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Degenerate Parabolic Differential Equations of Fourth Order and a Plasticity Model with Non-Local Hardening

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Abstract. By the means of energy estimates we prove existence and non-negativity results for degenerate parabolic partial differential equations of fourth order in arbitrary space dimensions. In addition, we present an elasto-viscoplastic model of Norton-Hoff type with isotropic, non-local hardening which takes the formation of dislocation patterns during plastic deformation into consideration. Finally, we give an existence result for this model.

Keywords: *Degenerate parabolic equations, existence of solutions of partial differential equations, non-negativity of solutions of partial differential equations, plasticity*

AMS subject classification: 35 K 35, 35 K 65, 73 E 05, 73 E 50

1. Introduction

In its first part this paper is concerned with existence and non-negativity results in arbitrary space dimensions for degenerate parabolic partial differential equations of fourth order of the following form:

$$\begin{aligned} \varrho_t + \operatorname{div}\Big(m(\varrho)\big(\nabla\Delta\varrho + \nabla A(\varrho)\big)\Big) &= f(\cdot,\cdot,\varrho) && \text{in } \Omega\times(0,T) \\ \frac{\partial}{\partial\nu}\varrho = \frac{\partial}{\partial\nu}\Delta\varrho &= 0 && \text{on } \partial\Omega\times(0,T) \\ \varrho(\cdot,0) &= \varrho_0(\cdot) && \text{in } \Omega \end{aligned} \tag{1.1}$$

We begin with a presentation of our general solution concept under comparatively strong requirements on the diffusion coefficient $m = m(\varrho)$ and the initial value ϱ_0. As applications we have in mind the Cahn-Hilliard equation and a model for the evolution of the dislocation density in the theory of plasticity. Afterwards, we shall investigate in which weaker sense we still can expect the existence of solutions of (1.1) if we do not impose the rather restrictive conditions mentioned above. By doing so, we generalize recent papers of F. Bernis and A. Friedman [3], and Yin Jingxue [16], which deal with similar problems in one space dimension, in the case of bounded diffusion coefficient m to arbitrary space dimensions, and in the case of unbounded diffusion coefficient m to

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the dimension $N = 2$ which is the relevant one for applications. Unfortunately, we still cannot answer the question about continuity or uniqueness of a solution of (1.1).

We devote the second part of this paper to a simple elasto-viscoplastic model with isotropic, non-local hardening. In our context, the attribute *non-local* refers to the fact that the growth of the hardening parameter does not depend directly on the plastic strain rate, but via a convolution operation on the dislocation density. We describe the evolution of the latter by a degenerate parabolic equation of type (1.1). Here, the gradient term has a destabilizing effect, and is therefore responsible for the expected pattern formation.

Let us now specify our requirements and results. We make the following assumptions on the data:

$$\Omega \subset \mathbb{R}^N \text{ is open with boundary } \partial\Omega \text{ of class } C^{1,1}. \tag{1.2}$$

The diffusion coefficient $m \in C^1(\mathbb{R}) \cap H^{1,\infty}(\mathbb{R})$ has the properties

$$m(s) \begin{cases} > 0 & \text{if } s > 0 \\ = 0 & \text{if } s = 0 \\ \geq 0 & \text{if } s < 0. \end{cases} \tag{1.3}$$

If $f \not\equiv 0$, we require in addition

$$\lim_{s \to \infty} m(s) > 0. \tag{1.4}$$

Let the function $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, fulfilling the estimate

$$|f(x,t,\varrho)| \leq C(1 + |\varrho|) \tag{1.5}$$

with a constant C independent of (x,t). Furthermore, we demand the existence of a constant $R_1 > 0$ with the property

$$f(\cdot,\cdot,\varrho) \geq 0 \quad \text{if } \varrho \leq R_1. \tag{1.6}$$

For a special constant $R > 0$ (we can choose $R = R_1$) let the function $G_0 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined by

$$G_0''(s) = \frac{1}{m(s)} \qquad \text{and} \qquad G_0'(R) = G_0(R) = 0. \tag{1.7}$$

For the initial value $\varrho_0 \in H^1(\Omega)$ we require

$$\int_\Omega G_0\big(\varrho_0(x)\big)\, dx < \infty. \tag{1.8}$$

Let the function $A : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and fulfilling the estimate

$$|A'(s)| \leq C \qquad \text{for every } s \in \mathbb{R}. \tag{1.9}$$

In Section 2 we shall prove the following existence result.

Theorem 1.1: *Under the assumptions* (1.2) – (1.9), *there exists a pair* (ϱ, J) *with the following properties:*

$$\begin{aligned} &\varrho \in W_2^1(I; H^1(\Omega), L^2(\Omega)) \cap L^2(I; H^2_*(\Omega)) \cap L^\infty(I; H^1(\Omega)) \\ &J \in L^2(\Omega_T; \mathbb{R}^N) \\ &\varrho_t = -\operatorname{div} J + f(\cdot,\cdot,\varrho) \quad \text{in } L^2(I; (H^1(\Omega))'). \end{aligned}$$

Moreover, J *fulfills the relation* $J = m(\varrho)(\nabla\Delta\varrho + \nabla A(\varrho))$ *in the following weak sense:*

$$\iint_{\Omega_T} J\eta = \iint_{\Omega_T} m(\varrho)\nabla A(\varrho)\eta - \iint_{\Omega_T} m(\varrho)\Delta\varrho \operatorname{div}\eta - \iint_{\Omega_T} m'(\varrho)\nabla\varrho\Delta\varrho\eta \qquad (1.10)$$

for all $\eta \in L^2(I; H^1(\Omega; \mathbb{R}^N)) \cap L^{2+N+\epsilon}(\Omega_T; \mathbb{R}^N)$ *with* $\eta\nu = 0$ *in* $\partial\Omega \times [0,T]$.

Eventually, the initial value ϱ_0 *is assumed* pre-continuously, *i.e., there exists a set* $E \subset I$ *with* $\mu(I - E) = 0$ *such that* $\varrho(\cdot, t)$ *converges to* ϱ_0 *in* $H^1(\Omega)$, *for* $t \in E, t \to 0$.

In contrast to non-degenerate parabolic differential equations of fourth order the solutions of which in general become negative even in the case of positive initial values, we expect for solutions of problem (1.1) under an appropriate choice of the initial values non-negativity (or even positivity) for (almost) all times. Here, the value of the growth exponent in the degeneracy is crucial. The larger n is, the stronger is the tendency towards positivity. We underline this in Section 3 by proving the following results.

Theorem 1.2: *Let* $n > 1$ *be the growth exponent of the diffusion coefficient* m *at zero. Under the same assumptions as in Theorem* 1.1 *we get the following statements:*

1) *For each* $n > 1$, ϱ *is non-negative almost everywhere on* Ω_T.

2) *If* $n \geq 2$, *then there does not exist any subset* $E \subset \Omega$ *with positive measure such that*

$$\int_E \varrho(x, t_0)\, dx = 0$$

for any $t_0 \in (0,T)$.

3) *If* $n = 2$ *or* $n > 2$, *then there exists a positive constant* C *depending only on the data and* T *which bounds for almost all* $t \in [0,T]$ *the integral*

$$\int_\Omega \log(\varrho(x,t))\, dx \qquad \text{or} \qquad \int_\Omega \varrho(x,t)^{2-n}\, dx,$$

respectively.

Moreover, depending on the dimension N we obtain the following positivity result.

Theorem 1.3: *Let* $N = 2$ *and* $n \geq 4$, *or* $N = 3$ *and* $n \geq 8$. *Under the assumptions of Theorem* 1 $\varrho(\cdot, t)$ *is positive on* Ω *for almost all* $t \in [0,T]$.

Let us remark that by appropriately choosing the initial values we can restrict the range of our solution of (1.1) to an intervall $[a, b]$, provided the diffusion coefficient m degenerates in a and b. This observation is crucial for the treatment of the *Cahn-Hilliard* *equation* (see, e.g., in [7] or [16]) which models phase separation in binary mixtures. In particular, we refer to the paper of C. M. Elliott and H. Garcke [7] which – as we learned after finishing the first part of our work – was developped parallelly in time and offers similar existence and non-negativity results.

Assumption (1.8) conceals a sign condition on ϱ_0 which for large n makes it impossible to apply our concept to problems with initial values of compact support. In Section 4 we free ourselves from the boundedness conditions on the diffusion coefficient m and on the integral $\int_\Omega G_0(\varrho_0)\,dx$, and under appropriate assumptions on A and f we shall obtain existence and non-negativity results in a weakened sense. In particular, we will see that for sufficiently large n the support of ϱ *does not shrink*. But on the other hand, we shall demonstrate that in general we cannot expect *growth* of the support in course of time.

In Section 5 we combine the classical equations for the description of elasto-viscoplastic material behaviour – as they are, e.g., expressed by the Norton-Hoff model with isotropic hardening – with a degenerate parabolic equation of similar type as (1.1). The latter is supposed to describe the evolution of the dislocation density and in particular to permit the modelling of the formation of patterns of cellular structure which is observed on the micro scale during plastic deformation. For more detailed information regarding modelling and mathematical results we refer the reader to Subsection 5.1.

Notation. By Ω_T we denote the space-time cylinder $\Omega \times (0,T)$, I stands for the time interval $(0,T)$, and μ denotes the Lebesgue measure. For subsets $E \subset \Omega$ we define the set

$$Q_\sigma(E,t_0) = \Big\{(x,t) \in \Omega_T \Big| \; x \in E \;\text{ and }\; t \in (t_0-\sigma, t_0+\sigma)\Big\}.$$

As usual, $⨍$ stands for mean-value integral, ν is the outer unit normal vector to Ω, and $W^{k,p}(\Omega)$ is the Sobolev space of k-times weakly differentiable functions in $L^p(\Omega)$, the weak derivatives of which are integrable to the power p. Furthermore, we make use of the following abbreviations:

- $H^m(\Omega) = W^{m,2}(\Omega)$
- $H^2_*(\Omega) = \Big\{u \in H^2(\Omega) : \dfrac{\partial}{\partial\nu}u = 0 \;\text{ on } \partial\Omega\Big\}$
- $\overline{H}^2_*(\Omega) = \Big\{u \in H^2_*(\Omega) : ⨍_\Omega u\,dx = 0\Big\}$
- $H^2_N(\Omega) = \Big\{\eta \in H^2(\Omega;\mathbb{R}^N) : \eta\nu = 0 \;\text{ on }\; \partial\Omega\Big\}$
- $V^3(\Omega) = \Big\{\eta \in H^3(\Omega;\mathbb{R}^2) : \;\operatorname{div}\eta \in H^2_*(\Omega) \;\text{ and }\; \eta\nu = 0 \;\text{ on }\; \partial\Omega\Big\}$
- $W^1_2(I;V,H) = \Big\{u \in L^2(I;V) : \; u_t \in L^2(I;V')\Big\}$, $V \subset H \subset V'$ an evolution triple.
- By the word *tensor* we always denote tensors of second order on $\mathbb{R}^3$. We do not make any distinction between covariant and contravariant components, and identify the set of tensors with $M^{3\times3}$, the set of (3×3)-matrices. Equipped with the trace

form, this set becomes an Euklidean vector space. $M^{3\times 3}_{\mathrm{sym}}$ is the subspace of the symmetric (3×3)-matrices.

- For a tensor field $\sigma \in H^1(\mathbb{R}^3; M^{3\times 3})$ we define the vector field $\operatorname{div}\sigma$ by the formula $(\operatorname{div}\sigma)_j = \sum_{i=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_i}$.
- By (u,v) we denote $\int_\Omega uv\,dx$. Here, the arguments u and v are either scalar, vector or tensor functions. We do not require that both factors are elements of $L^2(\Omega)$, but their product must be element of $L^1(\Omega)$.
- $\langle\cdot,\cdot\rangle$ denotes the dual pairing on $\left(H^1(\Omega)\right)' \times H^1(\Omega)$.
- $\langle\cdot,\cdot\rangle_{H^2_*(\Omega)}$ denotes the dual pairing on $\left(H^2_*(\Omega)\right)' \times H^2_*(\Omega)$.
- $[u>\delta] = \{x\in\Omega : u(x)>\delta\}$ and $[u>\delta]_T = \{(x,t)\in\Omega_T : u(x,t)>\delta\}$
- $\mathcal{L}(V,W)$ is the set of bounded linear operators from V to W.

2. Proof of existence for the model problem

Let us begin with an examination of the following auxiliary problems (P_δ) $(\delta>0)$ with non-degenerate non-linearity:

$$
\begin{aligned}
(\mathbf{P}_\delta)\quad \varrho^\delta_t + \operatorname{div}\Big(m_\delta(\varrho^\delta)\big(\nabla\Delta\varrho^\delta + \nabla A(\varrho^\delta)\big)\Big) &= f(\cdot,\cdot,\varrho^\delta) && \text{in } \ \Omega\times(0,T)\\
\frac{\partial}{\partial\nu}\varrho^\delta = \frac{\partial}{\partial\nu}\Delta\varrho^\delta &= 0 && \text{on } \ \partial\Omega\times(0,T)\\
m_\delta(x) &= m(x)+\delta.
\end{aligned}
$$

We shall solve the problems (P_δ) by use of the Faedo-Galerkin method. We observe that the eigenfunctions $(\Phi_k)_{k\in\mathbb{N}}$ of the Laplace operator with Neumann boundary values and meanvalue zero combined with the constant function $\Phi_0\equiv 1$ form a total and free subset of $H^2_*(\Omega)$. Under assumption (1.2) they are in particular smooth, and in addition the normal derivatives of $\Delta\Phi_k$ vanish on $\partial\Omega$. Thus they are a suitable choice as ansatz functions for the Galerkin method.

