{\omega}\Big\{F^{*}\widetilde{K}^{[-1]}-\widetilde{K}^{[-1]}F+\widetilde{K}^{[-1]}(F\widetilde{K}-\widetilde{K}F^{*})\widetilde{K}^{[-1]}\Big\}
$$

\n
$$
=(z-\overline{\omega})\widetilde{K}^{[-1]}+z\overline{\omega}\Big\{(I-\widetilde{K}^{[-1]}\widetilde{K})F^{*}\widetilde{K}^{[-1]}-\widetilde{K}^{[-1]}F(I-\widetilde{K}\widetilde{K}^{[-1]})\Big\}
$$

\n
$$
=(z-\overline{\omega})\widetilde{K}^{[-1]}
$$

which both with (4.17) leads to (4.8). From (4.7) and (4.8) we conclude that Θ and $\tilde{\Theta}$ belong to the class W, and hence, Θ belongs to W_{π} by Theorem 1.3

Remark 4.3: Since the functions Θ and $\widetilde{\Theta}$ are both *J*-unitary on the real axis, then by the symmetry principle, $\Theta^{-1}(z) = J\Theta(\overline{z})^*J$ and $\widetilde{\Theta}^{-1}(z) = J\widetilde{\Theta}(\overline{z})^*J$, and on account of (4.7) and (4.8), the relations

$$
= (z - \overline{\omega})\tilde{K}^{[-1]} + z\overline{\omega}\Big\{ (I - \tilde{K}^{[-1]}\tilde{K})F^*\tilde{K}^{[-1]} - \tilde{K}^{[-1]}F(I - \tilde{K}\tilde{K}^{[-1]}) \Big\}
$$

\n
$$
= (z - \overline{\omega})\tilde{K}^{[-1]}
$$

\nboth with (4.17) leads to (4.8). From (4.7) and (4.8) we conclude that Θ and $\widetilde{\Theta}$
\nto the class **W**, and hence, Θ belongs to **W** _{π} by Theorem 1.3 **1**
\nmark 4.3: Since the functions Θ and $\widetilde{\Theta}$ are both *J*-unitary on the real axis,
\n*t* the symmetry principle, $\Theta^{-1}(z) = J\Theta(\overline{z})^*J$ and $\widetilde{\Theta}^{-1}(z) = J\widetilde{\Theta}(\overline{z})^*J$, and on
\n*t* of (4.7) and (4.8), the relations
\n
$$
J - \Theta(\omega)^{-*}J\Theta^{-1}(z)
$$

\n
$$
= J(J - \Theta(\overline{\omega})J\Theta(\overline{z})^*)J
$$

\n
$$
= i(\overline{\omega} - z)\begin{pmatrix} e^* \\ e^*KF^* \end{pmatrix}(I - \overline{\omega}F^*)^{-1}K^{[-1]}(I - zF)^{-1}(e, FKe)
$$

\n
$$
\omega)^*JP(z) - \Theta(\omega)^{-*}P(\omega)^*JP(z)\Theta^{-1}(z)
$$

\n
$$
= i(\overline{\omega} - z)P(\omega)^*\begin{pmatrix} e^* \\ e^*K \end{pmatrix}(I - \overline{\omega}F^*)^{-1}\tilde{K}^{[-1]}(I - zF)^{-1}(e, Ke)P(z)
$$

\n
$$
= i(\overline{\omega} - z)P(\omega)^*\begin{pmatrix} e^* \\ e^*K \end{pmatrix}(I - \overline{\omega}F^*)^{-1}\tilde{K}^{[-1]}(I - zF)^{-1}(e, Ke)P(z)
$$

\n
$$
\tilde{\omega} = \tilde{\omega}(\tilde{\omega}
$$

and

$$
P(\omega)^* JP(z)-\Theta(\omega)^{-*}P(\omega)^*JP(z)\Theta^{-1}(z)
$$

= $i(\overline{\omega}-z)P(\omega)^*\begin{pmatrix} e^*\\ e^*K \end{pmatrix}(I-\overline{\omega}F^*)^{-1}\widetilde{K}^{[-1]}(I-zF)^{-1}(e,Ke)P(z)$ (4.19)

are true.

For the further purposes we need those J-forms of Θ and $\widetilde{\Theta}$ which are dual to (4.7) and (4.8).

Lemma 4.4: Let Θ *and* $\widetilde{\Theta}$ *be the functions defined by (4.5) and (4.6), respectively.***
** $\Theta(\omega)^* J \Theta(z) - J = i(\overline{\omega} - z) R^* K^{[-1]} (I - \overline{\omega} F)^{-1} K (I - zF^*)^{-1} K^{[-1]} R$ **(4.20)
 \tilde{\Theta}(\omega)^* J \widetilde{\Theta}(z) - J = i(\overline{\omega} - z) \widetilde{R}^* \widetilde{K}** *Then be the functions defined by* (4.5) and (4.6), respectively.
 $-z)R^*K^{[-1]}(I - \overline{\omega}F)^{-1}K(I - zF^*)^{-1}K^{[-1]}R$ (4.20)
 $-z)\widetilde{R}^*\widetilde{K}^{[-1]}(I - \overline{\omega}F)^{-1}\widetilde{K}(I - zF^*)^{-1}\widetilde{K}^{[-1]}\widetilde{R}$ (4.21)

$$
\Theta(\omega)^* J \Theta(z) - J = i(\overline{\omega} - z) R^* K^{[-1]} (I - \overline{\omega} F)^{-1} K (I - z F^*)^{-1} K^{[-1]} R \quad (4.20)
$$

$$
\tilde{\Theta}(\omega)^* J \tilde{\Theta}(z) - J = i(\overline{\omega} - z) \widetilde{R}^* \widetilde{K}^{[-1]} (I - \overline{\omega} F)^{-1} \widetilde{K} (I - zF^*)^{-1} \widetilde{K}^{[-1]} \widetilde{R} \quad (4.21)
$$

where

\n
$$
\text{Im}\mathbf{a} \cdot \mathbf{4} \cdot \text{Let } \Theta \text{ and } \widetilde{\Theta} \text{ be the functions defined by (4.5) and (4.6), respectively.}
$$
\n
$$
\Theta(\omega)^* J \Theta(z) - J = i(\overline{\omega} - z) R^* K^{[-1]} (I - \overline{\omega} F)^{-1} K (I - z F^*)^{-1} K^{[-1]} R \quad (4.20)
$$
\n
$$
\widetilde{\Theta}(\omega)^* J \widetilde{\Theta}(z) - J = i(\overline{\omega} - z) \widetilde{R}^* \widetilde{K}^{[-1]} (I - \overline{\omega} F)^{-1} \widetilde{K} (I - z F^*)^{-1} \widetilde{K}^{[-1]} \widetilde{R} \quad (4.21)
$$
\n
$$
R = (\mathbf{e}, F K \mathbf{e}) \Psi \qquad \text{and} \qquad \widetilde{R} = (\mathbf{e}, K \mathbf{e}) \widetilde{\Psi} \begin{pmatrix} I & 0 \\ -\mathbf{e}^* K^{[-1]} \mathbf{e} & I \end{pmatrix}. \qquad (4.22)
$$
\n

\n\n The original problem is given by:\n $\widetilde{R} = (\mathbf{e}, K \mathbf{e}) \widetilde{\Psi} \begin{pmatrix} I & 0 \\ -\mathbf{e}^* K^{[-1]} \mathbf{e} & I \end{pmatrix}.$ \n

Proof: Using the representation (4.7) of Θ and taking into account (4.10) and (4.21), we obtain

*O()*J0(z) - J = __iR*{z(I - zF*)_ ^l K] - JK ¹ (I - 7F)1 + z 3K[_ ¹](I - F)'{FK - KF*}(I - zF*)-1K[_11}R (4.23) = i(- z)R* Kt11 (I -F)'K(I - zF*)_lK(_hlR - izR*(I - KH']K)(I - zF)K1R + ic;YR*K(Th I(I - OF)-' (I - KK[-')R. (I - zF*)_ l =z'F', 1: (4.24)*

