Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 14 (1995), No. 3, 623–635

Solving Boundary Value Problems with "Average" Data Sampled on an Arc

Peter A. McCoy

Abstract. Function-theoretic methods are employed with appropriate regularity conditions to solve boundary value problems of the type

$$(\nabla^2 + P(|x|^2))u(x) = 0 \qquad (x \in \mathbf{D};\ u(x_0) = f(x_0),\ x_0 \in \partial\mathbf{D}).$$

The solutions are uniquely determined from the mean values of the boundary data $f(\mathbf{x})$ sampled at equally spaced points of the circumference; or, along an arc on the circumference of the disk $\mathbf{D}$. Equivalent problems are formulated and solved from the point of view of conformal equivalence of domains and transmutation of differential operators. Solutions to inverse problems are reconstructed from the mean values of data taken along the boundary.

Keywords: *Elliptic boundary value problems, mean boundary values, sampling, transmutations, inverse problems, function-theoretic methods*

AMS subject classification: Primary 35 J 05, 35 A 22, 35 A 35, secondary 41 A 10, 35 R 30

1. Introduction

Solutions of elliptic boundary value problems are typically specified as integrals of the boundary data. However, in many applications circumstances make it essential to expand or recover the solution from discrete measurements taken over a set of data limited to part of the boundary. The purpose of this article is to develop some general function-theoretic methods for handling these cases.

Function theory is frequently employed in the study of partial differential equations because of its utility in reformulating a problem in terms of the theory of analytic functions about which much is known. The results frequently resemble the form of their antecedents in analytic function theory. P. A. McCoy [12] used R. P. Gilbert's integral operators [1, 8] to develop a method along the lines of Shannon's theorem in signal processing for recovering a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the standard boundary value problem

$$\begin{aligned} (\nabla^2 + P(|x|^2))u(x) &= 0 \qquad (x \in \Omega) \\ u(x_0) &= f(x_0) \qquad (x_0 \in \partial\Omega) \end{aligned} \tag{1}$$

Peter A. McCoy: United States Naval Academy, Math. Dep., Annapolis, Maryland 21402-5002, USA.
Investigations supported, in part, by a grant from the Radar Division, U.S. Naval Research Laboratory, Washington, D.C. 20375.

ISSN 0232-2064 / \$ 2.50 © Heldermann Verlag

from sampled values of "band-limited" data $f(x_0)$ taken at predetermined points on the boundary. The domain $\Omega \subset E^2$ is star-shaped with respect to the origin and has smooth boundary $\partial\Omega \in C^2$. The coefficient $P(|x|^2)$ is a real-valued, analytic function that is negative on the closure $\overline{\Omega}$ of Ω. Recently, P. A. McCoy [13] extended the sampling method to boundary value problems for higher order elliptic partial differential equations on bounded simply connected domains in $\mathbb{E}^N$ with smooth boundaries. The notions of transmutation of partial differential operators found in H. Begehr and R. P. Gilbert [1, 2] and equivalence of domains were introduced to develop equivalence classes of solutions. The boundary data was "band-limited" and the sampling points were preassigned by the theory and distributed over the boundary at fixed lattice points.

A function theory exists for reconstructing solutions of generalized axially symmetric potentials from the mean values of data sampled on a point set distributed over the boundary of the domain (see P. A. McCoy [10, 11]). The boundary data need not be "band-limited", but must be sufficiently smooth. This paper incorporates aspects of both types of sampling methods to develop a general function-theoretic procedure for reconstructing solutions of equation (1) from smooth data sampled on an arc of the boundary of a plane domain. Equivalence classes of boundary value problems are addressed in the plane for domains that are conformally equivalent and for equivalence classes of differential operators. Inverse problems are solved from the point of view of reconstructing a solution from information contained in its mean boundary values taken from sampled values along the boundary.

2. Background

Important classes of elliptic boundary value problems are modelled by the equations (1) on domains $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, which are star-shaped with respect to the origin and have Lyapunov boundaries (see R. P. Gilbert [8], and M. Kracht and E. O. Kreyszig [9]). The present discussion is limited to domains that are star-shaped about the origin with C^2 boundaries. These domains are referred to as "appropriate". R. P. Gilbert [8] based his solution of the boundary value problem on a pair of integral transforms, the first of which maps harmonic functions $h \in C^2(\Omega) \cap C(\overline{\Omega})$ onto solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ by

$$u(\mathbf{x}) = (I + G)h(\mathbf{x}) = h(\mathbf{x}) + \int_0^1 \sigma G(r, 1 - \sigma^2) h(\mathbf{x}\sigma^2)\, d\sigma. \tag{2}$$

