On Solvability of a Parabolic System Arising in Physical Oceanography

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Abstract. In physical oceanography, thermocline theories are to explain the phenomenon of strong, vertical density gradient in a relatively shallow layer of water where transition occurs from the ocean's surface temperature to the colder abyss. We derive a nonlinear system of partial differential equations governing the motion of thermocline layer. Function spaces are set up to study properties of the solutions. By a local version of Banach's fixed point theorem, the existence and smoothness of solutions are established.

Keywords: *Physical oceanography, thermocline, Hölder spaces, Banach fixed point theorem*

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1. Introduction

In this paper we continue our study of the thermocline problem in [1]. Thermocline theories are to explain the phenomenon of strong, vertical *density* gradient in a relatively shallow layer of water where transition occurs from the ocean's surface temperature to the colder abyss. The ocean can be modelled as a thin spherical layer of fluid rotating with angular velocity Ω. Gravity g acts normal to the spherical surface. The ocean's motion is driven in part by seasonally changing wind stress. A basic assumption is that the ocean's direct response to wind stress is limited to the upper layer known as *Ekman layer*. The vertical velocity at bottom of Ekman layer can be calculated from observed data. Thus, on top of the thermocline region, vertical velocity is specified as a function of space and time.

The governing equations are derived by conservation principles of mass, momentum, and energy. This problem has been studied from various point of view. Salmon [8] considers a model with unequal diffusion coefficients in vertical and horizontal directions. He concentrates on finding steady-state similarity solutions subject to various boundary conditions. He also numerically investigates physical properties of the solutions. The existence of classical and weak steady-state solutions with other boundary conditions has been proved by Chen [1] and Kordzadze [3, 4] independently via different approaches. When the system is simplified by considering vertical diffusion only or no diffusion at all, it is called the *ideal thermocline problem*. Many similarity solutions have been found since 1960. A complete set of all similarity solutions is presented by Lie group method in Salmon and Hollerbach [9]. A survey is given by Huang [2].

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This paper is organized as follows. Sketches of formulation of the problem with initial boundary conditions are contained in Section 2. Section 3 gives definitions of function spaces and states a fixed point theorem. The existence theorem is then proved in Section 4.

2. Formulation of the problem

Let ρ be the density, p the pressure, and g the acceleration due to gravity of the Earth. We assume that the state equation takes a simple form

$$\rho = \rho_0\big(1 - \varepsilon(T - T_0)\big) \tag{1}$$

where T is the temperature, ρ_0 the mean density of sea water at mean temperature T_0, and ε the thermal expansion coefficient. Conservation of momentum in the vertical direction z is accurately assumed to be hydrostatic,

$$\frac{\partial p}{\partial z} = -g\rho. \tag{2}$$

Let u, v and w be the eastward, northward and vertical velocity, respectively. Since ocean's vertical scale of motion is much small compared to its horizontal scale, with the Coriolis force (rotation of the Earth) taken into account, conservation of momentum implies the so-called *geostrophic balance*

$$2\Omega\rho_0 u \sin\theta = -\frac{1}{R}\frac{\partial p}{\partial\theta} \tag{3}$$

$$2\Omega\rho_0 v \sin\theta = \frac{1}{R\cos\theta}\frac{\partial p}{\partial\varphi} \tag{4}$$

where φ measures longitude, θ the latitude, Ω is the angular velocity of the Earth's rotating, and R is the average radius of the Earth. Conservation of mass implies

$$\frac{\partial w}{\partial z} + \frac{1}{R\cos\theta}\frac{\partial}{\partial\theta}(v\cos\theta) + \frac{1}{R\cos\theta}\frac{\partial u}{\partial\varphi} = 0. \tag{5}$$

Considering thermal diffusion and using the state equation (1), conservation of energy gives

$$\begin{aligned}
\rho_t + w\frac{\partial\rho}{\partial z} &+ \frac{v}{R}\frac{\partial\rho}{\partial\theta} + \frac{u}{R\cos\theta}\frac{\partial\rho}{\partial\varphi} \\
&= \kappa\left(\frac{\partial^2\rho}{\partial z^2} + \frac{2}{R}\frac{\partial\rho}{\partial z} + \frac{1}{R^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\rho}{\partial\theta}\right) + \frac{1}{R^2\cos^2\theta}\frac{\partial^2\rho}{\partial\varphi^2}\right)
\end{aligned} \tag{6}$$

where t is the time and κ is a positive constant. For detail derivations see Pedlosky [6: Chapter 6] and [7]. In regions away from the equator and the pole in northern hemisphere, equations (2) - (6) may be simplified by introducing change of independent variables

$$\bar{x} = R\varphi, \qquad e^{\bar{y}/R} = \sin\theta, \qquad \bar{z} = z, \qquad \bar{t} = t.$$