Since

$$
(I - zF^*)^{-1} = \sum_{j=0}^{n} z^j F^{*j},
$$
(4.24)

$$
(I - K^{[-1]}K)(I - zF^*)^{-1}K^{[-1]} = 0 \quad \text{for all } z \in \mathbb{C}.
$$

then by (4.3)

$$
(I - K^{[-1]}K)(I - zF^*)^{-1}K^{[-1]} = 0 \quad \text{for all } z \in \mathbb{C}.
$$

Substituting this last equality into *(4.23)* we obtain (4.20). The equality *(4.21)* can be checked quite similarly I

Remark **4.5:** Using (4.22), *(4.13)* and the equalities

ee^{*}
$$
K\widetilde{K}^{[-1]}K\mathbf{e} + FK\mathbf{e} = K\widetilde{K}^{[-1]}K\mathbf{e} + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}
$$

 $\mathbf{e} - K\mathbf{e}\mathbf{e}^*K^{[-1]}\mathbf{e} = (I - KK^{[-1]})\mathbf{e} + \widetilde{K}F^*K^{[-1]}\mathbf{e}$

which follow from (4.9) , we obtain

$$
R = \left(\mathbf{e}, K\widetilde{K}^{[-1]}K\mathbf{e} + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}\right) \tag{4.25}
$$

$$
\widetilde{R} = \left((I - KK^{[-1]})\mathbf{e} + \widetilde{K}F^*K^{[-1]}\mathbf{e}, K\mathbf{e} \right). \tag{4.26}
$$

Note that in view of (4.5), (4.6) and (4.24), Θ and $\widetilde{\Theta}$ are matrix polynomials of degree $n + 1$.

5. Description of all solutions

In this section we parametrize the set $\mathcal{Z}(\mathbf{H}_{2n+1})$ of all solutions to the degenerate Stieltjes moment problem in terms of a linear fractional transformation. We begin with the following auxiliary results. ions

set $\mathcal{Z}(\mathbf{H}_{2n+1})$ of

of a linear fractiona

et
 $=\begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}$
 e^{2m} -valued function 1 solutions to the degenerate

ransformation. We begin with
 $\in \mathbf{W}_{\pi}$ into four $\mathbb{C}^{m \times m}$ -valued

s s of the system of inequalities
 ${s(z) \choose l_m} \ge 0$ (5.1)
 ${s(z) \choose l_m} \ge 0$ (5.2)

Theorem 5.1 (see [10: §3]): *Let*

$$
\Theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}
$$

be the block decomposition of a $\mathbb{C}^{2m \times 2m}$ *valued function* $\Theta \in \mathbf{W}_{\pi}$ *into four* $\mathbb{C}^{m \times m}$ *valued blocks. Then the following statements are true.*

(i) All $\mathbb{C}^{m \times m}$ -valued meromorphic in $\mathbb{C}\backslash\mathbb{R}_+$ solutions s of the system of inequalities

$$
(s(z)^*, I_m) \frac{\Theta(z)^{-*} J \Theta^{-1}(z)}{i(\overline{z}-z)} \begin{pmatrix} s(z) \\ I_m \end{pmatrix} \ge 0
$$
 (5.1)

liary results.
\n(see [10: §3]): Let
\n
$$
\Theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix}
$$
\nposition of a $\mathbb{C}^{2m \times 2m}$ -valued function $\Theta \in \mathbb{W}_{\pi}$ into four $\mathbb{C}^{m \times m}$ -valued
\nallowing statements are true.
\n-valued meromorphic in $\mathbb{C}\setminus\mathbb{R}_+$ solutions s of the system of inequalities
\n
$$
(s(z)^*, I_m) \frac{\Theta(z)^{-*} J \Theta^{-1}(z)}{i(\overline{z}-z)} \begin{pmatrix} s(z) \\ I_m \end{pmatrix} \ge 0
$$
\n
$$
(s(z)^*, I_m) \frac{\Theta(z)^{-*} P(z)^* J P(z) \Theta^{-1}(z)}{i(\overline{z}-z)} \begin{pmatrix} s(z) \\ I_m \end{pmatrix} \ge 0
$$
\nfor (5.2)
\nnear fractional transformation (1.10) when the parameter $\{p, q\}$ varies
\nall Stieltjes pairs and satisfies the condition
\n
$$
\det \left(\theta_{21}(z) p(z) + \theta_{22}(z) q(z)\right) \neq 0.
$$
\n
$$
\{p, q\} \text{ and } \{p_1, q_1\} \text{ lead by (1.10) to the same function s if and only\nequivalent.}
$$

are given by the linear fractional transformation (1.10) when the parameter {p, q} varies in the set \overline{S}_m of all Stieltjes pairs and satisfies the condition

$$
\det \left(\theta_{21}(z)p(z)+\theta_{22}(z)q(z)\right)\not\equiv 0. \tag{5.3}
$$

(ii) Two pairs $\{p,q\}$ and $\{p_1,q_1\}$ lead by (1.10) to the same function s if and only *if these pairs are equivalent.*

Lemma 5.2 Let F and G be two orthogonal subspaces in $\mathbb{C}^{1 \times m}$ (i.e. $f g^* = 0$ for *all* $f \in \mathcal{F}$ and $g \in \mathcal{G}$). Let $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ (5.2)

fractional transformation (1.10) when the parameter {p, q} varies

iteltjes pairs and satisfies the condition

det $(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \neq 0$. (5. $\{p_{12}(z)p(z) + \theta_{22}(z)q(z)\}\neq 0.$ (5.3)
 $\{p_{13}(z)p(z) + \theta_{22}(z)q(z)\} \neq 0.$ (5.3)
 $\{p_{24}(z)p(z) + \theta_{22}(z)q(z)\} \neq 0.$ (5.4)
 $\{p_{15}(z)p(z) \neq 0\}$ and $\{p_{15}(z)p(z)\} = \{p_{25}(z)p(z)\} = 0$ (5.4)
 $\{p_{15}(z)p(z)\} = \{p_{25}(z)p(z)\} = 0$ (5.5)

$$
\dim \mathcal{F} = \mu \qquad \qquad \text{and} \qquad \qquad \dim \mathcal{G} = \nu \qquad (5.4)
$$

and let $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ be the orthogonal projections onto $\mathcal F$ and $\mathcal G$, *respectively. Then every pair* $\{p,q\} \in \overline{S}_m$ *such that*

$$
P_{\mathcal{F}} p(z) \equiv P_{\mathcal{G}} q(z) \equiv 0 \tag{5.5}
$$

is equivalent to a pair $\{p_1, q_1\}$ *of the form*

e set
$$
\overline{S}_m
$$
 of all Stieltjes pairs and satisfies the condition
\n
$$
\det (\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \neq 0.
$$
\n(i) 10) Two pairs {p, q} and {p₁, q₁} lead by (1.10) to the same function s if and only
\nsee pairs are equivalent.
\nLemma 5.2 Let F and G be two orthogonal subspaces in $\mathbb{C}^{1 \times m}$ (i.e. $fg^* = 0$ for
\n $\in \mathbb{F}$ and $g \in \mathcal{G}$). Let
\n
$$
\dim \mathcal{F} = \mu
$$
 and $\dim \mathcal{G} = \nu$ (5.4)
\nlet $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ be the orthogonal projections onto F and G, respectively. Then every
\n $\{p,q\} \in \overline{S}_m$ such that
\n
$$
P_{\mathcal{F}} p(z) \equiv P_{\mathcal{G}} q(z) \equiv 0
$$
\n(i.5)
\nuivalent to a pair {p₁, q₁} of the form
\n
$$
p_1(z) = V \begin{pmatrix} \hat{p}(z) \\ 0_{\mu} \\ I_{\nu} \end{pmatrix}
$$
 and $q_1(z) = V \begin{pmatrix} \hat{q}(z) \\ I_{\mu} \\ 0_{\nu} \end{pmatrix}$ (5.6)
\n
$$
p_1(z) = V \begin{pmatrix} \hat{p}(z) \\ 0_{\mu} \\ I_{\nu} \end{pmatrix}
$$
 and $q_1(z) = V \begin{pmatrix} \hat{q}(z) \\ I_{\mu} \\ 0_{\nu} \end{pmatrix}$ (5.6)
\n
$$
p_1(z) = V \begin{pmatrix} \hat{p}(z) \\ 0_{\mu} \\ I_{\nu} \end{pmatrix}
$$
 and $q_1(z) = V \begin{pmatrix} \hat{q}(z) \\ I_{\mu} \\ 0_{\nu} \end{pmatrix}$ (5.6)
\n
$$
p_1(z) = V \begin{pmatrix} \hat{p}(z) \\ 0_{\mu} \\ I_{\nu} \end{pmatrix}
$$
 and $q_1(z) = V \begin{pmatrix} \hat{q}(z) \\ I_{\$