The kernel of the transform is the solution of the hyperbolic partial differential equation

$$2(1 - \sigma^2)G_{r\sigma} - G_r + r(G_{rr} + PG) = 0$$

with Goursat data

$$G(0, \sigma) = 0, \qquad G(r, 0) = \int_0^r \tau P(\tau^2)\, d\tau, \qquad P(|\mathbf{x}|^2) < 0 \quad (\mathbf{x} \in \overline{\Omega}).$$

The inverse transform given by R. P. Gilbert [8: p. 25] is

$$h(\mathbf{x}) = (I+G)^{-1}u(\mathbf{x}) = u(r,\theta) + r^{1/2}\int_0^r \tau^{1/2}\Gamma(r,\tau)u(\tau,\theta)\,d\tau. \tag{3}$$

The kernel is expressed as the series

$$\Gamma(r,\tau) = \sum_{j=1}^{\infty} K^{(j)}(r,\tau)$$

with

$$K^{(j)}(r,\tau) = \int_r^{\tau} K(r,s)K^{(j)}(s,\tau)\,ds \qquad \text{and} \qquad K(r,\tau) = \frac{r}{\tau}\frac{G\left(r,1-\frac{r}{\tau}\right)}{2\sqrt{r\tau}}.$$

The transform pair

$$\{I+G,(I+G)^{-1}\}$$

establishes a one-to-one map between solutions $u \in C^2(\Omega)\cap C(\overline{\Omega})$ of equations (1) and harmonic associates $h \in C^2(\Omega)\cap C(\overline{\Omega})$. Here, and throughout, we use the same notation $u(\mathbf{x}) = u(x,y) = u(z)$ for the functions. The interpretation is clear in context.

S. Bergman [3] developed the integral transform

$$u(\mathbf{z}) = B_2(f(\mathbf{z})) = \int_0^1 E(r^2,t)f\left(\frac{\mathbf{z}(1-t^2)}{2}\right)d\mu(t) \tag{4}$$

with $d\mu(t) = dt/\sqrt{1-t^2}$ (and its inverse) linking solutions $u(z)$ with analytic functions $f(z)$ on appropriate domains Ω in the plane. On these domains, the kernels of the Bergman and the Gilbert operators are connected by

$$E(r,t) = 1 + \frac{t^2}{2}\int_0^1 (1-s)^{1/2}G(r,st^2)\,ds$$

(see H. Begehr and R. P. Gilbert [1: p. 296]).

For the time being, let us consider functions that are defined on the unit disk $\mathbf{D} : |z| < 1$. Let $\{u,h\}$ be a pair of real-valued functions in $C^2(\mathbf{D})\cap C(\overline{\mathbf{D}})$ that are associated via equations (2) - (3). Expand the associate of $u(\mathbf{x})$ in the series

$$h(x,y) = \sum_{n=0}^{\infty} \gamma_n a_n h_n(x,y) \tag{5}$$

of real-valued circular harmonic polynomials

$$h_n(x,y) = (x^2+y^2)^{n/2} \cos\left(\frac{nx}{\sqrt{x^2+y^2}}\right)$$

and where the constant is $\gamma_n = \Gamma\left(\frac{1}{2}\right)\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$. Anticipating the sampling procedures, we seek the harmonic associate h of a solution $u = (I+G)h$ whose limit at the boundary point $(1^-, \theta_0)$ is

$$(I+G)h(r,\theta_0) \longrightarrow f(e^{i\theta_0}) \qquad \text{as} \quad r \longrightarrow 1^-$$

for each $0 \le \theta_0 < 2\pi$.

The construction requires an intermediate transformation to normalize equation (5) in reference to the analytic function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbf{D}) \tag{6}$$

and establish the connection between $h = h(x,y)$ and the harmonic function $f(x,y) = \operatorname{Re} F(z)$. The function $F = F(z)$ is subject to special requirements. Let the restriction $F(e^{i\theta})$ of $F(z)$ to $\partial\mathbf{D}$ be a smooth function. This condition is designated by $F \in B_\varepsilon(\partial\mathbf{D})$ and means that the Fourier coefficients a_n of $F(e^{i\theta})$ satisfy $a_n \sim O(1/n^{1+\varepsilon})$ $(\varepsilon > 0)$ asymptotically. Consequently, the restriction is in the set $F \in C(\partial\mathbf{D}) \cap B_\varepsilon(\partial\mathbf{D})$. The function is analytic since its Fourier coefficients satisfy $a_n = a_n(f) = 0$ $(n \in -I\!N)$, and, we write $F \in A(\partial\mathbf{D})$.