In southern hemisphere, we set $\bar{x} = R\varphi, e^{\bar{y}/R} = -\sin\theta, \bar{z} = z$ and $\bar{t} = t$. For dependent variables, we let

$$\bar{\rho} = \rho - \rho_0, \quad \bar{p} = p + g\rho_0 z, \quad \bar{u} = \frac{\sin^2\theta}{\cos\theta}u, \quad \bar{v} = (\sin\theta\cos\theta)v, \quad \bar{w} = (\sin^2\theta)w.$$

After dropping all over bars, equations (2) - (6) turn to be

$$\frac{\partial p}{\partial z} = -g\rho \qquad (7)$$

$$u = -\frac{1}{2\Omega\rho_0}\frac{\partial p}{\partial y} \qquad (8)$$

$$v = \frac{1}{2\Omega\rho_0}\frac{\partial p}{\partial x} \qquad (9)$$

$$\frac{\partial w}{\partial z} + \frac{1}{R}\left(\frac{\partial v}{\partial y} - v\right) + \frac{1}{R}\frac{\partial u}{\partial x} = 0 \qquad (10)'$$

$$-\rho_t + \kappa\Delta^*\rho - \kappa\frac{1+e^{2y/R}}{Re^{2y/R}}\frac{\partial\rho}{\partial y} + \kappa\frac{2}{R}\frac{\partial\rho}{\partial z} = e^{-2y/R}\left(w\frac{\partial\rho}{\partial z} + v\frac{\partial\rho}{\partial y} + u\frac{\partial\rho}{\partial x}\right) \qquad (11)$$

where Δ^* is the partial differential operator

$$\Delta^* = \frac{1}{1-e^{2y/R}}\frac{\partial^2}{\partial x^2} + \frac{(1-e^{2y/R})}{e^{2y/R}}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Suppose that ρ is twice differentiable. By virtue of (8) and (9) equation $(10)'$ takes a simpler form

$$\frac{\partial w}{\partial z} = \frac{v}{R}. \qquad (10)$$

Let $D \in \mathbb{R}^3$ be a smooth bounded thermocline region which has top surface Q_a : $z = z_a(x,y,t)$ and bottom surface Q_b : $z = z_b(x,y,t)$. The boundary of D is denoted by ∂D. Outside the thermocline region D, values of density are specified by

$$\rho = \rho_a(x,y,z,t) \qquad \text{on} \ \bar{G} \qquad (12)$$

where $\overline{G} = \partial D \times [0,\tau]$, and initially

$$\rho = \rho_b(x,y,z) \qquad \text{on} \ D. \qquad (13)$$

On the top boundary surface Q_a, the pressure field is given by

$$p(x,y,z_a) = p_a(x,y,t). \qquad (14)$$

Meanwhile, wind stress can be calculated to specify the vertical velocity entering or leaving the base of Ekman layer. Thus

$$w(x,y,z_a) = w_a(x,y,t). \qquad (15)$$

3. Function spaces and preliminaries

We give definitions of all function spaces suitable for this work and state a local version of Banach's fixed point theorem without proof.

If $0 < \alpha < 1$ and $i = 0, 1, 2$, then $C^{i+\alpha}(\overline{D})$ are the Banach spaces whose elements are continuous functions having in $\overline{D}$ continuous i-th derivatives which are Hölder continuous with exponent α. Their norms are denoted by $\| \cdot \|_{i+\alpha}$. Let $E = D \times (0, \tau)$ and $\overline{E}$ the closure of E. For $i = 0, 1, 2$ and $j = 0, 1$ the Banach spaces $C^{i+\alpha, j+\alpha/2}(\overline{E})$ and their norms $\| \cdot \|_{i+\alpha, j+\alpha/2}$ are defined as follows. We say that ρ belongs to $C^{\alpha, \alpha/2}(\overline{E})$ if

$$\|\rho\|_{\alpha, \alpha/2} = \sup_{s \in \overline{E}} |\rho(s)| + \|\rho\|_\alpha < \infty$$

where

$$\|\rho\|_\alpha = \sup_{s, s' \in \overline{E}} \frac{|\rho(s) - \rho(s')|}{d^\alpha(s, s')} \quad \text{and} \quad d(s, s') = \left(\sum_{j=1}^3 (x_j - x_j')^2 + |t - t'| \right)^{1/2}$$

and $s = (x, y, z, t) = (x_1, x_2, x_3, t)$. We say that ρ belongs to $C^{1+\alpha, \alpha/2}(\overline{E})$ if

$$\|\rho\|_{1+\alpha, \alpha/2} = \|\rho\|_{\alpha, \alpha/2} + \sum_{j=1}^3 \left\| \frac{\partial \rho}{\partial x_j} \right\|_\alpha < \infty.$$