2.1 Application of the Faedo-Galerkin method to the auxiliary problems $(\mathbf{P}_\delta)$. Let S^k be defined by $S^k = \operatorname{span}\{\Phi_0,\Phi_1,\dots,\Phi_k\}$ and P^k be the orthogonal projection on S^k in $L^2(\Omega)$. By use of the Peano existence theorem we obtain – at least locally in time – the existence of functions $w_0,\dots,w_k$ which solve the following explicit system of ordinary differential equations:

$$
\varrho^\delta_k(\cdot,t) = \sum_{j=0}^{k} w_j(t)\Phi_j(\cdot) \quad \text{for all } \ t\in[0,T] \tag{2.1}
$$

$$
\varrho^\delta_k(\cdot,0) = P^k\varrho_0 \tag{2.2}
$$

$$
\frac{d}{dt}(\varrho^\delta_k,\Phi_j) - \Big(m_\delta(\varrho^\delta_k)\big(\nabla\Delta\varrho^\delta_k + \nabla A(\varrho^\delta_k)\big),\nabla\Phi_j\Big) = \Big(f(\cdot,\cdot,\varrho^\delta_k),\Phi_j\Big) \tag{2.3}
$$

for all $j = 0, \dots, k$ and $t \in (0,T)$.

Global solvability for arbitrary, but fixed T can be proven by use of a priori estimates. For this purpose we multiply the system above by $\varrho_k^\delta - \Delta\varrho_k^\delta$. This yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\varrho_k^\delta\|^2_{L^2(\Omega)} + \|\nabla\varrho_k^\delta\|^2_{L^2(\Omega)}\Big) &+ \Big(m_\delta(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta, \nabla\Delta\varrho_k^\delta\Big) \\ &= \Big(m_\delta(\varrho_k^\delta)\nabla A(\varrho_k^\delta), \nabla\varrho_k^\delta\Big) + \Big(m_\delta(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta, \nabla(\varrho_k^\delta - A(\varrho_k^\delta))\Big) \\ &\quad + \Big(f(\cdot,\cdot,\varrho_k^\delta), \varrho_k^\delta - \Delta\varrho_k^\delta\Big).\end{aligned} \tag{2.4}$$

We take advantage of the growth conditions on m (see (1.3)), f (see (1.5)) and A (see (1.9)), and obtain, by the use of the Young inequality, standard absorption techniques and the Gronwall lemma, the following a priori estimates depending on δ, but independent of k:

$$\begin{aligned} \varrho_k^\delta &\quad \text{uniformly lies in a bounded subset of } L^\infty(I; H^1(\Omega)) \\ m_\delta(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta &\quad \text{uniformly lies in a bounded subset of } L^2(\Omega_T; \mathbb{R}^N). \end{aligned} \tag{2.5}$$

In order to prove compactness in time of our approximate solutions ϱ_k^δ, we multiply equation (2.3) by $P^k v$ for arbitrary $v \in H^1(\Omega)$, and obtain for $t_0 \in [0,T]$ the estimate

$$\begin{aligned}((\varrho_k^\delta)_t, v) &= \Big(m_\delta(\varrho_k^\delta)(\nabla\Delta\varrho_k^\delta + \nabla A(\varrho_k^\delta)), \nabla P^k v\Big) + \Big(f(\cdot, t_0, \varrho_k^\delta), P^k v\Big) \\ &\le C\bigg(\|\nabla\varrho_k^\delta\|_{L^2(\Omega)}(t_0) + \Big(m_\delta(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta, \nabla\Delta\varrho_k^\delta\Big)^{1/2}(t_0) \\ &\quad + \|f(\cdot, t_0, \varrho_k^\delta)\|_{L^2(\Omega)}\bigg)\|P^k v\|_{H^1(\Omega)}.\end{aligned}$$

Since S^k is invariant under application of the Laplacian, the projectors P^k are uniformly bounded in $\mathcal{L}(H^1(\Omega), H^1(\Omega))$, and with the a priori estimates above we conclude that the following estimate is valid independently of k:

$$(\varrho_k^\delta)_t \quad \text{uniformly lies in a bounded subset of } L^2(I; (H^1(\Omega))'). \tag{2.6}$$

Observing the relation

$$((\nabla\varrho_k^\delta)_t, \eta) = -((\varrho_k^\delta)_t, \operatorname{div}\eta) = -((\varrho_k^\delta)_t, P^k\operatorname{div}\eta)$$

which is true for all $\eta \in H^2_N(\Omega)$, we continue with the estimate:

$$(\nabla\varrho_k^\delta)_t \quad \text{uniformly lies in a bounded subset of } L^2(I; (H^2_N(\Omega))'). \tag{2.7}$$

Let us remark that the last result is meaningless for the limit process $k \to \infty$. But we shall need it in a subtle way when δ tends to zero.

For the sake of completeness, let us now recall the following parabolic compactness and imbedding results.

Lemma 2.1 (Aubin - Lions; see [13: p. 57]): *Let B_0, B and B_1 be reflexive Banach spaces with compact imbedding $B_0 \hookrightarrow\hookrightarrow B$ and continuous imbedding $B \hookrightarrow B_1$. Then for $1 < p, q < \infty$ the Banach space*

$$W = \Big\{v : \ v \in L^p(I; B_0) \ \ and \ \ v_t \in L^q(I; B_1)\Big\}$$

is compactly imbedded into $L^p(I; B)$.

Definition 2.2 (see [18: p. 416]): We call $V \subset H \subset V'$ an *evolution triple* if the following conditions are fulfilled:

- V is a real, separable, reflexive Banach space
- H is a real, separable Hilbert space
- The imbedding $V \hookrightarrow H$ is continuous, and V is dense in H.

Lemma 2.3 (see [18: p. 422]): *Let $V \subset H \subset V'$ be an evolution triple, $1 < p < \infty$ and p' the dual exponent to p. Then the space*

$$W_p^1(I; V, H) = \Big\{u : \ u \in L^p(I; V) \ \ and \ \ u_t \in L^{p'}(I; V')\Big\}$$

is continuously imbedded into $C(I; H)$.

Now we have gathered all the informations necessary to let $k \to \infty$. By time integration of equation (2.3) we get for arbitrary $v \in L^2(I; H^1(\Omega))$ the equality

$$\begin{aligned} \int_0^T \langle (\varrho_k^\delta)_t, P^k v\rangle dt - \int_0^T \Big(m_\delta(\varrho_k^\delta)(\nabla A(\varrho_k^\delta) + \nabla\Delta\varrho_k^\delta), \nabla P^k v\Big) dt \\ = \int_0^T \Big(f(\cdot, t, \varrho_k^\delta), P^k v\Big) dt. \end{aligned} \tag{2.8}$$

From the a priori estimates (2.5) - (2.7) and the assumptions (1.3), (1.5), (1.9) we finally infer the following convergence results for a subsequence $(\varrho_k^\delta)_{k\in \mathbb{N}}$:

$$\begin{array}{rcll} \varrho_k^\delta & \to & \varrho^\delta & \text{pointwise almost everywhere and strongly in } L^2(\Omega_T) \\ (\varrho_k^\delta)_t & \rightharpoonup & \varrho_t^\delta & \text{weakly in } L^2(I; (H^1(\Omega))') \\ (\nabla\varrho_k^\delta)_t & \rightharpoonup & \nabla\varrho_t^\delta & \text{weakly in } L^2(I; (H_N^2(\Omega))') \\ \nabla\Delta\varrho_k^\delta & \rightharpoonup & \nabla\Delta\varrho^\delta & \text{weakly in } L^2(\Omega_T; \mathbb{R}^N) \\ f(\cdot,\cdot,\varrho_k^\delta) & \to & f(\cdot,\cdot,\varrho^\delta) & \text{strongly in } L^2(\Omega_T) \\ m_\delta(\varrho_k^\delta)(1 + A'(\varrho_k^\delta))v^k & \to & m_\delta(\varrho^\delta)(1 + A'(\varrho^\delta))v & \text{strongly in } L^2(\Omega_T) \end{array}$$

if $v^k \to v$ strongly in $L^2(\Omega_T)$ (for the last point we have made essential use of the boundedness of m_δ, A' and the Vitali convergence theorem, see, e.g., in [1: p. 46]). Thus passing to the limit in equation (2.8) and using Lemma 2.2 we find a function

$$\varrho^\delta \in W_2^1(I; H^1(\Omega), L^2(\Omega)) \cap L^2(I; H^3(\Omega)) \cap L^2(I; H_*^2(\Omega)) \cap C(I; H^1(\Omega))$$

with the properties

$$\int_0^T \langle \varrho_t^\delta, v\rangle dt - \int_0^T \Big(m_\delta(\varrho^\delta)(\nabla\Delta\varrho^\delta + \nabla A(\varrho^\delta)), \nabla v\Big) dt = \int_0^T \big(f(\cdot, t, \varrho^\delta), v\big) dt \tag{2.9}$$

for every $v \in L^2\big(I; H^1(\Omega)\big)$ and

$$\varrho^\delta(\cdot, 0) = \varrho_0(\cdot) \tag{2.10}$$

in Ω.

2.2 A priori estimates independent of $\delta > 0$. For the limit process $\delta \to 0$ we have to improve our a priori estimates decisively. Just modifying (2.5), (2.6) or (2.7) would not be sufficient because in this case we would not get any information concerning the $L^2(\Omega_T)$-norm of $\Delta\varrho^\delta$. A uniform estimate of the latter, however, is essential in order to identify a weak limit of $\big(m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta\big)_{\delta\to 0}$ in $L^2(\Omega_T)$ with a limit of $(\varrho^\delta)_{\delta\to 0}$ in the sense of (1.10).

Now we shall take advantage of a concept of F. Bernis and A. Friedman used in [3] for proving non-negativity of solutions of (1.1) in space dimension $N = 1$. Using peculiar primitives of the inverse of m_δ as test functions the authors get rid of m_δ in the elliptic part of the equation. In combination with a convexity argument, which in this context seems to be new, we shall apply this idea for proving a uniform bound on the $L^2\big(I; H^2_*(\Omega)\big)$-norm of ϱ_δ.

For fixed $R > 0$ let us consider the functions

$$g_\delta(t) = \int_R^t \frac{1}{m_\delta(s)}\, ds \qquad \text{and} \qquad G_\delta(t) = \int_R^t g_\delta(s)\, ds.$$

Remark 2.4: It is possible to choose $R = R_1$ (cf. (1.6)).

We have

$$g_\delta(t) \begin{cases} > 0 & \text{if } t > R \\ = 0 & \text{if } t = R \\ < 0 & \text{if } t < R \end{cases} \qquad \text{and} \qquad G_\delta(t) \begin{cases} > 0 & \text{if } t \neq R \\ = 0 & \text{if } t = R. \end{cases} \tag{2.11}$$

The following estimates are obvious:

$$|g_\delta(t)| \le \frac{1}{\delta}|t - R| \qquad \text{and} \qquad G_\delta(t) \le \frac{1}{\delta}|t - R|^2.$$

Especially in the case $t > R$ we can – using inequalities (1.4) and $R > 0$ – refine our estimate, and obtain with a constant C independent of $\delta > 0$ the estimate

$$|g_\delta(t)| \le C(t - R). \tag{2.12}$$

For the convex functions G_δ we have the following growth depending on $0 < \delta_1 < \delta_2$:

$$G_{\delta_2}(t) \le G_{\delta_1}(t) \le G_0(t) \qquad \text{for each } t \in \mathbb{R}. \tag{2.13}$$

Let us proceed further by testing equation (2.9) with $\varrho^\delta - \Delta\varrho^\delta + g_\delta(\varrho^\delta)$. Under the premises

$$\int_0^T \langle \varrho_t^\delta, \varrho^\delta \rangle dt = \frac{1}{2}\Big(\|\varrho^\delta\|^2_{L^2(\Omega)}(T) - \|\varrho^\delta\|^2_{L^2(\Omega)}(0)\Big) \tag{2.14}$$

$$\int_0^T \langle \varrho_t^\delta, \Delta\varrho^\delta \rangle dt = \frac{1}{2}\Big(\|\nabla\varrho^\delta\|^2_{L^2(\Omega)}(0) - \|\nabla\varrho^\delta\|^2_{L^2(\Omega)}(T)\Big) \tag{2.15}$$

$$\int_0^T \langle \varrho_t^\delta, g_\delta(\varrho^\delta) \rangle dt = \int_\Omega G_\delta\big(\varrho^\delta(x,T)\big)\, dx - \int_\Omega G_\delta\big(\varrho_0(x)\big)\, dx \tag{2.16}$$

which we are going to justify later on we obtain

$$\begin{aligned}
&\frac{1}{2}\|\varrho^\delta\|^2_{H^1(\Omega)}(T) + \int_\Omega G_\delta\big(\varrho^\delta(x,T)\big)\, dx \\
&\qquad + \int_0^T \Big(m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta, \nabla\Delta\varrho^\delta\Big)\, dt + \int_0^T \|\Delta\varrho^\delta\|^2_{L^2(\Omega)}\, dt \\
&\qquad \le \frac{1}{2}\|\varrho_0\|^2_{H^1(\Omega)} + \int_\Omega G_\delta\big(\varrho_0(x)\big)\, dx + \int_0^T \Big(\big(m_\delta(\varrho^\delta) + 1\big)\nabla A(\varrho^\delta), \nabla\varrho^\delta\Big) dt \\
&\qquad + \int_0^T \Big(m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta, \nabla\big(\varrho^\delta - A(\varrho^\delta)\big)\Big)\, dt + \int_0^T \Big(f(\cdot,\cdot,\varrho^\delta), \varrho^\delta - \Delta\varrho^\delta\Big)\, dt \\
&\qquad + \iint_{[\varrho^\delta > R]_T} f(\cdot,\cdot,\varrho^\delta) g_\delta(\varrho^\delta)\, dx dt \\
&\qquad = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6.
\end{aligned}$$