Integral operators relevant to the solution of mean boundary value problem are constructed by modifying existing operators. The first step in the process is to relate equations (5) - (6). Let $h(z,0), z = x+iy$, be the analytic continuation of $h(x,y)$ to the disk $\mathbf{D}$. This continuation is valid since $\limsup_{n\to\infty} |\gamma_n|^{1/n} = 1$. Modifying the transform found in H. Begehr and R. P. Gilbert [1: p. 284] and noting that $F(z) = f(z,0)$ yields the transform pair

$$\begin{aligned} F(z) = B_1[h(z,0)] &= \frac{1}{2\pi}\int_{-1}^{+1} h\big(z(1-t^2),0\big)\frac{dt}{t^2} \\ h(z,0) = B_2[f(z)] &= \int_{-1}^{+1} F\big(z(1-t^2)\big)\,d\mu(t) \end{aligned} \tag{7}$$

for $z \in \mathbf{D}$. The contours connecting $\{-1,+1\}$ do not generally pass through the origin. The normalizations of $\{B_1, B_2\}$ differ by a factor of 2 or $\frac{1}{2}$ in the arguments of the associates from those usually taken. Moreover, if $h(z,0) \in C(\overline{\mathbf{D}})$ so is $F(z) \in C(\overline{\mathbf{D}})$ and conversely. The transforms are

$$\begin{aligned} f(x,y) = B^{-1}[h] &:= \frac{B_1[h(z,0)] + \overline{B_1[h(z,0)]}}{2} \\ h(z,0) = B[f] &:= \frac{B_2[f(z,0)] + \overline{B_2[f(z,0)]}}{2} \end{aligned} \tag{8}$$

where we identify $f(z) = f(z,0)$. The transform pair assigning the proper correspondence to the boundary values between associated functions in $C^2(\mathbf{D}) \cap C(\overline{\mathbf{D}})$ is

$$u(\mathbf{x}) = (I+G)Bf(z) \qquad \text{and} \qquad f(z) = B^{-1}(I+G)^{-1}u(\mathbf{x}). \tag{9}$$

Each integral operator represents an analytic function in $\mathbf{D}$. This fact follows from the singularities theorem established in D. L. Colton and R. P. Gilbert [4]. It is easy to verify that on account of uniform convergence, limits interchange and the boundary values correspond as required.

3. Sampling boundary data

The focus of this study is on problems with smooth boundary data $f \in A(\partial\mathbf{D}) \cap B_\epsilon(\partial\mathbf{D})$. These boundary functions are sampled on an arc $[\theta_1, \theta_2]$ of $\partial\mathbf{D}$. This arc is equivalent to the interval $[t_1, t_2]$. Either set is referred to as an "arc". We follow C. H. Ching and C. K. Chui [5: p. 175] and define the *standard set of interpolating angles* at the points

$$\theta_{k,n} = \frac{2\pi k(t_2 - t_1)}{n} + 2\pi t_1 \qquad (1 \le k \le n,\ n \in \mathbb{N})$$

located on the arc $[\theta_1, \theta_2]$. The function $f(e^{i\theta})$ is interpolated at the reference angles to obtain sets $\{f(e^{i\theta_{k,n}})\}_{k=1}^{n}$ $(n \in \mathbb{N})$ of sampled boundary values. It is convenient to introduce the auxiliary sums

$$\sigma_{n,m} = \sigma_{n,m}(f; t_1, t_2) = \frac{1}{n} \sum_{k=1}^{m} f(e^{i\theta_{k,n}}) \qquad (1 \le m \le n,\ n \in \mathbb{N}).$$

The *arithmetic means* of the function $f(e^{i\theta})$ taken at the standard set of interpolating angles on the arc $[t_1, t_2]$ are

$$s_n(f) = s_n(f; t_1, t_2) = \sigma_{n,n}(f; t_1, t_2)$$
$$s_n^{\sim}(f; t_1, t_2) = \frac{f(e^{i2\pi t_1}) + f(e^{i2\pi t_2})}{2n} + \sigma_{n,n-1}(f; t_1, t_2)$$

for $n \in \mathbb{N}$. Sampling over the circumference of the unit circle at the standard interpolating angles yields the usual arithmetic means

$$s_n(f) = \frac{1}{n} \sum_{k=1}^{n} f(e^{i2\pi k/n}) \qquad (n \in \mathbb{N})$$

with

$$s_\infty(f) = \lim_{n\to\infty} s_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta.$$

The *Riemann coefficients* are defined as

$$r_0(f) = r_0(f;t_1,t_2) = s_\infty(f;t_1,t_2)$$
$$r_n(f) = r_n(f;t_1,t_2) = s_n(f) - s_\infty(f) \qquad (n \in \mathbb{N}).$$