Similarly, $\rho \in C^{1+\alpha, 1+\alpha/2}(\overline{E})$ if

$$\|\rho\|_{1+\alpha, 1+\alpha/2} = \|\rho\|_{\alpha, \alpha/2} + \sum_{j=1}^3 \left\| \frac{\partial \rho}{\partial x_j} \right\|_\alpha + \left\| \frac{\partial \rho}{\partial t} \right\|_\alpha < \infty.$$

In the same fashion, ρ is an element of $C^{2+\alpha, \alpha/2}(\overline{E})$ if

$$\|\rho\|_{2+\alpha, \alpha/2} = \|\rho\|_{1+\alpha, \alpha/2} + \sum_{j=1}^3 \sum_{k=1}^3 \left\| \frac{\partial^2 \rho}{\partial x_j \partial x_k} \right\|_\alpha < \infty.$$

Finally, we say that $\rho \in C^{2+\alpha, 1+\alpha/2}(\overline{E})$ if

$$\|\rho\|_{2+\alpha, 1+\alpha/2} = |\rho\|_{2+\alpha, \alpha/2} + \left\| \frac{\partial \rho}{\partial t} \right\|_\alpha < \infty.$$

Same notations will be applied to the set $\overline{Q}$ which is the domain of definition of $\overline{Q}_a$.

Lemma: *Let (Y, d) be a complete metric space with the metric d and $B = \{y \in Y : d(y, y_0) < r\}$. Let $F : B \to Y$ be a contractive mapping with constant $a < 1$. If $d(F(y_0), y_0) < (1 - a)r$, then F has a fixed point.*

The proof of this lemma is trivial and omitted.

4. The existence theorem

Let $D \subset \mathbb{R}^3$ be a smooth, open, bounded region in (x, y, z)-coordinate, which in terms of (φ, θ, z) satisfies $0 < \theta_a \leq \theta \leq \theta_b < \frac{\pi}{2}$ with some constants θ_a and θ_b. Hence D represents a region away from the equator and the pole in northern hemisphere. Let $G = \partial D \times (0, \tau)$, $E = D \times (0, \tau)$, and $\overline{Q}$ the domain of $\overline{Q}_a$.

Theorem (Existence): *Let $\rho_a \in C^{2+\alpha, 1+\alpha/2}(\overline{G})$, $\rho_b \in C^{2+\alpha}(\overline{D})$, $p_a \in C^{1+\alpha, \alpha/2}(\overline{Q})$, $w_a \in C^{1+\alpha, \alpha/2}(\overline{Q})$, and $z_a \in C^{\alpha, \alpha/2}(\overline{Q})$. Define $H = \sup_{z_a \in Q_a, z_b \in Q_b} |z_a - z_b|$. Then there exist $r > 0$ and $r^* > 0$ such that, if*

$$\|\rho_a\|_{2+\alpha, 1+\alpha/2} + \|\rho_b\|_{2+\alpha} + \|p_a\|_{1+\alpha, \alpha/2} + \|w_a\|_{1+\alpha, \alpha/2} + \|z_a\|_{\alpha, \alpha/2} < r^*$$

and

$$\mathcal{B} = \left\{ \rho \in C^{2+\alpha, 1+\alpha/2}(\overline{E}) : \|\rho\|_{2+\alpha, 1+\alpha/2} < r \right\},$$

then solutions (ρ, p, u, v, w) of the problem $(7)-(11)$ with boundary conditions $(12)-(15)$ exist. Moreover, both ρ and p belong to $\mathcal{B}$ and $(u, v, w) \in \left(C^{1+\alpha, 1+\alpha/2}(\overline{E}) \right)^3$.

Proof : We define a nonlinear mapping $\mathcal{M} : \mathcal{B} \to C^{2+\alpha, 1+\alpha/2}(\overline{E})$ as follows. If $\rho \in \mathcal{B}$, then p, u, v, w are defined by (7) with (14), (8), (9), and (10) with (15), respectively. We insert them into the right-hand side of (11). The image $\rho^* = \mathcal{M}(\rho)$ under the mapping $\mathcal{M}$ is the solution of

$$-\rho_t^* + \kappa \Delta^* \rho^* - \kappa \frac{1 + e^{2y/R}}{Re^{2y/R}} \frac{\partial \rho^*}{\partial y} + \kappa \frac{2}{R} \frac{\partial \rho^*}{\partial z} = e^{-2y/R} \left(w \frac{\partial \rho}{\partial z} + v \frac{\partial \rho}{\partial y} + u \frac{\partial \rho}{\partial x} \right) \quad (16)$$