for some unitary $(m \times m)$ *-matrix V which depends only on F and G.*

Proof: Since the subspaces $\mathcal F$ and $\mathcal G$ are orthogonal, there exists a unitary matrix $V \in \mathbb{C}^{m \times m}$ such that

$$
P_{\mathcal{F}} p(z) \equiv P_{\mathcal{G}} q(z) \equiv 0 \qquad (5.5)
$$

uivalent to a pair $\{p_1, q_1\}$ of the form

$$
p_1(z) = V \begin{pmatrix} \hat{p}(z) & 0_{\mu} & 0 \\ 0_{\mu} & I_{\nu} \end{pmatrix} \qquad and \qquad q_1(z) = V \begin{pmatrix} \hat{q}(z) & 0_{\mu} & 0 \\ 0_{\mu} & 0_{\nu} \end{pmatrix} \qquad (5.6)
$$

some unitary $(m \times m)$ -matrix V which depends only on F and G.
Proof: Since the subspaces F and G are orthogonal, there exists a unitary matrix

$$
\mathbb{C}^{m \times m}
$$
 such that

$$
V^* P_{\mathcal{F}} V = \begin{pmatrix} 0_{m-\mu-\nu} & 0_{\mu} & 0 \\ 0_{\mu} & 0_{\nu} \end{pmatrix} \qquad and \qquad V^* P_{\mathcal{G}} V = \begin{pmatrix} 0_{m-\mu-\nu} & 0_{\mu} & 0 \\ 0_{\mu} & I_{\nu} \end{pmatrix}
$$

with μ and ν given by (5.4). Therefore, the Stieltjes pair $\{V^*p, V^*q\}$ satisfies

V. Bolotnikov
\nand
$$
\nu
$$
 given by (5.4). Therefore, the Stieltjes pair $\{V^*p, V^*q\}$ satisfies
\n
$$
(0 \quad I_{\mu} \quad 0_{\mu \times \nu}) V^*p(z) \equiv 0 \quad \text{and} \quad (0 \quad 0_{\nu \times \mu} \quad I_{\nu}) V^*q(z) \equiv 0
$$
\n
$$
P(\text{cos } (3) \text{ Isomme } 4, 3)
$$

and hence (see [3: Lemma 4.3]), it is equivalent to some Stieltjes pair of the form

by (5.4). Therefore, the Stieltjes pair
$$
\{V^*p, V^*p(z) \equiv 0 \text{ and } (0 \quad 0_{\nu \times \mu} I_{\nu})
$$

\nerman 4.3]), it is equivalent to some Stieltjes
\n
$$
\left\{ \begin{pmatrix} \hat{p}(z) & 0 \\ 0_{\mu} & I_{\nu} \end{pmatrix}, \begin{pmatrix} \hat{q}(z) & 0 \\ 0_{\nu} & 0_{\nu} \end{pmatrix} \right\}
$$

This in turn means that the initial pair $\{p, q\}$ is equivalent to a pair $\{p_1, q_1\}$ of the form (5.6)

The following theorem describes the set $S(H_{2n+1})$ under the assumption $\{K, \widetilde{K}\}\in$ \mathcal{K}^+ (or equivalently, Ker $K \subseteq \text{Ker } K$).

Theorem 5.3: Let $\{K, \tilde{K}\} \in \mathcal{K}^+$, let Q and \tilde{Q} be matrices satisfying (3.7) and (3.8), and let $K^{[-1]}$ and $\widetilde{K}^{[-1]}$ be the pseudoinverse matrices defined by (4.1). Then the *linear fractional transformation* (1.10) with the resolvent matrix Θ defined by (4.5) gives *a parametrization of the set* $S(H_{2n+1})$ when the parameter $\{p,q\}$ varies in \overline{S}_m (the set *of C"'-valued Stieltjes pairs) and is of the form* (1) I_i / (1) $\left(\begin{array}{c} I_{\nu} \end{array} \right)$

ans that the initial pair $\{p,q\}$ is equivalent to a p

g theorem describes the set $S(H_{2n+1})$ under the

ntly, Ker $K \subseteq$ Ker \widetilde{K}).

5.3: Let $\{K, \widetilde{K}\} \in \mathcal{K}^+$, let **Proposition 5.3:** Let and let $K^{[-1]}$ and fractional transformetrization of the x^m -valued Stieltjes
 $p(z) = V \begin{pmatrix} \hat{p}(z) \end{pmatrix}$ m describes the set $S(H_{2n+1})$ under the assumption $\{K, \tilde{I}\}\$
 $K \subseteq \text{Ker }\tilde{K}\}$.
 $\{K, \tilde{K}\} \in \mathcal{K}^+$, let Q and \tilde{Q} be matrices satisfying (3.7)
 $\tilde{K}^{[-1]}$ be the pseudoinverse matrices defined by (4.1 scribes the set $S(H_{2n+1})$ under the assumptio
 I Ker \widetilde{K}).
 \tilde{K} $\}\in \mathcal{K}^+$, let Q and \widetilde{Q} be matrices satisfyin
 J be the pseudoinverse matrices defined by (4.1

on (1.10) with the resolvent mat ⁴¹ and $K^{(-1)}$ be the pseudoinverse matrices defined by (4.1). Then the
ransformation (1.10) with the resolvent matrix Θ defined by (4.5) gives
of the set $S(H_{2n+1})$ when the parameter $\{p,q\}$ varies in \overline{S}_m (t

$$
p(z) = V \begin{pmatrix} \hat{p}(z) & 0_{\mu} & 0 \\ 0 & I_{\nu} \end{pmatrix} \quad \text{and} \quad q(z) = V \begin{pmatrix} \hat{q}(z) & 0 \\ I_{\mu} & 0_{\nu} \end{pmatrix} \quad (5.7)
$$

with a unitary matrix V (which depends only on the initial data H_{2n+1}) and a Stieltjes *pair* ${\hat{p}, \hat{q}} \in \overline{\mathcal{S}}_{m-\mu-\nu}$ where

$$
\mu = \text{rank } P_{\text{Ker } K} \text{ e} \qquad \text{and} \qquad \nu = \text{rank } P_{\text{Ker } \widetilde{K}} K \text{ e}. \tag{5.8}
$$

More precisely: every function $s \in S(H_{2n+1})$ is of the form (1.10) for some pair $\{p,q\} \in$ \overline{S}_m of the form (5.7). Conversely, for every pair $\{p,q\} \in \overline{S}_m$ of the form (5.7) the *transformation* (1.10) is well-defined $(\det(\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \neq 0)$ and leads to some $s \in S(\mathbf{H}_{2n+1})$. Two pairs lead by (1.10) to the same function s if and only if these *pairs are equivalent.*

Proof: According to Theorem 1.2 the set $S(H_{2n+1})$ coincides with the set of all solutions of the system of inequalities (1.12) and (1.13) which is equivalent, by Lemma 2.4, to the following system:

Proof: According to Theorem 1.2 the set
$$
S(H_{2n+1})
$$
 coincides with the set of all
olutions of the system of inequalities (1.12) and (1.13) which is equivalent, by Lemma
4, to the following system:

$$
\frac{s(z) - s(z)^*}{z - \overline{z}} - (es(z) + FKe)^* (I - zF)^{-*} K^{[-1]} (I - zF)^{-1} (es(z) + FKe) \ge 0
$$

$$
\frac{zs(z) - \overline{z}s(z)^*}{z - \overline{z}} - (zes(z) + Ke)^* (I - zF)^{-*} \widetilde{K}^{[-1]} (I - zF)^{-1} (\overline{z}es(z) + Ke) \ge 0
$$
(5.9)

and

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\n
$$
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$$
\n
$$
P_{\text{Ker }K}(I - zF)^{-1}(e, FKe) \binom{s(z)}{I} \equiv 0 \tag{5.10}
$$
\n
$$
P_{\text{Ker }\widetilde{K}}(I - zF)^{-1}(e, Ke) \binom{zs(z)}{I} \equiv 0. \tag{5.11}
$$

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\n
$$
P_{\text{Ker }K}(I - zF)^{-1}(\mathbf{e}, FKe) \begin{pmatrix} s(z) \\ I \end{pmatrix} \equiv 0 \tag{5.10}
$$
\n
$$
P_{\text{Ker }\widetilde{K}}(I - zF)^{-1}(\mathbf{e}, K\mathbf{e}) \begin{pmatrix} z s(z) \\ I \end{pmatrix} \equiv 0. \tag{5.11}
$$
\nequalities (5.9) can be written as

It is easy to see that inequalities (5.9) can be written as

$$
(s(z)^*, I) \left\{ \frac{J}{i(\overline{z} - z)} - \begin{pmatrix} e^* \\ e^* K F^* \end{pmatrix} \right\}
$$

$$
(I - zF^*)^{-1} K^{-1} I(I - zF)^{-1} (e, FKe) \left\{ \begin{pmatrix} s(z) \\ I \end{pmatrix} \ge 0 \right\}
$$

$$
(\overline{z}s(z)^*, I) \left\{ \frac{J}{i(\overline{z} - z)} - \begin{pmatrix} e^* \\ e^* K \end{pmatrix} \right\}
$$

$$
(I - zF^*)^{-1} \widetilde{K}^{-1} I(I - zF)^{-1} (e, Ke) \left\{ \begin{pmatrix} zs(z) \\ I \end{pmatrix} \ge 0 \right\}
$$

which in turn, on account of (4.17) and (4.18), can be represented in the form (5.1) and (5.2) with the function Θ defined by (4.5) which is of the class \mathbf{W}_{π} by Lemma 4.1. According to Theorem *5.1,* all solutions *s* of the system (5.9) are parametrized by the linear fractional transformation (1.10) when the parameter $\{p, q\}$ varies in the set \overline{S}_m of all Stieltjes pairs and satisfies (5.3). It remains to choose among these solutions all functions *s* which satisfy also identities *(5.10)* and (5.11). The further proof is divided into three steps which we now detail. *PH* $\left(\frac{1}{2}F\right)^{n-1}(P - 2F)^{n-1}(e, Re)^{n-1}(P - 2F)^{n-1}(e, Re)^{n-1}(P - 2F)^{n-1}(e, Re)^{n-1}(P - 2F)^{n-1}(P - 2F$

Step 1: The function s of the form (1.10) satisfies the identities (5.10) and (5.11) if and only if the corresponding parameter {p, q} satisfies

$$
P_{\text{Ker }K} \operatorname{ep}(z) \equiv P_{\text{Ker }\widetilde{K}} K \operatorname{eq}(z) \equiv 0. \tag{5.12}
$$

Step 2: *If a pair* $\{p,q\} \in \overline{S}_m$ satisfies (5.12), then it also satisfies (5.3).

Step 3: *There exists a unitary matrix* $V \in \mathbb{C}^{m \times m}$ *such that every pair* $\{p,q\} \in \overline{S}_m$ satisfying (5.12) is equivalent to some pair $\{p_1, q_1\}$ of the form (5.6) with μ and ν defined *by (5.8). it also satisfies* (5.3).
 uch that every pair $\{p,q\} \in \overline{S}_m$
 form (5.6) *with* μ and ν defined

some pair $\{p,q\} \in \overline{S}_m$ which
 $\theta_{22}(z)q(z)\Big)^{-1}$

equivalent to
 $p(z)$
 $p(z)$
 $p(z)$
 $q(z)$ $\bigg) \equiv 0$,

Proof of Step 1: Let *s* be of the form (1.10) for some pair $\{p,q\} \in \overline{S}_m$ which satisfies the condition (5.3). Then

$$
\begin{pmatrix} s(z) \\ I \end{pmatrix} = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \left(\theta_{21}(z)p(z) + \theta_{22}(z)q(z) \right)^{-1}
$$

4.6), the identities (5.10) and (5.11) are equivalent to

$$
P_{\text{Ker } K}(I - zF)^{-1}(\mathbf{e}, FK\mathbf{e})\Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0
$$

and, in view of (4.6), the identities *(5.10)* and *(5.11)* are equivalent to

quivalent to some pair {
$$
p_1, q_1
$$
} of the form (5.6) with μ and ν defined
\n1: Let *s* be of the form (1.10) for some pair { p, q } $\in \overline{S}_m$ which
\non (5.3). Then
\n
$$
\binom{z}{I} = \Theta(z) \binom{p(z)}{q(z)} \left(\theta_{21}(z)p(z) + \theta_{22}(z)q(z) \right)^{-1}
$$
\n1, the identities (5.10) and (5.11) are equivalent to
\n
$$
P_{\text{Ker } K}(I - zF)^{-1}(e, FKe)\Theta(z) \binom{p(z)}{q(z)} \equiv 0
$$
\n(5.13)
\n
$$
P_{\text{Ker } K}(I - zF)^{-1}(e, Ke)\widetilde{\Theta}(z)P(z) \binom{p(z)}{q(z)} \equiv 0,
$$
\n(5.14)

$$
P_{\text{Ker }\widetilde{K}}(I-zF)^{-1}(\mathbf{e},K\mathbf{e})\widetilde{\Theta}(z)P(z)\begin{pmatrix}p(z)\\q(z)\end{pmatrix}\equiv 0,
$$
\n(5.14)

respectively. To simplify (5.13) and (5.14) we begin with identities

$$
\begin{aligned}\n\text{cov} \\
\text{SUSY} \text{SUSY} \text{CUSY} \text{CUSY
$$

which follow from (4.10) . Substituting (4.11) and (4.12) into (5.13) and (5.14) , respectively, and using (5.15) , we rewrite (5.13) and (5.14) as

$$
= \widetilde{K}(I - zF^*)^{-1} - (I - zF)^{-1}\widetilde{K}
$$

0). Substituting (4.11) and (4.12) into (5.13) and (5.14), respec-
), we rewrite (5.13) and (5.14) as

$$
P_{\text{Ker }K}(I - zF)^{-1}(I - KK^{[-1]})R\left(\frac{p(z)}{q(z)}\right) \equiv 0 \qquad (5.16)
$$

$$
P_{\text{Ker }\widetilde{K}}(I-zF)^{-1}(I-\widetilde{K}\widetilde{K}^{[-1]})\widetilde{R}P(z)\begin{pmatrix}p(z)\\q(z)\end{pmatrix}\equiv 0,\qquad(5.17)
$$

respectively, where R and \widetilde{R} are matrices given by (4.22). Using (2.4) one can rewrite (5.16) as

$$
P_{\text{Ker }K}(I - zF)^{-1}(I - KK^{[-1]})R\begin{pmatrix}p(z)\\q(z)\end{pmatrix} \equiv 0
$$

$$
P_{\text{Ker }\widetilde{K}}(I - zF)^{-1}(I - \widetilde{K}\widetilde{K}^{[-1]})\widetilde{R}P(z)\begin{pmatrix}p(z)\\q(z)\end{pmatrix} \equiv 0,
$$

