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Conformally Covariant Operators in Clifford Analysis

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Abstract. Following recent work of Peetre and Qian, the conformal covariance of convolution operators involving the fundamental solutions to iterates of the Dirac operator in Euclidean space is described using Vahlen matrices. This conformal covariance is applied to a number of problems, including Dirichlet problems over unbounded domains.

Keywords: *Conformal group, Dirac operator, intertwining operators, Möbius transformation, left Clifford holomorphic functions*

AMS subject classification: Primary 30 G 35, 31 B 20, 35 G 15, secondary 42 B 99

1. Introduction

The fact that a Möbius transformation over $\mathbb{R}^n$ with $n > 2$ can be expressed as $\varphi(x) = (ax+b)(cx+d)^{-1}$, where $x \in \mathbb{R}^n$ and a,b,c,d belong to a Clifford algebra and satisfy certain constraints, was first pointed out by Vahlen [24]. This approach was pursued by Maass in [17], but for the most part the approach remained forgotten until Ahlfors re-introduced the idea in a number of papers in the 1980's (see, for instance, [1]). Since this re-introduction, a growing number of authors (see [4, 6 - 8, 14, 19 - 21]) have found this approach to be very useful.

In [21], we show that if $D_{\underline{y}}$ is the Dirac operator with respect to the variable $\underline{y} \in \mathbb{R}^n$ and $f\big((a\underline{x}+b)(c\underline{x}+d)^{-1}\big)$ is a Clifford algebra-valued function satisfying the equation

$$D_{\underline{y}}^k f\big((a\underline{x}+b)(c\underline{x}+d)^{-1}\big) = 0$$

where $k \in \{1, 2, \ldots\}$ and $\underline{y} = (a\underline{x}+b)(c\underline{x}+d)^{-1}$, then there is a function $J_k(\varphi, \underline{x})$ such that

$$J_k(\varphi, \underline{x}) f\big((a\underline{x}+b)(c\underline{x}+d)^{-1}\big)$$

is annihilated by $D_{\underline{x}}$, the Dirac operator acting with respect to the variable $\underline{x}$. The result in [21] is established using a local argument involving Almansi decompositions of $f(\underline{y})$. In [4], Bojarski uses an adaptation of this argument to show that if $g(y)$ is a

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Clifford algebra-valued $\mathbb{C}^k$-function defined over a domain in $\mathbb{R}^n$, then there is a function $J_{-k}(\varphi, \underline{x})$ such that

$$J_{-k}(\varphi, \underline{x}) D^k_{\underline{y}} f\big((a\underline{x}+b)(c\underline{x}+d)^{-1}\big) = D^k_{\underline{x}} J_k(\varphi, \underline{x}) f\big((a\underline{x}+b)(c\underline{x}+d)^{-1}\big). \tag{1}$$

Consequently, J_{-k} and J_k are intertwining operators for the Dirac operator. An analogous identity to (1) has previously been established in [13] for the Dirac operator over Minkowski space using the group SU(2,2).

More recently, Peetre and Qian [20] have obtained a more direct and geometric proof of the identity (1). It is this approach that we shall use here.

After setting the stage with some preliminary results in Section 1, we move on in Section 2 and use the method employed in [20] to show that the fundamental solutions to the operators $D^k_{\underline{x}}$ are conformally covariant. Specifically, we show that

$$J_k(\varphi, \underline{x}) G_k\big(\varphi(\underline{x}) - \varphi(\underline{y})\big) * f\big(\varphi(\underline{y})\big) = G_k(\underline{x} - \underline{y}) * J_{-k}(\varphi, \underline{x}) f\big(\varphi(\underline{y})\big) \tag{2}$$

where $k = 1, 2, \ldots$ when n is odd and $k = 1, 2, \ldots, n-1$ when n is even, $G_k(x)$ is the fundamental solution of $D^k_{\underline{x}}$, $\varphi(\underline{x}) = (a\underline{x}+b)(c\underline{x}+d)^{-1}$ and $*$ denotes convolution over $\mathbb{R}^n$.

For the case $k = 1$, this result can immediately be combined with (1) to exhibit the conformal covariance of the Pompeiu representation [12] for $\mathbb{C}^1$-functions on $\mathbb{R}^n$. Until now, the Pompeiu representation has only been described over bounded regions. However, as a Möbius transformation can transform a bounded region into an unbounded region, it follows that we can obtain this formula over unbounded regions. The conformal weight J_{-k} ensures the appropriate decay at infinity.

We then turn our attention to the differential operator $D^k_{\underline{y}} + A(\underline{y})$, where $A(\underline{y})$ is some Clifford algebra-valued potential. We first show that this operator transforms to

$$D^k_{\underline{x}} + \frac{\widetilde{(c\underline{x}+d)} A(\underline{y}) (c\underline{x}+d)}{\|c\underline{x}+d\|^{4+4l}} \qquad \text{when } k = 2l-1$$

and it transforms to

$$D^k_{\underline{x}} + \frac{A(\underline{y})}{\|c\underline{x}+d\|^{2+4l}} \qquad \text{when } k = 2l.$$

During this process we also exhibit a conformal covariance for solutions to the equation

$$\big(D^k_{\underline{y}} + A(\underline{y})\big) f(\underline{y}) = 0.$$

We are also able to show that generalized solutions to this equation are conformally covariant.

