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A Fully Discrete Galerkin Method for Integral and Pseudodifferential Equations on Closed Curves

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Abstract. We propose a cheep fully discrete version of the trigonometric Galerkin method of optimal accuracy order for integral and pseudodifferential equations on closed curves. A practical implementation of the method leads to a band system of linear algebraic equations. The error analysis is based on a thorough study of related integral operators in Sobolev spaces of periodic functions.

Keywords: *Galerkin method, boundary integral equations*

AMS subject classification: Primary 65 R 20, secondary 65 N 38, 45 E 99, 35 S 99

1. Introduction

This paper is devoted to methods of solving integral and pseudodifferential equations in Sobolev spaces of periodic functions. These equations arise using a potential type representation of the solution of boundary value problems on a region $\Omega \subset \mathbb{R}^2$ with a smooth Jordan curve $\Gamma = \partial\Omega$ as the boundary of Ω.

Let us present some examples of boundary integral equations.

Example 1.1. The Symm's integral equation

$$-\frac{1}{2\pi}\int_\Gamma \log|x-y|v(y)\,d\Gamma_y = g(x) \qquad (x \in \Gamma)$$

arises solving the Dirichlet problem for the Laplace equation in Ω (see, e.g., [7: p. 303]). It is of interest also when a conformal map of Ω onto the unit disc is constructed (see, e.g., [8]). Introducing a smooth 1-periodic parametrization $x : \mathbb{R} \to \Gamma$ of Γ such that $|x'(t)| > 0$ for all $t \in \mathbb{R}$, the equation reduces to

$$\int_0^1 \log|x(t)-x(s)|u(s)\,ds = f(t) \qquad (t \in [0,1])$$

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or

$$\int_0^1 \kappa_0(t-s)u(s)\,ds + \int_0^1 a_1(t,s)u(s)\,ds = f(t) \qquad (t \in [0,1]) \tag{1.1}$$

where

$$u(t) = v(x(t))|x'(t)|, \qquad f(t) = -2\pi g(x(t)), \qquad \kappa_0(t) = \log|\sin \pi t|$$

and

$$a_1(t,s) = \begin{cases} \log \dfrac{|x(t)-x(s)|}{|\sin \pi(t-s)|} & \text{for } t \neq s \pmod 1 \\ \log \dfrac{1}{\pi}|x'(t)| & \text{for } t = s \pmod 1). \end{cases}$$

The kernel a_1 is smooth and 1-biperiodic, the Fourier coefficients of κ_0 are known: $\hat{\kappa}_0(0) = -\log 2$ and $\hat{\kappa}_0(m) = -\frac{1}{2}|m|^{-1}$ $(0 \neq m \in Z)$.

In [7: p. 326] and [9] one can find also boundary integral equations of the Neumann problem for the Laplace equation.

Example 1.2. In the case of the exterior Dirichlet boundary value problem for the Helmholz equation one has a boundary integral equation of the type (see [11])

$$\begin{aligned} &\int_0^1 \kappa_0(t-s)u(s)\,ds + \\ &\int_0^1 \kappa_1(t-s)a_1(t,s)u(s)\,ds + \int_0^1 a_2(t,s)u(s)\,ds = f(t) \end{aligned} \qquad (t \in [0,1]) \tag{1.2}$$

where a_1 and a_2 are smooth 1-biperiodic functions, $\kappa_0(t) = \log|\sin \pi t|$ and $\kappa_1(t) = (\sin \pi t)^2 \log|\sin \pi t|$. For the Fourier coefficients of κ_0 and κ_1 we have $\hat{\kappa}_0(m) \sim |m|^{-1}$ as in Example 1.1 and $\hat{\kappa}_1(m) \sim |m|^{-3}$ (cf. Example 1.4).

Example 1.3. The exterior Neumann boundary value problem for the Helmholz equation can be reduced to an equation of the type (see [18])

$$\begin{aligned} &\frac{\pi}{|x'(t)|} \int_0^1 \kappa_0(t-s)u(s)\,ds + iu(t) \\ &+ \int_0^1 \kappa_2(t-s)a_2(t,s)u(s)\,ds + \int_0^1 a_3(t,s)u(s)\,ds = f(t) \end{aligned} \qquad (t \in [0,1]) \tag{1.3}$$

where a_2 and a_3 are smooth 1-biperiodic functions, $\kappa_0(t) = 1/\sin^2 \pi t$ and $\kappa_2(t) = \log|\sin \pi t|$ whereby the first integral in (1.3) is understood in the sense of the Hadamard finite part. For the Fourier coefficients of κ_0 and κ_2 we have $\hat{\kappa}_0(0) = 0$, $\hat{\kappa}_0(m) = -2|m|$ $(0 \neq m \in Z)$ and $|\hat{\kappa}_2(m)| \sim |m|^{-1}$ (see Example 1.1).

Example 1.4. Solving boundary value problems for biharmonic equations the boundary integral equation

$$\int_\Gamma |x-y|^2 \log|x-y| v(y)\, d\Gamma_y = g(x) \qquad (x \in \Gamma) \tag{1.4}$$

is of interest (see [4] and [7: p. 336]). It reduces to the integral equation

$$\int_0^1 \kappa_0(t-s) a_0(t,s) u(s)\, ds + \int_0^1 a_1(t,s) u(s) ds = f(t) \qquad (t \in [0,1])$$

with $u(s) = v(x(s))|x'(s)|, f(t) = g(x(t))$, 1-biperiodic smooth functions

$$a_0(t,s) = \begin{cases} \dfrac{|x(t)-x(s)|^2}{\sin^2 \pi(t-s)} & \text{for } t \neq s \pmod 1 \\ \dfrac{1}{\pi^2}|x'(t)|^2 & \text{for } t = s \pmod 1 \end{cases}$$

$$a_1(t,s) = |x(t)-x(s)|^2 \log \frac{|x(t)-x(s)|}{|\sin \pi(t-s)|}$$

and 1-periodic function

$$\kappa_0(t) = (\sin \pi t)^2 \log|\sin \pi t|$$

with known Fourier coefficients:

$$\hat{\kappa}_0(0) = -\frac{1}{2}\log 2 + \frac{1}{4}, \qquad \hat{\kappa}_0(1) = \kappa_0(-1) = \frac{1}{4}\log 2 - \frac{3}{16}$$

$$\hat{\kappa}_0(m) = \left(4|m|(m^2-1)\right)^{-1} \qquad (m \in Z,\ |m| \geq 2).$$

Note that $a_0(t,t) \neq 0$.

Example 1.5. In this example we identify $\mathbb{R}^2$ with $\boldsymbol{C}$ treating $x = x_1 + ix_2$ and $y = y_1 + iy_2$ as complex numbers. Consider the singular integral equation

$$\frac{b(x)}{\pi i}\int_\Gamma \frac{v(y)}{y-x}\, dy + \int_\Gamma K(x,y) v(y)\, d\Gamma_y = g(x) \qquad (x \in \Gamma)$$

where Γ is a smooth Jordan curve in the complex plain, and b and K are given smooth functions on Γ and $\Gamma \times \Gamma$, respectively. A smooth 1-periodic parametrization $x = x_1 + ix_2 : \mathbb{R} \to \Gamma$ of Γ satisfying $x'(t) = x_1'(t) + ix_2'(t) \neq 0$ for all $t \in \mathbb{R}$ yields the equation

$$\int_0^1 \kappa_0(t-s) a_0(t,s) u(s) ds + \int_0^1 a_1(t,s) u(s) ds = f(t) \tag{1.5}$$

where $\kappa_0(t) = \frac{2}{1-e^{i2\pi t}} = 1 + i\cot \pi t$,

$$a_0(t,s) = \begin{cases} \dfrac{b(x(t))}{2\pi i}\,\dfrac{1-e^{i2\pi(t-s)}}{x(s)-x(t)}\, x'(s) & \text{for } t \neq s \pmod 1 \\ b(x(t)) & \text{for } t = s \pmod 1 \end{cases}$$

and

$$a_1(t,s) = K\big(x(t), x(s)\big)x'(s), \qquad u(s) = v(x(s)), \qquad f(t) = g(x(t)).$$

The functions a_0 and a_1 are 1-biperiodic and smooth (note that $x(s) \neq x(t)$ for $s \neq t \pmod 1$ since Γ as a Jordan curve does not cut itself). Further, for the Fourier coefficients of κ_0 we have $\hat{\kappa}_0(m) = 1$ for $m \geq 0$ and $\hat{\kappa}_0(m) = -1$ for $m < 0$. The first integral in (1.5) is understood in the sense of the Cauchy main value. Note that $a_0(t,t) \neq 0$ for $t \in I\!R$ if $b(x) \neq 0$ for $x \in \Gamma$.

The integral equations (1.1) – (1.5) are of the form

$$\sum_{p=0}^{q} \int_0^1 \kappa_p(t-s) a_p(t,s) u(s)\, ds = f(t) \tag{1.6}$$

where a_p are 1-biperiodic smooth functions and

$$\int_0^1 \kappa_p(t-s) e^{im2\pi s} ds = \hat{\kappa}_p(m) e^{im2\pi t} \qquad (m \in Z) \tag{1.7}$$

with known Fourier coefficients $\hat{\kappa}_p(m)$ $(m \in Z,\, p = 0, \ldots, q)$ of the order $|\hat{\kappa}_p(m)| \sim |m|^{-\alpha_p}$ $(\alpha_0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q)$. Usually $\alpha_p > 0$ but in Example 1.3 $\alpha_0 = -1$ and in Example 1.5 $\alpha_0 = 0$. Note that κ_p is continuous if $\alpha_p > 1$ and may have weak singularity if $0 < \alpha_p \leq 1$. For $\alpha_p \leq 0$, the corresponding integral in (1.6) should be understood in the sense of distributions; a more familiar understanding follows from (1.7) using the Fourier expansions of a_p and u (see Sections 2 and 3 for more details). Moreover, if $\hat{\kappa}_p(m)$ depends on m in a "smooth" way, namely, if $\hat{\kappa}_p(m) = |m|^{\alpha_p}$ or $\hat{\kappa}_p(m) = |m|^{\alpha_p}\text{sgn}(m)$, it is possible to construct a representation

$$\kappa_p(t-s)a_p(t,s) = \sum_{k=0}^{n} b_{p,k}(t)\kappa_{p,k}(t-s) + a_{p,n+1}(t,s)\kappa_{p,n+1}(t-s) \quad (n \geq 0) \tag{1.8}$$

where $|\hat{\kappa}_{p,k}(m)| \sim |m|^{\alpha_p - k}$ $(k = 0, \ldots, n+1)$, the coefficients $b_{p,k}$ are smooth and 1-periodic, and $a_{p,n+1}$ is smooth and 1-biperiodic. Unfortunately, this decomposition is somewhat impractical due to the complexity of the construction and involving high order derivatives of the parametrization function $x = x(t)$ in Examples 1.2 – 1.5. We shall solve (1.6) directly avoiding possible decopositions of the type (1.8). This also allows to avoid the "smoothness" assumptions for $\hat{\kappa}_p(m)$.