with initial boundary conditions (12) and (13). In this fashion, the fixed point of the mapping $\mathcal{M}$ would be a solution of $(7) - (11)$ with $(12) - (15)$. Since $\rho \in C^{2+\alpha, 1+\alpha/2}(\overline{E})$, the functions u, v, w are in $C^{1+\alpha, 1+\alpha/2}(\overline{E})$. Consequently, u, v, w, and first derivatives of ρ are in $C^{\alpha, \alpha/2}(\overline{E})$. Thus, the right-hand side of (16) belongs to $C^{\alpha, \alpha/2}(\overline{E})$. The unique solution $\rho^* \in C^{2+\alpha, 1+\alpha/2}(\overline{E})$ of equation (16) can be obtained by Theorem 5.2 in Ladyzhenskaya, Solonnikov and Ural'ceva [5: p. 320]. Therefore, $\mathcal{M}$ maps $\mathcal{B}$ into $C^{2+\alpha, 1+\alpha/2}(\overline{E})$.

Next, we need to verify that $\mathcal{M}$ is a contractive mapping. Let ρ_1 and ρ_2 be two elements in $\mathcal{B}$, $\rho_1^* = \mathcal{M}(\rho_1)$ and $\rho_2^* = \mathcal{M}(\rho_2)$. We shall show that

$$\|\rho_1^* - \rho_2^*\|_{2+\alpha, 1+\alpha/2} \leq \gamma \|\rho_1 - \rho_2\|_{2+\alpha, 1+\alpha/2} \quad (17)$$

for some constant $0 < \gamma < 1$. Let

$$f(\rho, p, u, v, w) = e^{-2y/R} \left(w \frac{\partial \rho}{\partial z} + v \frac{\partial \rho}{\partial y} + u \frac{\partial \rho}{\partial x} \right).$$

If ρ_i is given, then

$$p_i = \int\limits_z^{z_a} g\rho_i(x,y,z',t)\,dz' + p_a(x,y,t)$$

$$u_i = -\frac{1}{2\Omega\rho_0}\frac{\partial p_i}{\partial y}$$

$$v_i = \frac{1}{2\Omega\rho_0}\frac{\partial p_i}{\partial x}$$

$$w_i = \int\limits_{z_a}^z \frac{v_i(x,y,z',t)}{R}\,dz' + w_a(x,y,t)$$

and $f_i = f(\rho_i,p_i,u_i,v_i,w_i)$ for $i=1,2$. Since $\rho_1^* - \rho_2^*$ satisfies

$$-(\rho_1^* - \rho_2^*)_t + \kappa\Delta^*(\rho_1^* - \rho_2^*)$$

$$-\kappa\frac{1+e^{2y/R}}{R}\frac{\partial(\rho_1^* - \rho_2^*)}{\partial y} + \kappa\frac{2e^{2y/R}}{R}\frac{\partial(\rho_1^* - \rho_2^*)}{\partial z} = f_1 - f_2$$

with zero initial boundary conditions, from regularity results of Theorem 5.2 in [5: page 320], we have

$$\|\rho_1^* - \rho_2^*\|_{2+\alpha,1+\alpha/2} \le c\|f_1 - f_2\|_{\alpha,\alpha/2} \tag{18}$$

for some constant c. From now on and throughout the paper, we let c be a generic positive constant which varies from line to line unless otherwise specified. Among many terms in $f_1 - f_2$, we first examine

$$\left(u_1\frac{\partial\rho_1}{\partial x} - u_2\frac{\partial\rho_2}{\partial x}\right).$$

If $\|\cdot\|_0$ denotes the sup norm in $\overline{E}$, then we have

$$\left\|u_1\frac{\partial\rho_1}{\partial x} - u_2\frac{\partial\rho_2}{\partial x}\right\|_{\alpha,\alpha/2} \le c\left(\left\|\frac{\partial\rho_1}{\partial x}\right\|_0\|u_1 - u_2\|_{\alpha,\alpha/2} + \|u_2\|_0\|\rho_1 - \rho_2\|_{\alpha,\alpha/2}\right). \tag{19}$$

The following inequalities are established by definitions and straightforward computations:

$$\left\|\frac{\partial\rho_1}{\partial x}\right\|_0 \le \|\rho_1\|_{1+\alpha,\alpha/2} \le \|\rho_1\|_{2+\alpha,1+\alpha/2} \le r \tag{20}$$

$$\|u_2\|_0 \le \frac{g}{2\Omega\rho_0}\left\|\int\limits_z^{z_a}\frac{\partial\rho_2}{\partial y}(x,y,z',t)\,dz'\right\|_0 + \|p_a\|_0 \le c(rH + r^*). \tag{21}$$