Following arguments presented in [10], we use the Cayley transform to exhibit a power series expansion of solutions to the equation $\big(D^k_{\underline{y}} + A(\underline{y})\big) f(\underline{y}) = 0$ on upper half space, under the assumption that $A(\underline{y})$ satisfies certain reasonable constraints.

Using the formulae (1) and (2), we are easily able to establish the identity

$$D_{\underline{y}}^k G_k(\underline{z}-\underline{y}) * h(\underline{z}) = (-1)^k h(\underline{y})$$

over unbounded domains, provided $h(\underline{y})$ satisfies certain smoothness and decay conditions. Consequently, we can construct a solution to the equation $D_y^k g(\underline{y}) = h(\underline{y})$ under these conditions. Solutions to this equation have only previously been constructed using a Runge approximation theorem (see [5]).

The convolution $G_k(\underline{z}-\underline{y}) * h(\underline{z})$ is sometimes called the *T-transform* (see [12]). Having established the conformal covariance of the T-transform, we are, in Section3, in a position to adapt arguments given by Gürlebeck and Sprössig [11] to characterize the space of L^2-integrable functions over an unbounded domain which are orthogonal to the space of L^2-integrable solutions to the Dirac equation over the same domain. This enables one to extend the arguments given in [11], and to study Dirichlet problems over unbounded domains. The study of Dirichlet and related problems over unbounded domains using Clifford analysis has recently been developed by a number of authors (see [3, 15, 16]. However, none of those approaches make use of the conformal covariance of the T-operator.

Much of Section 3 is taken up with establishing nine lemmas. The purpose of these rather technical lemmas is to set the stage for solving the Dirichlet problem over unbounded domains with sufficiently smooth boundaries. These lemmas enable one to carry over results from the bounded setting to the unbounded one. In order to do this, one needs to extend some basic results on the T-operator deduced over bounded domains in [12]. In particular, while in [12] it is assumed that the T-operator acts on bounded $\mathbb{C}^1$-functions, it is necessary here to assume that the underlying function may admit a singularity of certain order at one point. The first four lemmas of Section 3 show that the results on T-operators obtained in [12] do indeed carry over to this context.

Preliminaries. Let A_n be the real 2^n-dimensional Clifford algebra generated from $\mathbb{R}^n$, subject to the anticommutation relationship $e_ie_j + e_je_i = -2\delta_{ij}$ where $1 \le i,j \le n$, $\{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathbb{R}^n$ and δ_{ij} is the Kroneker delta. Consequently, the algebra has as basis elements $1, e, ..., e_n, ..., e_j, ..., e_{j_r}, ..., e_1, ..., e_n$ where $1 \le r \le n$ and $j_1 < ... < j_r$. Basic properties of the algebra A_n are described in [2: Part 1] and elsewhere. One important property of A_n is that if $\underline{x} = x_1e_1 + ... + x_ne_n \in \mathbb{R}^n \setminus \{0\}$, then $\underline{x}$ has a multiplicative. The inverse is $-\underline{x}/\|\underline{x}\|^2$,where $\|\underline{x}\|^2 = x_1^2 + ... + x_n^2$. The vector $\underline{x}/\|\underline{x}\|^2$ is called the *Kelvin inverse* of the vector $\underline{x}$. Another important property of A_n is that it contains the pin group

$$\mathrm{Pin}(n) = \left\{a \in A_n : a = a_1 \cdots a_p, \text{ where } p \in \mathbb{N} \text{ and } a_j \in S^{n-1} \subseteq \mathbb{R}^n \ \ (1 \le j \le p)\right\}$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. For each element $a = a_1 \cdots a_p \in \mathrm{Pin}(n)$ we may denote the element $a_p \cdots a_1 \in \mathrm{Pin}(n)$ by $\tilde{a}$ and note that the action $a\underline{x}\tilde{a}$ describes orthogonal transformation over $\mathbb{R}$. As the action $\underline{a}\,\underline{x}\,\underline{a}$ describes a reflection in the direction of the vector $\underline{a} \in S^{n-1}$ and each orthogonal transformation can be expressed as a finite product of reflections, it follows that there is a surjective group homomorphism

$$\theta : \mathrm{Pin}(n) \longrightarrow 0(n).$$

It may easily be determined that $\{\pm 1\} \subseteq \operatorname{Ker}\theta$. In [2] it is shown that $\operatorname{Ker}\theta = \{\pm 1\}$. So $\operatorname{Pin}(n)$ is a double covering of $0(n)$.

In the discussion on $\operatorname{Pin}(n)$, we introduced the element $\tilde{a}$. More formally we may introduce $\sim$ as the anti-automorphism $\sim: A_n \longrightarrow A_n : e_{j_1} \cdots e_{j_r} \mapsto e_{j_r} \cdots e_{j_1}$. Instead of writing $\sim (A)$, we shall write $\tilde{A}$ for each $A \in A_n$. It may easily be verified that $\tilde{A}B = \tilde{B}A$ for all $A, B \in A_n$.