where *R* and \widetilde{R} are matrices given by (4.22). Using (2.4) one

$$
\left(I + zP_{\text{Ker }K}F(I - zF)^{-1}(I - KK^{[-1]})\right)P_{\text{Ker }K}R\begin{pmatrix}p(z)\\q(z)\end{pmatrix} \equiv 0
$$

and since

$$
\det\Bigl(I+zP_{\text{Ker }K}F(I-zF)^{-1}(I-KK^{[-1]})\Bigr)\not\equiv 0,
$$

then (5.16) is equivalent to

 \mathcal{A}

matrices given by (4.22). Using (2.4) one can rewrite
\n
$$
zF)^{-1}(I - KK^{[-1]}) \qquad P_{\text{Ker }K}R\left(\frac{p(z)}{q(z)}\right) \equiv 0
$$
\n
$$
r \cdot KF(I - zF)^{-1}(I - KK^{[-1]}) \neq 0,
$$
\n
$$
P_{\text{Ker }K}R\left(\frac{p(z)}{q(z)}\right) \equiv 0. \tag{5.18}
$$
\nquivalent to
\n
$$
c_{\text{er }K}\widetilde{R}P(z)\left(\frac{p(z)}{q(z)}\right) \equiv 0. \tag{5.19}
$$
\n
$$
ad (4.26), (1.9) into (5.19) we get
$$
\n
$$
c_{\text{er }K}F(I - \widetilde{K}K^{[-1]})Keq(z) = 0 \tag{5.20}
$$
\n
$$
-KK^{[-1]})ep(z) + Keq(z) = 0. \tag{5.21}
$$

Similarly, the identity (5.17) is equivalent to

Hint to

\n
$$
P_{\text{Ker }K}R\left(\frac{p(z)}{q(z)}\right) \equiv 0. \tag{5.18}
$$
\n(5.17) is equivalent to

\n
$$
P_{\text{Ker }\widetilde{K}}\widetilde{R}P(z)\left(\frac{p(z)}{q(z)}\right) \equiv 0. \tag{5.19}
$$
\nto (5.18) and (4.26), (1.9) into (5.19) we get

\n
$$
P_{\text{Ker }K}\left(\mathbf{ep}(z) + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}q(z)\right) \equiv 0 \tag{5.20}
$$
\n
$$
P_{\text{Ker }\widetilde{K}}\left(z(I - KK^{[-1]})\mathbf{ep}(z) + K\mathbf{e}q(z)\right) \equiv 0. \tag{5.21}
$$
\nwhere

\n
$$
P_{\text{Ker }\widetilde{K}}\left(I - KK^{[-1]}\right) \text{ from the left, subtracting the obtained}
$$

Substituting (4.25) into (5.18) and (4.26), (1.9) into (5.19) we get

$$
P_{\text{Ker }K}\left(\mathbf{e}p(z) + F(I - \widetilde{K}\widetilde{K}^{[-1]})K\mathbf{e}q(z)\right) \equiv 0 \tag{5.20}
$$

$$
P_{\text{Ker }\widetilde{K}}\left(z(I - KK^{[-1]})\mathbf{e}p(z) + Keq(z)\right) \equiv 0. \tag{5.21}
$$

Multiplying (5.20) by $zP_{\text{Ker }\widetilde{K}}(I - KK^{[-1]})$ from the left, subtracting the obtained identity from (5.21) and using the assumption Ker $K \subseteq \text{Ker } \widetilde{K}$, we obtain $F(F(X - \tilde{K}\tilde{K}^{[-1]})Keg(z)) \equiv 0$ (5.20)
 $F(K^{[-1]})ep(z) + Keq(z) = 0.$ (5.21)
 $F(K^{[-1]})$ from the left, subtracting the obtained

assumption Ker $K \subseteq \text{Ker } \tilde{K}$, we obtain
 $(F - \tilde{K}\tilde{K}^{[-1]})\bigg\}P_{\text{Ker } \tilde{K}}Keg(z) \equiv 0.$ (5.22)

$$
\left\{I - zP_{\text{Ker }K}F(I - \widetilde{K}\widetilde{K}^{[-1]})\right\}P_{\text{Ker }\widetilde{K}}K\text{eq}(z) \equiv 0
$$

which is equivalent to

$$
P_{\text{Ker }\widetilde{K}} K \text{eq}(z) \equiv 0. \tag{5.22}
$$

Substituting (5.22) into (5.20) we conclude that $P_{\text{Ker }K}ep(z) \equiv 0$, which ends the proof of Step I.

Proof of Step 2: Let a pair $\{p,q\} \in \overline{S}_m$ satisfy conditions (5.12). We introduce a pair $\{x, y\}$ by

Degenerate Stieltjes Moment Problem 463

\nwe conclude that
$$
P_{\text{Ker } K}ep(z) \equiv 0
$$
, which ends the proof

\nair $\{p,q\} \in \overline{S}_m$ satisfy conditions (5.12). We introduce a

\n
$$
\begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}
$$
\n(5.23)

\nindeed, suppose that the point $\lambda \in \mathbb{C}_+$ and the non-zero

\ndet $\Theta(\lambda) \neq 0$ and

\n
$$
y(\lambda)h = 0.
$$
\n(5.24)

and show that det $y(z) \neq 0$. Indeed, suppose that the point $\lambda \in \mathbb{C}_+$ and the non-zero vector $h \in \mathbb{C}^{m \times 1}$ are such that det $\Theta(\lambda) \neq 0$ and

$$
y(\lambda)h = 0.\tag{5.24}
$$

Since

1.
$$
p, q \in \overline{S}_m
$$
 satisfy conditions (5.12). We into y , y by

\n
$$
\begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}
$$
\nNow that $\det y(z) \neq 0$. Indeed, suppose that the point $\lambda \in \mathbb{C}_+$ and the $h \in \mathbb{C}^{m \times 1}$ are such that $\det \Theta(\lambda) \neq 0$ and

\n
$$
y(\lambda)h = 0.
$$
\n
$$
h^*(p(\lambda)^*, q(\lambda)^*)\Theta(\lambda)^* J\Theta(\lambda) \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h = h^*(x(\lambda)^*, 0) J\begin{pmatrix} x(\lambda) \\ 0 \end{pmatrix} h = 0,
$$
\n
$$
0 \le h^*(p(\lambda)^*, q(\lambda)^*) J\begin{pmatrix} p(\lambda) \\ p(\lambda) \end{pmatrix} h
$$

then

$$
h^*(p(\lambda)^*, q(\lambda)^*)\Theta(\lambda)^* J\Theta(\lambda) \left(\frac{p(\lambda)}{q(\lambda)}\right) h = h^*(x(\lambda)^*, 0) J\left(\frac{x(\lambda)}{0}\right) h = 0,
$$

then

$$
0 \le h^*(p(\lambda)^*, q(\lambda)^*) J\left(\frac{p(\lambda)}{q(\lambda)}\right) h
$$

$$
= h^*(p(\lambda)^*, q(\lambda)^*) \{J - \Theta(\lambda)^* J\Theta(\lambda)\} \left(\frac{p(\lambda)}{q(\lambda)}\right) h
$$

$$
\le 0.
$$

Substituting (4.20) (with $z = \omega = \lambda$) into the last inequality we conclude that

$$
K(I - \lambda F^*)^{-1} K^{[-1]} R\left(\frac{p(\lambda)}{q(\lambda)}\right) h = 0
$$
 (5.25)
(where *R* is the matrix given by (4.22)). It follows from (4.11), (4.22) and (5.24) that

$$
K(I - \lambda F^*)^{-1} K^{[-1]} R\begin{pmatrix}p(\lambda) \\ q(\lambda)\end{pmatrix} h = 0
$$
\n(5.25)

(where R is the matrix given by (4.22)). It follows from (4.11) , (4.22) and (5.24) that

$$
= h^{(p(\lambda)^{*}, q(\lambda)^{*})}\{J - \Theta(\lambda)^{*}J\Theta(\lambda)\}\left(\frac{1}{q(\lambda)}\right)h
$$