Here we have already made use of the non-positivity of the integral

$$\iint_{[\varrho^\delta \le R]_T} f(x,t,\varrho^\delta) g_\delta(\varrho^\delta)\, dx dt.$$

By the means of (1.8) and (2.13) it is possible to estimate $\mathcal{I}_1$ and $\mathcal{I}_2$ by a constant C just depending on ϱ_0. For the other terms we obtain – using the Young inequality, (2.12) and the assumptions (1.3) - (1.5) and (1.9) – the following inequalities with a constant C independent of $\delta > 0$:

$$\mathcal{I}_3 \le C \int_0^T \|\nabla \varrho^\delta\|^2_{L^2(\Omega)}\, dt$$

$$\mathcal{I}_4 \le \frac{1}{2} \int_0^T \left(m_\delta(\varrho^\delta) \nabla\Delta\varrho^\delta, \nabla\Delta\varrho^\delta \right) dt + C \int_0^T \|\nabla \varrho^\delta\|^2_{L^2(\Omega)}\, dt$$

$$\mathcal{I}_5 + \mathcal{I}_6 \le \frac{1}{2} \int_0^T \|\Delta \varrho^\delta\|^2_{L^2(\Omega)}\, dt + C \int_0^T \|\varrho^\delta\|^2_{L^2(\Omega)}\, dt.$$

Absorbing the terms containing $\nabla\Delta\varrho^\delta$ and $\Delta\varrho^\delta$ on the left-hand side, taking into account the non-negativity of G_δ and applying the Gronwall lemma, we finally arrive at the following a priori estimates independent of $\delta > 0$ [1]:

$$\left.\begin{array}{rl} \varrho^\delta & \text{uniformly lies in a bounded subset of} \\ & L^\infty(I; H^1(\Omega)) \cap L^2(I; H^2_*(\Omega)) \\ m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta & \text{uniformly lies in a bounded subset of } L^2(\Omega_T) \\ \displaystyle\int_\Omega G_\delta(\varrho^\delta(x,t))\, dx & \text{is uniformly bounded for } t \in [0,T]. \end{array}\right\} \qquad (2.17)$$

In the same way as in (2.6) and (2.7) we proceed, respectively, with the following estimates:

$$\begin{array}{rl} \varrho^\delta_t & \text{uniformly lies in a bounded subset of } L^2(I; (H^1(\Omega))') \\ \nabla\varrho^\delta_t & \text{uniformly lies in a bounded subset of } L^2(I; (H^2_N(\Omega))'). \end{array} \qquad (2.18)$$

Let us now present the following lemma as basis for our estimates above.

Lemma 2.5: *Let $u \in W^1_2(I; H^1(\Omega), L^2(\Omega))$ and let $G \in C^2(\mathbb{R}; \mathbb{R})$ be convex with linear growth of its derivative g. Then*

$$\int_{\tau_1}^{\tau_2} \langle u_t, g(u)\rangle\, dt = \int_\Omega G(u(x,\tau_2))\, dx - \int_\Omega G(u(x,\tau_1))\, dx$$

for arbitrary elements $\tau_1, \tau_2 \in I$.

Proof: By the convexity of G we have the inequalities

$$\frac{1}{h} \int_\Omega \Big(G(u(x,t)) - G(u(x,t-h)) \Big)\, dx \ge \int_\Omega \partial_t^h(u)\Big|_{t-h} g(u(x,t-h))\, dx$$

$$\frac{1}{h} \int_\Omega \Big(G(u(x,t)) - G(u(x,t-h)) \Big)\, dx \le \int_\Omega \partial_t^{-h}(u)\Big|_t g(u(x,t))\, dx.$$

---

[1] From the regularity theory for elliptic boundary value problems and the boundary regularity of Ω we infer that on $H^2_*(\Omega)$ the H^2-seminorm is equivalent to the L^2-norm of the Laplacian.

Here $\partial_t^h(u)$ and $\partial_t^{-h}(u)$ denote the forward and backward difference quotient of u with regard to time, respectively. Time integration of the first inequality from τ_1+h to τ_2+h and of the second inequality from τ_1 to τ_2 yields

$$\frac{1}{h}\left(\int_{\tau_2}^{\tau_2+h}\int_\Omega G(u)\,dxdt - \int_{\tau_1}^{\tau_1+h}\int_\Omega G(u)\,dxdt\right) \geq \int_{\tau_1}^{\tau_2}\int_\Omega \partial_t^h(u)g(u)\,dxdt$$
$$\frac{1}{h}\left(\int_{\tau_2-h}^{\tau_2}\int_\Omega G(u)\,dxdt - \int_{\tau_1-h}^{\tau_1}\int_\Omega G(u)\,dxdt\right) \leq \int_{\tau_1}^{\tau_2}\int_\Omega \partial_t^{-h}(u)g(u)\,dxdt.$$

Passing to the limit $h \to 0$ in these inequalities, using the linear growth of g and the regularity of u we finally obtain the result ∎

The premises (2.14) and (2.16) are simple consequences of Lemma 2.5. By minor modifications in its proof and by the fact that $\nabla\varrho^\delta$ is element of $C(I; L^2(\Omega; \mathbb{R}^N))$ we arrive at (2.15).

2.3 Limit process $\delta \to 0$. From the first a priori estimates of (2.17), the last one of (2.18) and the Aubin-Lions lemma we conclude that a subsequence $(\nabla\varrho^\delta)_{\delta\to 0}$ converges pointwise almost everywhere and strongly in $L^2(\Omega_T; \mathbb{R}^N)$ to a function $\nabla\varrho$. Using standard imbedding results for anisotropic Sobolev spaces we can improve this result furthermore. For this purpose we recall the following imbedding result (see, e.g., in [6: p. 10]).

Lemma 2.6: *Let Ω be of class $C^{1,1}$ and $v \in L^\infty(I; L^p(\Omega)) \cap L^p(I; W^{1,p}(\Omega))$. Then $v \in L^r(I; L^q(\Omega))$ and the estimate*

$$\|v\|_{q,r;\Omega_T} \leq \gamma\left(1+\frac{T}{|\Omega|^{p/N}}\right)^{1/r}\left(\|v\|_{p,\infty;\Omega_T} + \|\nabla v\|_{p;\Omega_T}\right)$$

is true. Here the constant γ only depends on N, p and the structure of $\partial\Omega$. Moreover, r, p, q and N have to fulfill the relations

$$\frac{1}{r} = \frac{N}{p^2} - \frac{N}{pq}$$

$$q \in \left[p, \frac{Np}{N-p}\right] \quad \textit{and} \quad r \in [p,\infty) \qquad \textit{if } 1 < p < N$$
$$q \in [p,\infty) \quad \textit{and} \quad r \in \left(\frac{p^2}{N}, \infty\right] \qquad \textit{if } 1 < N \leq p.$$

Combining this lemma with our convergence results for $\nabla\varrho^\delta$ and the Vitali convergence theorem we immediately obtain the following lemma.

Lemma 2.7: *There exists a subsequence $(\nabla \varrho^\delta)_{\delta\to 0}$ which for every $p < (2N+4)/N$ converges strongly in $L^p(\Omega_T; \mathbb{R}^N)$ to $\nabla\varrho$.*

Since there are simple counterexamples illustrating that the integrability of $|\Delta u|^2$ and $m(u)|\nabla\Delta u|^2$ does not imply that of $|\nabla\Delta u|$, we cannot expect – using only the estimates derived so far – the existence of spacial derivatives up to the order three for solutions of (1.1). Hence we proceed further in the following way.

Let

$$J_\delta = m_\delta(\varrho^\delta)\big(\nabla A(\varrho^\delta) + \nabla\Delta\varrho^\delta\big)$$

be the flux associated to ϱ^δ. From our a priori estimates (2.17) - (2.18) we derive the following convergence results for a subsequence $(\varrho^\delta)_{\delta\to 0}$:

$$\begin{array}{rcll} \varrho_t^\delta & \rightharpoonup & \varrho_t & \text{weakly in } L^2(I;(H^1(\Omega))') \\ \varrho^\delta & \to & \varrho & \text{strongly in } L^2(\Omega_T) \\ \nabla\varrho^\delta & \to & \nabla\varrho & \text{strongly in } L^p(\Omega_T;\mathbb{R}^N) \text{ if } p<(2N+4)/N \\ \nabla\varrho_t^\delta & \rightharpoonup & \nabla\varrho_t & \text{weakly in } L^2(I;(H_N^2(\Omega))') \\ J_\delta & \rightharpoonup & J & \text{weakly in } L^2(\Omega_T;\mathbb{R}^N) \\ f(\cdot,\cdot,\varrho^\delta) & \to & f(\cdot,\cdot,\varrho) & \text{strongly in } L^2(\Omega_T). \end{array}$$

Writing equation (2.9) in the form

$$\int_0^T \langle \varrho_t^\delta, v\rangle\, dt - \int_0^T (J_\delta, \nabla v)\, dt = \int_0^T \big(f(\cdot,t,\varrho^\delta), v\big)\, dt$$

we can pass to the limit for $\delta \to 0$, and obtain in $L^2(I;(H^1(\Omega))')$

$$\varrho_t = -\operatorname{div} J + f(\cdot,\cdot,\varrho).$$

In order to relate J to ϱ we note the identity

$$\int_0^T (J_\delta, \eta)\, dt = \int_0^T \Big(m_\delta(\varrho^\delta)\nabla A(\varrho^\delta), \eta\Big)\, dt$$
$$- \int_0^T \Big(m_\delta(\varrho^\delta)\Delta\varrho^\delta, \operatorname{div}\eta\Big)\, dt - \int_0^T \Big(m'(\varrho^\delta)\Delta\varrho^\delta\nabla\varrho^\delta, \eta\Big)\, dt$$

which is true for all vector functions $\eta \in L^2(I;(H^1(\Omega))^N) \cap L^q(\Omega_T)$ fulfilling $\eta\nu = 0$ on $\partial\Omega \times [0,T]$ and $q > 2+N$. Using once more the a priori estimates (2.17) - (2.18) we infer the convergences

$$\begin{array}{rcll} \Delta\varrho^\delta & \rightharpoonup & \Delta\varrho & \text{weakly in } L^2(\Omega_T) \\ m_\delta(\varrho^\delta)A'(\varrho^\delta)\eta & \to & m(\varrho)A'(\varrho)\eta & \text{strongly in } L^2(\Omega_T) \\ m'(\varrho^\delta)\eta & \to & m'(\varrho)\eta & \text{strongly in } L^q(\Omega_T;\mathbb{R}^N) \\ m_\delta(\varrho^\delta)\operatorname{div}\eta & \to & m(\varrho)\operatorname{div}\eta & \text{strongly in } L^2(\Omega_T). \end{array}$$

Thus J fulfills the relation

$$J = m(\varrho)\big(\nabla\Delta\varrho + \nabla A(\varrho)\big)$$

in the sense of (1.10).

For the proof of the *pre-continuity* of $\nabla\varrho(\cdot,t)$ in $t=0$ we first observe that $\nabla\varrho(\cdot,t)$ converges for $t\to 0$ weakly in $L^2(\Omega;\mathbb{R}^N)$ to $\nabla\varrho_0$. On the other hand, testing equation (2.9) with $\Delta\varrho^\delta$ and passing to the limit $\delta\to 0$ yields the estimate

$$\limsup_{\delta\to 0}\|\nabla\varrho^\delta(\cdot,t)\|^2_{L^2(\Omega)} \le \|\nabla\varrho_0\|^2_{L^2(\Omega)} + C\int_0^t \|\nabla\varrho\|^2_{L^2(\Omega)}dt + \int_0^t \big(f(\cdot,t,\varrho),\Delta\varrho\big)\,dt$$

with a constant C independent of t. Since $\nabla\varrho^\delta$ converges strongly in $L^2(\Omega_T;\mathbb{R}^N)$ to $\nabla\varrho$, there exists a set $E\subset I$ with the property $\mu(I-E)=0$ such that $\nabla\varrho^\delta(\cdot,t)$ converges for $t\in E$ and a sequence $\delta\to 0$ strongly in $L^2(\Omega,\mathbb{R}^N)$ to $\nabla\varrho(\cdot,t)$. Thus passing to the limit $t\to 0$ in the estimate above yields for $t\in E$

$$\limsup_{t\to 0}\|\nabla\varrho(\cdot,t)\|_{L^2(\Omega)} \le \|\nabla\varrho_0\|_{L^2(\Omega)}$$

hence $\lim_{t\to 0}\|\nabla\varrho(\cdot,t)\|_{L^2(\Omega)} = \|\nabla\varrho_0\|_{L^2(\Omega)}$ which proves Theorem 1.1.