Definition 1. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in A_n$ and satisfying the conditions

(i) $a = a_1 \cdots a_l, \quad b = b_1 \cdots b_m, \quad c = c_1 \cdots c_p, \quad d = d_1 \cdots d_q$ with $l, m, p, q \in \mathbb{N}$ and $a, b, c, d \in \mathbb{R}^n$ for $1 \leq i \leq l, 1 \leq j \leq m, 1 \leq k \leq p$ and $1 \leq h \leq l$

(ii) $a\tilde{c}, \tilde{c}d, d\tilde{b}, \tilde{b}a \in \mathbb{R}^n$

(iii) $a\tilde{d} - b\tilde{c} = 1$

is called a *Vahlen matrix.*

It is reasonably straightforward to show that the expression

$$(a\underline{x} + b)(c\underline{x} + d)^{-1}$$

is well-defined on $\mathbb{R}^n \cup \{\infty\}$ and describes a Möbius transformation over this set (see, for instance, [1]). It is also the case (see [1]) that any Möbius transformation over $\mathbb{R}^n \cup \{\infty\}$ can be expressed as $(a\underline{x} + b)(c\underline{x} + d)^{-1}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix. This way of describing Möbius transformations over $\mathbb{R}^n \cup \{\infty\}$ was introduced by Vahlen in [24].

We also have

Theorem 1 (see [1]). *Under matrix multiplication the set $V(n)$ of Vahlen matrices over $\mathbb{R}^n$ is a group.*

The group $V(n)$ is a direct analogue of the group $\mathrm{SU}(2,2)$ used to describe Möbius transformations over Minkowski space (see [13]).

We also will need towards the end of Section 3 the anti-automorphism

$$- : A_n \longrightarrow A_n : \quad e_{j_1} \cdots e_{j_r} \mapsto (-1)^r e_{j_r} \cdots e_{j_1}.$$

We denote $-(A)$ by $\bar{A}$ for each $A \in A_n$.

We now turn to some of the basic function theory associated to Dirac operators in $\mathbb{R}^n$. We begin with

Definition 2. Suppose that U is a domain in $\mathbb{R}^n$ and $f : U \longrightarrow A_n$ is a $\mathbb{C}^1$-function which satisfies the equation $\sum_{j=1}^n e_j \frac{\partial f}{\partial x_j}(\underline{x}) = 0$ for each $\underline{x} \in U$. Then f is called a *left-Clifford holomorphic function*, or a *left-monogenic function.* If f satisfies the equation $\sum_{j=1}^n \frac{\partial f}{\partial x_j}(\underline{x}) e_j = 0$, then f is called a *right-Clifford holomorphic function*, or a *right-monogenic function.* The differential operator $\sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$ is called the *Dirac operator*, in Euclidean space, and it is denoted by D.

We shall often write Df for $\sum_{j=1}^{n} e_j \frac{\partial f}{\partial x_j}$ and fD for $\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} e_j$. It is worth noting that $-D^2 = \Delta$, the Laplacian in $\mathbb{R}^n$.

Function-theoretic properties of functions which are either left- or right-Clifford holomorphic have been extensively studied over the years (see, for instance [3, 5, 8 - 12, 14 - 16, 18 - 21]). An example of a function which is both left- and right-Clifford holomorphic is the function $G_1(\underline{x}) = \omega_n^{-1}\underline{x}\|\underline{x}\|^{-n}$ where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

One basic result for the Dirac operator over $\mathbb{R}^n$ is the following generalization of Cauchy's integral formula.

Theorem 2 (see [12]). *Suppose that $f : U \longrightarrow A_n$ is a left-Clifford holomorphic function and M is a bounded Lipschitz domain lying in U. Suppose furthermore that $\underline{x}_0 \in \mathring{M}$. Then*

$$f(\underline{x}_0) = \int_{\partial M} G_1(\underline{x} - \underline{x}_0) n(\underline{x}) f(\underline{x})\, d\sigma(\underline{x})$$

where $n(\underline{x})$ is the outward-pointing unit vector to the surface ∂M at $\underline{x}$ and $d\sigma(\underline{x})$ is the Lebesgue measure on ∂M.

As M is a Lipschitz domain, it follows (see [22]) that $n(\underline{x})$ is defined almost everywhere on ∂M. Similarly, we have that if $f(\underline{x})$ is right-Clifford holomorphic, then

$$f(\underline{x}_0) = \int_{\partial M} f(\underline{x}) n(\underline{x}) G_1(\underline{x} - \underline{x}_0)\, d\sigma(\underline{x}).$$

The proof of Theorem 2 is a very simple adaptation of the proof of the classical Cauchy integral formula.