To estimate $\|u_1 - u_2\|_{\alpha,\alpha/2}$, we note that

$$\|u_1 - u_2\|_0 \le H\|\rho_1 - \rho_2\|_{1+\alpha,\alpha/2} \le H\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2} \tag{22}$$

$$\Big|(u_1 - u_2)(x,y,z,t) - (u_1 - u_2)(x,y,\bar{z},t)\Big| v \le |z - \bar{z}|\, \|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2} \qquad (23)$$

$$\Big|(u_1 - u_2)(x,y,z,t) - (u_1 - u_2)(\bar{x},y,z,t)\Big| \le A_1 + A_2 \qquad (24)$$

where

$$A_1 = \left| \int_{z_a(\bar{x},y,t)}^{z_a(x,y,t)} \frac{\partial}{\partial y}(\rho_1 - \rho_2)(x,y,z',t)\, dz' \right|$$

$$A_2 = \left| \int_{z_a(\bar{x},y,t)}^{z_a(\bar{x},y,t)} \frac{\partial}{\partial y}\Big(\rho_1(x,y,z',t) - \rho_2(\bar{x},y,z',t)\Big)\, dz' \right|.$$

Hence, from (23) and (24), and similar estimates for

$$\big|(u_1 - u_2)(x,y,z,t) - (u_1 - u_2)(x,\bar{y},z,t)\big|$$

$$\big|(u_1 - u_2)(x,y,z,t) - (u_1 - u_2)(x,y,z,\bar{t})\big|$$

we obtain

$$\frac{\big|(u_1 - u_2)(s) - (u_1 - u_2)(s')\big|}{d^\alpha(s,s')} \le c\Big(H^{1-\alpha} + \|z_a\|_{\alpha,\alpha/2}\Big)\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2} \qquad (25)$$

for all $s, s' \in \overline{E}$. Inequalities (22) and (25) imply that

$$\|u_1 - u_2\|_{\alpha,\alpha/2} \le c\Big(H + H^{1-\alpha} + \|z_a\|_{\alpha,\alpha/2}\Big)\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2}. \qquad (26)$$

Define $\beta = rH + rH^{1-\alpha} + rr^* + r^*$. Since $\|z_a\|_{\alpha,\alpha/2} < r^*$ and $\|p_a\|_{1+\alpha,\alpha/2} < r^*$, we have, by inserting (20), (21), and (26) into the right-hand side of (19),

$$\left\| u_1 \frac{\partial \rho_1}{\partial x} - u_2 \frac{\partial \rho_2}{\partial x} \right\|_{\alpha,\alpha/2} \le c\beta\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2}. \qquad (27)$$

For the remaining terms in $f_1 - f_2$ similar results hold. Thus

$$\left\| v_1 \frac{\partial \rho_1}{\partial y} - v_2 \frac{\partial \rho_2}{\partial y} \right\|_{\alpha,\alpha/2} \le c\beta\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2} \qquad (28)$$

$$\left\| w_1 \frac{\partial \rho_1}{\partial z} - w_2 \frac{\partial \rho_2}{\partial z} \right\|_{\alpha,\alpha/2} \le c\Big(\beta H + \|w_a\|_{1+\alpha,\alpha/2}\Big)\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2}. \qquad (29)$$

By virtue of (27) - (29) we obtain

$$\|f_1 - f_2\|_{\alpha,\alpha/2} \le c\big(\beta + \beta H + r^*\big)\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2}.$$

Inequality (18) now becomes

$$\|\rho_1^* - \rho_2^*\|_{2+\alpha,1+\alpha/2} \le c\big(\beta + \beta H + r^*\big)\|\rho_1 - \rho_2\|_{2+\alpha,1+\alpha/2}.$$

From Theorem 5.2 in [5: p. 320], we have

$$\|\mathcal{M}(0) - 0\|_{2+\alpha,1+\alpha/2} \le c\Big(\|\rho_a\|_{2+\alpha,1+\alpha/2} + \|\rho_b\|_{2+\alpha}\Big) \le cr^*$$

for some constant c which is independent of ρ_a and ρ_b. Choosing r and r^* so that

$$c\big(\beta + \beta H + r^*\big) = \gamma < 1 \qquad \text{and} \qquad cr^* < (1 - \gamma)r.$$

Thus (17) is true and the proof is complete by the Lemma in Section 3 ∎

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