The idea is to imitate the Fourier series expansion of the function $f(e^{i\theta})$ by replacing the Fourier coefficients $\{a_n\}_{n=0}^{\infty}$ and basis $\{e^{in\theta}\}_{n=0}^{\infty}$ with a series whose coefficients are the Riemann coefficients $\{r_n\}_{n=0}^{\infty}$. The appropriate basis for such a Riemann series expansion is the set of polynomials $\{p_n\}_{n=1}^{\infty}$ (see C. H. Ching and C. K. Chui [6: p. 326]). The polynomials are defined as

$$p_0(z) = 1 \qquad \text{and} \qquad p_n(z) = \sum_{s/n} \mu\left(\frac{n}{s}\right) z^s \quad (n \in \mathbb{N})$$

where $\mu = \mu(s)$ is the Möbius function (see C. H. Ching and C. K. Chui [7: p. 269]). According to C. H. Ching and C. K. Chui [6: p. 325], given a function $f \in A(\partial\mathbf{D}) \cap B_\varepsilon(\partial\mathbf{D})$ with Riemann coefficients $\{r_n(f;t_1,t_2)\}_{n=0}^{\infty}$, the operator $R : f \longrightarrow R(f)$ sending a function onto its Riemann series expansion is defined as

$$f(z) = R(f(z)) = \sum_{n=0}^{\infty} r_n(f) p_n(z). \tag{10}$$

This expansion is the unique analytic continuation of the function $f(e^{i\theta})$ to the disk $\mathbf{D}$. The series is continuous on the closure of this disk and its restriction to $\partial\mathbf{D}$ is precisely the function $f(e^{i\theta})$.

The expansion of a function $f \in A(\partial\mathbf{D}) \cap C^2(\partial\mathbf{D})$ in terms of its mean values taken on an arc $[0,\delta]$ is a more detailed construction requiring the shifted means

$$t_n(f;0,\delta) = \frac{f(e^{2\pi i\delta})}{2n} + \sigma_{n,n-1}(\theta_{2k-1,2n-1}[0,\delta]) \qquad (0 \le k \le n,\ n \in \mathbb{N})$$

of the function $f(e^{i\theta})$. Let

$$p_n^{\sim}(e^{i2\pi t}) = \operatorname{Re} p_n(e^{i2\pi t}) \qquad (n \in \mathbb{N}_0).$$

Following C. K. Ching and C. H. Chui [5: p. 182], we define the series

$$\sigma(f(t)) = \sum_{n=1}^{\infty} \left\{ s_n^{\sim}(f;0,\delta) - s_\infty^{\sim}(f;0,\delta) \right\} p_{2n}^{\sim}(e^{i\pi(t/\delta-1)})$$
$$+ \sum_{n=1}^{\infty} \left\{ \frac{2n}{2n-1} t_n(f;0,\delta) - s_\infty^{\sim}(f;0,\delta) \right\} p_{2n-1}^{\sim}(e^{i\pi(t/\delta-1)}) + s_\infty^{\sim}(f;0,\delta).$$

This series converges uniformly to an auxiliary function $\sigma(f(t))$. The analytic function that we seek is constructed from the auxiliary function with the aid of D. J. Patil's generalized Cauchy integral formula [14: p. 617 ff.]. The function $f(z)$ under discussion is determined as the uniform limit

$$f(z) = K(f) = \lim_{\lambda\to\infty} K_\lambda(\sigma(f), z) \tag{11}$$

of the sequence of functions $f_\lambda(z) = K_\lambda(f)$ defined by the operators $K_\lambda : f \longrightarrow K_\lambda(f)$ given by

$$\left.\begin{aligned} K_\lambda(f(t), z) &= \lambda \int_0^\delta \frac{k_\lambda(z)k_\lambda(e^{i2\pi t})}{1 - ze^{-i2\pi t}}\sigma(f(t))\,dt \\ k_\lambda(z) &= \exp\left\{\frac{-\log(1+\lambda)}{4\pi}\int_0^\delta \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\}. \end{aligned}\right. \qquad (\lambda > 0)$$

With this machinery in place, we proceed to the applications.

4. The boundary value problem

Expansion theorems are constructed on the disk and extended to conformally equivalent domains. Certain regularity requirements on the boundary values are essential.

We say that $u \in C^\epsilon(\overline{\mathbf{D}})$ $(\varepsilon > 0)$ whenever for its restriction to the boundary $f = u\big|_{\partial\mathbf{D}}$ we have $f \in A_\epsilon(\overline{\mathbf{D}}) := A(\partial\mathbf{D}) \cap B_\epsilon(\partial\mathbf{D})$. Application of the Riemann operator to a function in this class yields the unique analytic function $R(f)$ whose restriction to the boundary of the disk is the function "f". Operating on the Riemann series expansion $R(f(z))$ with the ascending operator equation (9) generates the unique Riemann series expansion of the solution as required.

Theorem 1: *Let $u \in C^2(\mathbf{D}) \cap C^\epsilon(\overline{\mathbf{D}})$ $(\varepsilon > 0)$ be the solution of the standard boundary value problem with boundary values $f(\mathbf{x})$. Let $\{s_n(f)\}_{n=1}^\infty$ be the mean values formed from sampled boundary values taken at the standard interpolating angles, and let*

$$r_0(f) = s_\infty(f) \qquad \text{and} \qquad r_n(f) = s_n(f) - s_\infty(f) \quad (n \in \mathbb{N})$$

be the Riemann coefficients. Then the solution $u(\mathbf{z})$ is uniquely represented by the generalized Riemann series

$$u(z) = \sum_{n=0}^\infty r_n(f)P_n(z) \qquad (z \in \overline{\mathbf{D}}) \tag{12}$$

where $P_n(z) = (I+G)Bp_n(z)$.