Now consider the following functions $G_k(x)$ for $k \in \mathbb{N}$:

$$G_k(\underline{x}) = \begin{cases} B_k \dfrac{\underline{x}}{\|\underline{x}\|^{n-2l}} & \text{when } k = 2l+1 \text{ and } l \in \mathbb{N}_0 \\ B_k \dfrac{1}{\|\underline{x}\|^{n-2m}} & \text{when } k = 2m \text{ and } m \in \mathbb{N} \end{cases} \qquad (n \text{ odd})$$

$$\begin{cases} \underline{B}'_k \dfrac{\underline{x}}{\|\underline{x}\|^{n-2l}} & \text{when } k = 2l+1 \text{ and } l = 0, 1, \ldots, \dfrac{n}{2} \\ \underline{B}'_k \dfrac{1}{\|\underline{x}\|^{n-2m}} & \text{when } k = 2m \text{ and } m = 1, \ldots, \dfrac{n}{2} - 1 \end{cases} \qquad (n \text{ even})$$

while

$$G_k(\underline{x}) = \underline{B}'_k(C_k \underline{x}^p - \underline{x}^p ln\|\underline{x}\|) \qquad \text{where } p = k - n \text{ and } k \geq n.$$

Here, $B_1 = \omega_n^{-1}$. Also, $B'_1 = \omega_n^{-1}$, and for $k > 1$ the real numbers B_k, B'_k and C_k are chosen so that $DG_k(\underline{x}) = G_{k-1}(\underline{x})$.

Using Stokes' theorem and a simple homogeneity argument, we have

Theorem 3. *Suppose that $g : U \longrightarrow A_n$ is a $\mathbb{C}^k$-function. Then*

$$g(\underline{x}_0) = \int_{\partial M} \sum_{m=1}^{k} (-1)^{k-m} G_{k-m+1}(\underline{x} - \underline{x}_0) n(\underline{x}) D^{m-1} g(\underline{x})\, d\sigma(\underline{x}) + \int_M G_k(\underline{x} - \underline{x}_0) D^k g(\underline{x})\, d\underline{x}^n \tag{3}$$

for each point $\underline{x}_0 \in \mathring{M}$, where M is a bounded Lipschitz subdomain of U.

Corollary. *Suppose that $g : \mathbb{R}^n \longrightarrow A_n$ is a $\mathbb{C}^k$-function with compact support. Then*

$$g(\underline{x}_0) = \int_{\mathbb{R}^n} G_k(\underline{x} - \underline{x}_0) D^k g(\underline{x})\, d\underline{x}^n. \tag{4}$$

When $k = 1$, formula (3) gives a generalization to $\mathbb{R}^n$ of the classical Pompeiu representation formula for $\mathbb{C}^1$-functions defined in the complex plane.

We also have via an elementary convolution argument

Theorem 4. *Suppose that U is a bounded domain in $\mathbb{R}^n$ and $g : U \longrightarrow A_n$ is a bounded $\mathbb{C}^k$-function. Then*

$$D^k \int_U G_k(\underline{x} - \underline{x}_0) g(\underline{x})\, d\underline{x}^n = g(\underline{x}_0).$$

Corollary. *Suppose that $g : U \longrightarrow A_n$ is a $\mathbb{C}^k$-function with compact support. Then*

$$D^k \int_{\mathbb{R}^n} G_k(\underline{x} - \underline{x}_0) g(\underline{x})\, d\underline{x}^n = g(\underline{x}_0). \tag{5}$$

Let us denote the right A_n-module of A_n-valued $\mathbb{C}^k$-functions with compact support by $\mathcal{F}_k(A_n)$. Also, let us denote the right A_n-module

$$\left\{ h : \mathbb{R}^n \longrightarrow A_n : \quad h = G_k * g \ \text{ for some } \ g \in \mathcal{F}_k(A_n) \right\}$$

by $\mathcal{K}_k(A_n)$. Putting the corollaries to Theorems 3 and 4 together, we have

Theorem 5. *The operator $G_k : \mathcal{F}_k(A_n) \longrightarrow \mathcal{K}_k(A_n) : \quad g \mapsto G_k * g$ is the inverse of the operator $D^k : \mathcal{K}_k(A_n) \longrightarrow \mathcal{F}_k(A_n) : \quad h \mapsto D^k h(\underline{x})$.*

It follows that the operator G_1 corresponds to the operator D^{-1} (see [20]). Consequently, we can also refer to the operator G_k as D^{-k}.

Using the identities (4) and (5), the identity $\underline{x}^{-1} - \underline{y}^{-1} = \underline{x}^{-1}(\underline{y} - \underline{x})\underline{y}^{-1}$ and the fact that the Jacobian of the Möbius transformation $(a\underline{x} + b)(c\underline{x} + d)^{-1}$ is $\|c\underline{x} + d\|^{-2n}$, where $\|c\underline{x} + d\|^2$ is the absolute value of the real number $(c\underline{x} + d)\widetilde{(c\underline{x} + d)}$, Peetre and Qian [20] show that

$$J_{-k}(\varphi, \underline{x}) D^k_{\underline{y}} f\big((a\underline{x} + b)(c\underline{x} + d)^{-1}\big) = D^k_{\underline{x}} J_k(\varphi, \underline{x}) f\big((a\underline{x} + b)(c\underline{x} + d)^{-1}\big) \tag{6}$$

where $\underline{y} = (a\underline{x} + b)(c\underline{x} + d)^{-1}$, $D_{\underline{y}}$ is the Dirac operator acting with respect to the $\underline{y}$ variable, and

$$J_k(\varphi, \underline{x}) = \begin{cases} \dfrac{\widetilde{c\underline{x} + d}}{\|c\underline{x} + d\|^{n-2l}} & \text{for } k = 2l + 1 \text{ with } l \in \mathbb{N}_0 \\ \dfrac{1}{\|c\underline{x} + d\|^{n-2m}} & \text{for } k = 2m \text{ with } m \in \mathbb{N} \end{cases}$$

$$J_{-k}(\varphi, \underline{x}) = \begin{cases} \dfrac{\widetilde{c\underline{x} + d}}{\|c\underline{x} + d\|^{n+2m}} & \text{for } k = 2m - 1 \text{ with } m \in \mathbb{N} \\ \dfrac{1}{\|c\underline{x} + d\|^{n+2m}} & \text{for } k = 2m \text{ with } m \in \mathbb{N}. \end{cases}$$