The trigonometric Galerkin and collocation methods, and fully discretized versions of those have been extensively examined for Symm's integral equation and other integral equation of type (1.1) with $|\hat{\kappa}_0(m)| \sim |m|^{-\alpha}$ $(\alpha \in I\!R)$ and smooth a_1 (see, e.g., [2, 10, 13, 21, 23, 24]. More general problems have been treated in [1, 14 - 16, 18, 20, 22]. We refer also to [3] where a fast algorithm is proposed for the generalized airfoil equation with constant coefficients in a non-periodical case. In [11], the product integration was incorporated into the collocation method to solve the integral equation (1.2). This idea was used also in [4] and [18] for equations (1.3) and (1.4) and in [25] for general integral equations of type (1.6) with a_0 depending on both t and s.

In this paper we exploit another idea. We start from the trigonometric Galerkin method and obtain a full discretization using trigonometric interpolants of a_p $(p = 0, \ldots, q)$. This idea seems to orginate from works of P. Henrici (see, e.g., [8]). The order M of interpolants may be essentially smaller than the order N of the approximate solution u_N, e.g. $M \sim N^\sigma$ $(0 < \sigma \leq 1)$ is acceptable. This leads to a band systems of linear algebraic equations which can be solved in $\mathcal{O}(N^{1+2\sigma})$ arithmetical operations. The proposed method is of optimal convergence order in Sobolev norms. Beside (1.6) we discuss the case with the "full" main part A_0 (see (2.19)).

The paper is organized as follows. Section 2 contains the description of the method and formulation of main results. In Section 3 we discuss a matrix form of the method. After some preliminaries in Section 4, we examine the properties of integal operators trying to minimize the smoothness assumptions on the coefficients a_p. After that, in Section 6, we are ready to present the proofs of the main results formulated in Section 2. Numerical examples are presented in Section 7.

2. Problem, methods, and formulation of the main results

Now we formulate more precisely the conditions posed on the problem

$$\mathcal{A}u = f \tag{2.1}$$

where f is a given 1-periodic function and u is a 1-periodic function which is looked for. The operator has the form

$$\mathcal{A} = \sum_{p=0}^{q} A_p, \qquad (A_p u)(t) = \int_0^1 \kappa_p(t-s) a_p(t,s) u(s)\, ds \tag{2.2}$$

where the coefficients a_p are assumed to be C^∞-smooth and 1-biperiodic, and the Fourier coefficients

$$\hat{\kappa}_p(m) = \int_0^1 \kappa_p(t) e^{-im2\pi t} dt = \langle \kappa_p(t), e^{-im2\pi t} \rangle \qquad (m \in Z)$$

are assumed to be known and to satisfy the inequalities

$$\begin{aligned} c_1 \underline{m}^{-\alpha} \leq |\hat{\kappa}_0(m)| &\leq c_2 \underline{m}^{-\alpha} && (|m| \geq m_0) \\ |\hat{\kappa}_0(m) - \hat{\kappa}_0(m-1)| &\leq c\underline{m}^{-\alpha-\beta} && (m \in Z) \\ |\hat{\kappa}_p(m)| &\leq c\underline{m}^{-\alpha-\beta} && (p = 1, \ldots, q,\ m \in Z) \end{aligned} \tag{2.3}$$

with some $\alpha \in I\!R$, $\beta > 0$, $m_0 \in I\!N$, and positive constants c_1, c_2 and c. Here

$$\underline{m} := \begin{cases} |m| & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases} \qquad (m \in Z).$$

Since α may be also negative, problem (2.1) actually includes a variety of pseudodifferential equations, and for smooth u, the integral in (2.2) actually means the duality product between elements of appropriate Sobolev spaces $H^{-\lambda}$ and H^{λ}:

$$(A_p u)(t) = \langle \kappa_p(t-s), a_p(t,s)u(s) \rangle.$$

An equivalent understanding was outlined in Section 1 on the basis of (1.7) (see also Section 3).

The Sobolev space H^{μ} ($\mu \in \mathbb{R}$) consists of 1-periodic functions (distributions)

$$u(s) = \sum_{m \in Z} \hat{u}(m) e^{im2\pi s}, \qquad \hat{u}(m) = \int_0^1 u(s) e^{-im2\pi s} ds,$$

such that

$$\|u\|_{\mu} = \left(\sum_{m \in Z} \underline{m}^{2\mu} |\hat{u}(m)|^2 \right)^{1/2} < \infty.$$

Similarly, the Sobolev space $H^{\lambda,\mu}$ ($\lambda, \mu \in \mathbb{R}$) consists of 1-biperiodic functions (distributions)

$$v(t,s) = \sum_{l,m \in Z} \hat{v}(l,m) e^{il2\pi t} e^{im2\pi s}, \qquad \hat{v}(l,m) = \int_0^1 \int_0^1 v(t,s) e^{-il2\pi t} e^{-im2\pi s} dt ds$$

such that

$$\|v\|_{\lambda,\mu} = \left(\sum_{l,m \in Z} \underline{l}^{2\lambda} \underline{m}^{2\mu} |\hat{v}(l,m)|^2 \right)^{1/2} < \infty.$$

Proposition 2.1. *Under conditions* (2.3), *for* $\lambda \in \mathbb{R}$,

$$A_0 \in \mathcal{L}(H^{\lambda}, H^{\lambda+\alpha}) \qquad \text{and} \qquad A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda+\alpha+\beta}) \quad (1 \le p \le m).$$

If $a_0(t,t) \neq 0$ *for all* $t \in \mathbb{R}$, *then*

$$\mathcal{A} = \sum_{p=0}^{q} A_p \in \mathcal{L}(H^{\lambda}, H^{\lambda+\alpha})$$

is a Fredholm operator of index 0 *whereby*

$$\mathcal{N}(\mathcal{A}) := \{ u \in H^{\lambda} : \mathcal{A}u = 0 \} \subset \cap_{\mu} H^{\mu} =: C_1^{\infty}.$$

The proof of Proposition 2.1 is given in Section 6.

Our further assumptions about problem (2.1) are as follows:

$$b(t) := a_0(t,t) \neq 0 \quad \text{for all} \quad t \in \mathbb{R} \tag{2.4}$$

$$W(b) := \frac{1}{2\pi} \arg b(t)\Big|_{t=0}^{t=1} = 0 \tag{2.5}$$

$$v \in C_1^\infty, \quad \mathcal{A}v = 0 \quad \text{implies} \quad v = 0. \tag{2.6}$$

According to Proposition 2.1 assumptions (2.3), (2.4) and (2.6) guarantee the existence of the bounded inverse $\mathcal{A}^{-1} \in \mathcal{L}(H^{\lambda+\alpha}, H^\lambda)$ for any $\lambda \in \mathbb{R}$. Assumption (2.5) is needed for the convergence of the Galerkin method.

For $N \in \mathbb{N}$ denote

$$Z_N = \left\{ k \in Z : -\frac{N}{2} < k \le \frac{N}{2} \right\} \quad \text{and} \quad \mathcal{T}_N = \left\{ u_N = \sum_{k \in Z_N} c_k e^{ik2\pi t} : c_k \in \mathbf{C} \right\}.$$

In other words, $\mathcal{T}_N$ is the linear span of $e^{ik2\pi t}$ $(k \in Z_N)$. Denote by P_N the corresponding Fourier projection:

$$(P_N u)(t) = \sum_{k \in Z_N} \hat{u}(k) e^{ik2\pi t} \qquad (u \in H^\lambda).$$

Obviously, the projection P_N is orthogonal in H^λ for all $\lambda \in \mathbb{R}$. It is also clear that

$$\|u - P_N u\|_\lambda \le \left(\frac{N}{2}\right)^{\lambda-\mu} \|u\|_\mu \qquad (\lambda \le \mu). \tag{2.7}$$

First we approximate problem (2.1) by the Galerkin method

$$u_N \in \mathcal{T}_N, \qquad P_N \mathcal{A} u_N = P_N f. \tag{2.8}$$

Theorem 2.2. *Assume* (2.3) – (2.6) *and* $f \in H^{\mu+\alpha}$ $(\mu \in \mathbb{R})$. *Then there is a* N_0 *such that for* $N \ge N_0$ *the Galerkin method* (2.8) *determines a unique polynomial* $u_N \in \mathcal{T}_N$ *such that*

$$\|u_N - u\|_\lambda \le c_\lambda \|u - P_N u\|_\lambda \le c_\lambda \left(\frac{N}{2}\right)^{\lambda-\mu} \|u\|_\mu \qquad (\lambda \le \mu) \tag{2.9}$$

where $u = \mathcal{A}^{-1} f \in H^\mu$ *is the (unique) solution of problem* (2.1), *and the constant* c_λ *is independent of* N *and* u *(or* f*).*

The proof of Theorem 2.2 is presented in Section 6.

The Galerkin method performs only a semidiscretization of the problem (2.1). Now we construct a fully discretized version of the method. Let us denote by $Q_N u$ the interpolation projection of $u \in H^\mu$ $(\mu > \frac{1}{2})$ to $\mathcal{T}_N$ on the uniform grid:

$$(Q_N u)(t) = \sum_{k \in Z_N} c_k e^{ik2\pi t}, \qquad (Q_N u)\left(\frac{n}{N}\right) = u\left(\frac{n}{N}\right) \quad (n = 0, 1, \dots, N-1).$$

The approximation properties of Q_N are similar to those of P_N (cf. (2.7)) but on a more restricted scale of Sobolev norms [1, 22]:

$$\|u - Q_N u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad \left(0 \le \lambda \le \mu,\ \mu > \frac{1}{2}\right). \tag{2.10}$$

Introduce also the interpolation projection $Q_{N,M}$ for functions of two variables:

$$(Q_{N,M} v)(t,s) = \sum_{k \in Z_N} \sum_{l \in Z_M} c_{kl} e^{ik2\pi t} e^{il2\pi s} \in \mathcal{T}_N \otimes \mathcal{T}_M$$

$$(Q_{N,M} v)\left(\frac{n}{N}, \frac{m}{M}\right) = v\left(\frac{n}{N}, \frac{m}{M}\right) \qquad \begin{pmatrix} n = 0,1,\ldots,N-1 \\ m = 0,1,\ldots,M-1 \end{pmatrix}.$$

The coefficients a_p in (2.2) may be approximated simply by $Q_{M,M} a_p \in \mathcal{T}_M \otimes \mathcal{T}_M$ but, without a loss of the approximation order, we may drop a part of the Fourier coefficients of $Q_{M,M} a_p$ putting

$$a_{p,M}(t,s) = \sum_{|k|+|l| \le M/2} (\widehat{Q_{M,M} a_p})(k,l) e^{ik2\pi t} e^{il2\pi s} \qquad (p = 0,\ldots,q). \tag{2.11}$$

Due to the C^∞-smoothness of the coefficients a_p, we obtain a suitable approximation $a_{p,M}$ already in dimensions M which may be much smaller than N. We put

$$M \sim N^\sigma \qquad (0 < \sigma \le 1) \tag{2.12}$$

where $M \sim N^\sigma$ means that $c_1 \le MN^{-\sigma} \le c_2$ as $N \to \infty$ with some positive constants c_1 and c_2. Introduce the operators (cf. (2.2))

$$\mathcal{A}_M = \sum_{p=0}^{q} A_{p,M}, \qquad (A_{p,M} u)(t) = \int_0^1 \kappa_p(t-s) a_{p,M}(t,s) u(s) ds. \tag{2.13}$$

Instead of $P_N f$ we shall use some approximation $f_N \in \mathcal{T}_N$ assuming that $\hat{f}_N$ is computable. We are ready to introduce a modified Galerkin method which is fully discrete:

$$u_N \in \mathcal{T}_N, \qquad P_N \mathcal{A}_M u_N = f_N \tag{2.14}$$

(see Section 3 about the implementation of the method).