We remark that for functions $u \in \big(C^2(\mathbf{D})\cap C(\overline{\mathbf{D}})\big)\cap\big(A(\partial\mathbf{D})\cap B_\epsilon(\partial\mathbf{D})\big)$ $(\varepsilon > 0)$ there is a Riemann sampling operator $R : u \longrightarrow R(u)$ that maps $u(\mathbf{z})$ onto its generalized Riemann series. The operator is constructed as

$$f = Rf = RB^{-1}(I+G)^{-1}u$$

and, since $u = (I+G)Bf$,

$$u = \big((I+G)B\big)R\big(B^{-1}(I+G)^{-1}\big)u.$$

To summarize the argument, consider the following

Corollary 1.1: *The operator mapping of a solution* $u \in C^2(\mathbf{D}) \cap C^\varepsilon(\overline{\mathbf{D}})$ $(\varepsilon > 0)$ *with boundary data* $f \in C(\partial\mathbf{D}) \cap B_\varepsilon(\partial\mathbf{D})$ *onto its generalized Riemann series expansion is given by* $R = \big((I+G)B\big)R\big((I+G)B\big)^{-1}$.

The next problem is to recover a solution from boundary data sampled on an arc of the boundary. The basic procedures are the same. We consider a function $f \in C^2(\mathbf{D}) \cap A(\partial\mathbf{D})$ and use Patil's kernel function to construct the approximation $f_\lambda(z)$ of $f(z)$ in terms of the kernel $K_\lambda(f(t), z)$ generated from the boundary data. The corresponding approximations of the solutions are $u_\lambda(z) = \big((I+G)B\big)K_\lambda(f(t), z)$ for $\lambda > 0$. Recalling that the integral transforms and series are uniformly convergent, we pass to the limit $\lambda \to \infty$. The end result follows.

Theorem 2: *Let* $u \in C^2(\overline{\mathbf{D}}) \cap A(\partial\mathbf{D})$ *be a solution of the standard boundary value problem with boundary values* $f \in C^2(\partial\mathbf{D}) \cap A(\partial\mathbf{D})$. *Let* $\{s_n^\sim(f; 0, \delta)\}_{n=1}^\infty$ *and* $\{t_n(f; 0, \delta)\}_{n=1}^\infty$ *be the means and shifted means of the boundary values sampled at the standard interpolating angles on the arc* $[0, 2\pi\delta]$ $(0 < \delta < 1)$. *Then the solution is represented by*

$$u(z) = \lim_{\lambda\to\infty} \big((I+G)B\big)K_\lambda(f(t), z) \tag{13}$$

uniformly for $z \in \overline{\mathbf{D}}$. *The operator mapping* $u(z)$ *onto its sampling expansion is precisely*

$$K = \lim_{\lambda\to\infty} \big((I+G)B\big)K_\lambda\big((I+G)B\big)^{-1}.$$

The question now arises of extending the construction to problems defined conformally equivalent domains. Consider an appropriate domain Ω located in the w-plane. Let the conformal map $M : \mathbf{D} \subset \mathbb{C}_z \longrightarrow \Omega \subset \mathbb{C}_w$ be designated by $w = \varphi(z)$ and let the inverse map be $M^{-1} : z = \varphi^{-1}(w)$. We are considering domains Ω for which the maps φ and φ^{-1} extend to the boundaries as C^2 diffeomorphisms. The boundary value problem conformally equivalent to the standard boundary value problem is expressed in the w-plane

$$\begin{aligned} L_w v(w) := \big(\nabla_w^2 + P_w(|w|^2)\big)v(w) &= 0 \qquad (w \in \Omega) \\ v(w_0) &= g(w_0) \qquad (w_0 \in \partial\Omega). \end{aligned} \tag{14}$$

The subscripts "w" are introduced to distinguish the domains in problem (14) and problem (1). The Laplacian's are connected by $\nabla^2 = |\varphi'(z)|^2\nabla_w^2$. The corresponding factor is absorbed into the function P_w. However, it is apparent in the following construction that the partial differential operators need not be connected by such a transformation. Just the fact that the associate has conformal invariance is sufficient and no such "conformal" requirement is in place for the operators.