The identity (6) was first established by Bojarski [4] using different techniques. This identity shows that the operator D^k is intertwined by the operators J_k and J_{-k}.

2. Conformal covariance

We begin with the following

Theorem 6. *Suppose that $(a\underline{x} + b)(c\underline{x} + d)^{-1}$ is a Möbius transformation over $\mathbb{R}^n \cup \{\infty\}$ and $g : \mathbb{R}^n \longrightarrow A_n$ is a $\mathbb{C}^k$-function with compact support. Then for n odd and $k \in \mathbb{N}$, and for n even and $k \in \{1, ..., n-1\}$ we have*

$$J_k(\varphi, \underline{x}) \int_{\mathbb{R}^n} G_k(\varphi(\underline{x}) - \underline{z}) g(\underline{z})\, d\underline{z}^n = \int_{\mathbb{R}^n} G_k(\underline{x} - \underline{y}) J_{-k}(\varphi, \underline{y}) g(\varphi(\underline{y}))\, d\underline{y}^n$$

where $\underline{z} = \varphi(\underline{y}) = (a\underline{y} + b)(c\underline{y} + d)^{-1}$.

Proof. First, we may note that for n even we have that

$$G_k(\underline{x}) = \begin{cases} (-1)^{n/2+l} \underline{x}^{-n/2+l+1} & \text{for } k = 2l + 1 \text{ and } l = 0, \ldots, \frac{n}{2} - 1 \\ (-1)^{n/2+m} \underline{x}^{-n/2+m} & \text{for } k = 2m \text{ and } m = 1, \ldots, \frac{n}{2} - 1. \end{cases}$$

Also, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix with $c \neq 0$, it follows that

$$(a\underline{x} + b)(c\underline{x} + d)^{-1} = ac^{-1} - (c\underline{x}\tilde{c} + d\tilde{c})^{-1},$$

so that

$$\begin{aligned} \varphi(\underline{x}) - \varphi(\underline{y}) &= (c\underline{y}\tilde{c} + d\tilde{c})^{-1} - (c\underline{x}\tilde{c} + d\tilde{c})^{-1} \\ &= \tilde{c}^{-1}(\underline{y} + c^{-1}d)^{-1}c^{-1} - \tilde{c}^{-1}(\underline{x} + c^{-1}d)^{-1}c^{-1} \\ &= \tilde{c}^{-1}(\underline{x} + c^{-1}d)^{-1}(\underline{y} - \underline{x})(\underline{y} + c^{-1}d)^{-1}c^{-1} \\ &= (\underline{x}\tilde{c} + \tilde{d})(\underline{y} - \underline{x})(c\underline{y} + d)^{-1} \\ &= (\widetilde{c\underline{x} + d})^{-1}(\underline{y} - \underline{x})(c\underline{y} + d)^{-1}. \end{aligned}$$

Similarly, we obtain that

$$\varphi(\underline{x}) - \varphi(\underline{y}) = (\widetilde{c\underline{x} + d})^{-1}(\underline{y} - \underline{x})(c\underline{y} + d)^{-1}$$

whenever $c = 0$. It follows that

$$G_j(\varphi(\underline{x}) - \varphi(\underline{y})) = J_k(c\underline{x} + d)^{-1} G_k(\underline{y} - \underline{x})\tilde{J}_k(c\underline{y} + d)^{-1}$$

for n even and $k = 1, \ldots, n-1$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^n} G_k(\varphi(\underline{x}) - \underline{z}) g(\underline{z}) d\underline{z}^n \\
&\quad = J_k(\varphi, \underline{x})^{-1} \int_{\mathbb{R}^n} G_k(\underline{y} - \underline{x}) \tilde{J}_k(\varphi, \underline{y})^{-1} g(\varphi(\underline{y})) \frac{d\underline{y}^n}{\|c\underline{y} + d\|^{2n}} \\
&\quad = J_k(\varphi, \underline{x})^{-1} \int_{\mathbb{R}^n} G_k(\underline{y} - \underline{x}) J_{-k}(\varphi, \underline{y}) g(\varphi(\underline{y})) d\underline{y}^n.
\end{aligned}$$

A similar argument holds for all k when n is odd ∎

As G_k contains a log function when n is even and $k \geq n$, it is clear that the previous argument breaks down in these cases. It is also clear that the proof of the previous theorem does not rely on k being an integer.