Theorem 2.3. *Assume* (2.3) − (2.6) *and* $f \in H^{\mu+\alpha}$ ($\mu \in \mathbb{R}$). *Let the operator* $\mathcal{A}_M$ *be defined by* (2.11) − (2.13). *Then there is an* N_0 *such that for* $N \ge N_0$ *the modified Galerkin method* (2.14) *determines a unique polynomial* $u_N \in \mathcal{T}_N$, *and*

$$\|u_N - u\|_\lambda \le c_\lambda \Big(\|u - P_N u\|_\lambda + \|f_N - P_N f\|_{\lambda+\alpha} + c_{\lambda,r} N^{-r} \|u\|_\lambda \Big) \quad (\lambda \le \mu) \tag{2.15}$$

where $r > 0$ *is arbitrary,* $u = \mathcal{A}^{-1} f \in H^\mu$ *is the solution of problem* (2.1) *and the constants* c_λ *and* $c_{\lambda,r}$ *are independent of* N *and* u.

The proof of Theorem 2.3 is given in Section 6.

We complete Theorem 2.3 specifying the error estimate (2.15) for some choices of $f_N \in \mathcal{T}_N$.

1. Case: $f_N = Q_N f$ and $\mu + \alpha > \frac{1}{2}$. Using (2.7) and (2.10) we obtain from (2.15)

$$\|u_N - u\| \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad (-\alpha \le \lambda \le \mu). \tag{2.16}$$

This is the most standard way to approximate f.

2. Case $f_N = \hat{f}(0) + (\frac{d}{dt})^k Q_N f^{(-k)}$ $(k \in \mathbb{N})$ and $\mu + \alpha > \frac{1}{2}$, where $f^{(0)}(t) = f(t)$ and

$$\left.\begin{aligned} c_{-j+1} &= \int_0^1 f^{(-j+1)}(s)\,ds \\ f^{(-j)}(t) &= \int_0^t \left(f^{(-j+1)}(s) - c_{-j+1}\right) ds \end{aligned}\right\} \qquad (j = 1, \ldots, k).$$

This time (2.15) yields the estimate

$$\|u_N - u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad (-\alpha - k \le \lambda \le \mu) \tag{2.17}$$

which differs from (2.16) by a more wide scale of Sobolev norms towards negative λ. Optimal estimates of type (2.17) for strongly negative λ are of interest when the solution of a boundary value problem at a point $x \in \Omega$ is determined from the solution of the boundary integral equation on $\Gamma = \partial\Omega$. Unfortunately, the approximation f_N proposed here is practical only in the case where the values of $f^{(-k)}$ on the grid $\{\frac{j}{N} : j = 0, \ldots, N-1\}$ are available, e.g. if the integrations to find $f^{(-k)}$ can be performed analytically.

3. Case with known Fourier coefficients of f, and $\mu \in \mathbb{R}$ arbitrary. Then one may put $f_N = P_N f$, and (2.15) yields

$$\|u_N - u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad (-\infty < \lambda \le \mu). \tag{2.18}$$

Remark 2.4. For simplicity, we have assumed C^∞-smoothness of the coefficients a_p $(p = 0, \ldots, q)$. Actually, the arguments of Sections 3 – 6 allow us to point out a finite smoothness of a_p $(p = 0, 1, \ldots, q)$ under which the estimates (2.16) – (2.18) remain true for a fixed λ, or for all λ from a finite interval. For instance, in the case $\mu + \alpha > \frac{1}{2}$ and $f_N = Q_N f$, estimate (2.16) remains true for all $\lambda \in [-\alpha, \mu]$ if $a_p \in H^{\lambda_1, \lambda_2}$ $(p = 0, \ldots, q)$, with

$$\lambda_1 = \nu + \frac{\mu + \alpha}{\sigma} \qquad \text{and} \qquad \lambda_2 = \max(|\alpha|, \nu) + \frac{\mu + \alpha}{\sigma}$$

where $\nu > \frac{1}{2}$ is arbitrary and $\sigma \in (0, 1]$ is the parameter from condition (2.12).

Remark 2.5. All results can be extended to the case where the operator A_0 has a structure

$$(A_0 u)(t) = \int_0^1 \Big(\kappa_+(t-s) a_+(t,s) + \kappa_-(t-s) a_-(t,s)\Big) u(s)\,ds \tag{2.19}$$

with smooth 1-biperiodic coefficients $a_\pm$ satisfying the conditions (cf. (2.4) and (2.5))

$$\begin{aligned} b_1(t) &:= a_+(t,t) + a_-(t,t) \neq 0 \\ b_2(t) &:= a_+(t,t) - a_-(t,t) \neq 0 \end{aligned} \qquad (t \in \mathbb{R}) \tag{2.20}$$

and (see (2.5))

$$W(b_1) = W(b_2) = 0 \tag{2.21}$$

and with 1-periodic functions (distributions) $\kappa_\pm$ such that for their Fourier coefficients we have

$$\hat{\kappa}_+(m) = |m|^{-\alpha} \quad \text{and} \quad \hat{\kappa}_-(m) = |m|^{-\alpha}\text{sign}(m) \qquad (0 \neq m \in Z). \tag{2.22}$$

Assumptions (2.3) now reduces to

$$|\hat{\kappa}_p(m)| \leq c\underline{m}^{-\alpha-\beta} \qquad (p = 1, \ldots, q;\ m \in Z) \tag{2.23}$$

where $\beta > 0$. Of course, we have to assume again (2.6).

Remark 2.6. Every classical elliptic pseudodifferential operator $\mathcal{A}$ of negative integer order r on a closed smooth Jordan curve (see, e.g., [1]) can be represented in the form $\mathcal{A} = A_0 + B$ where A_0 is defined by (2.19), (2.20), (2.22) with $\alpha = |r|$, and B is a smoothing operator, i.e.

$$(Bu)(t) = \int_0^1 b(t,s)u(s)\,ds$$

with a smooth 1-biperiodic function b. Unfortunately, for a given problem (2.1) and (2.2), an explicit construction of the functions $a_\pm$ and b for the representation $\mathcal{A} = A_0 + B$ is rather complicated and impractical. We preferred to solve problem (2.1) and (2.2) directly.

3. Matrix form of the method and computational cost

Using the Fourier representations of $u = u(s)$ and $a_p = a_p(t,s)$ it is easy to find the Fourier representation of $\mathcal{A}u$ where the operator $\mathcal{A}$ is defined by (2.2):

$$(\mathcal{A}u)(t) = \sum_{k\in Z}\sum_{j\in Z}\sum_{p=0}^{q}\sum_{m\in Z} \hat{a}_p(k-m, m-j)\hat{\kappa}_p(m)\hat{u}(j)e^{ik2\pi t}.$$

For $u_N \in \mathcal{T}_N$ we have

$$(P_N\mathcal{A}u_N)(t) = \sum_{k\in Z_N}\sum_{j\in Z_N}\sum_{p=0}^{q}\sum_{m\in Z} \hat{a}_p(k-m, m-j)\hat{\kappa}_p(m)\hat{u}_N(j)e^{ik2\pi t}.$$

Thus the (pure) Galerkin method (2.8) is equivalent to the system of linear equations

$$\sum_{j\in Z_N} g_{kj}\hat{u}_N(j) = \hat{f}(k) \qquad (k \in Z_N) \tag{3.1}$$

where

$$g_{kj} = \sum_{p=0}^{q}\sum_{m\in Z} \hat{a}_p(k-m, m-j)\hat{\kappa}_p(m) \qquad (k, j \in Z_N).$$

Similarly, the modified Galerkin methods (2.14) is equivalent to the system of linear equations

$$\sum_{j\in Z_N} g_{kj}^M \hat{u}_N(j) = \hat{f}_N(k) \qquad (k \in Z_N) \tag{3.2}$$

where, for $k, j \in Z_N$,

$$g_{kj}^M = \begin{cases} \sum_{p=0}^{q}\sum_{m:\,|m-k|+|m-j|\le \frac{M}{2}} (\widehat{Q_{M,M}a_p})(k-m, m-j)\hat{\kappa}_p(m) & \text{for } |k-j|\le \frac{M}{2} \\ 0 & \text{for } |k-j| > \frac{M}{2}. \end{cases} \tag{3.3}$$

We obtained a band system of a band width $M+1$. By the standard Gauss elimination method, with pivoting along columns under the main diagonal, system (3.2) can be solved in $\mathcal{O}(M^2N)$ arithmetical operations, or taking into account (2.12), $\mathcal{O}(N^{1+2\sigma})$ arithmetical operations. Using the fast Fourier transformation, $(\widehat{Q_N f})$ or $(\widehat{Q_N f^{(-k)}})$ and $(\widehat{Q_{M,M}a_p})$ can be found in $\mathcal{O}(N\log N)$ and $\mathcal{O}(M^2\log M) = \mathcal{O}(N^{2\sigma}\log N)$ arithmetical operations, respectively. Making use of the convolution structure of (3.3) along a fixed diagonal $k-j = \text{const}$, we can use the fast Fourier transformation also to compute the sums over m for k and j on the diagonal. This costs $\mathcal{O}(N\log N)$ arithmetical operations for one diagonal and $\mathcal{O}(MN\log N) = \mathcal{O}(N^{1+\sigma}\log N)$ arithmetical operations for the whole matrix of system (3.2).

Let us summarize.

Proposition 3.1. *Under the relation $M \sim N^\sigma$ $(0 < \sigma \le 1)$ the computational cost of the fully discretized Galerkin method* (2.14) *is as follows:*

(i) $\mathcal{O}(N^{1+\sigma}\log N)$ *aritmetical operations for the construction of system* (3.2) *using the fast Fourier transformation.*

(ii) $\mathcal{O}(N^{1+2\sigma})$ *arithmetical operations for the solution of system* (3.2) *by the Gauss elimination.*

As we saw in Section 2, in the case of sufficiently smooth coefficients a_p $(p = 0, 1, \dots, q)$ the parameter σ does not influence on the optimal convergence rate of the method (2.14) but this result is only asymptotical as $N \to \infty$. For moderate N one has to compromise between the accuracy (great σ) and a cheap implementation of the method (small σ).

4. Product of biperiodic functions

It is relatively easy to show (see, e.g., [25]) that, for $\lambda \in I\!R$ and $\nu > \frac{1}{2}$,

$$\|au\|_\lambda \le c_{\lambda,\nu}\|a\|_{\max(|\lambda|,\nu)}\|u\|_\lambda \qquad \left(a \in H^{\max(|\lambda|,\nu)},\ u \in H^\lambda\right). \tag{4.1}$$

Here we prove a two-dimensional counterpart of (4.1).