The solution of problem (14) is developed by transforming the associate $k(w) = (I + G)_w^{-1}v(w) \in C^2(\Omega) \cap C(\overline{\Omega})$ of the solution $v(w)$ onto a harmonic function $h(z)$ on the disk $\mathbf{D}$ in the z-plane, sampling the boundary values, and then reversing the process. The harmonic function $h(z) = k(\varphi(z))$ is determined by the map $M : k \circ \varphi$. This map and its inverse put the harmonic functions $k(w) = h(\varphi^{-1}(w))$ and the boundary value problems into a one-to-one correspondence. The appropriate form of the associate is produced by $f(x,y) = B^{-1}(k \circ \varphi)(z)$ and sampled as a Riemann expansion $R : f \longrightarrow R(f)$ by $f(z) = RB^{-1}(k \circ \varphi)(z)$. The expansion is

$$R(f)(z) = \sum_{n=0}^{\infty} r_n(B^{-1}(k \circ \varphi))p_n(z).$$

We note that properties of the Riemann series are invariant under conformal mappings (see C. H. Ching and C. K. Chui [5: p. 178]). The map is reserved to determine the Riemann series expansion of the "original" associate in the w-plane. This function is $k(w) = R(f)(\varphi^{-1}(w))$. The final step in the construction applies the operator $(I+G)_w$ to generate the Riemann series expansion $v(w) = (I+G)_w k(w)$ of the solutions of equations (14). The Riemann sampling operator may be developed under these criteria along previous lines of argument. The summary follows.

Theorem 3: *Let $v \in C^2(\overline{\Omega}) \cap A(\partial\Omega)$ be the solution of the equivalent boundary value problem under the conformal map $w = \varphi(z) : D \rightarrow \Omega$. Let the map extend to the boundary as a C^2-diffeomorphism. And, let $k(w)$ be the harmonic function associated with $v(z)$ by the operator $(I+G)_w$. Then the generalized Riemann series expansion of the solution is*

$$v(w) = \sum_{n=0}^{\infty} r_n(B^{-1}(k \circ \varphi))P_n(w) \qquad \textit{where} \;\; P_n(w) = (I+G)_w p_n(\varphi^{-1}(w)). \qquad (15)$$

The operator mapping $v(w)$ onto its generalized Riemann series expansion is

$$((I+G)_w M^{-1}B)R((I+G)_w M^{-1}B)^{-1}.$$

5. Transmutations

The method of transmutations is a powerful tool for generating solutions of equivalence classes of boundary value problems. We apply some of the results on transmuations found in H. Begehr and R. P. Gilbert ([1: pp. 229 - 259] and [2: pp. 99 - 100]) in conjunction with the methods from the previous section to solve boundary value problems equivalent to problem (1). Briefly, let $P(D_x)$ and $Q(D_y)$ be partial differential operators acting on suitable spaces X and Y, respectively. The operators need not be of the same order. An operator Σ (which is usually an integral operator) is said to *transmute* $P(D_x)$ into $Q(D_y)$, if it is an isomorphism from X onto Y and satisfies the commutation relation $\Sigma P(D_x) = Q(D_y)\Sigma$. The operator Σ is referred to as a *transmutation.*

In this study, we identify the operator in equations (1) as $P(D_x) = \nabla^2 + P(|x|^2)$. Two equivalent boundary value problems are under consideration, namely

Problem (A) $P(D_x)u(x) = 0 \;\; (x \in D), \;\; u(x_0) = f(x_0) \;\; (x_0 \in \partial D)$

Problem (B) $Q(D_y)v(y) = 0 \;\; (y \in D), \;\; v(y_0) = g(y_0) \;\; (y_0 \in \partial D).$

These problems are *equivalent* if both the partial differential operarors and the boundary values correspond one to the other under the transmutation Σ.

In order to construct a transmutation that maintains the proper correspondence, R. P. Gilbert develops an expansion formula (refer to H. Begehr and R. P. Gilbert [1: pp. 249ff]) that connects the Green functions Γ_P and Γ_Q for problems (A) and (B), respectively. By working with Gilbert's expansion, we find that the relationship can be written as

$$\Gamma_Q = \Lambda_Q \Gamma_P \qquad \text{where } \Lambda_Q = \big(P(D_x)\big)\Sigma^{-1}\big(Q(D_y)\big).$$

This relationship transmutes the Green function from problem (A) into that for problem (B). It has the property that when the solution $u \in C^2(D) \cap C^\varepsilon(\overline{D})$ $(\varepsilon > 0)$, then the transmutation is $v \in C^2(D) \cap C^\varepsilon(\overline{D})$. The operator $\Gamma_P = \Lambda_P \Gamma_Q$ with $\Lambda_P = \Lambda_Q^{-1}$ inverts the transmutation and maintains the regularity conditions between the solutions. Thus, the problems (A) and (B) and their solutions are put into a one-to-one correspondence. Here, the notation $\big(P(D_x)\big)^{-1}$ designates the integral operator solving problem (A) as in equations (9). Analogous notation holds in the second case. Working with the relationships between the Green functions, it is easy to transmute the Riemann series $R_x(u)$ solving problem (A) from the Riemann solution $R_y(v)$ solving problem (B), and conversely.