3. Proof of the non-negativity results

Our proof of the Theorems 1.2 and 1.3 is based on the observation that the last a priori estimate of (2.17) conserves the non-negativity properties of the initial value for (almost) all times. For this reason let us have a closer look on G_0 in a positive neighbourhood of the origin. Consider

$$m(s) = |s|^n f_0(s) \qquad (n>1)$$

and

$$f_0 \in C\big([0,\infty)\big)\cap C\big((-\infty,0)\big) \qquad \text{with} \quad f_0\begin{cases} >0 & \text{in } [0,\infty) \\ \ge 0 & \text{in } (-\infty,0).\end{cases}$$

We cite from [3]:

$$G_0(t) = \begin{cases} A_0 + O(t^{2-n}) & \text{if } 1<n<2 \\ -A_1\log t + O(1) & \text{if } n=2 \\ A_2 t^{2-n} + R(t) & \text{if } n>2. \end{cases} \tag{3.1}$$

Here A_0, A_1 and A_2 are positive constants, only depending on f_0 and n, and $R(t)$ has the growth properties

$$R(t) = \begin{cases} O(t^{3-n}) & \text{if } n>3 \\ O(-\log t) & \text{if } n=3 \\ O(1) & \text{if } n<3. \end{cases}$$

Proof of Theorem 1.2: ad 1): Using the fact that with exception of a set of Lebesgue measure zero each point $x \in \Omega$ is a Lebesgue point of a function $u \in L^1(\Omega)$, it is sufficient to show that there does not exist any set $E \subset \Omega$ having for any $t \in [0,T]$ the property

$$\fint_E \varrho(x,t)\,dx < 0.$$

Let us argue by contradiction. Suppose there exist $E \subset \Omega$ and $t_0 \in (0,T)$ such that

$$\fint_E \varrho(x,t_0)\,dx = -\varepsilon < 0.$$

The inclusion $\varrho \in C\big(I; L^2(\Omega)\big)$ implies the existence of $\sigma > 0$ fulfilling

$$\fint\!\!\fint_{Q_\sigma(E,t_0)} \varrho(x,t)\,dxdt \le -\varepsilon' < 0.$$

Let $S \subset Q_\sigma(E,t_0)$ be defined as the set of all $(x,t) \in Q_\sigma(E,t_0)$ which fulfill $\varrho(x,t) \le -\varepsilon'$. Obviously, $\mu(S) > 0$.

Since the functions ϱ^δ converge for $\delta \to 0$ strongly in $L^2(\Omega_T)$ to ϱ, we can – perhaps after the selection of a subsequence – apply the Egorov theorem, and get for arbitrary $\gamma > 0$ a subset $S_\gamma \subset S$ with $\mu(S - S_\gamma) < \gamma$ and the property that $(\varrho^{\delta'})_{\delta' \to 0}$ converges uniformly on S_γ to ϱ. This yields for sufficiently small $\delta_0' > 0$

$$\varrho^{\delta'}(x,t) < -\varepsilon'' < 0 \qquad \text{for all } \; (x,t) \in S_\gamma,\; \delta' < \delta_0'.$$

On the other hand, we obtain by (2.11)

$$\begin{aligned}\iint_{Q_\sigma(E,t_0)} G_{\delta'}(\varrho^{\delta'}) &\ge \iint_{S_\gamma} G_{\delta'}(\varrho^{\delta'}) \\ &= \iint_{S_\gamma} \int_R^{\varrho^{\delta'}(x,t)} g_{\delta'}(s)\,dsdxdt \ge \mu(S_\gamma) \int_R^{-\varepsilon''} g_{\delta'}(s)\,ds.\end{aligned}$$

Using the monotone convergence theorem we conclude that the last term converges for $\delta' \to 0$ to

$$\mu(S_\gamma) \int_R^{-\varepsilon''} g_0(s)\,ds = +\infty.$$

This contradicts the third a priori estimate of (2.17).

ad 2): Suppose there exist nevertheless a subset $E \subset \Omega$ and a point $t_0 \in [0,T]$ having the properties described in Theorem 1.2. Then for each $\varepsilon > 0$ we could find $\sigma(\varepsilon) > 0$ such that

$$\fint\!\!\fint_{Q_{\sigma(\varepsilon)}(E,t_0)} \varrho(x,t)\,dxdt \le \frac{\varepsilon}{2}.$$

Statement 1) of Theorem 1.2 implies that the measure of the set $[\varrho \geq \varepsilon] \cap Q_{\sigma(\varepsilon)}(E, t_0)$ is bounded by $\sigma(\varepsilon)\mu(E)$. For this reason the set $S_\varepsilon = [\varrho < \varepsilon] \cap Q_{\sigma(\varepsilon)}(E, t_0)$ at least has measure $\sigma(\varepsilon)\mu(E)$. Applying the Egorov theorem once more we make ourselves sure of the existence of a subset $S'_\varepsilon \subset S_\varepsilon$ with measure $\frac{1}{2}\sigma(\varepsilon)\mu(E)$ such that a subsequence $(\varrho^{\delta'})_{\delta' \to 0}$ is uniformly convergent on S'_ε to ϱ. For sufficiently small δ'_0 we have on S'_ε that $0 \leq \varrho^{\delta'} < \varepsilon'$ for all $\delta' < \delta'_0$. Hence

$$\iint_{Q_{\sigma(\varepsilon)}(E,t_0)} G_{\delta'}\left(\varrho^{\delta'}(x,t)\right) dxdt \geq \frac{1}{2}\sigma(\varepsilon)\mu(E) \int_R^{\varepsilon'} g_{\delta'}(s)\, ds. \tag{3.2}$$

The right-hand side of this inequality converges for $\delta' \to 0$ to

$$\frac{1}{2}\sigma(\varepsilon)\mu(E) \int_R^{\varepsilon'} g_0(s)\, ds \geq \frac{c}{2}\sigma(\varepsilon)\mu(E) \begin{cases} \varepsilon'^{2-n} & \text{if } n > 2 \\ -\log \varepsilon' & \text{if } n = 2. \end{cases}$$

Taking into account that in inequality (3.2) we have integrated over $Q_{\sigma(\varepsilon)}(E, t_0)$ and that the last a priori estimate of (2.17) is valid for all times $t \in [0, T]$, we obtain a contradiction in the limit $\varepsilon, \varepsilon' \to 0$.

ad 3): First, we observe that on the set $[\varrho > 0]_T$ the convergence $\varrho^\delta(x,t) \to \varrho(x,t)$ implies the convergence $G_\delta(\varrho^\delta(x,t)) \to G_0(\varrho(x,t))$ for $\delta \to 0$. From statement 2) of Theorem 1.2 we infer that this convergence takes place for almost all $(x,t) \in \Omega_T$. Since we deduce from the last estimate of (2.17)

$$\fint_{\tau-\varepsilon}^{\tau+\varepsilon} \int_\Omega G_\delta(\varrho^\delta)\, dxdt \leq C$$

we can apply the Fatou lemma, and end up with

$$\fint_{\tau-\varepsilon}^{\tau+\varepsilon} \int_\Omega G_0(\varrho)\, dxdt \leq C.$$

Combined with (3.1) this yields the theorem ∎

Remark 3.1: Since G_0 does not generate any singularity in the case $n < 2$, the integrability of $G_0(\varrho)$ is obvious.

Proof of Theorem 1.3: Exemplarily, we prove the first result: As the function $\varrho(\cdot, t)$ is element of $H^2_*(\Omega)$, the Sobolev imbedding theorem implies its Lipschitz continuity. If $\varrho(\cdot, t)$ vanished at a point $x_0 \in \Omega$, we could conclude with statement 3) of Theorem 1.2 that

$$\infty > \int_\Omega \varrho(x,t)^{2-n}\, dx \geq c_1 \int_\Omega |x - x_0|^{2-n} dx \geq c_2 \int_0^1 r^{3-n} dr.$$

For $n \geq 4$ this is a contradiction ∎

4. Perspectives and limitations of the presented solution concept

The boundedness conditions on the integral $\int_\Omega G_0(\varrho_0)\,dx$, on the diffusion coefficient m and on its derivative m' were essential ingredients of the existence and non-negativity results presented above. Unfortunately, these requirements make it impossible to use our concept as soon as the diffusion coefficient m has polynomial growth or as soon as $n \geq 2$ (cf. (3.1)) and the initial value ϱ_0 has compact support. But for some applications like, e.g., the spreading of a viscous droplet on a plain surface (cf., e.g., [3] and the references therein), one should be able to handle these difficulties.

In this section we derive existence and non-negativity results in a weakened sense. For the sake of simplicity, we confine ourselves to the following problem:

$$
\begin{aligned}
&(\mathrm{P}') \quad \varrho_t + \operatorname{div}(m(\varrho)\nabla\Delta\varrho) = 0 && \text{in} \quad \Omega \times (0,T)\\
&\frac{\partial}{\partial\nu}\varrho = \frac{\partial}{\partial\nu}\Delta\varrho = 0 && \text{on} \quad \partial\Omega \times (0,T)\\
&\varrho(\cdot,0) = \varrho_0(\cdot) && \text{in} \quad \Omega.
\end{aligned}
$$

4.1 Initial values with compact support. In this subsection, for $n \geq 2$ we free ourselves from the boundedness requirements on the integral $\int_\Omega G_0(\varrho_0)\,dx$. For merely technical reasons we require in addition to condition (1.3) that

$$m(s) = 0 \quad \text{if} \;\; s \leq 0. \tag{4.1}$$

Let us remark that it is also possible to handle the more general equation considered in Theorem 1, provided the regularity of the inhomogeneity $f(\cdot,\cdot,\cdot)$ permits integration by parts in the last term of equation (2.4). In that case – already by testing with $\varrho^\delta - \Delta\varrho^\delta$ – we obtain a priori estimates independent of δ for the $W_2^1(I;H^1(\Omega),L^2(\Omega))$-norm of ϱ^δ and for the $L^2(\Omega_T)$-norm of J_δ, respectively.

Having applied the procedure of Subsection 2.1 for proving the existence of weak solutions ϱ^δ of non-degenerate auxiliary problems (as in (2.9) and (2.10)) we pass to the limit $\delta \to 0$ and get a pair

$$(\varrho, J) \in W_2^1(I;H^1(\Omega),L^2(\Omega)) \cap L^\infty(I;H^1(\Omega)) \times L^2(\Omega_T;\mathbb{R}^N)$$

with the property

$$\varrho_t = -\operatorname{div} J \quad \text{in} \quad L^2(I;(H^1(\Omega))').$$

Globally, we cannot expect a relation between J and ϱ in the sense of (1.10) since it is not possible to control the $L^2(\Omega_T)$-norm of $\Delta\varrho^\delta$. Neither do we obtain a global non-negativity result. But the following statement of merely local character may be derived.

Theorem 4.1: *Assume conditions* (1.2), (1.3) *and* (4.1) *are fulfilled. Let ϱ^δ be a weak solution of the auxiliary problem* (P'_δ) *associated to problem* (P'). *Let $\xi_0 \in C_0^\infty(\Omega)$ be non-negative having the property*

$$\int_\Omega \xi_0(x) G_0(\varrho_0(x))\,dx < \infty.$$

Then for each subset $S \subset\subset [\xi_0 > 0]$ there exists a constant $C(S)$ which is independent of t and δ such that

$$\int_0^T \int_S |\Delta \varrho^\delta(x,t)|^2 \, dx dt < C(S) \qquad \text{and} \qquad \int_S G_\delta(\varrho^\delta(x,t)) \, dx < C(S)$$

for all $t \in [0,T]$ and $\delta > 0$.

Remark 4.2: The results of the Theorems 1.2 and 1.3 apply to the interior of $[\xi_0 > 0]$, i.e., in this region the non-negativity and positivity properties of ϱ_0 are conserved for almost all times t. In particular, the support of ϱ does not shrink in course of time. Unfortunately, the second estimate stated in Theorem 4.1 does not suffice to identify J even only locally with $m(\varrho)\nabla\Delta\varrho$ in the sense of (1.10).

Proof of Theorem 4.1: In the weak formulation of problem (P'_δ) analogous to (2.9) we choose $v(x,t) = \xi(x) g_\delta(\varrho^\delta(x,t))$ as test function. Here the function $\xi \in C_0^\infty(\Omega)$ is non-negative, and its support is compactly contained in $[\xi_0 > 0]$. Using the smoothness and time independency of ξ we can apply Lemma 2.5 and obtain

$$\begin{aligned}
&\int_\Omega \xi G_\delta(\varrho^\delta(T)) \, dx + \int_0^T (\Delta\varrho^\delta, \xi\Delta\varrho^\delta) \, dt \\
&\quad = \int_\Omega \xi G_\delta(\varrho_0) \, dx + \int_0^T (h_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta, \nabla\xi) \, dt - \int_0^T (\Delta\varrho^\delta \nabla\xi, \nabla\varrho^\delta) \, dt
\end{aligned}$$

(here h_δ is defined by $h_\delta(s) = m_\delta(s) g_\delta(s)$). Integration by parts yields

$$\begin{aligned}
&\int_\Omega \xi G_\delta(\varrho^\delta(T)) \, dx + \int_0^T (\Delta\varrho^\delta, \xi\Delta\varrho^\delta) \, dt \\
&\quad = \int_\Omega \xi G_\delta(\varrho_0) \, dx - \int_0^T (\Delta\varrho^\delta \nabla\xi, \nabla\varrho^\delta) \, dt \\
&\qquad + \int_0^T (h'_\delta(\varrho^\delta)\Delta\varrho^\delta \nabla\varrho^\delta, \nabla\xi) \, dt + \int_0^T (h_\delta(\varrho^\delta)\Delta\varrho^\delta, \Delta\xi) \, dt.
\end{aligned} \tag{4.2}$$

For the derivative of h_δ we calculate

$$h'_\delta(s) = m'(s) \int_R^s \frac{d\tau}{m_\delta(\tau)} + 1.$$

As $m'(s)$ and $\int_R^s m(\tau)^{-1} d\tau$ grow in a positive neighbourhood of the origin like s^{n-1} and s^{1-n}, respectively, the following estimate is valid with a constant C independent of $\delta > 0$:

$$|h'_\delta(s)| \begin{cases} = 0 & \text{if } s \le 0 \\ \le C & \text{if } s > 0. \end{cases}$$

Inserting this result into (4.2) yields

$$\int_\Omega \xi G_\delta(\varrho^\delta(T))\,dx + \int_0^T (\Delta\varrho^\delta, \xi\Delta\varrho^\delta)\,dt$$
$$\leq \int_\Omega \xi G_\delta(\varrho_0)\,dx + \iint_{\Omega_T} \Big(|\Delta\varrho^\delta|\,|\nabla\xi|\,|\nabla\varrho^\delta| + |\Delta\varrho^\delta|\,|\varrho^\delta|\,|\Delta\xi|\Big)\,dxdt.$$

Now we choose – following the concept in [3] – $\xi = \zeta^s$ for some $s \geq 4$ and a test function $\zeta \in C_0^\infty(\Omega)$. Since ϱ^δ is uniformly bounded in $L^\infty\big(I; H^1(\Omega)\big)$, we get the result ∎

Remark 4.3: It is not possible to conclude from this result, which by minor modifications can also be applied to equations of the form

$$\varrho_t + \operatorname{div}\Big(m(\varrho)\big(\lambda^2\nabla\varrho + \nabla\Delta\varrho\big)\Big) = 0$$

with positive λ and boundary conditions as in problem (P′), that the support increases in course of time. In one space dimension this can already be illustrated by the following example of a steady state solution with compact support:

Let Ω be the interval $\big[-\frac{2\pi}{\lambda}, +\frac{2\pi}{\lambda}\big]$ and M be a positive constant. Consider the function

$$\varrho(x) = \begin{cases} M + M\cos(\lambda x) & \text{if } x \in \big(-\frac{\pi}{\lambda}, +\frac{\pi}{\lambda}\big) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, ϱ is element of $H^2(\Omega) \cap C^\infty\big(-\frac{\pi}{\lambda}, +\frac{\pi}{\lambda}\big) \cap C^\infty\big(\Omega \setminus \big[-\frac{\pi}{\lambda}, +\frac{\pi}{\lambda}\big]\big)$ and fulfills the equation above in the sense of Theorem 1.1, respectively in the sense of Theorem 3.1 in [3].