Lemma 4.1. *For any $\lambda, \mu \in I\!R$, $\nu > \frac{1}{2}$, $u \in H^{\lambda,\mu}$ and $a \in H^{\max(|\lambda|,\nu),\max(|\mu|,\nu)}$, the inequality*

$$\|au\|_{\lambda,\mu} \le c_{\lambda,\mu,\nu}\|a\|_{\max(|\lambda|,\nu),\max(|\mu|,\nu)}\|u\|_{\lambda,\mu} \tag{4.2}$$

holds where the constant $c_{\lambda,\mu,\nu}$ is independent of a and u.

Proof. Denoting $a_{l,m} = \hat{a}(l,m)$ and $u_{p,q} = \hat{u}(p,q)$ we have

$$(au)(t,s) = \sum_{j,k\in Z}\left(\sum_{p,q\in Z} a_{j-p,k-q}u_{p,q}\right)e^{ij2\pi t}e^{ik2\pi s}$$

and

$$\|au\|_{\lambda,\mu} \le \left\{\sum_{j,k\in Z}\left(\sum_{p,q\in Z}\underline{j}^\lambda\underline{k}^\mu|a_{j-p,k-q}|\,|u_{p,q}|\right)^2\right\}^{1/2}. \tag{4.3}$$

(i) The case $\lambda > \frac{1}{2}$ and $\mu > \frac{1}{2}$. Using the inequalities

$$\underline{j}^\lambda \le 2^\lambda\left((\underline{j-p})^\lambda + \underline{p}^\lambda\right) \qquad \text{and} \qquad \underline{k}^\mu \le 2^\mu\left((\underline{k-q})^\mu + \underline{q}^\mu\right) \tag{4.4}$$

we obtain from (4.3)

$$\|au\|_{\lambda,\mu} \le 2^{\lambda+\mu} \left\{ \sum_{j,k\in Z} \left(\sum_{p,q\in Z} (\underline{j-p})^{\lambda}(\underline{k-q})^{\mu}|a_{j-p,k-q}|\,|u_{p,q}| \right.\right.$$

$$+ \sum_{p,q\in Z} \underline{p}^{\lambda}(\underline{k-q})^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|$$

$$+ \sum_{p,q\in Z} (\underline{j-p})^{\lambda}\underline{q}^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|$$

$$\left.\left. + \sum_{p,q\in Z} \underline{p}^{\lambda}\underline{q}^{\mu}|a_{j-p,k-q}|\,|u_{p,q}| \right)^2 \right\}^{1/2}$$

$$= 2^{\lambda+\mu}\|\sum_{n=1}^{4} b_n v_n\|_{0,0}$$

$$\le 2^{\lambda+\mu}\sum_{n=1}^{4}\|b_n v_n\|_{0,0}$$

where the functions b_n and v_n are defined by their Fourier coefficients

$$\hat{b}_1(j,k) = \underline{j}^{\lambda}\underline{k}^{\mu}|a_{j,k}| \qquad \hat{v}_1(p,q) = |u_{p,q}|$$

$$\hat{b}_2(j,k) = \underline{k}^{\mu}|a_{j,k}| \qquad \hat{v}_2(p,q) = \underline{p}^{\lambda}|u_{p,q}|$$

$$\hat{b}_3(j,k) = \underline{j}^{\lambda}|a_{j,k}| \qquad \hat{v}_3(p,q) = \underline{q}^{\mu}|u_{p,q}|$$

$$\hat{b}_4(j,k) = |a_{j,k}| \qquad \hat{v}_4(p,q) = \underline{p}^{\lambda}\underline{q}^{\mu}|u_{p,q}|.$$

Let us estimate the norms $\|b_n v_n\|_{0,0}$. Clearly,

$$\|b_1 v_1\|_{0,0} = \left(\int_0^1\int_0^1 |b_1(t,s)|^2|v_1(t,s)|^2 dtds\right)^{1/2}$$

$$\le \|b_1\|_{0,0}\sup_{t,s}|v_1(t,s)| \le \|a\|_{\lambda,\mu}\sum_{p,q\in Z}|u_{p,q}| \le \gamma_\lambda\gamma_\mu\|a\|_{\lambda,\mu}\|u\|_{\lambda,\mu}$$

where

$$\gamma_\nu = \left(\sum_{p\in Z}\underline{p}^{-2\nu}\right)^{1/2} = \left(1+2\sum_{p=1}^{\infty}p^{-2\nu}\right)^{1/2} < \infty \qquad \text{for} \quad \nu > \frac{1}{2}. \tag{4.5}$$

A similar argument yields also $\|b_4 w_4\|_{0,0} \le \gamma_\lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}$. Further,

$$\|b_2 v_2\|_{0,0} = \left(\int_0^1 \int_0^1 |b_2(t,s)|^2 |v_2(t,s)|^2 dt ds \right)^{1/2}$$
$$\le \left(\int_0^1 \sup_t |b_2(t,s)|^2 ds \right)^{1/2} \left(\int_0^1 \sup_s |v_2(t,s)|^2 dt \right)^{1/2}.$$

Since

$$\sup_t |b_2(t,s)| = \sup_t \left| \sum_{j\in Z} \left(\sum_{k \in Z} \hat{b}_2(j,k) e^{ik2\pi s} \right) e^{ij2\pi t} \right|$$
$$\le \sum_{j\in Z} \left| \sum_{k\in Z} \hat{b}_2(j,k) e^{ik2\pi s} \right| \le \gamma_\lambda \left(\sum_{j \in Z} \underline{j}^{2\lambda} \left| \sum_{k\in Z} \hat{b}_2(j,k) e^{ik2\pi s} \right|^2 \right)^{1/2}$$

we have

$$\int_0^1 \sup_t |b_2(t,s)|^2 ds \le \gamma_\lambda^2 \sum_{j\in Z} \underline{j}^{2\lambda} \int_0^1 \left| \sum_{k\in Z} \hat{b}_2(j,k) e^{ik2\pi s} \right|^2 ds$$
$$= \gamma_\lambda^2 \sum_{j\in Z} \underline{j}^{2\lambda} \sum_{k\in Z} |\hat{b}_2(j,k)|^2 = \gamma_\lambda^2 \|a\|_{\lambda,\mu}^2.$$

In a similar way, or simply using a symmetry argument, we find

$$\int_0^1 \sup_s |v_2(t,s)|^2 dt \le \gamma_\mu^2 \|u\|_{\lambda,\mu}^2.$$

This results to $\|b_2 v_2\|_{0,0} \le \gamma_\lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}$. Exploiting the symmetry between b_2, v_2 and b_3, v_3 we immediately obtain also $\|b_3 v_3\|_{0,0} \le \gamma_\lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}$. Summing up we obtain (4.2) in the case $\lambda > \frac{1}{2}$ and $\mu > \frac{1}{2}$: $\|au\|_{\lambda,\mu} \le 2^{\lambda+\mu+2} \gamma_\lambda \gamma_\mu \|a\|_{\lambda,\mu} \|u\|_{\lambda,\mu}$.

(ii) The case $\lambda < -\frac{1}{2}$ and $\mu > \frac{1}{2}$. Rewrite (4.3) as

$$\|au\|_{\lambda,\mu} \le \left\{ \sum_{j,k\in Z} \underline{j}^{2\lambda} \left(\sum_{p,q\in Z} \underline{k}^{\mu} |a_{j-p,k-q}| \underline{p}^{|\lambda|} \underline{p}^{\lambda} |u_{p,q}| \right)^2 \right\}^{1/2}.$$

Using the inequality $p^{|\lambda|} \le 2^{|\lambda|}((\underline{j-p})^{|\lambda|} + \underline{j}^{|\lambda|})$ and the second one of inequalities (4.4) we obtain

$$\|au\|_{\lambda,\mu} \le 2^{|\lambda|+\mu}\left\{\sum_{j,k\in Z}\left(\sum_{p,q\in Z}\underline{p}^{\lambda}(\underline{k-q})^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|\right)^2\right\}^{1/2}$$

$$+\,2^{|\lambda|+\mu}\left\{\sum_{j,k\in Z}\left(\sum_{p,q\in Z}\underline{p}^{\lambda}\underline{q}^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|\right)^2\right\}^{1/2}$$

$$+\,2^{|\lambda|+\mu}\left\{\sum_{j,k\in Z}\underline{j}^{2\lambda}\left(\sum_{p,q\in Z}\underline{p}^{\lambda}(\underline{j-p})^{|\lambda|}(\underline{k-q})^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|\right)^2\right\}^{1/2}$$

$$+\,2^{|\lambda|+\mu}\left\{\sum_{j,k\in Z}\underline{j}^{2\lambda}\left(\sum_{p,q\in Z}\underline{p}^{\lambda}(\underline{j-p})^{|\lambda|}\underline{q}^{\mu}|a_{j-p,k-q}|\,|u_{p,q}|\right)^2\right\}^{1/2}$$

$$= 2^{|\lambda|+\mu}\Big(\|b_2v_2\|_{0,0} + \|b_4v_n\|_{0,0} + S_1 + S_2\Big)$$

where b_2, v_2, b_4, v_4 are the same as in case (i), with similar estimates

$$\|b_2v_2\|_{0,0} \le \gamma_{|\lambda|}\gamma_{\mu}\|a\|_{|\lambda|,\mu}\|u\|_{\lambda,\mu} \qquad \text{and} \qquad \|b_4v_4\|_{0,0} \le \gamma_{|\lambda|}\gamma_{\mu}\|a\|_{|\lambda|,\mu}\|u\|_{\lambda,\mu}$$

and

$$S_1 = \left\{\sum_{j,k\in Z}\underline{j}^{2\lambda}\left(\sum_{q\in Z}(\underline{k-q})^{\mu}\sum_{p\in Z}(\underline{j-p})^{|\lambda|}|a_{j-p,k-q}|\underline{p}^{\lambda}|u_{p,q}|\right)^2\right\}^{1/2}$$

$$S_2 = \left\{\sum_{j,k\in Z}\underline{j}^{2\lambda}\left(\sum_{q\in Z}\underline{q}^{\mu}\sum_{p\in Z}(\underline{j-p})^{|\lambda|}|a_{j-p,k-q}|\underline{p}^{\lambda}|u_{p,q}|\right)^2\right\}^{1/2}.$$

The sum over p in S_1 and S_2 we estimate by the Cauchy inequality:

$$\sum_{p\in Z}(\underline{j-p})^{|\lambda|}|a_{j-p,k-q}|\cdot\underline{p}^{\lambda}|u_{p,q}|$$

$$\le\left(\sum_{p\in Z}(\underline{j-p})^{2|\lambda|}|a_{j-p,k-q}|^2\right)^{1/2}\left(\sum_{p\in Z}\underline{p}^{2\lambda}|u_{p,q}|^2\right)^{1/2} = d_{k-q}w_q$$

where

$$d_k = \left(\sum_{p\in Z}\underline{p}^{2|\lambda|}|a_{p,k}|^2\right)^{1/2} \qquad \text{and} \qquad w_q = \left(\sum_{p\in Z}\underline{p}^{2\lambda}|u_{pq}|^2\right)^{1/2}$$

are independent of j. Now we can sum up over j:

$$S_1 \leq \gamma_{|\lambda|} \left\{ \sum_{k\in Z} \left(\sum_{q\in Z} \underline{(k-q)}^{\mu} d_{k-q} w_q \right)^2 \right\}^{1/2} = \gamma_{|\lambda|} \|g_1 w_1\|_0$$

$$S_2 \leq \gamma_{|\lambda|} \left\{ \sum_{k\in Z} \left(\sum_{q\in Z} \underline{q}^{\mu} d_{k-q} w_q \right)^2 \right\}^{1/2} = \gamma_{|\lambda|} \|g_2 w_2\|_0$$

where the functions g_1, w_1 and g_2, w_2 are determined by their Fourier coefficients

$$\hat{g}_1(k) = \underline{k}^{\mu} d_k, \quad \hat{w}_1(q) = w_q \qquad \text{and} \qquad \hat{g}_2(k) = d_k, \quad \hat{w}_2(q) = \underline{q}^{\mu} w_q.$$