Theorem 4: *Let $u(x)$ and $v(y)$ be functions in $C^2(D) \cap C^\varepsilon(\overline{D})$ $(\varepsilon > 0)$ that solve the boundary value problems* (A) *and* (B), *respectively, and let the operator Σ transmute problem* (A) *into problem* (B). *Then the Riemann series expansion of the solutions are related by $R_y v(y) = \Lambda_Q R_x u(x)$ and the transmutation of the Riemann series of problem* (B) *from the Riemann series of problem* (A) *is*

$$v(y) = \Big((\Lambda_P Q)^{-1} R_x (\Lambda_P Q)\Big) v(y)$$

where $\Lambda_P = \big(Q(D_y)\big)^{-1}\Sigma\big(P(D_x)\big)^{-1}$.

This problem gives an application of the basic construction procedure in the context of transmutations. The expansion problem for the arc is similar.

6. The inverse problem

Previous sections have dealt with the representation of certain classes of solutions in terms of expansions determined from the arithmetic means of specified boundary data. In this section, we consider the inverse problem which is to reconstruct a solution with suitable properties given a sequence of arithmetic means; rather, than to compute the means from specified boundary data and then expand the function.

There are two distinct differences that must be accounted for. The first difference is that the Riemann coefficients decrease to zero slower than the Fourier coefficients. This property forces a stronger regularity condition on the solution at the boundary. The second difference is that the arithmetic means annihilate the "odd" component of the boundary data. This is compensated for by imposing a second boundary condition involving the tangential derivative along the circumference.

Let us recall that the operators in equations (9) were constructed to retain the regularity conditions of the associates are boundaries. We further observe that the kernels in the operators $(I+G)B$ and $B^{-1}(I+G)^{-1}$ are independent of $\arg z$. Thus, by differentiating at the boundary ∂D in the direction $\hat{\theta}$ at the point $(1,\theta)$ yields

$$\partial_\theta u(1,\theta) = ((I+G)B)\partial_\theta f(1,\theta)$$

$$\partial_\theta f(1,\theta) = ((I+G)B)^{-1}\partial_\theta u(1,\theta).$$

Following C. H. Ching and C. K. Chui [6: pp. 330ff], let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences, $\alpha_n \to \alpha$ and $\beta_n \to \beta$, such that

$$\alpha_n - \alpha = O\left(\frac{1}{n^{3+\varepsilon}}\right) \qquad \text{and} \qquad \beta_n - \beta = O\left(\frac{1}{n^{2+\varepsilon}}\right).$$

Then the series

$$\sum_{n=1}^{\infty}(\alpha_n - \alpha)\, u_n(r,\theta) + \sum_{n=1}^{\infty}\beta_n v_n(r,\theta) + \alpha$$

converges uniformly to a unique harmonic function $w \in C^{2+\varepsilon}(\overline{D})$ for which

$$s_n\left(w\left(1,\frac{2\pi k}{n}\right)\right) = \alpha_n \qquad \text{and} \qquad s_n\left(w_\theta\left(1,\frac{2\pi k}{n}\right)\right) = \beta_n.$$

Here,

$$u_n(r,\theta) = \operatorname{Re} p_n(z) \qquad \text{and} \qquad v_n(r,\theta) = \operatorname{Im} p_n(z) \qquad (n \in \mathbb{N}_0).$$

Expand the boundary function into its even and odd components as

$$f(e^{i\theta}) = \frac{f(e^{i\theta}) + f(e^{-i\theta})}{2} + \frac{f(e^{i\theta}) - f(e^{-i\theta})}{2}$$

written as $f_e(e^{i\theta})$ and $f_o(e^{i\theta})$. Recalling that the arithmetic means of $s_n(f_o) = 0$ $(n \in \mathbb{N})$. Similarly, if the odd part $f_o \in C^{2+\varepsilon}(\partial D)$ $(\varepsilon > 0)$, then $s_n(f_e) = 0$ $(n \in \mathbb{N})$. We introduce a second operator to account for the appearance of the harmonic polynomials

$$v_n(r,\theta) = \operatorname{Im} p_n^*(z) \qquad \text{where } p_n^*(z) = \sum_{k/n} \frac{\mu(\frac{n}{k})}{k} z^k \quad (n \in \mathbb{N}).$$

The operator and its inverse are

$$\begin{aligned} f_o(x,y) = B_*^{-1}[v] &= \frac{B_1[v(z,0)] - \overline{B_1[v(z,0)]}}{2i} \\ v(x,y) = B_*[f_o] &= \frac{B_1[v(z,0)] - \overline{B_1[v(z,0)]}}{2i}. \end{aligned} \tag{16}$$

The reconstruction of the even component of the boundary data is handled with the operator in equations (8). At this point, the reasoning proceeds along the lines of Theorem 1 and C. H. Ching and C. K. Chui [6: pp. 331ff] to establish the solution of the following inverse problem.