Consider the functions

$$H_\alpha(\underline{x}) = \frac{1}{\|\underline{x}\|^\alpha} \qquad \text{and} \qquad I_\alpha(\underline{x}) = \underline{x} H_\alpha(\underline{x})$$

where $\alpha \in \mathbb{R}$. We have via identical arguments to those given in the proof of Theorem 6

Theorem 7. *Whenever the convolutions*

$$\int_{\mathbb{R}^n} H_\alpha(\underline{x} - \underline{y}) \eta(\underline{y})\, d\underline{y}^n \qquad \textit{and} \qquad \int_{\mathbb{R}^n} I_\alpha(\underline{x} - \underline{y}) \psi(\underline{y})\, d\underline{y}^n$$

are well-defined, we have

$$K_\alpha(\varphi, \underline{x}) \int_{\mathbb{R}^n} H_\alpha(\varphi(\underline{x}) - \underline{z}) \eta(\underline{z})\, d\underline{z}^n = \int_{\mathbb{R}^n} H_\alpha(\underline{x} - \underline{y}) K_{-\alpha}(\varphi, \underline{x}) \eta(\varphi(\underline{y}))\, dy^n \tag{7}$$

and

$$L_\alpha(\varphi, \underline{x}) \int_{R^n} I_\alpha(\varphi(\underline{x}) - \underline{z}) \psi(\underline{z})\, d\underline{z}^n = \int_{\mathbb{R}^n} I_\alpha(\underline{x} - \underline{y}) L_{-\alpha}(\varphi, \underline{x}) \psi(\varphi(\underline{y}))\, dy^n \tag{8}$$

where

$$K_\alpha(\varphi, \underline{x}) = \frac{1}{\|c\underline{x} + d\|^\alpha} \qquad K_{-\alpha}(\varphi, \underline{x}) = \frac{1}{\|c\underline{x} + d\|^{2n-\alpha}}$$

and

$$L_\alpha(\varphi, \underline{x}) = \frac{c\underline{x} + d}{\|c\underline{x} + d\|^\alpha} \qquad L_{-\alpha}(\varphi, \underline{x}) = \frac{c\underline{x} + d}{\|c\underline{x} + d\|^{2n-\alpha}}$$

and

$$\underline{z} = \varphi(\underline{y}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}.$$

The operators H_α and I_α are examples of conformally covariant operators, as they satisfy the identities (7) and (8). When $\alpha = n+1$, we get that $I_\alpha(\underline{x}) = \underline{x}/\|\underline{x}\|^{n+1}$. In this case, $I_\alpha(\underline{x}) = \sum_{j=1}^n R_j(\underline{x})e_j$, where $R_j(\underline{x})$ is the jth Riesz potential in $\mathbb{R}^n$ (see [23]). The conformal covariance of $\sum_{j=1}^n R_j(\underline{x})e_j$ has previously been observed in [8].

Let $A_n(\mathbb{C})$ denote the complexification of A_n, and let $L^2(A_n(\mathbb{C}), \mathbb{R}^n)$ denote the right $A_n(\mathbb{C})$-module of $A_n(\mathbb{C})$-valued functions which are L^2-bounded over $\mathbb{R}^n$. As the Fourier transform of I_{n+1} is $i\underline{\xi}/\|\underline{\xi}\|$, where $\underline{\xi} \in \mathbb{R}^n$ (see [23]), it follows from Plancherel's theorem and arguments given in [8] that we have

Proposition 1. *The operator $\underline{\sigma}I + \delta I_{n+1} : L^2(A_n(\mathbb{C}), \mathbb{R}^n) \longrightarrow L^2(A_n(\mathbb{C}), \mathbb{R}^n)$ is a conformally covariant operator, where $\sigma, \delta \in \mathbb{C}$ and I is the identity map.*

The cases $\sigma = \pm\frac{1}{2}$ and $\delta = 1$ have previously been described in [8]. We may also observe that Theorem 7 remains valid if we assume that α is a complex number.

We now turn to establish a conformal covariance for the formula (4) for the case $k = 1$. We begin by noting that by similar arguments to ones given in [20], and elsewhere, we obtain

Proposition 2. *Suppose that S is a smooth, orientable surface in $\mathbb{R}^n$, and $f, g : S \longrightarrow A_n(\mathbb{C})$ are integrable functions. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix, then*

$$\int_S f(\underline{y})n(\underline{y})g(\underline{y})\,d\sigma(\underline{y}) = \int_{\varphi^{-1}(S)} f(\varphi(\underline{x}))\tilde{J}_1(\varphi, \underline{x})n(\underline{x})J_1(\varphi, \underline{x})g(\varphi(\underline{x}))\,d\sigma(\underline{x})$$

where $\underline{y} = \varphi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ and $\varphi^{-1}(S) = \{\underline{x} \in \mathbb{R}^n : \varphi(\underline{x}) \in S\}$.