4.2 Diffusion coefficients with polynomial growth. In this subsection we present existence and non-negativity results for problem (P′) under the assumption

$$m(s) = |s|^\alpha \qquad \text{for an} \quad \alpha > 1. \tag{4.3}$$

Since we are mainly interested in modelling the spreading of a viscous droplet, we confine ourselves to the case $N = 2$. By modification of the proof of Theorem 1.1 we obtain the following statement.

Theorem 4.4: *Under the assumptions* (1.2), (1.8) *and* (4.3) *there exists a pair* (ϱ, J) *with the properties*

$$\varrho \in W_2^1\big(I; H_*^2(\Omega), L^2(\Omega)\big) \cap L^\infty\big(I; H^1(\Omega)\big)$$
$$J \in L^{2-\epsilon}\big(\Omega_T; \mathbb{R}^2\big)$$

and

$$\int_0^T \langle \varrho_t, v\rangle_{H_*^2(\Omega)}\,dt = \int_0^T (J, \nabla v)\,dt \tag{4.4}$$

for all $v \in L^2(I; H^2_*(\Omega)) \cap L^\infty(I; H^1(\Omega))$. *Moreover, J fulfills the relation*

$$J = m(\varrho)\nabla\Delta\varrho$$

in the following weak sense:

$$\iint_{\Omega_T} J\eta\,dxdt = -\iint_{\Omega_T} m(\varrho)\Delta\varrho\,\operatorname{div}\eta\,dxdt - \iint_{\Omega_T} m'(\varrho)\nabla\varrho\Delta\varrho\eta\,dxdt \tag{4.5}$$

for all $\eta \in L^{2+\varepsilon}(I; W^{1,2+\varepsilon}(\Omega; \mathbb{R}^2)) \cap L^\infty(\Omega_T; \mathbb{R}^2)$ *with* $\eta\nu = 0$ *on* $\partial\Omega$.

Sketch of the proof: As in Section 2, we obtain approximate solutions ϱ_k^δ of problem (P'_δ) by application of the Faedo-Galerkin method. By testing with $\Delta\varrho_k^\delta$ we immediately derive the following estimates independent of δ and k:

$$\begin{array}{rl} m^{1/2}(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta & \text{uniformly lies in a bounded subset of } L^2(\Omega_T; \mathbb{R}^2) \\ \varrho_k^\delta & \text{uniformly lies in a bounded subset of } L^\infty(I; H^1(\Omega)). \end{array} \tag{4.6}$$

For proving time compactness of $(\rho_k^\delta)_{k\in\mathbb{N}}$ it is necessary to take into account the growth of the diffusion coefficient m. We make use of the uniform boundedness of the projections P^k in $\mathcal{L}(H^2_*(\Omega), H^2_*(\Omega))$, and obtain for arbitrary $v \in H^2_*(\Omega)$ the estimate

$$\left((\varrho_k^\delta)_t, P^k v\right)\Big|_{t_0} \le C\left\|m^{1/2}(\varrho_k^\delta)\nabla\Delta\varrho_k^\delta\right\|_{L^2(\Omega)}\Big|_{t_0}\left\|m^{1/2}(\varrho_k^\delta)\right\|_{L^{2+\varepsilon}(\Omega)}\Big|_{t_0}\|v\|_{H^2_*(\Omega)}.$$

The estimates (4.6) and the Sobolev imbedding theorem (N=2!) then imply

$$\begin{array}{rl} (\varrho_k^\delta)_t & \text{uniformly lies in a bounded subset of } L^2(I; (H^2_*(\Omega))') \\ (\nabla\varrho_k^\delta)_t & \text{uniformly lies in a bounded subset of } L^2(I; (V^3(\Omega))'). \end{array} \tag{4.7}$$

Passing to the limit $k \to \infty$ we find a function

$$\varrho^\delta \in W_2^1(I; H^2_*(\Omega), L^2(\Omega)) \cap L^\infty(I; H^1(\Omega)) \cap L^2(I; H^3(\Omega))$$

which solves problem (P'_δ) in the following sense:

$$\int_0^T \langle \varrho_t^\delta, v\rangle_{H^2_*(\Omega)}\,dt = \int_0^T \left(m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta, \nabla v\right) dt \tag{4.8}$$

for all $v \in L^2(I; H^2_*(\Omega)) \cap L^\infty(I; H^1(\Omega))$.

From the estimates (4.6) we easily conclude that for a subsequence $\delta \to 0$ the flux $J_\delta = m_\delta(\varrho^\delta)\nabla\Delta\varrho^\delta$ weakly converges in $L^{2-\varepsilon}(\Omega_T; \mathbb{R}^2)$ to J. Combined with the first inclusion of (4.7) this yields (4.4). For the proof of (4.5) we observe that $g_\delta(\varrho^\delta)$ is an admissible test function in (4.8), and we obtain as in the first and third statement of (2.17)

$$\begin{array}{rl} \Delta\varrho^\delta & \text{uniformly lies in a bounded subset of } L^2(\Omega_T) \\ \int_\Omega G_\delta(\varrho^\delta)\,dx & \text{is uniformly bounded for } t \in [0,T]. \end{array} \tag{4.9}$$

respectively. Using the last statement of (4.7), the first one of (4.9) and Lemma 2.7 it is clear that for a subsequence $(\varrho^\delta)_{\delta\to 0}$ the gradient $\nabla\varrho^\delta$ converges pointwise almost everywhere and strongly in $L^q(\Omega_T;\mathbb{R}^2)$ to $\nabla\varrho$ for arbitrary $q<4$. Thus passing to the limit in (4.8) we find a function

$$\varrho \in W_2^1\big(I; H_*^2(\Omega), L^2(\Omega)\big) \cap L^\infty\big(I; H^1(\Omega)\big)$$

which solves problem (P′) in the following sense:

$$\int_0^T \langle \varrho_t, v\rangle_{H_*^2(\Omega)}\, dt = -\int_0^T \big(m(\varrho)\Delta\varrho, \Delta v\big)\, dt - \int_0^T \big(m'(\varrho)\nabla\varrho, \Delta\varrho\nabla v\big)\, dt$$

for all $v \in L^{2+\varepsilon}\big(I; W^{2,2+\varepsilon}(\Omega)\big) \cap L^\infty\big(I; W^{1,\infty}(\Omega)\big)$ with $(\nabla v)\nu = 0$ on $\partial\Omega$. This proves Theorem 4.4 ∎

A direct consequence of the last statement of (4.9) is the following

Corollary 4.5: *Under the assumptions of Theorem* 4.4 *the non-negativity results of the Theorems* 1.2 *and* 1.3 *apply to weak solutions of problem* (P′).

5. A simplified Norton-Hoff model with isotropic, non-local hardening

5.1 Modelling and statement of the results. In recent years material scientists and engineers became more and more occupied with modelling and explaining the formation of cellular patterns in metals during plastic deformation. A decisive influence on phenomena like this is exerted by dislocations, i.e., perturbances in the lattice structure which are generated to a smaller extent during crystal growth, but mainly during plastic deformation. Usually, one distinguishes between dislocations of *edge-type* and of *screw-type* (cf. the monography of Hirth and Lothe [8]). The curve in the crystal along which the lattice structure is disturbed is called the *dislocation line*. The *Burgers vector* $\vec{b}$ describes in which direction and how far atoms have to be moved in order to reorganize the lattice to the original configuration. In the case of dislocations of edge type, it is orientated perpendicularly, in the case of dislocations of screw type parallelly to the dislocation line. For energetical reasons dislocations often organize to more stable configurations, so called *dislocation dipols*. Especially for the phenomenon mentioned above *dipolar loops* (i.e., dipols made of dislocations with a circular dislocation line) of edge type dislocations are of great importance since experiments indicate that in the cell walls their concentration is incomparatively higher than in the cell interior. Observing that especially high concentrations of dipolar loops hinder plastic glide and therefore contribute to the hardening of the material, it is natural to relate the loop density to the parameter of isotropic hardening used in classical models of phenomenological plasticity.

A simple model describing the pattern formation mentioned above has been proposed by J. Kratochvíl and M. Saxlová [10].

For these authors the dislocation population consists – in an idealized way – of glide dislocations of screw type and dipolar loops of edge type. As the size of the latter is negligible in comparison to the diameter of the cells, they may be considered as point objects and be described by an internal parameter ϱ which stands for their density. In contrast, the glide dislocations enter the model only implicitely via the plastic strain rate.

The authors propose – confining themselves at first to one glide system – the one-dimensional evolution equation

$$\dot{\varrho} + D\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} M(x-x')\varrho(x')\,dx' = A_0|\dot{\varepsilon_p}|.$$

Here $M \in C_0^\infty(I\!R)$ is non-negative and symmetric, D and A_0 are positive constants and $\dot{\varepsilon_p}$ denotes the plastic strain rate.

In a successive paper J. Kratochvíl and M. Saxlová present a method to generalize this model to the case of more than one active glide system [11]. Assuming an isotropic interaction between the different glide systems they obtain

$$\dot{\varrho} + D\sum_{j=1}^{N}\mathrm{div}^j\,\mathrm{grad}^j\int_{-\infty}^{+\infty} M(\alpha)\varrho(x+\alpha b^j,t)\,d\alpha = A_0|\dot{\varepsilon_p}|. \tag{5.1}$$

Here b^j denotes the Burgers vector of the j^{th} glide system. Unfortunately, this formulation still withstands a successful mathematical analysis. Since ϱ is supposed to play the role of a density, we have to require its non-negativity. But even in the case of positive initial values solutions of equation (5.1) in general do not remain non-negative in course of time. Since the introduction of a diffusion coefficient with degeneracy in zero which seems to guarantee – as numerical experiments in [12] indicate – the non-negativity of ϱ poses severe problems for the mathematical analysis, we free ourselves from the convolution formulation in equation (5.1). Instead of postulating finitely many glide systems we assume infinitely many of them being uniformly distributed over the unit sphere. Hence we substitute the summation over the different glide systems by an integration over the unit sphere. This yields after Taylor expansion of ϱ up to order two and after neglecting the terms of higher order

$$\dot{\varrho} + \mathrm{div}\Big(D\big(H_0\nabla\varrho + H_1\nabla\Delta\varrho\big)\Big) = A_0|\dot{\varepsilon}_p|.$$

Here H_0 and H_1 depend in the following manner on M:

$$H_0 = \frac{4}{3}\pi\int_{-\infty}^{+\infty} M(x)\,dx \qquad \text{and} \qquad H_1 = \frac{2}{5}\pi\int_{-\infty}^{+\infty} M(x)x^2\,dx.$$

In order to guarantee the non-negativity of ϱ we choose a diffusion coefficient $m(\varrho)$ which depends on ϱ and obeys the conditions (1.3) and (1.4), and obtain finally as evolution equation for the density of dipolar loops an equation of type (1.1) with right-hand side $f(\cdot,\cdot,\varrho) = |\dot{\varepsilon}_p|$ (we have chosen $A_0 \equiv 1$).

The non-local character of the hardening process is taken into account in the following way. Let us define by

$$h(\varrho)(x,t) = h_0(x) + \int_{I\!R^3} \hat{M}(x-y)\hat{\varrho}(y,t)\,dy \tag{5.2}$$

with

$$\hat{\varrho}(x,\cdot) = \begin{cases} \varrho(x,\cdot) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

the hardening parameter $h(\varrho)$ associated to ϱ. The function $h_0 \in W^{1,\infty}(\Omega)$ is non-negative, $\widehat{M} \in C_0^\infty(I\!R^3)$ is radially symmetric and non-negative, and we have the equality $\int_{R^3} \widehat{M}(x)\,dx = 1$. In particular, for arbitrary $1 \le p \le \infty$ the estimates

$$\begin{aligned} \|h(\varrho)\|_{L^p(\Omega)} &\le C\Big(1 + \|\varrho\|_{L^p(\Omega)}\Big) \\ \|\nabla h(\varrho)\|_{L^p(\Omega)} &\le \|\nabla M\|_{L^1(\Omega)}\|\varrho\|_{L^p(\Omega)} + \|\nabla h_0\|_{L^p(\Omega)} \end{aligned} \tag{5.3}$$

are true. Let us now combine the evolution equations derived above with the continuum mechanical equations describing plastic behaviour with hardening as they were treated, e.g., in [9]. For technical reasons we confine ourselves to an elasto-viscoplastic model of Norton-Hoff type which we are going to present now.