Clearly,

$$\begin{aligned} \|g_1 w_1\|_0 &\leq \|g_1\|_0 \sup_t |w_1(t)| \leq \left(\sum_{k\in Z} \underline{k}^{2\mu} d_k^2 \right)^{1/2} \sum_{q\in Z} |w_q| \\ &\leq \|a\|_{|\lambda|,\mu} \gamma_\mu \left(\sum_{q\in Z} \underline{q}^{2\mu} w_q^2 \right)^{1/2} = \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu} \\ \|g_2 w_2\|_0 &\leq \sup_t |g_2(t)| \|w_2\|_0 \leq \sum_{k\in Z} d_k \|u\|_{\lambda,\mu} \leq \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu}, \end{aligned}$$

therefore $S_1, S_2 \leq \gamma_{|\lambda|} \gamma_\mu \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu}$. Summing up we obtain inequality (4.2) for the case $\lambda < -\frac{1}{2}$ and $\mu > \frac{1}{2}$: $\|au\|_{\lambda,\mu} \leq 2^{|\lambda|+\mu+2} \|a\|_{|\lambda|,\mu} \|u\|_{\lambda,\mu}$.

(iii) The case $\lambda > \frac{1}{2}$ and $\mu < -\frac{1}{2}$. It is symmetrical to the case (ii), and a symmetry argument yields immediately $\|au\|_{\lambda,\mu} \leq 2^{\lambda+|\mu|+2} \|a\|_{\lambda,|\mu|} \|u\|_{\lambda,\mu}$.

(iv) The case $\lambda < -\frac{1}{2}$ and $\mu < -\frac{1}{2}$. It can be treated by a duality argument. Take an element $v \in H^{-\lambda,-\mu}$ such that $\|v\|_{-\lambda,-\mu} = 1$ and $\|au\|_{\lambda,\mu} = \langle au, v \rangle$. Then, due to the estimate proved in the case (i),

$$\begin{aligned} \|au\|_{\lambda,\mu} &= \langle au, v \rangle = \langle u, av \rangle \leq \|u\|_{\lambda,\mu} \|av\|_{-\lambda,-\mu} \\ &\leq \|u\|_{\lambda,\mu} 2^{-\lambda-\mu+2} \gamma_{-\lambda} \gamma_{-\mu} \|a\|_{-\lambda,-\mu} \|v\|_{-\lambda,-\mu} \\ &= 2^{|\lambda|+|\mu|+2} \gamma_{|\lambda|} \gamma_{|\mu|} \|a\|_{|\lambda|,|\mu|} \|u\|_{\lambda,\mu}. \end{aligned}$$

The summary of the cases (i) - (iv) sounds as follows:

$$\|au\|_{\lambda,\mu} \leq 2^{|\lambda|+|\mu|+2} \gamma_{|\lambda|} \gamma_{|\mu|} \|a\|_{|\lambda|,|\mu|} \|u\|_{\lambda,\mu} \qquad \text{for} \quad |\lambda| > \frac{1}{2} \text{ and } |\mu| > \frac{1}{2}. \tag{4.6}$$

(v) The case $|\lambda| \le \frac{1}{2}$ and $|\mu| > \frac{1}{2}$. Take a $\nu > \frac{1}{2}$. According to (4.6),

$$\|au\|_{\nu,\mu} \le 2^{\nu+|\mu|+2}\gamma_\nu\gamma_{|\mu|}\|a\|_{\nu,|\mu|}\|u\|_{\nu,\mu}$$

$$\|au\|_{-\nu,\mu} \le 2^{\nu+|\mu|+2}\gamma_\nu\gamma_{|\mu|}\|a\|_{\nu,|\mu|}\|u\|_{\nu,\mu}.$$

Using the interpolation theorem for an operator in scales of Hilbert spaces (see [12, 27]) we obtain herefrom

$$\|au\|_{\lambda,\mu} \le 2^{\nu+|\mu|+2}\gamma_\nu\gamma_{|\mu|}\|a\|_{\nu,|\mu|}\|u\|_{\lambda,\mu} \qquad (-\nu \le \lambda \le \nu). \tag{4.7}$$

We obtained inequality (4.2) in the case $|\lambda| \le \frac{1}{2}$ and $|\mu| > \frac{1}{2}$.

(vi) The case $|\lambda| > \frac{1}{2}$ and $|\mu| \le \frac{1}{2}$. It is symmetrical to the case (v) yielding (4.2):

$$\|au\|_{\lambda,\mu} \le 2^{|\lambda|+\nu+2}\gamma_{|\lambda|}\gamma_\nu\|a\|_{|\lambda|,\nu}\|u\|_{\lambda,\mu}. \tag{4.8}$$

(vii) The case $|\lambda| \le \frac{1}{2}$ and $|\mu| \le \frac{1}{2}$. Take again a $\nu > \frac{1}{2}$. According to (4.7)

$$\|au\|_{\lambda,\nu} \le 2^{2(\nu+1)}\gamma_\nu^2\|a\|_{\nu,\nu}\|u\|_{\lambda,\nu}$$

$$\|au\|_{\lambda,-\nu} \le 2^{2(\nu+1)}\gamma_\nu^2\|a\|_{\nu,\nu}\|u\|_{\lambda,-\nu}.$$

The interpolation theorem yields inequality (4.2):

$$\|au\|_{\lambda,\mu} \le 2^{2(\nu+1)}\gamma_\nu^2\|a\|_{\nu,\nu}\|u\|_{\lambda,\mu} \qquad (-\nu \le \mu \le \nu). \tag{4.9}$$

Now inequality (4.2) is established in all possible cases ∎

Inequalities (4.6) – (4.9) give some information about the constant $c_{\lambda,\mu,\nu}$ in inequality (4.2) in different cases. These values can be somewhat reduced.

Corollary 4.2. *For any $\lambda, \mu \in \mathbb{R}$ the inequality*

$$\|a(t,s)u(s)\|_{\lambda,\mu} \le c_{\lambda,\mu,\nu}\|a\|_{\max(|\lambda|,\nu),\max(|\mu|,\nu)}\|u\|_\mu \tag{4.10}$$

holds with any $\nu > \frac{1}{2}$.

Proof. This follows immediately from inequality (4.2) considering $u = u(s)$ as a biperiodic function (distribution) which is constant with respect to t. Note that $\|u(s)\|_{\lambda,\mu} = \|u\|_\mu$ for any $\lambda \in \mathbb{R}$ ∎

5. Bounded integral operators in Sobolev spaces

Consider an integal operator

$$(Au)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s)\,ds \tag{5.1}$$

where the coefficient a is smooth and 1-biperiodic, and the function (or distribution) κ is 1-periodic and its Fourier coefficients satisfy the inequality

$$|\hat{\kappa}(m)| \le \underline{m}^{-\alpha} \qquad (m \in Z) \tag{5.2}$$

with an $\alpha \in I\!R$. In the case of a constant coefficient $a(t,s) = 1$ we have

$$(Au)(t) = \int_0^1 \kappa(t-s)u(s)\,ds = \sum_{m\in Z} \hat{\kappa}(m)\hat{u}(m)e^{im2\pi t},$$

and due to (5.2) A is a bounded operator from any H^λ to $H^{\lambda+\alpha}$ $(\alpha \in I\!R)$. Moreover, if κ satisfies also the inverse inequality $|\hat{\kappa}(m)| \ge c_0\underline{m}^{-\alpha}$ $(m \in Z)$ with a $c_0 > 0$, then A is an isomorphism between H^λ and $H^{\lambda+\alpha}$ $(\lambda \in I\!R)$. Our purpose is to show that the operator A remains bounded from H^λ to $H^{\lambda+\alpha}$ also in the presense of a smooth 1-biperiodic coefficient a. We deduce this result from Corollary 4.2 and the following lemma.

Lemma 5.1. *Assume inequality* (5.2). *Then, for any* $\lambda \in I\!R$,

$$\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \le \begin{cases} 2^{\lambda+\alpha+1}\gamma_{\lambda+\alpha}\|v\|_{\lambda+\alpha,\lambda}, & \text{if } \lambda+\alpha > \frac{1}{2} \\ 2^{\lambda+\alpha+1}\gamma_\nu\|v\|_{\nu,\lambda}, & \text{if } 0 \le \lambda+\alpha \le \frac{1}{2} \\ 2^{|\lambda+\alpha|}\gamma_\nu\|v\|_{|\lambda+\alpha|+\nu,\lambda} & \text{if } \lambda+\alpha \le 0 \end{cases} \tag{5.3}$$

where $\nu > \frac{1}{2}$ *is arbitrary,* γ_ν *is defined by* (4.5), *and* v *is any* 1*-biperiodic function (distribution) of a finite Sobolev norm indicated on the right hand of the inequality.*

To prove (5.3) we first formulate an elementary inequality.