\n
$$
\leq 0.
$$

\n
$$
\text{ating (4.20) (with } z = \omega = \lambda \text{) into the last inequality we conclude that}
$$

\n
$$
K(I - \lambda F^{*})^{-1} K^{[-1]}R\left(\frac{p(\lambda)}{q(\lambda)}\right)h = 0
$$
\n
$$
R \text{ is the matrix given by (4.22)). It follows from (4.11), (4.22) and (5.24) that}
$$

\n
$$
x(\lambda)h = (I_{m}, O_{m})\Theta(\lambda)\left(\frac{p(\lambda)}{q(\lambda)}\right)h
$$

\n
$$
= \left\{(I, e^{*}K\tilde{K}^{[-1]}Ke) + \lambda e^{*}KF^{*}(I - \lambda F^{*})^{-1}K^{[-1]}R\}\left(\frac{p(\lambda)}{q(\lambda)}\right)h.
$$

\n
$$
\lambda F^{*}(I - \lambda F^{*})^{-1} = (I - \lambda F^{*})^{-1} - I,
$$

\naccount of (5.25)
\n
$$
x(\lambda)h = \left\{(I, e^{*}K\tilde{K}^{[-1]}Ke) - e^{*}KK^{[-1]}R\}\left(\frac{p(\lambda)}{q(\lambda)}\right)h.
$$
\n(5.26)
\n
$$
\text{values}
$$

Since

$$
\lambda F^*(I - \lambda F^*)^{-1} = (I - \lambda F^*)^{-1} - I,
$$

then on account of (5.25)

$$
x(\lambda)h = \left\{ (I, e^*K\tilde{K}^{[-1]}Ke) - e^*KK^{[-1]}R \right\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h.
$$
 (5.26)
ss

$$
K^{[-1]}F(I - \tilde{K}\tilde{K}^{[-1]}) = 0
$$
 and
$$
K^{[-1]}K\tilde{K}^{[-1]} = \tilde{K}^{[-1]}
$$

The equalities

$$
K^{\{-1\}}F(I - \tilde{K}\tilde{K}^{\{-1\}}) = 0
$$
 and $K^{\{-1\}}K\tilde{K}^{\{-1\}} = \tilde{K}^{\{-1\}}$

(see (4.5)) both with (5.23) lead to

$$
KK^{[-1]}R = \left(KK^{[-1]}\mathbf{e}, K\widetilde{K}^{[-1]}K\mathbf{e}\right).
$$

Substituting this last equality into (5.26) and using (2.4) we obtain

$$
x(\lambda)h = \left\{ e^*(I - KK^{[-1]})e, 0 \right\} \begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h
$$

= $e^*(I - KK^{[-1]})P_{\text{Ker }K}ep(\lambda)h$
= 0.

Since det $\Theta(\lambda) \neq 0$, the equality $x(\lambda)h = 0$ both with (5.24) and (5.23) implies

$$
\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} h = \Theta^{-1}(\lambda) \begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix} h = 0
$$

which contradicts (since λ is an arbitrary point in \mathbb{C}_+) the non-degeneracy of the Stieltjes pair $\{p,q\}$. $\neq 0$, the equality $x(\lambda)h = 0$ both with (5.24) and (5.23) implies
 $\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}h = \Theta^{-1}(\lambda)\begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix}h = 0$.
 \therefore its (since λ is an arbitrary point in \mathbb{C}_+) the non-degeneracy of the Stieltj

Proof of Step 3: Let us consider the subspaces $\mathcal F$ and $\mathcal G$ defined by

$$
\mathcal{F} = \text{Ran}(P_{\text{Ker }K} \mathbf{e}) \quad \text{and} \quad \mathcal{G} = \text{Ran}\left(P_{\text{Ker }\widetilde{K}} K \mathbf{e}\right). \quad (5.27)
$$

For such a choice of F and G the equalities (5.4) and (5.5) are equivalent to (5.8) and (5.12), respectively. Meaning to apply Lemma 5.2 we show that the subspaces $\mathcal F$ and G in (5.27) are orthogonal. Indeed, let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then $f = \tilde{f}e$ and $g = \tilde{g}K\mathbf{e}$ for some vectors $\tilde{f} \in \text{Ker } K$ and $\tilde{q} \in \text{Ker } \tilde{K}$. Using (4.9) we obtain

$$
fg^* = \tilde{f}ee^*K\tilde{g}^* = \tilde{f}K\tilde{g}^* - \tilde{f}(I - ee^*)K\tilde{g}^* = \tilde{f}K\tilde{g}^* - \tilde{f}F\tilde{K}\tilde{g}^* = 0.
$$

The application of Lemma 5.2 to subspaces (5.27) finishes the proof of Step 3.

By Theorem 5.1, different pairs lead under the transformation (1.10) to the same s if and only if they are equivalent. Hence, instead of all Stieltjes pairs satisfying (5.12) we can substitute in (1.10) all pairs of the form (5.7) . This ends the proof of Theorem 5.31

In view of Remark 2.8, the condition $\{K_n, \widetilde{K}_n\} \in \mathcal{K}_n^+$ is not restrictive and hence, the description obtained in Theorem 5.3 is applicable to the general situation $\{K_n, \tilde{K}_n\} \in$ \mathcal{K}_n .

6. Even Stieltjes moment problem

As it was mentioned above the Stieltjes moment problem for $N = 2n$ is solvable if and only if the *information* matrices K_n and \widetilde{K}_{n-1} defined by **Degenerate Stieltjes Moment Problem**
 ieltjes moment problem

ioned above the Stieltjes moment problem for $N = 2n$ is solvable if and
 Knation matrices K_n and \widetilde{K}_{n-1} defined by
 $K_n = (s_{i+j})_{i,j=0}^n$ and \widetilde

$$
K_n = (s_{i+j})_{i,j=0}^n \quad \text{and} \quad \widetilde{K}_{n-1} = (s_{i+j+1})_{i,j=0}^{n-1} \quad (6.1)
$$

are non-negative. Similarly to the odd case we denote by $\widetilde{\mathcal{K}}_n$ the set of all such pairs ${K_n, \tilde{K}_{n-1}}$. If, moreover, K_{n-1} admits a non-negative Hankel extension (i.e. if there exists a matrix $s_{2n+1} \in \mathbb{C}^{m \times m}$ such that the block matrix $\widetilde{K}_n = (s_{i+j+1})_{i,j=0}^n$ is still non-negative), then we say that the pair $\{K_n, \widetilde{K}_{n-1}\}$ belongs to $\widetilde{K}_n^+ \subset \widetilde{K}_n$. **(I)** the Stieltjes moment problem for $N = 2n$ is solvable if and

ices K_n and \widetilde{K}_{n-1} defined by
 $\widetilde{K}_{n-1} = (s_{i+j+1})_{i,j=0}^{n-1}$ (6.1)

to the odd case we denote by \widetilde{K}_n the set of all such pairs
 K_{n-1 *C* is a non-negative Hankel extension (i.e. if there
 C is a non-negative Hankel extension (i.e. if there

ic is still

ir $\{K_n, \tilde{K}_{n-1}\}$ belongs to $\tilde{K}_n^+ \subset \tilde{K}_n$.

ices K_n and \tilde{K}_{n-1} are of struct

only if

$$
(I - \mathbf{ee}^*)K_n B^* - C\widetilde{K}_{n-1} = 0 \tag{6.2}
$$

where

From the following equations:

\nFrom the following equations:

\nFrom the easily checked that matrices
$$
K_n
$$
 and K_{n-1} are of structure (6.1) if and only if

\n
$$
(I - ee^*)K_nB^* - C\widetilde{K}_{n-1} = 0
$$
\nwhere

\n
$$
e = \begin{pmatrix} I_m \\ 0_{mn \times m} \end{pmatrix}, \qquad B = \begin{pmatrix} 0_{m \times mn} \\ I_{mn} \end{pmatrix}, \qquad C = (I_{mn}, 0_{m \times mn})
$$
\n(6.3)

\n(an "even" analogue of Lemma 2.1). Note that $F_{m,n} = CB$, where $F_{m,n}$ is a shift given

by (1.14).