Under the influence of time dependent volume forces f and surface forces F , an elasto-plastic body, which occupies at time $t = 0$ the reference configuration $\Omega \subset I\!R^3$, is deformed. The Cauchy stress tensor σ, the hardening parameter $h(\varrho)$, and the displacement u describe the evolution of the body. Assuming small deformations and a quasistatic evolution, we identify the actual configuration with the reference configuration and neglect in the equation of motion such terms in u which contain time derivatives of order two. This permits to formulate the following relation between forces and stresses:

$$\begin{aligned} \operatorname{div}\sigma + f = 0 &\quad \text{in } \Omega \\ \sigma\nu = F &\quad \text{on } \Gamma_1 \subset \partial\Omega. \end{aligned}$$

Let us proceed further with the *constitutive law* which prescribes the interdependencies between stress tensor and hardening parameter on the one hand and strain tensor on the other hand. As long as the *von Mises condition* is fulfilled, which requires that the norm of the deviatoric of the stress tensor in each point $(x,t) \in \Omega_T$ is bounded by the value of the hardening parameter $h(\varrho(x,t))$, we assume the validity of *Hooke's law*, i.e., a linear dependency between the stress and strain rate. Otherwise, in addition to the elastic part of the strain rate there appears a plastic part which mathematically we shall obtain by penalizing the von Mises-condition. To get more into detail let the *von Mises cone* $K \subset M^{3\times 3}_{\text{sym}} \times I\!R$ be defined by

$$K = \Big\{(\tau,h):\ |\tau^D(\tau,h)| \le h\Big\}.$$

Here

$$\tau^D = \tau - \frac{1}{3}\mathrm{tr}(\tau) Id$$

denotes the deviatoric of τ where Id is the identity on $I\!R^3$. For fixed $\alpha > 0$, $d(\tau, h)$ stands for

$$d(\tau, h) = \alpha^{-1} \mathrm{dist}\,\big((\tau, h), K\big).$$

Using a result of E. H. Zarantonello [17] we observe that $d^2(\tau, h)$ is differentiable with differential

$$\gamma(\tau, h) = 2\alpha^{-2}\left(\begin{pmatrix}\tau^D \\ h\end{pmatrix} - \Pi_K\begin{pmatrix}\tau^D \\ h\end{pmatrix}\right) =: \begin{pmatrix}\gamma_{\mathrm{ten}}(\tau, h) \\ \gamma_{\mathrm{skal}}(\tau, h)\end{pmatrix}.$$

Here Π_K denotes the orthogonal projection onto K in $M^{3\times 3}_{\mathrm{sym}} \times I\!R$. Elementary geometrical considerations yield

$$\gamma_{\mathrm{ten}}(\tau, h) = \begin{cases} \alpha^{-2}(|\tau^D| - h)\dfrac{\tau^D}{|\tau^D|} & \text{if } |\tau^D| \geq |h| \\ 2\alpha^{-2}\tau^D & \text{if } |\tau^D| < -h \wedge h < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_{\mathrm{skal}}(\tau, h) = \begin{cases} \alpha^{-2}(h - |\tau^D|) & \text{if } |\tau^D| \geq |h| \\ 2\alpha^{-2}h & \text{if } |\tau^D| < -h \wedge h < 0 \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Hence, there exists a constant $C(\alpha)$ such that

$$|\gamma_{\mathrm{ten}}(\tau, h)| + |\gamma_{\mathrm{skal}}(\tau, h)| \leq C(\alpha)\big(|\tau^D| + |h|\big). \tag{5.4}$$

By $\Gamma(\tau, h) = \int_\Omega d^2(\tau, h)\,dx$ we define a real-valued functional on $L^2(\Omega; M^{3\times 3}_{\mathrm{sym}} \times I\!R)$. According to the estimate above it is finite everywhere, convex, continuous, and Gateaux differentiable. With γ we again denote its derivative. From the theory of monotone operators we infer that γ is a monotone, hemi-continuous operator on $L^2(\Omega; M^{3\times 3}_{\mathrm{sym}} \times I\!R)$.

In the sequel we shall be concerned with the following

Problem: Find a triple $(\sigma, \varrho, u) \in S_{\mathrm{ad}}(f, F) \times H^2(\Omega; I\!R_0^+) \times C_{\mathrm{ad}}(U)$ which solves in an appropriate weak sense the following system:

$$\begin{aligned} A\dot{\sigma} + \gamma_{\mathrm{ten}}\big(\sigma, h(\varrho)\big) &= \varepsilon(\dot{u}) && \text{in } \Omega \times (0,T) \\ \dot{\varrho} + \mathrm{div}\Big(m(\varrho)(H\nabla\varrho + \nabla\Delta\varrho)\Big) &= -\gamma_{\mathrm{skal}}\big(\sigma, h(\varrho)\big) && \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial\nu}\varrho = \frac{\partial}{\partial\nu}\Delta\varrho &= 0 && \text{on } \partial\Omega \times (0,T) \\ \big(\sigma(\cdot,0), \varrho(\cdot,0), u(\cdot,0)\big) &= (\sigma_0, \varrho_0, u_0). \end{aligned} \tag{5.5}$$

Here, we have used the following abbreviations:

- $C_{\rm ad}(U_0) = \{u \in H^1(\Omega, I\!R^3) :\ u = U_0 \text{ on } \Gamma_0\}$ is the set of the kinetically admissible displacement fields. $C^0_{\rm ad}$ is equivalent to $C_{\rm ad}(0)$.

- $S_{\rm ad}(f_0, F_0) = \{\sigma \in L^2(\Omega; M^{3\times 3}_{\rm sym}) :\ \operatorname{div}\sigma + f_0 = 0 \text{ in } \Omega \text{ and } \sigma\nu = F_0 \text{ on } \Gamma_1\}$ is the set of the statically admissible stress tensor fields. $S^0_{\rm ad}$ stands for $S_{\rm ad}(0,0)$.

- $\partial\Omega = \Gamma_0 \cup \Gamma_1$, and the surface measure of Γ_0 does not vanish.

- $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearized strain tensor.

- A^{-1} is the symmetric, positive definite Lamé-Navier operator of linearized elasticity. En detail, $(A\sigma)_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\operatorname{tr}(\sigma)\delta_{ij}$ for all $\sigma \in M^{3\times 3}_{\rm sym}$ with Lamé constants λ, μ.
- H is a positive constant.

Remark 5.1: In the sense of the usual decomposition of the strain tensor in a plastic and an elastic part the latter of which is – according to the equations of linearized elasticity – given by $A\sigma$ we can interprete $\gamma_{\rm ten}(\sigma, h(\varrho))$ in problem (5.5) as plastic strain rate. Our results of the previous sections indicate that ϱ will be non-negative. Hence we have

$$-\gamma_{\rm skal}(\sigma, h(\varrho)) = \left|\gamma_{\rm ten}(\sigma, h(\varrho))\right|.$$

For this reason problem (5.5) is equivalent to

$$\dot{\varrho} + \operatorname{div}\Big(m(\varrho)\big(H\nabla\varrho + \nabla\Delta\varrho\big)\Big) = |\dot{\varepsilon}_p|.$$

Assuming a homogeneous, non-negative initial distribution ϱ_0 we expect the following course of the deformation process: At first $\gamma_{\rm ten} = 0$, and ϱ remains constant. From a time t_0 on locally in Ω there occurs plastic deformation which initiates the evolution of ϱ. On account of the positivity of H the gradient term in problem (5.5) has a destabilizing effect, and a linear stability analysis – neglecting the degeneracy m – indicates that small deviations of ϱ will grow exponentially in course of time.

Unfortunately, this framework does not allow to confine the evolution of ϱ to such regions in Ω_T where plastic deformation takes place. For this reason it seems to be impossible to simulate cyclic loading using this model. Nevertheless, it is worthwile to mention that we do not expect an immediate spreading all over Ω because of the degeneracy of m and the smoothing properties of h. In this context we refer to the results of F. Bernis, L. A. Peletier and S. M. Williams concerning source type solutions of degenerate parabolic equations of fourth order (see [4]).

Finally, let us mention that our choice of the boundary conditions for ϱ was motivated by the assumption that the flux of the dipolar loops vanishes at the surface of the body.

In order to relate surface forces on the boundary $\partial\Omega$ to the stress tensor σ we direct the reader's attention to the following result of R. Temam for domains with boundary of class C^1.

Lemma 5.2: *If* $\tau \in L^2(\Omega; M^{3\times3}_{\text{sym}})$ *and* $\operatorname{div}\tau \in L^2(\Omega; \mathbb{R}^3)$, *we can define the trace of* $\tau\nu$ *on* $\partial\Omega$ *as element of* $H^{-1/2}(\partial\Omega)$, *and obtain the following estimate with a constant* C *depending only on* Ω:

$$\|\tau\nu\|_{H^{-1/2}(\partial\Omega)} \le C\Big(\|\tau\|_{L^2(\Omega)} + \|\operatorname{div}\tau\|_{L^2(\Omega)}\Big). \tag{5.6}$$

Moreover, the following generalized Green formula is valid for all $v \in H^1(\Omega, \mathbb{R}^3)$ *and* $\tau \in L^2(\Omega; M^{3\times3}_{\text{sym}})$ *with* $\operatorname{div}\tau \in L^2(\Omega; \mathbb{R}^3)$:

$$\int_\Omega \tau\varepsilon(v)\,dx = \int_{\partial\Omega} \tau(v\otimes\nu)\,d\Gamma - \int_\Omega v\operatorname{div}\tau\,dx. \tag{5.7}$$

Here the first term on the right-hand side denotes the dual pairing between the spaces $H^{1/2}(\partial\Omega)$ *and* $H^{-1/2}(\partial\Omega)$.

For the data we require the following regularity properties:

$$\left.\begin{array}{l}
U,\dot U,\ddot U \in L^\infty(I; H^1(\Omega;\mathbb{R}^3)),\quad U = 0 \quad \text{on } \Gamma_1\times[0,T]\\
u_0, u_1 \in H^1(\Omega;\mathbb{R}^3),\quad u_0 = U(0),\ u_1 = \dot U(0) \quad \text{on } \Gamma_1\\
\sigma_0 \in L^2(\Omega; M^{3\times3}_{\text{sym}}),\quad \sigma_0^D \in K \quad \text{a.e. on } \Omega\\
f,\dot f \in L^\infty(I; L^2(\Omega;\mathbb{R}^3)) \quad \text{and} \quad F,\dot F \in L^\infty(I; L^2(\Gamma_1;\mathbb{R}^3))\\
\varrho_0 \in H^1(\Omega) \quad \text{and} \quad \displaystyle\int_\Omega G_0(\varrho_0(x))\,dx < \infty.
\end{array}\right\} \tag{5.8}$$

Finally, we assume the validity of the following condition ascertaining that the set of admissible tensor fields contains elements with sufficient regularity.

Safe-Load Condition: There exists a tensor field χ_{ten} with the property that, for each $t \in [0,T]$,

$$\chi_{\text{ten}}, \dot\chi_{\text{ten}} \in L^\infty(\Omega_T; M^{3\times3}_{\text{sym}}) \qquad \text{and} \qquad \chi_{\text{ten}}(\cdot,t) \in S_{\text{ad}}(f(t), F(t)). \tag{5.9}$$

We prove the following existence result for problem (5.5).

Theorem 5.3: *Under the assumptions* (1.2) – (1.4), (5.2), (5.3), (5.8) *and* (5,9) *there exists a quadruple* (σ, u, ϱ, J) *which solves system* (5.5) *in the following sense:*

$$A(\sigma(t)-\sigma_0) + \int_0^t \gamma_{\text{ten}}(\sigma(s), h(\varrho(s)))\,ds = \varepsilon(u(t)) - \varepsilon(u_0) \quad \forall\, t \in [0,T] \tag{5.10}$$

$$\int_0^T \langle\dot\varrho,\eta\rangle dt - \int_0^T (J,\nabla\eta)dt = -\int_0^T \big(\gamma_{\text{skal}}(\sigma, h(\varrho)),\eta\big)dt \quad \forall\, \eta \in L^2(I; H^1(\Omega))$$

$$\sigma(\cdot,t) \in S_{\text{ad}}(f(t), F(t)) \quad \forall\, t \in [0,T].$$

For ϱ the non-negativity results of Theorems 1.2 *and* 1.3 *apply, and J fulfills the relation* $J = m(\varrho)(H\nabla\varrho + \nabla\Delta\varrho)$ *in the sense of* (1.10).

Furthermore, ϱ and J have the same regularity properties as in Theorem 1.1, $\sigma \in W_2^1(I; L^2(\Omega))$ *and* $u \in L^2(I; H^1(\Omega))$.

Remark 5.4: Since the admissible set K does not have a sufficiently smooth boundary, it is not straight forward to relate $\dot\sigma$ and $\varepsilon(\dot u)$ in the sense of problem (5.5). Furthermore, to the best of our knowledge, there are no results offering more than p-integrability of the deviatoric of σ if γ has p-growth in σ^D. Indeed, by the use of duality methods J. Frehse and A. Bensoussan show in [2] $L^\infty(I; H^1_{\mathrm{loc}}(\Omega))$-regularity of the stress tensor for a Norton-Hoff model without hardening parameter. Unfortunately, in our case this result is not sufficient. For these reasons we only permit linear growth of γ and confine ourselves to the results of Theorem 5.3.

As in the case of Section 2, the uniqueness of the solution remains an open problem.