Lemma 5.2. *For any* $\nu_1 \ge 0$ *and* $\nu_2 \ge 0$ *with* $\nu_1 + \nu_2 > \frac{1}{2}$ *and any* $k, l \in Z$,

$$\left(\sum_{m\in Z} (\underline{m-k})^{-2\nu_1}(\underline{m-l})^{-2\nu_2} \right)^{1/2} \le \gamma_{\nu_1+\nu_2}. \tag{5.4}$$

Proof. In the cases $\nu_1 = 0$ or $\nu_2 = 0$ inequality (5.4) is trivial (cf. (4.5)). Let $\nu_1 > 0$ and $\nu_2 > 0$. The Hölder inequality with $p_1 = \frac{\nu_1+\nu_2}{\nu_1}$ and $p_2 = \frac{\nu_1+\nu_2}{\nu_2}$ (obviously,

$\frac{1}{p_1} + \frac{1}{p_2} = 1$) yields

$$\begin{aligned}
&\sum_{m\in Z} (\underline{m-k})^{-2\nu_1}(\underline{m-l})^{-2\nu_2} \\
&\le \left(\sum_{m\in Z} (\underline{m-k})^{-2(\nu_1+\nu_2)}\right)^{\nu_1/(\nu_1+\nu_2)} \left(\sum_{m\in Z} (\underline{m-l})^{-2(\nu_1+\nu_2)}\right)^{\nu_2/(\nu_1+\nu_2)} \\
&= \sum_{m\in Z} \underline{m}^{-2(\nu_1+\nu_2)} = \gamma^2_{\nu_1+\nu_2}.
\end{aligned}$$

Thus the lemms is shown ∎

Proof of Lemma 5.1. We have

$$\int_0^1 \kappa(t-s)v(t,s)\,ds = \sum_{k\in Z}\left(\sum_{m\in Z} \hat{\kappa}(m)\hat{v}(k-m,m)\right) e^{ik2\pi t}$$

and due to (5.2)

$$\begin{aligned}
&\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \\
&\le \left\{ \sum_{k\in Z} \underline{k}^{2(\lambda+\alpha)} \left(\sum_{m\in Z} \underline{m}^{-\alpha} |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2} \qquad (5.5) \\
&= \left\{ \sum_{k\in Z} \left(\sum_{m\in Z} \underline{k}^{\lambda+\alpha} \underline{m}^{-\alpha} |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}.
\end{aligned}$$

In the case $\lambda+\alpha \ge 0$ we may use the inequality

$$\underline{k}^{\lambda+\alpha} \le 2^{\lambda+\alpha}\left((\underline{k-m})^{\lambda+\alpha} + \underline{m}^{\lambda+\alpha}\right).$$

Noticing that for $(c_{km})_{k,m\in Z}$ the expression $\left\{\sum_{k\in Z}\left(\sum_{m\in Z}|c_{km}|\right)^2\right\}^{1/2}$ has the norm properties we obtain

$$\begin{aligned}
&\left\| \int_0^1 \kappa(t-s)v(t,s)\,ds \right\|_{\lambda+\alpha} \\
&\le 2^{\lambda+\alpha} \left\{ \sum_{k\in Z} \left(\sum_{m\in Z} (\underline{k-m})^{\lambda+\alpha} \underline{m}^{-\alpha} |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2} \qquad (5.6) \\
&\quad + 2^{\lambda+\alpha} \left\{ \sum_{k\in Z} \left(\sum_{m\in Z} \underline{m}^{\lambda} |\hat{v}(k-m,m)| \right)^2 \right\}^{1/2}.
\end{aligned}$$

(i) The case $\lambda+\alpha>\frac{1}{2}$. Rewrite (5.6) as follows:

$$\left\|\int_0^1 \kappa(t-s)v(t,s)\,ds\right\|_{\lambda+\alpha}$$
$$\le 2^{\lambda+\alpha}\left\{\sum_{k\in Z}\left(\sum_{m\in Z}\underline{m}^{-(\lambda+\alpha)}\cdot(\underline{k-m})^{\lambda+\alpha}\underline{m}^{\lambda}|\hat{v}(k-m,m)|\right)^2\right\}^{1/2}$$
$$+2^{\lambda+\alpha}\left\{\sum_{k\in Z}\left(\sum_{m\in Z}(\underline{k-m})^{-(\lambda+\alpha)}\cdot(\underline{k-m})^{\lambda+\alpha}\underline{m}^{\lambda}|\hat{v}(k-m,m)|\right)^2\right\}^{1/2}.$$

Estimating the sums over m by the Cauchy inequality we obtain (5.3).

(ii) The case $0\le\lambda+\alpha\le\frac{1}{2}$. This time we rewrite (5.6) in the form

$$\left\|\int_0^1 \kappa(t-s)v(t,s)\,ds\right\|_{\lambda+\alpha}$$
$$\le \quad 2^{\lambda+\alpha}\left\{\sum_{k\in Z}\left(\sum_{m\in Z}\underline{m}^{-(\lambda+\alpha)}(\underline{k-m})^{\lambda+\alpha-\nu}\cdot(\underline{k-m})^{\nu}\underline{m}^{\lambda}|\hat{v}(k-m,m)|\right)^2\right\}^{1/2}$$
$$+2^{\lambda+\alpha}\left\{\sum_{k\in Z}\left(\sum_{m\in Z}(\underline{k-m})^{-\nu}\cdot(\underline{k-m})^{\nu}\underline{m}^{\lambda}|\hat{v}(k-m,m)|\right)^2\right\}^{1/2}$$

where $\nu>\frac{1}{2}$. Estimating the sums over m again by the Cauchy inequality and taking into account Lemma 5.2 (with $\nu_1=\lambda+\alpha$ and $\nu_2=\nu-(\lambda+\alpha)$) we obtain (5.3).

(iii) The case $\lambda+\alpha<0$. Using Peetre's inequality

$$\underline{k}^{\lambda+\alpha}\le 2^{|\lambda+\alpha|}\underline{m}^{\lambda+\alpha}(\underline{k-m})^{|\lambda+\alpha|}\qquad (k,m\in Z)$$

we continue (5.5) as follows:

$$\left\|\int_0^1 \kappa(t-s)v(t,s)\,ds\right\|_{\lambda+\alpha}$$
$$\le 2^{|\lambda+\alpha|}\left\{\sum_{k\in Z}\left(\sum_{m\in Z}(\underline{k-m})^{|\lambda+\alpha|}\underline{m}^{\lambda}|\hat{v}(k-m,m)|\right)^2\right\}^{1/2}.$$

The Cauchy inequality yields again (5.3) ∎

Proposition 5.3. *Assume that $a = a(t,s)$ is C^∞-smooth and 1-biperiodic, and $\kappa = \kappa(t)$ satisfies (5.2). Then the integral operator A defined by (5.1) is bounded from H^λ to $H^{\lambda+\alpha}$ for any $\lambda \in \mathbb{R}$. With any $\nu > \frac{1}{2}$, the following estimates of the norm of A hold:*

(i) *If $\alpha \geq 0$, then for all $\lambda \in \mathbb{R}$*

$$\|A\|_{\lambda,\lambda+\alpha} := \|A\|_{\mathcal{L}(H^\lambda,H^{\lambda+\alpha})} \leq c_{\lambda,\alpha,\nu}\|a\|_{\max(|\lambda+\alpha|,\nu),\max(|\lambda|,\nu)}. \tag{5.7}$$

(ii) *If $\alpha < 0$, then (5.7) holds for $\lambda \leq 0$ and for $\lambda \geq -\alpha$, whereas for $0 < \lambda < -\alpha$*

$$\|A\|_{\lambda,\lambda+\alpha} \leq c_{\lambda,\alpha,\nu} \min\left(\|a\|_{|\lambda+\alpha|+\nu,\max(\lambda,\nu)}, \|a\|_{\max(|\lambda+\alpha|,\nu),\lambda+\nu}\right). \tag{5.8}$$

Proof. Take any $u \in H^\lambda$ and denote $v(t,s) = a(t,s)u(s)$. According to Lemma 5.1 and Corollary 4.2 we have

$$\|Au\|_{\lambda+\alpha} \leq c_{\lambda,\alpha,\nu} \left\{ \begin{array}{ll} \|a\|_{\lambda+\alpha,\max(|\lambda|,\nu)} & \text{if } \lambda+\alpha > \frac{1}{2} \\ \|a\|_{\nu,\max(|\lambda|,\nu)} & \text{if } 0 \leq \lambda+\alpha \leq \frac{1}{2} \\ \|a\|_{|\lambda+\alpha|+\nu,\max(|\lambda|,\nu)} & \text{if } \lambda+\alpha < 0 \end{array} \right\} \|u\|_\lambda,$$

i.e. the operator A is bounded from H^λ to $H^{\lambda+\alpha}$ whereby, for any $\lambda \in \mathbb{R}$,

$$\|A\|_{\lambda,\lambda+\alpha} \leq c_{\lambda,\alpha,\nu} \left\{ \begin{array}{ll} \|a\|_{\max(\lambda+\alpha,\nu),\max(|\lambda|,\nu)} & \text{if } \lambda+\alpha \geq 0 \\ \|a\|_{|\lambda+\alpha|+\nu,\max(|\lambda|,\nu)} & \text{if } \lambda+\alpha \leq 0. \end{array} \right. \tag{5.9}$$

The Banach dual (or "transposed") operator $A' \in \mathcal{L}(H^{-(\lambda+\alpha)}, H^{-\lambda})$ to the operator $A \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is defined by

$$(A'u)(t) = \int_0^1 \kappa(s-t)a(s,t)u(s)\,ds = \int_0^1 \kappa'(t-s)a'(t,s)u(s)\,ds$$

where $\kappa'(t) = \kappa(-t)$ and $a'(t,s) = a(s,t)$. Since $\hat{\kappa}'(m) = -\hat{\kappa}(-m)$, the dual operator A' also satisfies the conditions of the proposition, and (5.9) is true for it:

$$\|A'\|_{-\lambda-\alpha,-\lambda} \leq c_{\lambda,\alpha,\nu} \left\{ \begin{array}{ll} \|a'\|_{\max(-\lambda,\nu),\max(|\lambda+\alpha|,\nu)} & \text{if } -\lambda \geq 0 \\ \|a'\|_{|\lambda|+\nu,\max(|\lambda+\alpha|,\nu)} & \text{if } -\lambda \leq 0. \end{array} \right.$$

Since the norms of an operator and its dual are equal we obtain

$$\|A\|_{\lambda,\lambda+\alpha} \leq c_{\lambda,\alpha,\nu} \left\{ \begin{array}{ll} \|a\|_{\max(|\lambda+\alpha|,\nu),\max(|\lambda|,\nu)} & \text{if } \lambda \leq 0 \\ \|a\|_{\max(|\lambda+\alpha|,\nu),\lambda+\nu} & \text{if } \lambda \geq 0. \end{array} \right.$$

Together with (5.9) this covers the assertions of the proposition ∎

For $\alpha < 0$ and $0 < \lambda < -\alpha$, inequality (5.7) may fail. For instance, for $A = A_N$ with $\hat{\kappa}(m) = |m|$ $(\alpha = -1, m \in Z)$

$$\hat{a}_N(k,l) = \begin{cases} 1 & \text{for } k + l = 0, \quad 1 \le l \le N \\ 0 & \text{for other } k, l \in Z \end{cases}$$

and for $e_0(t) \equiv 1$ we have

$$A_N e_0 = \left(\sum_{m \in Z} \hat{a}_N(-m,m)|m| \right) e_0 = \left(\sum_{m=1}^{N} m \right) e_0 \sim N^2 e_0$$

$$\|A_N e_0\|_{\lambda+\alpha} \sim N^2 \qquad (\lambda \in \mathbb{R}).$$

For $\lambda = \frac{1}{2}$, inequality (5.7) fails since $\max(|\lambda + \alpha|, \nu) = \nu = \max(|\lambda|, \nu)$ and

$$\|a_N\|_{\nu,\nu} = \left(\sum_{k=1}^{N} k^{4\nu} \right)^{1/2} \sim N^{(4\nu+1)/2} << N^2 \qquad \left(\frac{1}{2} < \nu < \frac{3}{4} \right).$$

Of course, we are interested to obtain estimates of the type $\|A\|_{\lambda,\lambda+\alpha} \le c\|a\|_{\lambda_1,\lambda_2}$ with possibly small λ_1 and λ_2. In this sense, estimates (5.7) and (5.8) are the best we know for moderate $|\lambda|$. For great $|\lambda|$, the result can be improved if we allow a sum of two different Sobolev norms in the right-hand side. In the following estimates only one of the indices λ_1 and λ_2 increases linearly in $|\lambda|$ as $|\lambda| \to \infty$.