We ommit proofs of the following lemmas which are obtained closely to the corresponding results from Sections 2 - 5.

Lemma 6.2: *Let* $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n$ *and let* $\widetilde{\mathcal{L}}$ *be the subspace of* $\mathbb{C}^{1 \times m}$ *defined as* $\widetilde{\mathcal{L}} = \left\{ f \in \mathbb{C}^{1 \times m} \middle| \begin{array}{c} (f_1, \ldots, f_{n-1}, f) \in \text{Ker } \widetilde{K}_{n-1} \\ \text{for some } f, \quad f \in \mathbb{C$

$$
(I - ee^*)K_nB^* - C\tilde{K}_{n-1} = 0
$$
 (6.2)

$$
\begin{pmatrix} I_m \\ 0_{mn \times m} \end{pmatrix}, B = \begin{pmatrix} 0_{m \times mn} \\ I_{mn} \end{pmatrix}, C = (I_{mn}, 0_{m \times mn})
$$
 (6.3)
gue of Lemma 2.1). Note that $F_{m,n} = CB$, where $F_{m,n}$ is a shift given
roots of the following lemmas which are obtained closely to the corre-
from Sections 2 - 5.
: Let $\{K_n, \tilde{K}_{n-1}\} \in \tilde{K}_n$ and let \tilde{L} be the subspace of $\mathbb{C}^{1 \times m}$ defined as

$$
\tilde{L} = \begin{cases} f \in \mathbb{C}^{1 \times m} \middle| (f_1, \ldots, f_{n-1}, f) \in \text{Ker } \tilde{K}_{n-1} \\ \text{for some } f_1, \ldots, f_{n-1} \in \mathbb{C}^{1 \times m} \end{cases}
$$
 (6.4)
ng statements are equivalent:

$$
-1 \in \tilde{K}_n^+.
$$

$$
\begin{cases} (s_{n+1}, \ldots, s_{2n+1})^T = 0 \\ \text{for } s \in \mathbb{C}^{n+1} \end{cases}
$$
 (6.5)
gative matrix $\tilde{L} \in \mathbb{C}^{m \times m}$ which vanishes on the subspace \tilde{L} and does
se choice of $K^{[-1]}$

Then the following statements are equivalent:

- (i) ${K_n, \widetilde{K}_{n-1}} \in \widetilde{\mathcal{K}}_n^+$.
- (ii) $\mathbf{P}_{K_{\text{err}}\widetilde{K}_{-}}$, $(s_{n+1},...,s_{2n+1})^{\top} = 0$
- *(iii) The block S* **2n** *is of the form*

$$
s_{2n} = (s_n, \dots, s_{2n-1}) K_{n-1}^{[-1]} (s_n, \dots, s_{2n-1})^* + \widetilde{L}
$$
 (6.6)

for some non-negative matrix $\widetilde{L} \in \mathbb{C}^{m \times m}$ *which vanishes on the subspace* \mathcal{L} and does not depend on the choice of $K_{n-1}^{[-1]}$.

 (iv) *There exists a measure* $d\sigma(\lambda) \geq 0$ *such that*

$$
a \text{triz } L \in \mathbb{C} \text{ which vanishes on}
$$
\n
$$
of K_{n-1}^{[-1]}.
$$
\n
$$
measure d\sigma(\lambda) \ge 0 \text{ such that}
$$
\n
$$
\int_{0}^{\infty} \lambda^{k} d\sigma(\lambda) = s_{k} \qquad (k = 0, \ldots, 2n).
$$

Lemma 6.3: Let $\{K_n, \widetilde{K}_{n-1}\}\in \widetilde{\mathcal{K}}_n$, let s_{2n} be of the form (6.6), let $\widetilde{\mathcal{L}}$ be the subspace defined by (6.4) and let s be an arbitrary $(m \times m)$ -matrix. Then order $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n$, let s_{2n} be of the form (6.6), let $\widetilde{\mathcal{L}}$ be the
 $s = \int_0^\infty \lambda^{2n} d\sigma(\lambda)$

(a) if and only if s admits a representation
 $s = (s_n, \ldots, s_{2n-1}) K_{n-1}^{[-1]}(s_n, \ldots, s_{2n-1})^* + \widetilde$

$$
s=\int\limits_{0}^{\infty}\lambda^{2n}d\sigma(\lambda)
$$

for some $\sigma \in \mathcal{Z}(\mathbf{H}_{2n})$ *if and only if s admits a representation*

$$
s = (s_n, \dots, s_{2n-1}) K_{n-1}^{[-1]} (s_n, \dots, s_{2n-1})^* + \widetilde{L}_0
$$
 (6.7)

for some non-negative matrix $\widetilde{L}_0 \in \mathbb{C}^{m \times m}$ *such that* $\widetilde{L}_0 \leq \widetilde{L}$ *and* $\widetilde{L}_0 |_{\widetilde{L}} = 0$.

If, moreover, $\widetilde{L}_0|_{\widetilde{L}^\perp} = \widetilde{L}|_{\widetilde{L}^\perp}$, then the Stieltjes moment problems associated with the *sets* $s = (s_n, \ldots, s_{2n-1})K_{n-1}^{[-1]}(s_n, \ldots, s_{2n-1})^* + \tilde{L}_0$

on-negative matrix $\tilde{L}_0 \in \mathbb{C}^{m \times m}$ such that $\tilde{L}_0 \leq \tilde{L}$ and $\tilde{L}_0|_{\tilde{L}} = 0$.

eover, $\tilde{L}_0|_{\tilde{L}^{\perp}} = \tilde{L}|_{\tilde{L}^{\perp}}$, then the Stieltj

$$
H_{2n} = \{s_0, \ldots, s_{2n-1}, s_{2n}\} \text{ and } H_{2n}^1 = \{s_0, \ldots, s_{2n-1}, s\}
$$

have the same solutions: $\mathcal{Z}(\mathbf{H}_{2n}) = \mathcal{Z}(\mathbf{H}_{2n}^1)$.

Lemma **6.4:** Let $\{K_n, \widetilde{K}_{n-1}\}\in \widetilde{\mathcal{K}}_n^+$, let rank $K_n = r$ and rank $\widetilde{K}_{n-1} = \widetilde{r}$, and *let B and C be matrices given by* (6.3). Then there exist matrices $Q \in \mathbb{C}^{r \times m(n+1)}$ and $\widetilde{Q} \in \mathbb{C}^{\widetilde{r} \times mn}$ such that *matrix* $\tilde{L}_0 \in \mathbb{C}^{m \times m}$ *such that* $\tilde{L}_0 \leq \tilde{L}$ and $\tilde{L}_0 | \tilde{\zeta} = 0$.
 $= \tilde{L} | \tilde{\zeta}_\perp$, then the Stieltjes moment problems associated with the
 \ldots, s_{2n-1}, s_{2n} and $\mathbf{H}^1_{2n} = \{s_0, \ldots, s_{2n-$ *Latrix* $L_0 \in \mathbb{C}^{m \times m}$ such that $L_0 \leq L$ and $L_0|_{\widetilde{\mathcal{L}}} = 0$.
 $=\widetilde{L}|_{\widetilde{\mathcal{L}}^{\perp}}$, then the Stieltjes moment problems associated with the
 L_0, S_{2n-1}, S_{2n} and $H_{2n}^1 = \{s_0, \ldots, s_{2n-1}, s\}$
 \mathcal{Z} *s* given by (6.3). Then there exist matrices $Q \in \mathbb{C}^{r \times m(n+1)}$ and
 $QK_nQ^* > 0$ and $\widetilde{Q}\widetilde{K}_{n-1}\widetilde{Q}^* > 0$ (6.8)
 $QC = N\widetilde{Q}$ and $\widetilde{Q}B = \widetilde{N}Q$ (6.9)
 \vdots $\mathbb{C}^{r \times \tilde{r}}$ and $\widetilde{N} \in \mathbb{C}^{\tilde{r$

$$
QK_nQ^* > 0 \qquad \text{and} \qquad \widetilde{Q}\widetilde{K}_{n-1}\widetilde{Q}^* > 0 \tag{6.8}
$$

$$
QC = N\widetilde{Q} \qquad \text{and} \qquad \widetilde{Q}B = \widetilde{N}Q \tag{6.9}
$$

for some matrices $N \in \mathbb{C}^{r \times \tilde{r}}$ and $\tilde{N} \in \mathbb{C}^{\tilde{r} \times r}$. In other words, there exist subspaces $Q = \text{Ran } Q$ and $\widetilde{Q} = \text{Ran } \widetilde{Q}$ such that $QC \subseteq \widetilde{Q}$ and $\widetilde{Q}B \subseteq Q$, and and $\widetilde{Q}\widetilde{K}_n$.