Theorem 5: *Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences of real numbers converging to α and β, respectively, with the rates*

$$\alpha_n - \alpha = O\left(\frac{1}{n^{3+\varepsilon}}\right) \qquad \text{and} \qquad \beta_n - \beta = O\left(\frac{1}{n^{2+\varepsilon}}\right) \qquad (\varepsilon > 0).$$

Then there exists a unique solution $u \in C^{2+\varepsilon'}(\overline{D})$ $(\varepsilon' > 0)$ to the equation

$$\left(\nabla^2 + P(|x|^2)\right) u(z) = 0 \qquad (z \in D)$$

with arithmetic means

$$s_n\left(u\left(1, \frac{2\pi k}{n}\right)\right) = \alpha_n \qquad \text{and} \qquad s_n\left(\partial_\theta u\left(1, \frac{2\pi k}{n}\right)\right) = \beta_n \qquad (n \in \mathbb{N}).$$

The solution is represented by the uniformly convergent generalized Riemann series

$$u(z) = \sum_{n=1}^{\infty}(\alpha_n - \alpha)P_n(z) + \sum_{n=1}^{\infty}\beta_n P_n^*(z) + \alpha P_0(z) \qquad (z \in \overline{D})$$

where $P_n(z) = (I+G)Bp_n(z)$ and $P_n^*(z) = (I+G)B^*p_n^*(z)$ for $n \in \mathbb{N}_0$).

7. Remarks

The procedures introduced here extend to axially symmetric problems in $\mathbb{E}^N$ by renormalizing the transform pair equations (7) - (8) as in R. P. Gilbert [8: pp. 27ff] to account for the dimension. The G-functions in the kernel of R. P. Gilbert's operators are independent of the dimension. The only required modification of the transform equations (9) is to insert a power (depending on the dimension) of the integrating variable. The current theory is not extended to non-axially symmetric problems. The reason is that the integral transforms necessary to handle the existing mean value theory for analytic functions of several complex variables are not presently known. And, the transforms appear difficult to construct.

References

[1] Begehr, H. and R. P. Gilbert: *Transformations, Transmutations, and Kernel Functions*, Volume 1 (Pitman Mon. Surveys in Pure Appl. Math.: Vol. 58). New York: Longman Sci. & Techn., John Wiley & Sons 1992.

[2] Begehr, H. and R. P. Gilbert: *Transformations, Transmutations, and Kernel Functions*, Volume 2 (Pitman Mon. Surveys in Pure Appl. Math.: Vol. 58). New York: Longman Sci. & Techn., John Wiley & Sons 1992.

[3] Bergman, S.: *Integral Operators in the Theory of Linear Partial Differential Equations* (Ergebn. Math. und ihrer Grenzgeb.: Volume 23). New York: Springer - Verlag 1969.

[4] Colton, D. L. and R. P. Gilbert: *Singularities of solutions to elliptic partial differential equations.* Quarterly J. Math. 19 (1968), 391 - 396.

[5] Ching, C. H. and C. K. Chui: *Analytic functions characterized by their means on an arc.* Trans. Amer. Math. Soc. 184 (1973), 175 - 183.

[6] Ching, C. H. and C. K. Chui: *Mean boundary value problems and Riemann series.* J. Approx. Theory 10 (1974), 324 - 336.

[7] Ching, C.H. and C. K. Chui: *Uniqueness theorems determined by function values at the roots of unity.* J. Approx. Theory 9 (1973), 267 - 271.

[8] Gilbert, R. P.: *Constructive Methods for Elliptic Equations.* Lect. Notes Math. 365 (1974), 1 - 397.

[9] Kracht, M. and E. O. Kreyszig: *Methods of Complex Analysis in Partial Differential Equations with Applications* (Canadian Math. Soc. Series Monogr. Adv. Texts). New York: John Wiley & Sons 1988.

[10] McCoy, P. A.: *Mean boundary value problems for a class of elliptic equations in* $\mathbb{E}^3$. Proc. Amer. Math. Soc. 76 (1979), 123 - 128.

[11] McCoy, P. A.: *A mean boundary value problem for a generalized axisymmetric potential on doubly connected regions.* J. Math. Anal. Appl. 76 (1980), 213 - 221.

[12] McCoy, P. A.: *Sampling theorems associated with boundary value problems for elliptic partial differential equations in* $\mathbb{R}^N$. Compl. Variables 23 (1993), 269 - 281.

[13] McCoy, P. A.: *Reconstructing solutions of elliptic boundary value problems in* $\mathbb{R}^n$. Submitted.

[14] Patil, D. J.: *Representation of* H^p *functions.* Bull. Amer. Math. Soc. 78 (1972), 617 - 620.

Received 21.07.1994