5.2 Proof of the existence result. Our proof combines our techniques of Section 2 with the methods R. Temam used in his paper [15] for the pure Norton-Hoff model without hardening. Let us begin with the following weak formulation for the inclusion property $\sigma(\cdot,t) \in S_{\mathrm{ad}}(f(t), F(t))$:

$$(\sigma(\cdot,t), \varepsilon(w)) = \int_\Omega f(x,t)w(x)\,dx + \int_{\partial\Omega} F(x,t)w(x)\,d\Gamma =: L(t,w)$$

for all $w \in C^0_{\mathrm{ad}}$.

By the means of the Faedo-Galerkin method we prove the existence of solutions

$$(\sigma_{\delta\lambda}, \varrho_{\delta\lambda}, u_{\delta\lambda})$$

for the following doubly regularized problem $(\mathrm{P}_{\delta\lambda})$ with $\delta, \lambda > 0$.

$$(\mathbf{P}_{\delta\lambda})\quad A\dot\sigma_{\delta\lambda} + \gamma_{\mathrm{ten}}(\sigma_{\delta\lambda}, h(\varrho_{\delta\lambda})) = \varepsilon(\dot u_{\delta\lambda})$$

$$\int_0^T \langle \dot\varrho_{\delta\lambda}, \eta\rangle dt - \int_0^T \Big(m_\delta(\varrho_{\delta\lambda})(H\nabla\varrho_{\delta\lambda} + \nabla\Delta\varrho_{\delta\lambda}), \nabla\eta\Big)dt$$

$$= -\int_0^T \Big(\gamma_{\mathrm{skal}}(\sigma_{\delta\lambda}, h(\varrho_{\delta\lambda})), \eta\Big)dt \quad \text{for all}\quad \eta \in L^2(I; H^1(\Omega)).$$

$$\lambda(\varepsilon(\dot u_{\delta\lambda}), \varepsilon(w)) + (\sigma_{\delta\lambda}, \varepsilon(w)) = L(t,w) \quad \text{for all}\quad w \in C^0_{\mathrm{ad}}.$$

Moreover, we require that $(\sigma_{\delta\lambda}, \varrho_{\delta\lambda}, u_{\delta\lambda})$ fulfill the initial and boundary conditions of problem (5.5) and that $u_{\delta\lambda} \in C^{\mathrm{ad}}(U)$.

Let $(\tau_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ be total and free subsets of $L^2(\Omega; M^{3\times3}_{\mathrm{sym}})$ and C^0_{ad}, respectively, and Φ_j, S^k and P^k as in Section 2. Further, T^k denotes span $\{\tau_1, ..., \tau_k\}$ and

P_{T^k} is the orthogonal projection in $L^2(\Omega; M^{3\times 3})$ onto T^k. In a similar way we define W^k and P_{W^k}. With the help of the Peano existence theorem we find functions

$$\sigma_k(\cdot,t) = \sum_{j=1}^{k} \alpha_{1j}(t)\tau_j(\cdot), \quad \varrho_k(\cdot,t) = \sum_{j=0}^{k} \alpha_{2j}(t)\Phi_j(\cdot), \quad v_k(\cdot,t) = \sum_{j=1}^{k} \alpha_{3j}(t)w_j(\cdot)$$

solving locally in time the system of ordinary differential equations

$$\left.\begin{aligned} \frac{d}{dt}(A\sigma_k,\tau_j) + \Big(\gamma_{\text{ten}}(\sigma_k, h(\varrho_k)),\tau_j\Big) &= (\varepsilon(v_k + \dot U),\tau_j) \\ &\quad \text{for } j = 1,\dots,k \\ \frac{d}{dt}(\varrho_k,\Phi_j) - \Big(m_\delta(\varrho_k)(H\nabla\varrho_k + \nabla\Delta\varrho_k),\nabla\Phi_j\Big) &= -\Big(\gamma_{\text{skal}}(\sigma_k, h(\varrho_k)),\Phi_j\Big) \\ &\quad \text{for } j = 0,\dots,k \\ \lambda\big(\varepsilon(v_k + \dot U),\varepsilon(w_j)\big) + \big(\sigma_k,\varepsilon(w_j)\big) &= L(t,w_j) \\ &\quad \text{for } j = 1,\dots,k \end{aligned}\right\} \tag{5.11}$$

to the initial values

$$\sigma_k(\cdot,0) = P_{T^k}\sigma_0(\cdot), \quad \varrho_k(\cdot,0) = P^k\varrho_0(\cdot), \quad v_k(\cdot,0) = P_{W^k}\big(u_1(\cdot) - \dot U(\cdot,0)\big).$$

For simplicity, we have omitted the indices λ and δ.

In order to obtain the a priori-estimates necessary for the proof of global solvability of (5.11) on $[0,T]$ and the limit process $k \to \infty$ we test the equations of system (5.11) with σ_k, $\varrho_k - \Delta\varrho_k$ and v_k, respectively. Summing up we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{(A\sigma_k,\sigma_k) + (\varrho_k,\varrho_k) + (\nabla\varrho_k,\nabla\varrho_k)\Big\} \\ &\quad + \Big(m_\delta(\varrho_k)\nabla\Delta\varrho_k,\nabla\Delta\varrho_k\Big) + \lambda\|\varepsilon(v_k)\|^2_{L^2(\Omega)} \\ &= L(t,v_k) + \big(\varepsilon(\dot U),\sigma_k\big) - \lambda\big(\varepsilon(\dot U),\varepsilon(v_k)\big) \\ &\quad + \Big(\gamma_{\text{skal}}(\sigma_k, h(\varrho_k)),\Delta\varrho_k - \varrho_k\Big) \\ &\quad - \Big(\gamma_{\text{ten}}(\sigma_k, h(\varrho_k)),\sigma_k\Big) + \Big(m_\delta(\varrho_k)H\nabla\varrho_k,\nabla\varrho_k\Big) \\ &\quad + \Big(m_\delta(\varrho_k)\nabla\Delta\varrho_k,(1-H)\nabla\varrho_k\Big) \\ &= \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV} + \mathcal{V} + \mathcal{VI} + \mathcal{VII}. \end{aligned}$$

As in [15] it is possible to estimate $\mathcal{I}, \mathcal{II}$ and $\mathcal{III}$ by $2(A\sigma_k,\sigma_k) + c(1 + \lambda + \frac{1}{\lambda})$ with a constant c depending only on T and the data. Using the first relation of (5.3) and (5.4) we may handle $\mathcal{IV}, \mathcal{V}, \mathcal{VI}$ and $\mathcal{VII}$ as in Section 2, and applying the Gronwall lemma we finally end up with the following estimates depending only on δ and λ:

ϱ_k	uniformly lies in a bounded subset of $L^\infty(I; H^1(\Omega))$
$m_\delta(\varrho_k)\nabla\Delta\varrho_k$	uniformly lies in a bounded subset of $L^2(\Omega_T; \mathbb{R}^3)$
σ_k	uniformly lies in a bounded subset of $L^2(\Omega_T; M^{3\times 3}_{\text{sym}})$
$\gamma(\sigma_k, h(\varrho_k))$	uniformly lies in a bounded subset of $L^2(\Omega_T; M^{3\times 3}_{\text{sym}} \times \mathbb{R})$
$\dot u_k = v_k + \dot U$	uniformly lies in a bounded subset of $L^2(I; H^1(\Omega; \mathbb{R}^3))$.

For the last result we have made use of the Korn inequality (see, e.g., in [5: p. 291]).

Proceeding in a similar way as in Section 2, the following additional estimates may easily be derived:

$$\begin{array}{ll} \dot\varrho_k & \text{uniformly lies in a bounded subset of } L^2(I;(H^1(\Omega))') \\ \dot\sigma_k & \text{uniformly lies in a bounded subset of } L^2(\Omega_T;M^{3\times 3}_{\text{sym}}) \\ (\nabla\varrho_k)_t & \text{uniformly lies in a bounded subset of } L^2(I;(H^2_N(\Omega))'). \end{array}$$

Thus we obtain – after time integration of system (5.5) and passing to the limit $k\to\infty$ – a quadruple $(\sigma_{\delta\lambda},\dot u_{\delta\lambda},\varrho_{\delta\lambda},\Sigma)$ with the inclusion properties

$$(\sigma_{\delta\lambda},\dot u_{\delta\lambda},\Sigma)\in W^1_2(I;L^2(\Omega))\times L^2(I;H^1(\Omega;\mathbb{R}^3))\times L^2(\Omega_T;M^{3\times 3}_{\text{sym}}\times\mathbb{R})$$

$$\varrho_{\delta\lambda}\in W^1_2(I;H^1(\Omega),L^2(\Omega))\cap L^2(I;H^3(\Omega))\cap L^2(I;H^2_*(\Omega))\cap C(I;H^1(\Omega))$$

and the equalities

$$\int_0^T(A\dot\sigma_{\delta\lambda},\tau)dt+\int_0^T(\Sigma_{\text{ten}},\tau)dt=\int_0^T\big(\varepsilon(\dot u_{\delta\lambda}),\tau\big)dt \tag{5.14}$$

$$\int_0^T\langle\dot\varrho_{\delta\lambda},\eta\rangle dt-\int_0^T\Big(m_\delta(\varrho_{\delta\lambda})(H\nabla\varrho_{\delta\lambda}+\nabla\Delta\varrho_{\delta\lambda}),\nabla\eta\Big)dt=-\int_0^T(\Sigma_{\text{skal}},\eta)dt \tag{5.15}$$

$$\lambda\int_0^T\big(\varepsilon(\dot u_{\delta\lambda}),\varepsilon(w)\big)dt+\int_0^T\big(\sigma_{\delta\lambda},\varepsilon(w)\big)dt=\int_0^T L(t,w)\,dt \tag{5.16}$$

for all

$$\tau\in L^2(\Omega_T;M^{3\times 3}_{\text{sym}}),\qquad \eta\in L^2(I;H^1(\Omega)),\qquad w\in C^0_{\text{ad}}$$

respectively. Moreover, the initial values are assumed by $(\sigma_{\delta\lambda},\dot u_{\delta\lambda},\varrho_{\delta\lambda})$.

Identification of Σ and $\gamma(\sigma_{\delta\lambda},h(\varrho_{\delta\lambda}))$ by the monotonicity of γ. Let J_k and $J_{\delta\lambda}$ be defined by

$$J_k=m_\delta(\varrho_k)(H\nabla\varrho_k+\nabla\Delta\varrho_k)\qquad\text{and}\qquad J_{\delta\lambda}=m_\delta(\varrho_{\delta\lambda})(H\nabla\varrho_{\delta\lambda}+\nabla\Delta\varrho_{\delta\lambda}).$$

We notice that J_k converges weakly in $L^2(\Omega_T;\mathbb{R}^3)$ to $J_{\delta\lambda}$ as $k\to\infty$; at the same time (5.3) imply that $h(\varrho_{k'})$ tends strongly in $L^2(I;H^1(\Omega))$ to $h(\varrho_{\delta\lambda})$ for a subsequence $k'\to\infty$. Testing the first and second equation of system (5.11) with σ_k and $h(\varrho_k)$, respectively, and subsequently integrating in time from 0 to T yields

$$\begin{aligned}&\int_0^T\left(\gamma(\sigma_k,h(\varrho_k)),\begin{pmatrix}\sigma_k\\ h(\varrho_k)\end{pmatrix}\right)dt\\ &\quad=-\frac12(A\sigma_k,\sigma_k)\Big|_0^T+\int_0^T\big(\varepsilon(\dot u_k),\sigma_k\big)dt+\int_0^T\langle\dot\varrho_k,h(\varrho_k)\rangle-\big(J_k,\nabla h(\varrho_k)\big)dt.\end{aligned} \tag{5.17}$$

Similarly, by multiplying equations (5.14) and (5.15) by $\sigma_{\delta\lambda}$ and $h(\varrho_{\delta\lambda})$, respectively, we obtain

$$\int_0^T \left(\Sigma, \begin{pmatrix} \sigma_{\delta\lambda} \\ \varrho_{\delta\lambda} \end{pmatrix}\right) dt$$

$$= -\frac{1}{2}\left(A\sigma_{\delta\lambda}, \sigma_{\delta\lambda}\right)\Big|_0^T + \int_0^T \left(\varepsilon(\dot{u}_{\delta\lambda}), \sigma_{\delta\lambda}\right) dt + \int_0^T \left\langle \dot{\varrho}_{\delta\lambda}, h(\varrho_{\delta\lambda})\right\rangle - \left(J_{\delta\lambda}, \nabla h(\varrho_{\delta\lambda})\right) dt. \tag{5.18}$$

The monotonicity of γ implies that

$$X_k := \int_0^T \left(\gamma(\sigma_k, h(\varrho_k)) - \gamma(\hat{\sigma}, \hat{\varrho}), \begin{pmatrix} \sigma_k - \hat{\sigma} \\ h(\varrho_k) - \hat{\varrho} \end{pmatrix}\right) dt \tag{5.19}$$

is non-negative for each $k \in \mathbb{N}$ and arbitrary pairs $(\hat{\sigma}, \hat{\varrho}) \in L^2\left(\Omega_T; M^{3\times 3}_{\mathrm{sym}} \times \mathbb{R}\right)$. Inserting (5.17) into equation (5.19) and passing to the limes superior yields

$$\begin{aligned} \limsup_{k\to\infty} X_k \le & -\frac{1}{2}\left(A\sigma_{\delta\lambda}, \sigma_{\delta\lambda}\right)\Big|_0^T + \int_0^T \left\langle \dot{\varrho}_{\delta\lambda}, h(\varrho_{\delta\lambda})\right\rangle dt \\ & - \int_0^T \left(J_{\delta\lambda}, \nabla h(\varrho_{\delta\lambda})\right) dt + \int_0^T \left(\varepsilon(\dot{u}_{\delta\lambda}), \sigma_{\delta\lambda}\right) dt \\ & - \int_0^T \left(\Sigma, \begin{pmatrix} \hat{\sigma} \\ \hat{\varrho} \end{pmatrix}\right) dt - \int_0^T \left(\gamma(\hat{\sigma}, \hat{\varrho}), \begin{pmatrix} \sigma_{\delta\lambda} - \hat{\sigma} \\ h(\varrho_{\delta\lambda}) - \hat{\varrho} \end{pmatrix}\right) dt. \end{aligned} \tag{5.20}$$

Here we have used the fact that, for fixed $t \in [0, T]$, $\sigma_k(\cdot, t)$ converges weakly in $L^2(\Omega)$ to $\sigma_{\delta\lambda}(\cdot, t)$. Furthermore, we have concluded from the identity

$$\int_0^T \left(\varepsilon(\dot{u}_k), \sigma_k\right) dt = \int_0^T \Big\{\left(\varepsilon(\dot{U}), \sigma_k\right) + L(t, v_k) - \lambda\left(\varepsilon(\dot{U}), \varepsilon(v_k)\right) - \lambda\left(\varepsilon(v_k), \varepsilon(v_k)\right)\Big\} dt$$

and the weak convergence of v_k in $L^2\left(I; H^1(\Omega)\right)$ that

$$\limsup_{k\to\infty} \int_0^T \left(\varepsilon(\dot{u}_k), \sigma_k\right) dt \le \int_0^T \left(\varepsilon(\dot{u}_{\delta\lambda}), \sigma_{\delta\lambda}\right) dt.$$

Combining (5.17) and (5.19) we arrive at

$$\int_0^T \left(\Sigma - \gamma(\hat{\sigma}, \hat{\varrho}), \begin{pmatrix} \sigma_{\delta\lambda} - \hat{\sigma} \\ h(\varrho_{\delta\lambda}) - \hat{\varrho} \end{pmatrix}\right) dt \ge 0.$$

Now applying standard methods for monotone, hemicontinuous operators we succeed in identifying Σ and $\gamma(\sigma_{\delta\lambda}, h(\varrho_{\delta\lambda}))$.