and $\widetilde{Q}B =$
 $\mathbb{C}^{\tilde{r}\times r}$. In otl
 $C \subseteq \widetilde{Q}$ and $\widetilde{Q}I$

and \mathbb{C}^{m} ,

seudoinverse r

and $\widetilde{K}_{n-1}^{[-1]}$

atrix given by

$$
\mathbb{C}^{m(n+1)} = \text{Ker } K_n \dot{+} \mathcal{Q} \qquad \text{and} \qquad \mathbb{C}^{mn} = \text{Ker } \widetilde{K}_{n-1} \dot{+} \widetilde{\mathcal{Q}}. \tag{6.10}
$$

Lemma $\mathbf{6.5:} \,\,\, Let \,\{K_n,\widetilde{K}_{n-1}\} \in \widetilde{\mathcal{K}}_n^+, \,\, let \,Q \,\, and \,\, \widetilde{Q} \,\, be \,\,matrices \,\, satisfying \,\, conditions$ (6.8) and (6.9) , let $K_n^{[-1]}$ and $\widetilde{K}_{n-1}^{[-1]}$ be pseudoinverse matrices defined as *K* and $Q = \text{Ran } Q$ such that $Q($
 $\mathbb{C}^{m(n+1)} = \text{Ker } K_n + Q$
 na 6.5 : Let $\{K_n, \widetilde{K}_{n-1}\} \in \widetilde{K}_n^+$

(6.9), let $K_n^{[-1]}$ and $\widetilde{K}_{n-1}^{[-1]}$ be p
 $K_n^{[-1]} = Q^*(QK_nQ^*)^{-1}Q$

$$
K_n^{[-1]} = Q^*(QK_nQ^*)^{-1}Q \qquad \text{and} \qquad \widetilde{K}_{n-1}^{[-1]} = \widetilde{Q}^*(\widetilde{Q}\widetilde{K}_{n-1}\widetilde{Q}^*)^{-1}\widetilde{Q}
$$

and let $\Psi \in \mathbb{C}^{2m \times 2m}$ be the *J*-unitary matrix given by

Let
$$
\{K_n, \tilde{K}_{n-1}\} \in \tilde{K}_n^+
$$
, let Q and \tilde{Q} be matrices satisfying conditions
\n $t K_n^{[-1]}$ and $\tilde{K}_{n-1}^{[-1]}$ be pseudoinverse matrices defined as
\n $Q^*(QK_n Q^*)^{-1}Q$ and $\tilde{K}_{n-1}^{[-1]} = \tilde{Q}^*(\tilde{Q}\tilde{K}_{n-1}\tilde{Q}^*)^{-1}\tilde{Q}$
\n 2m be the J-unitary matrix given by
\n $\Psi = \begin{pmatrix} I_m & e^* K_n B^* \tilde{K}_{n-1}^{[-1]} B K_n e \\ 0 & I_m \end{pmatrix}$
\n $= \begin{pmatrix} I_m & (s_0, \ldots, s_{n-1}) \tilde{K}_{n-1}^{[-1]} (s_0, \ldots, s_{n-1})^\top \\ 0 & I_m \end{pmatrix}$
\n-valued function
\n $n + z \begin{pmatrix} e^* K_n F_m^* \\ -e^* \end{pmatrix} (I - z F_{m,n}^*)^{-1} K_n^{[-1]} (e, F_{m,n} K_n e) \end{pmatrix} \Psi$ (6.11)

Then the $\mathbb{C}^{2m \times 2m}$ *valued function*

$$
= \left(\begin{array}{cc} I_m & (s_0, \ldots, s_{n-1}) \widetilde{K}_{n-1}^{[-1]}(s_0, \ldots, s_{n-1})^\top \\ 0 & I_m \end{array}\right).
$$
\nthe $\mathbb{C}^{2m \times 2m}$ -valued function

\n
$$
\Theta(z) = \left\{ I_{2m} + z \begin{pmatrix} e^* K_n F_m^* \\ -e^* \end{pmatrix} (I - z F_{m,n}^*)^{-1} K_n^{[-1]}(e, F_{m,n} K_n e) \right\} \Psi \qquad (6.11)
$$

is of class W_{π} .

Theorem 6.6: Let $\{K_n, \widetilde{K}_{n-1}\}\in \widetilde{\mathcal{K}}_n^+$ (or equivalently, let $H_{2n}\in \mathcal{H}^+$). Then the *linear fractional transformation (1.10) with the resolvent matrix 0 defined by (6.10) gives a parametrization of the set* $S(H_{2n})$ when the parameter $\{p,q\}$ varies in \overline{S}_m (the set of $\mathbb{C}^{m \times m}$ -valued Stieltjes pairs) and is of the form (5.6) with a unitary matrix V (which **depends only 1.** Degenerate Stieltjes Moment Problem
 dependent on the initial data Fig. (3)
 depends only on the set $S(\mathbf{H}_{2n})$ *when the parameter {p, q} varies in* \overline{S}_m *(the
* $\mathbb{C}^{m \times m}$ *-valued Stieltjes pa* **p** Degenerate Stieltjes Moment Problem 467
 p 0.6: Let $\{K_n, \tilde{K}_{n-1}\} \in \tilde{K}_n^+$ (or equivalently, let $H_{2n} \in \mathcal{H}^+$). Then the

transformation (1.10) with the resolvent matrix Θ defined by (6.10) gives

n o *depends only on the initial data* H_{2n} *) and a Stieltjes pair* $\{\hat{p}, \hat{q}\} \in \overline{S}_{m-\mu-\nu}$ where Degenerate Stieltjes Moment Problem
 B $\in \widetilde{K}_n^+$ (or equivalently, let $H_{2n} \in \mathcal{H}^+$).

(0) with the resolvent matrix Θ defined by (6.
 b when the parameter $\{p,q\}$ varies in \overline{S}_m (t
 s of the form

$$
\mu = \operatorname{rank} P_{\operatorname{Ker} K_n} \mathbf{e} = \operatorname{rank} (I_m, 0, \dots, 0) P_{\operatorname{Ker} K}
$$
(6.12)

$$
\nu = \text{rank } P_{\text{Ker }\widetilde{K}_{n-1}} B K_n \mathbf{e} = \text{rank } (s_0, \dots, s_{n-1}) P_{\text{Ker }\widetilde{K}}.
$$
 (6.13)

In conclusion we note that Lemmas *1.6* and *1.7* follows immediately from Lemmas 2.7, 6.3 and 2.5, 6.2, respectively. Theorem 1.5 is a consequence of Theorems 5.3 and 6.6. Distinction of parameters in (1.11) and (5.6) is not essential: the linear fractional transformation with the resolvent matrix Θ *6.6.* Distinction of parameters in (1.11) and (5.6) is not essential: the linear fractional transformation with the resolvent matrix $\Theta \in W_{\pi}$ and parameters $\{p, q\}$ of the form (5.6) is equivalent to the linear fractional transformation with the resolvent matrix

$$
\hat{\Theta}(z) = \Theta(z) \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}
$$

and parameters of the form (1.11) . Since the matrix V is unitary, it is easy to see that the function $\ddot{\Theta}$ is still of the class \mathbf{W}_{π} .

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