From Proposition 2, formula (3) and Theorem 6 we immediately obtain

Theorem 8. *Suppose that $g : U \longrightarrow A_n(\mathbb{C})$ is a $\mathbb{C}^1$-function and M is a bounded region in U with Lipschitz continuous boundary. Then for any Vahlen matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and for any point $\underline{y} = \varphi(\underline{x}_0) = (a\underline{x}_0 + b)(c\underline{x}_0 + d)^{-1}$ in $\mathring{M}$*

$$\begin{aligned}
&\int_{\partial M} G_1(\underline{y} - \underline{y}_0)n(\underline{y})g(\underline{y})\,d\sigma(\underline{x}) - \int_M G(\underline{y} - \underline{y}_0)D_{\varphi(\underline{y})}g(\underline{y})\,d\underline{y}^n \\
&\quad = J_1(\varphi, \underline{x})^{-1}\left(\int_{\partial(\varphi^{-1}M)} G_1(\underline{x} - \underline{x}_0)n(\underline{x})J_1(\varphi, \underline{x})g(\varphi(\underline{x}))\,d\sigma(\underline{x})\right. \\
&\qquad \left. - \int_{\varphi^{-1}(M)} G_1(\underline{x} - \underline{x}_0)D_{\underline{x}}J_1(\varphi, \underline{x})g(\varphi(\underline{x}))\,d\underline{x}^n\right)
\end{aligned}$$

where $\underline{y} = \varphi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$.

Theorem 8 shows that the Pompeiu representation formula from Clifford analysis is conformally covariant.

One crucial observation is that though we have assumed that $\mathring{M}$ is a bounded domain, we do not need to assume that $\varphi^{-1}(\mathring{M})$ is also bounded. For instance, we could take M to be the unit disk $\mathrm{cl}B(0,1) = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| \leq 1\}$ in $\mathbb{R}^n$ and $\varphi(\underline{x})$ to be the Cayley transform so that

$$\varphi^{-1}\big(\mathrm{cl}B(0,1)\big) = H^n = \big\{\underline{y} \in \mathbb{R}^n : y_1 \geq 0\big\},$$

where cl denotes topological closure.

We shall now take a closer look at the formula (3) when $k = 2$. In this case, we get

$$\begin{aligned} g(\underline{y}_0) = &\int_{\partial M} \Big(G_1(\underline{y} - \underline{y}_0)n(\underline{y})g(\underline{y}) - G_2(\underline{y} - \underline{y}_0)n(\underline{y})D_{\varphi(\underline{x})}g(\underline{y})\Big)d\sigma(\underline{x}) \\ &- \int_M G_2(\underline{y} - \underline{y}_0)D^2_{\varphi(\underline{y})}g(\underline{y})\,dy^n. \end{aligned}$$

Now,

$$\begin{aligned} &\int_M G_2(\underline{y} - \underline{y}_0)D^2_{\varphi(\underline{y})}g(\underline{y})\,d\underline{y}^n \\ &\quad = J_2(\varphi, \underline{x}_0)^{-1} \int_{\varphi^{-1}(M)} G_2(\underline{x} - \underline{x}_0)D^2_{\underline{x}}J_2(\varphi, \underline{x})g\big(\varphi(\underline{x})\big)\,d\underline{x}^n. \end{aligned} \tag{9}$$

Upon realizing that $G_1\big(\varphi(\underline{x}) - \varphi(\underline{x}_0)\big) = G_2\big(\varphi(\underline{x}) - \varphi(\underline{x}_0)\big)D_{\underline{y}}$, we may observe that

$$\begin{aligned} &\int_{\partial M} \Big(G_1(\underline{y} - \underline{y}_0)n(\underline{y})g(\underline{y}) - G_2(\underline{y} - \underline{y}_0)n(\underline{y})D_{\varphi(\underline{y})}g(\underline{y})\Big)d\sigma(\underline{y}) \\ &\quad = J_2(\varphi, \underline{x}_0)^{-1}\Bigg(\int_{\partial\varphi^{-1}(M)} \Bigg(G_2(\underline{x} - \underline{x}_0)\frac{(c\underline{x} + d)}{\|c\underline{x} + d\|^2}D_x\Bigg)n(x)\frac{(cx + d)}{\|cx + d\|^{n-2}}g\big(\varphi(x)\big) \\ &\qquad - G_2(\underline{x} - \underline{x}_0)(\varphi, \underline{x})n(\underline{x})D_{\underline{x}}J_1(\varphi, \underline{x})g\big(\varphi(\underline{x})\big)\Bigg)d\sigma(\underline{x}). \end{aligned}$$

It follows from (9) that when $k = 2$, then the right-hand side of (3) is equal to

$$g\big(\varphi(\underline{x}_0)\big) + J_2(\varphi, \underline{x}_0)^{-1}\int_{\varphi^{-1}(M)} G_2(\underline{x} - \underline{x}_0)D^2_{\underline{x}}J_2(\varphi, \underline{x})g\big(\varphi(\underline{x})\big)\,d\underline{x}^n.$$

When $\varphi^{-1}(M)$ is a bounded set, then

$$\begin{aligned} J_2(\varphi, \underline{x}_0)g\big((\varphi(x_0)\big) + &\int_{\varphi^{-1}(M)} G_2(\underline{x} - \underline{x}_0)D^2_{\underline{x}}J_2(\varphi, \underline{x})g\big(\varphi(\underline{x})\big)\,d\underline{x}^n \\ &= \int_{\partial\varphi^{-1}(M)} \Big(G_1(\underline{x} - \underline{x}_0)n(\underline{x})J_2(\varphi, \underline{x})g\big(\varphi(\underline{x})\big) \\ &\qquad - G_2(\underline{x} - \underline{x}_0)n(\underline{x})D_{\underline{x}}J_2(\varphi, \underline{x})g\big(\varphi(\underline{x})\big)d\sigma(\underline{x})\Big). \end{aligned}$$