Estimates independent of λ. Since $\tau_{\delta\lambda} := \sigma_{\delta\lambda} - \chi_{\text{ten}}$ is not contained in S^0_{ad}, the method used in [15: Section 4.1] is not correct, and we have to modify the arguments in order to obtain estimates which are independent of λ. For this purpose we begin with multiplying equation (5.13) by $\tau_{\delta\lambda}$. Using (5.9), (5.15) and the Green formula (5.7) we obtain

$$\begin{aligned}
&\frac{1}{2}\big(A\tau_{\delta\lambda}, \tau_{\delta\lambda}\big)\Big|_T + \int_0^T \Big(\gamma_{\text{ten}}\big(\sigma_{\delta\lambda}, h(\varrho_{\delta\lambda})\big), \tau_{\delta\lambda}\Big)dt \\
&\quad = \frac{1}{2}\big(A\tau_{\delta\lambda}, \tau_{\delta\lambda}\big)\big|_0 - \int_0^T \big(A\dot{\chi}_{\text{ten}}, \tau_{\delta\lambda}\big)dt + \int_0^T \big(\varepsilon(\dot{U}), \tau_{\delta\lambda}\big)dt \\
&\quad\quad + \lambda \int_0^T \big(\varepsilon(\dot{u}_{\delta\lambda}), \varepsilon(\dot{U})\big)dt - \lambda \int_0^T \|\varepsilon(\dot{u}_{\delta\lambda})\|^2_{L^2(\Omega)} dt \\
&\quad = \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV} + \mathcal{V}.
\end{aligned}$$

With the help of the first relation of (5.8) and (5.9) we can estimate $\mathcal{II}$ and $\mathcal{III}$ in the following way:

$$\mathcal{II} + \mathcal{III} \leq C + \int_0^T \big(A\tau_{\delta\lambda}, \tau_{\delta\lambda}\big)dt.$$

In order to estimate $\mathcal{IV}$ independently of λ, but still dependent of δ, we take advantage of the fact that (5.12) implies with a constant C_1 independent of λ the estimates

$$\|\varepsilon(\dot{u}_{\delta\lambda})\|_{L^2(\Omega_T)} \leq \frac{C_1}{\lambda}.$$

It follows for the right-hand side in the equation above

$$R.H.S. \leq \frac{1}{2}\big(A\tau_{\delta\lambda}, \tau_{\delta\lambda}\big)\big|_0 + C\left(1 + \int_0^T \big(A\tau_{\delta\lambda}, \tau_{\delta\lambda}\big)dt\right).$$

In order to handle the second term on the left-hand side we test (5.15) with $\varrho_{\delta\lambda} - \Delta\varrho_{\delta\lambda}$, add this equation to the inequality above and obtain by standard absorption techniques the following a priori estimates independent of λ, but still depending on δ:

$$\begin{aligned}
(\sigma_{\delta\lambda}, \varrho_{\delta\lambda}) \quad & \text{uniformly lies in a bounded subset of } L^\infty\big(I; L^2(\Omega; M^{3\times 3}_{\text{sym}}) \times H^1(\Omega)\big) \\
m_\delta(\varrho_{\delta\lambda})\nabla\Delta\varrho_{\delta\lambda} \quad & \text{uniformly lies in a bounded subset of } L^2(\Omega_T; \mathbb{R}^3) \\
(\dot{\varrho}_{\delta\lambda}, \nabla\dot{\varrho}_{\delta\lambda}) \quad & \text{uniformly lies in a bounded subset of } L^2\big(I; (H^1(\Omega))' \times (H^2_N(\Omega))'\big).
\end{aligned}$$

Now testing equation (5.14) with $\tau \in L^2(I; S^0_{\mathrm{ad}})$ we derive by the use of the Green formula

$$\int_0^T (A\dot{\tau}_{\delta\lambda}, \tau)\,dt + \int_0^T \Big(\gamma_{\mathrm{ten}}\big(\sigma_{\delta\lambda}, h(\varrho_{\delta\lambda})\big), \tau\Big)\,dt$$
$$= \int_0^T \int_{\Gamma_0} (\dot{U} \otimes \nu)\tau \, d\Gamma dt - \int_0^T (A\dot{\chi}_{\mathrm{ten}}, \tau)\,dt.$$

Being equipped with the L^2-scalar product, S^0_{ad} becomes a Hilbert space, and we end up with the a priori estimates

$$\dot{\tau}_{\delta\lambda} \quad \text{uniformly lies in a bounded subset of } L^2\big(I; (S^0_{\mathrm{ad}})'\big). \tag{5.21}$$

Limit process $\lambda \to 0$. Proceeding as in [15] we obtain by time integration in the strong version of (5.14) a quadruple $(\sigma_\delta, \varrho_\delta, u_\delta, \Sigma^\delta)$ with the inclusions

$$\sigma_\delta \in L^\infty\big(I; L^2(\Omega)\big), \qquad \Sigma^\delta \in L^2\big(\Omega_T; M^{3\times 3}_{\mathrm{sym}} \times I\!R\big), \qquad u_\delta \in L^\infty\big(I; H^1(\Omega)\big)$$

and ϱ_δ as in Subsection 2.1, and the following properties:

$$\begin{gathered}
A\big(\sigma_\delta(t) - \sigma_0\big) + \int_0^t \Sigma^\delta_{\mathrm{ten}}(s)\,ds = \varepsilon(u_\delta)(t) - \varepsilon(u_0) \\
\int_0^T \langle \dot{\varrho}_\delta, \eta\rangle dt - \int_0^T \Big(m_\delta(\varrho_\delta)\big(H\nabla\varrho_\delta + \nabla\Delta\varrho_\delta\big), \nabla\eta\Big)dt = -\int_0^T (\Sigma^\delta_{\mathrm{skal}}, \eta)\,dt \\
\text{for all } \ \eta \in L^2\big(I; H^1(\Omega)\big) \\
\big(\sigma_\delta(t), \varepsilon(w)\big) = L(t, w) \quad \text{for all } w \in C^0_{\mathrm{ad}}.
\end{gathered} \tag{5.22}$$

It is worth mentioning that the first and last relations of (5.22) imply the inclusion $\tau_\delta = \sigma_\delta - \chi_{\mathrm{ten}} \in L^\infty\big(I; S^0_{\mathrm{ad}}\big)$. With (5.21) we conclude the inclusion

$$\tau_\delta \in W^1_2\big(I; S^0_{\mathrm{ad}}\big).$$

In particular, we have for all $\tau \in L^2\big(I; S^0_{\mathrm{ad}}\big)$

$$\int_0^T (A\dot{\tau}_\delta, \tau)\,dt + \int_0^T (\Sigma^\delta_{\mathrm{ten}}, \tau)\,dt = \int_0^T \int_{\Gamma_0} (\dot{U} \otimes \nu)\tau\, d\Gamma dt - \int_0^T (A\dot{\chi}_{\mathrm{ten}}, \tau)\,dt. \tag{5.23}$$

Combining our methods with the techniques R. Temam used in [15] we easily identify Σ^δ and $\gamma\big(\sigma_\delta, h(\varrho_\delta)\big)$.

Estimates independent of δ. We test (5.23) with τ_δ, the second relation of (5.22) with $\varrho_\delta + g_\delta(\varrho_\delta) - \Delta\varrho_\delta$ and add the equations. Using the absorption techniques of Subsection 2.2 we end up with the following inequality containing a constant C_0 depending only on the initial values:

$$\begin{aligned}
&\frac{1}{2}\Big\{(A\tau_\delta,\tau_\delta) + \|\varrho_\delta\|^2_{H^1(\Omega)}\Big\}\Big|_T + \int_\Omega G_\delta(\varrho_\delta(T))\,dx \\
&+ \int_0^T \Big(m_\delta(\varrho_\delta)\nabla\Delta\varrho_\delta, \nabla\Delta\varrho_\delta\Big)dt + \int_0^T \|\Delta\varrho_\delta\|^2_{L^2(\Omega)}dt \\
&\le C_0\left(1 + \int_0^T (\nabla\varrho_\delta,\nabla\varrho_\delta)dt\right) - \int_0^T \Big(\gamma_{\mathrm{skal}}\big(\sigma_\delta, h(\varrho_\delta)\big), \varrho_\delta + g_\delta(\varrho_\delta) - \Delta\varrho_\delta\Big)dt \\
&- \int_0^T \Big(\gamma_{\mathrm{ten}}\big(\sigma_\delta, h(\varrho_\delta)\big), \tau_\delta\Big)dt - \int_0^T (A\dot\chi_{\mathrm{ten}}, \tau_\delta)dt + \int_0^T\int_{\Gamma_0} (\dot U \otimes \nu)\tau_\delta\, d\Gamma dt.
\end{aligned} \tag{5.24}$$

Using the first growth condition of (5.3), (5.4) and (5.9), the non-positivity of γ_{skal} and the imbedding (5.6) we obtain in the case of the coupled problem the same estimates for ϱ as in (2.17) and (2.18). Moreover, for τ_δ we have independently of $\delta > 0$ that

$$\begin{aligned}
&\tau_\delta \quad \text{uniformly lies in a bounded subset of } L^\infty(I; S^0_{\mathrm{ad}}) \\
&\dot\tau_\delta \quad \text{uniformly lies in a bounded subset of } L^2(I; (S^0_{ad})').
\end{aligned}$$

Remark 5.5: Unfortunately, the term $\gamma(\sigma_\delta, h(\varrho_\delta))$ cannot be absorbed on the left-hand side of (5.24). Thus we do not obtain estimates which are independent of the growth constant of γ. For this reason it seems to be impossible to apply conventional methods in order to prove existence results for a *Prandtl-Reuss model* with a similar hardening parameter as considered in this paper.

Limit process $\delta \to 0$. Using the a priori estimates derived above and repeating the monotonicity arguments of the previous subsection we obtain – passing to the limit $\delta \to 0$ – a quadruple (τ, u, ϱ, J) with the following properties:

$$\int_0^T \langle\dot\varrho, \eta\rangle dt - \int_0^T (J, \nabla\eta)dt = -\int_0^T \Big(\gamma_{\mathrm{skal}}\big(\tau + \chi_{\mathrm{ten}}, h(\varrho)\big), \eta\Big)dt$$

for all $\eta \in L^2(I; H^1(\Omega))$, $J = m(\varrho)(H\nabla\varrho + \nabla\Delta\varrho)$ in the sense of (1.10), $\tau \in C(I; S^0_{\mathrm{ad}})$ and

$$\int_0^T (A\dot\tau, \hat\tau)\,dt + \int_0^T \Big(\gamma_{\mathrm{ten}}\big(\tau + \chi_{\mathrm{ten}}, h(\varrho)\big), \hat\tau\Big)dt = \int_0^T\int_{\Gamma_0} (\dot U \otimes \nu)\hat\tau\, d\Gamma dt - \int_0^T (A\dot\chi_{\mathrm{ten}}, \hat\tau)\,dt$$

for all $\hat{\tau} \in L^2(I; S^0_{\rm ad})$.

Now we choose $\sigma = \tau + \chi_{\rm ten}$. As both τ_δ and $\int_0^t \gamma_{\rm ten}(\sigma_\delta, h(\varrho_\delta))\, ds$ converge weakly$*$ in $L^\infty(I; L^2(\Omega; M^{3\times3}_{\rm sym}))$ to τ and $\int_0^t \gamma_{\rm ten}(\sigma, h(\varrho))\, ds$, respectively, we conclude by the first relation of (5.22), Korn's inequality and the inclusion $u_\delta \in C_{\rm ad}(U_0)$ that u_δ converges weakly$*$ in $L^\infty(I; H^1(\Omega; \mathbb{R}^3))$ to u. Passing to the limit in the first equation of (5.22) gives (5.10). Furthermore, from the last relaton of (2.17) we infer the non-negativity results conjectured for ϱ. This proves Theorem 5.3.

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