Proposition 5.4. *Assume the conditions of Proposition 5.3. Then ther following estimates for the norm of A hold:*

(i) *For $\lambda + \alpha > \frac{1}{2}$, with any $\frac{1}{2} < \nu \le \lambda + \alpha$,*

$$\|A\|_{\lambda,\lambda+\alpha} \le c_{\lambda,\alpha,\nu} \Big(\|a\|_{\lambda+\alpha, \max(|\nu-\alpha|,\nu)} + \|a\|_{\nu, \max(|\lambda|,\nu)} \Big). \tag{5.10}$$

(ii) *For $\lambda < -\frac{1}{2}$, with any $\frac{1}{2} < \nu \le |\lambda|$,*

$$\|A\|_{\lambda,\lambda+\alpha} \le c_{\lambda,\alpha,\nu} \Big(\|a\|_{\max(|\nu-\alpha|,\nu),|\lambda|} + \|a\|_{\max(|\lambda+\alpha|,\nu),\nu} \Big). \tag{5.11}$$

Proof. Let $\lambda + \alpha > \frac{1}{2}$ with $\frac{1}{2} < \nu \le \lambda + \alpha$. From (5.6) we obtain

$$\left\| \int_0^1 \kappa(t-s) v(t,s)\, ds \right\|_{\lambda+\alpha} \le 2^{\lambda+\alpha} \gamma_\nu \big(\|v\|_{\lambda+\alpha,\nu-\alpha} + \|v\|_{\nu,\lambda} \big).$$

Note that $\nu - \alpha \le \lambda$. Now estimate (5.10) follows with the help of inequality (4.10). Estimate (5.11) can be proved by the dual argument already used in the proof of Proposition 5.3 ∎

Proposition 5.5. *Assume that $a = a(t,s)$ is C^∞-smooth, 1-biperiodic, and vanishes on the diagonal: $a(t,t) = 0$ for all $t \in \mathbb{R}$. Let $\kappa = \kappa(t)$ satisfy (cf. (5.2))*

$$|\hat{\kappa}(m) - \hat{\kappa}(m-1)| \le c\underline{m}^{-\alpha-\gamma} \qquad (m \in Z)$$

where $\gamma > 0$. Then the integral operator A defined by (5.1) is bounded from any H^λ to $H^{\lambda+\alpha+\gamma}$ $(\lambda \in \mathbb{R})$.

Proof. We represent A in the form

$$(Au)(t) = \int_0^1 \kappa_1(t-s)a_1(t,s)\,ds$$

where

$$\kappa_1(t) = \kappa(t)(1 - e^{i2\pi t}) \qquad \text{and} \qquad a_1(t,s) = \frac{a(t,s)}{1 - e^{i2\pi(t-s)}}.$$

Note that a_1 is also 1-biperiodic and C^∞-smooth. Further, we have

$$\hat{\kappa}_1(m) = \hat{\kappa}(m) - \hat{\kappa}(m-1) \qquad \text{and} \qquad |\hat{\kappa}_1(m)| \le c\underline{m}^{-\alpha-\gamma} \qquad (m \in Z).$$

Now the assertion follows from Proposition 5.3 ∎

Estimates (5.10) and (5.11) allow to weaken the smoothness assumptions on a_p $(p = 0, \ldots, q)$ formulated in Remark 2.4 and based on inequality (5.7). Unfortunately, the formulation becomes more sophisticated.

Estimates (5.10) and (5.11) are rather useful also in the analysis of the numerical stability of the methods discribed in Section 2 and other approximate methods for problem (2.1). This analysis will be presented in another paper.

6. Proof of Theorems 2.2 and 2.3

Denote by P_+ and P_- the projections

$$(P_+u)(t) = \sum_{k\ge0} \hat{u}(k)e^{ik2\pi t} \qquad \text{and} \qquad (P_-u)(t) = \sum_{k<0} \hat{u}(k)e^{ik2\pi t}.$$

Lemma 6.1. *Let b_1 and b_2 be C^∞-smooth 1-periodic functions such that $b_1(t) \ne 0$ and $b_2(t) \ne 0$ for all $t \in \mathbb{R}$ and (see (2.5))*

$$W(b_1) = W(b_2) = 0.$$

Then there is an N_0 such that for all $N \ge N_0$, $v_N \in \mathcal{T}_N$, and $\lambda \in \mathbb{R}$ the inequality

$$\|v_N\|_\lambda \le c_\lambda \big\|P_N(b_1P_+ + b_2P_-)v_N\big\|_\lambda$$

holds with a constant c_λ independent of N and v_N.

Proof. This stability result is well known in the case $\lambda = 0$; a proof can be found in the books [5, 6, 17, 19]. A similar result holds in the case of Hölder spaces (see, e.g., [16, 17]). The proof of [16] can be repeated in our case of Sobolev spaces H^λ without changes. We omit the details ∎

Putting $b_1 = b_2 = b$, remembering the definition (2.5) of $W(b)$ and taking into account that $P_+ + P_- = I$ (the identity operator) we obtain

Corollary 6.2. *Let b be a C^∞-smooth 1-periodic function such that $b(t) \neq 0$ for all $t \in I\!R$ and $W(b) = 0$. Then there is an N_0 such that for all $N \geq N_0$, $v_N \in \mathcal{T}_N$, and $\lambda \in I\!R$ the inequality*

$$\|v_n\|_\lambda \leq c_\lambda \big\|P_N(bv_N)\big\|_\lambda$$

holds with a constant c_λ independent of N and v_N.

Proof of Proposition 2.1. The first assertion of Proposition 2.1 follows immediately from Proposition 5.3. Let us assume (2.4) and prove that $\mathcal{A}$ is a Fredholm operator of index 0. Represent it in the form

$$\mathcal{A} = B + K \tag{6.1}$$

where

$$(Bu)(t) = b(t) \int_0^1 \kappa_0(t-s)u(s)\,ds \qquad \text{with } b(t) = a_0(t,t)$$

and

$$(Ku)(t) = \int_0^1 \kappa_0(t-s)\big(a(t,s) - a(t,t)\big)u(s)\,ds + \sum_{p=1}^{q} \int_0^1 \kappa_p(t-s)a_p(t,s)u(s)\,ds.$$

Propositions 5.3 and 5.5 confirm us that, due to (2.3), $K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha+\beta})$ for all $\lambda \in I\!R$. Since the immbeding $H^\mu \hookrightarrow H^\lambda$ is compact for $\lambda < \mu$, the operator $K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is compact. According to (2.3) some of the Fourier coefficients $\hat{\kappa}_0(m)$ ($|m| \leq m_0$) may vanish but we do not lose in generality assuming that $\hat{\kappa}_0(m) \neq 0$ for all $m \in Z$ (in the opposite case we redefine κ_0 that causes a slight change in the structure of the operator K). In other words, we may assume that the first one of inequalities (2.3) holds for all $m \in Z$. Thus, we have

$$B = b(t)\Lambda \qquad \text{with} \qquad (\Lambda u)(t) = \int_0^1 \kappa_0(t-s)u(s)\,ds. \tag{6.2}$$

The operator Λ is an isomorphism between H^λ and $H^{\lambda+\alpha}$. Since $b(t) \neq 0$ for all $t \neq 0$, the operator B itself is also an isomorphism between H^λ and $H^{\lambda+\alpha}$, and $\mathcal{A} = B + K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is a Fredholm operator of index 0 for all $\lambda \in I\!R$.

If $\mathcal{A}u = 0$, then $u + B^{-1}Ku = 0$ and $u = (-1)^n(B^{-1}K)^n u$ ($n \in I\!N$). Since $B^{-1}K \in \mathcal{L}(H^\lambda, H^{\lambda+\beta})$ for all $\lambda \in I\!R$, we obtain $u \in \cap_\lambda H^\lambda$. This proves the last assertion of Proposition 2.1 concerning $\mathcal{N}(\mathcal{A})$ ∎

In the case of the operator A_0 defined by (2.19) – (2.22) we put

$$\begin{aligned}(Bu)(t) &= a_+(t,t) \int_0^1 \kappa_+(t-s)u(s)\,ds + a_-(t,t) \int_0^1 \kappa_-(t-s)u(s)\,ds \\ &= \big(b_1(t)P_+ + b_2(t)P_-\big)\Lambda^{-\alpha}u\end{aligned}$$

where the functions b_1 and b_2 are defined by (2.20) and

$$(\Lambda^{-\alpha}u)(t) = \sum_{k\in Z} \underline{k}^{-\alpha}\hat{u}(k)e^{ik2\pi t}$$

is an isometry between H^λ and $H^{\lambda+\alpha}$ $(\lambda \in \mathbb{R})$. Under conditions (2.20) and (2.21), the operator $b_1P_+ + b_2P_- \in \mathcal{L}(H^\lambda, H^\lambda)$ is known to be an isomorphism for all $\lambda \in \mathbb{R}$: in the case $\lambda = 0$ we refer again, e.g., to [5, 6, 17]; the case $\lambda \neq 0$ can be easily reduced to the case $\lambda = 0$ (see, e.g., [26]). Consequently, $B \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ is an isomorphism for all $\lambda \in \mathbb{R}$, and Proposition 2.1 holds true also in the case (2.19) - (2.23) ∎

Proof of Theorem 2.2. We represent equation (2.1) in the form (see (6.1) and (6.2)) $b\Lambda u + Ku = f$. The Galerkin method (2.8), in these notations, reads as follows:

$$u_N \in \mathcal{T}_N, \qquad P_Nb\Lambda u_N + P_NKu_N = P_Nf.$$

Notice that the isomorphism $\Lambda \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$ has the property that $\Lambda\mathcal{T}_N \subset \mathcal{T}_N$ and $\Lambda^{-1}\mathcal{T}_N \subset \mathcal{T}_N$. Due to Corollary 6.2, for any $v_n \in \mathcal{T}_N$

$$\|\Lambda v_N\|_{\lambda+\alpha} \le c'_\lambda \|P_Nb\Lambda v_N\|_{\lambda+\alpha} \qquad (N \ge N_0)$$

or

$$\|v_N\|_\lambda \le c''_\lambda \|P_NBv_N\|_{\lambda+\alpha}, \qquad (N \ge N_0,\ \lambda \in \mathbb{R}). \tag{6.3}$$

A standard argument (see, e.g., [28]) enables us to extend this stability inequality for $P_N\mathcal{A} = P_NB + P_NK$:

$$\|v_N\|_\lambda \le c_\lambda \|P_N\mathcal{A}v_N\|_{\lambda+\alpha} \qquad (N \ge N_1,\ v_N \in \mathcal{T}_N,\ \lambda \in \mathbb{R}). \tag{6.4}$$

Indeed, assume that (6.4) is violated: there are $v_N \in \mathcal{T}_N$ such that $\|v_N\|_\lambda = 1$ and $\|P_NBv_N + P_NKv_N\|_{\lambda+\alpha} \to 0$ as $N \to \infty$ along some subsequence. Due to the compactness of the operator $K \in \mathcal{L}(H^\lambda, H^{\lambda+\alpha})$, we may assume that, along the subsequence, P_NKv_N convergences in $H^{\lambda+\alpha}$ to some limit $w \in H^{\lambda+\alpha}$, and consequently P_NBv_N converges in $H^{\lambda+\alpha}$ to $-w$. Now it follows from the stability inequality (6.3) that the corresponding subsequence of v_N converges in H^λ to $v = -B^{-1}w \in H^\lambda$ and $\|v\|_\lambda = 1$. The limiting in the relation $\|P_N\mathcal{A}v_N\|_{\lambda+\alpha} \to 0$ yields $\mathcal{A}v = 0$. Thus, $\mathcal{N}(\mathcal{A}) \neq \{0\}$ contradicting the condition (2.6) of the theorem and proving the stability inequality (6.4). With the help of the interpolation theorem one can even prove that the minimal constant c_λ in (6.4) is uniformly bounded in λ on every (finite) interval $[\lambda_1, \lambda_2]$.

Thus, (2.8) determines for $N \ge N_1$ a unique $u_N \in \mathcal{T}_N$, the Galerkin approximation to $u = \mathcal{A}^{-1}f$. The stability inequality (6.4) yields

$$\begin{aligned}\|u_N - P_Nu\|_\lambda &\le c_\lambda\|P_N\mathcal{A}(u_N - P_Nu)\|_{\lambda+\alpha} = c_\lambda\|P_Nf - P_N\mathcal{A}P_Nu\|_{\lambda+\alpha}\\ &= c_\lambda\|P_N\mathcal{A}u - P_N\mathcal{A}P_Nu\|_{\lambda+\alpha} \le c_\lambda\|\mathcal{A}\|_{\lambda,\lambda+\alpha}\|u - P_Nu\|_\lambda\end{aligned}$$

and

$$\|u_N - u\|_\lambda \le \|u_N - P_Nu\|_\lambda + \|u - P_Nu\|_\lambda \le \left(c_\lambda\|\mathcal{A}\|_{\lambda,\lambda+\alpha} + 1\right)\|u - P_Nu\|_\lambda.$$

Together with (2.7) this proves the error estimate (2.9) ∎

In the case of an operator A_0 of the form (2.19) we use Lemma 6.1 instead of Corollary 6.2. Note that $\Lambda^{-\alpha} v_N \in \mathcal{T}_N$ for $v_N \in \mathcal{T}_N$.

Proof of Theorem 2.3. For the operators $\mathcal{A}$ and $\mathcal{A}_M$ defined in (2.2) and (2.13), respectively, we have according to Proposition 5.3

$$\|\mathcal{A}_M - \mathcal{A}\|_{\lambda,\lambda+\alpha} \le c_{\lambda,\nu} \sum_{p=0}^{q} \|a_{p,M} - a_p\|_{|\lambda+\alpha|+\nu,\max(|\lambda|,\nu)}, \qquad \left(\nu > \frac{1}{2}\right)$$

(we used here the coarser but more universal inequality (5.8); in the cases indicated in Proposition 5.3, the more sharp inequality (5.7) may be used). Now our task is to show that, with any $r > 0$,

$$\|a_{p,M} - a_p\|_{\lambda,\mu} \le c_{\lambda,\mu,r} M^{-r} \|a_p\|_{\lambda+r,\mu+r} \qquad \left(\lambda, \mu > \frac{1}{2}\right) \tag{6.5}$$

resulting to

$$\|\mathcal{A}_M - \mathcal{A}\|_{\lambda,\lambda+\alpha} \le c_{\lambda,\alpha,r} M^{-r} \sum_{p=0}^{q} \|a_p\|_{|\lambda+\alpha|+\nu+r,\max(|\lambda|,\nu)+r} \qquad (\lambda \in \mathbb{R}). \tag{6.6}$$

We have (see (2.11))

$$a_{p,M} - a = \Pi_M Q_{M,M} a - a = \Pi_M (Q_{M,M} a - a) + (\Pi_M a - a)$$

where Π_m is the two-dimensional Fourier projection defined by

$$(\Pi_M a)(t,s) = \sum_{|k|+|l| \le M/2} \hat{a}(k,l) e^{ik2\pi t} e^{il2\pi s}.$$

It is clear that Π_m is orthogonal in any space $H^{\lambda,\mu}$ and

$$\|a - \Pi_M a\|_\mu \le \left(\frac{M}{2} - 1\right)^{-r} \|a\|_{\lambda+r,\mu+r} \qquad (r > 0).$$

Similarly to (2.10) we have (see [26] for a detailed proof)

$$\|a - Q_{M,M}\|_{\lambda,\nu} \le c_{\lambda,\mu,r} M^{-r} \|a\|_{\lambda+r,\mu+r} \quad \left(\lambda \ge 0,\, \mu \ge 0,\, \lambda + r > \frac{1}{2},\, \mu + r > \frac{1}{2}\right).$$

This proves (6.5) and (6.6).

Due to (6.6), the stability property (6.4) extends to the modified Galerkin method (2.14): there is an N_2 such that, for any $N \ge N_2$, $v_N \in \mathcal{T}_N$ and $\lambda \in \mathbb{R}$,

$$\|v_n\|_\lambda \le c_\lambda \|P_N \mathcal{A}_M v_n\|_{\lambda+\alpha}.$$

Consequently, for $N \geq N_2$ (2.14) provides a unique approximation $u_N \in \mathcal{T}_N$, and with $u = \mathcal{A}^{-1} f$ we have

$$\begin{aligned}
&\|u_N - P_N u\|_\lambda \\
&\leq c_\lambda \big\|P_N \mathcal{A}_M (u_N - P_N u)\big\|_{\lambda+\alpha} \\
&= c_\lambda \big\|f_n - P_N \mathcal{A}_M P_N u\big\|_{\lambda+\alpha} \\
&\leq c_\lambda \Big(\big\|f_N - P_N f\big\|_{\lambda+\alpha} + \big\|P_N \mathcal{A}(u - P_N u)\big\|_{\lambda+\alpha} + \big\|(\mathcal{A} - \mathcal{A}_M) P_N u\big\|_{\lambda+\alpha}\Big) \\
&\leq c'_\lambda \Big(\big\|u - P_N u\big\|_\lambda + \big\|f_N - P_N f\big\|_{\lambda+\alpha} + c_{\lambda,r} M^{-r} \|u\|_\lambda\Big)
\end{aligned}$$

resulting to the error estimate (2.15). Note that since $r > 0$ is arbitrary and $M \sim N^\sigma$ $(0 < \sigma \leq 1)$, we may replace M^{-r} by N^{-r} in the last estimate ∎

No changes in the argument are needed in the case of an operator A_0 of the form (2.19).

7. Numerical illustration

Let us illustrate the behaviour of the actual error $u_N - u$ of the method (2.14) with $f_N = Q_N f$ for different M and N.

Example 7.1. We took the model problem

$$\int_0^1 \kappa_0(t-s) a_0(t,s) u(s)\, ds = f(t) \tag{7.1}$$

with $\alpha = 2$,

$$\begin{aligned}
\hat{\kappa}_0(m) &= \frac{1}{\pi} \frac{4}{4m^2 - 1} \qquad (m \in Z) \\
a_0(t,s) &= a(t) a(s), \qquad a(t) = 3 + \sum_{0 \neq m \in Z} 2^{-4|m|} e^{im2\pi t}.
\end{aligned}$$

We put $u(t) = \sum_{0 \neq m \in Z} m^{-3} e^{im2\pi t}$ and computed with a high accuracy $f = f(t)$ from (7.1). After that we applied method (2.14) with $f_N = Q_N f$ to recover $u_N \in \mathcal{T}_N$. Constructing the system (3.2) we used only grid values

$$a_0(j_1 M^{-1}, j_2 M^{-1}) \quad (j_1, j_2 = 0, \ldots, M) \qquad \text{and} \qquad f(jN^{-1}) \quad (j = 0, \ldots, N)$$

and did not use the Fourier coefficients of a_0 and f (which are available in this example). The error $\|u_N - u\|_0$ for some N and M is presented in Table 7.1. Note that $u \in H^\mu$ $(\mu < \frac{5}{2})$ and $u \notin H^{5/2}$. The estimate (2.16) takes the form $\|u_N - u\|_0 \leq c_\mu N^{-\mu} \|u\|_\mu$ $(\mu < \frac{5}{2})$. The experimental convergence order with $M \sim N^{1/2}$, as seen from Table 7.1, is approximately $cN^{-5/2}$.

	N=8	N=16	N=32	N=64	N=128	N=256
$M=2$	$1.82 \cdot 10^{-1}$	$1.79 \cdot 10^{-1}$	$1.80 \cdot 10^{-1}$			
$M=4$	$4.01 \cdot 10^{-2}$	$2.44 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$	$2.40 \cdot 10^{-2}$		
$M=8$	$3.04 \cdot 10^{-2}$	$4.43 \cdot 10^{-3}$	$7.92 \cdot 10^{-4}$	$2.96 \cdot 10^{-4}$	$2.67 \cdot 10^{-4}$	$2.66 \cdot 10^{-4}$
$M=16$			$7.46 \cdot 10^{-4}$	$1.30 \cdot 10^{-4}$	$2.28 \cdot 10^{-5}$	$4.01 \cdot 10^{-6}$

Table 7.1: The error $\|u_N - u\|_0$ in Example 7.1

Example 7.2. This example is similar to Example 7.1 but this time we took more slowly decaying Fourier coefficients of $a_0(t,s) = a(t)a(s)$, namely

$$a(t) = 3 + \sum_{0 \neq m \in Z} 2^{-|m|} e^{im2\pi t}.$$

The theoretical estimate $\|u_N - u\|_0 \leq c_\mu N^{-\mu} \|u\|_\mu$ $(\mu < \frac{5}{2})$ is the same as in Example 7.1 but now its numerical realization can be seen beginning from greater N and M. We also present the errors $\|u_N - u\|_{-2}$. According to (2.16), the theoretical error estimate is $\|u_N - u\|_{-2} \leq c_\mu N^{-\mu-2}$ $(\mu < \frac{5}{2})$.

	N=16	N=32	N=64	N=128	N=256	N=512
$M=4$	$1.85 \cdot 10^{0}$	$1.80 \cdot 10^{0}$	$1.80 \cdot 10^{0}$			
	$3.17 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$			
$M=8$		$1.20 \cdot 10^{0}$	$1.20 \cdot 10^{0}$	$1.20 \cdot 10^{0}$		
		$7.00 \cdot 10^{-2}$	$7.00 \cdot 10^{-2}$	$7.00 \cdot 10^{-2}$		
$M=16$			$1.26 \cdot 10^{0}$	$2.70 \cdot 10^{-1}$	$2.70 \cdot 10^{-1}$	
			$7.03 \cdot 10^{-2}$	$4.13 \cdot 10^{-3}$	$4.13 \cdot 10^{-3}$	
$M=32$					$4.11 \cdot 10^{-3}$	$4.11 \cdot 10^{-3}$
					$1.61 \cdot 10^{-5}$	$1.61 \cdot 10^{-5}$
$M=64$					$4.33 \cdot 10^{-6}$	$7.81 \cdot 10^{-7}$
					$3.47 \cdot 10^{-10}$	$2.45 \cdot 10^{-10}$

Table 7.2: The errors $\|u_N - u\|_0$ and $\|u_N - u\|_{-2}$ in Example 7.2

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