Consequently,

$$\int_{\partial\varphi^{-1}(M)} \Big(G_1(\underline{x}-\underline{x}_0)n(\underline{x})J_2(\varphi,\underline{x})g\big(\varphi(\underline{x})\big)$$

$$- G_2(\underline{x}-\underline{x}_0)n(\underline{x})D_{\underline{x}}J_2(\varphi,\underline{x})g\big(\varphi(\underline{x})\big)\Big)\,d\sigma(\underline{x})$$

$$= \int_{\partial\varphi^{-1}(M)} \left(\left(G_2(\underline{x}-\underline{x}_0)\frac{(c\underline{x}+d)}{\|c\underline{x}+d\|^2}D_{\underline{x}}\right) n(\underline{x})\frac{(c\underline{x}+d)}{\|c\underline{x}+d\|^{n-1}}g\big(\varphi(\underline{x})\big) \right.$$

$$\left. - G_2(\underline{x}-\underline{x}_0)(\varphi,\underline{x})n(\underline{x})D_{\underline{x}}J_{-1}(\varphi,\underline{x})g\big(\varphi(\underline{x})\big)\right) d\sigma(\underline{x}).$$

It follows that when $k=2$ and $\varphi^{-1}(\mathring{M})$ is a bounded domain, then the integral formula (3) is conformally covariant.

Suppose now that $\varphi^{-1}(\mathring{M})$ is no longer a bounded domain. Then we may consider a domain sequence $\{M\}_{l=1}^{\infty}$ such that

(i) $\mathring{M}_l$ is bounded

(ii) $M_l \subseteq \mathring{M}_{l+1}$

(iii) $M_l \uparrow M$.

By applying the previous calculation to each of the bounded domains $\mathring{M}_l$, it follows that the identity (3) remains valid even if $\varphi^{-1}(M)$ is not bounded. Consequently, we have deduced that when $k=2$, then the integral formula (3) is conformally covariant. Similar arguments may be used to show that the integral formula (3) is conformally covariant for any positive integer k.

We now turn to look at the differential equation

$$D_{\underline{y}}^k f(\underline{y}) + A(\underline{y})f(\underline{y}) = 0 \tag{10}$$

where $A(\underline{y})$ is some $A_n(\mathbb{C})$-valued function defined on a domain U. As

$$D_{\underline{y}}^k = J_{-k}(c\underline{x}+d)^{-1}D_{\underline{x}}^k J_k(c\underline{x}+d),$$

we have

Proposition 3. *Suppose that $f(\underline{y})$ is a solution to the differential equation*

$$D_{\underline{y}}^k f(\underline{y}) + A(\underline{y})f(\underline{y}) = 0$$

on the domain U, where $A(\underline{y})$ is a $\mathbb{C}^k$-function with values in $A_n(\mathbb{C})$. Then

$$J_k(c\underline{x}+d)f\big(\varphi(\underline{x})\big)\,.$$

is annihilated by the operator

$$D_{\underline{x}}^k + (\widetilde{c\underline{x}+d})\frac{A(\varphi(\underline{x}))}{\|c\underline{x}+d\|^{2k}}(c\underline{x}+d)$$

when k is odd, and it is annihilated by

$$D_{\underline{x}}^k + \frac{A(\varphi(\underline{x}))}{\|c\underline{x}+d\|^{2k}}$$

when k is even.

We also have

Proposition 4. *Suppose that $f : U \longrightarrow A_n(\mathbb{C})$ is a generalized solution to the equation $\big(D_{\underline{y}}^k + A(\varphi(\underline{x}))\big)f(\varphi(\underline{x})) = 0$. Then $J_k(c\underline{x}+d)f(\varphi(\underline{x}))$ is a generalized solution to the equation*

$$\left(D_{\underline{x}}^k + \frac{(\widetilde{c\underline{x}+d})A(\varphi(\underline{x}))(c\underline{x}+d)}{\|c\underline{x}+d\|^{2k}}\right)g(\underline{x}) = 0 \tag{11}$$

when k is odd, and it is a generalized solution to the equation

$$\left(D_{\underline{x}}^k + \frac{A(\varphi(\underline{x}))}{\|c\underline{x}+d\|^{2k}}\right)g(\underline{x}) = 0$$

when k is even.

Proof. Suppose that $\psi : U \longrightarrow A_n(\mathbb{C})$ is a $\mathbb{C}^\infty$-function with compact support, and such that the support of $\psi(\varphi(\underline{x}))$ on $\varphi^{-1}(U)$ is also compact. Suppose also that $f : U \longrightarrow A_n(\mathbb{C})$ is a generalized solution to the equation $D_{\underline{y}}^k f(\varphi(\underline{x})) + A(\varphi(\underline{x}))f(\varphi(\underline{x})) = 0$, and k is odd. Then we have

$$\int_U \big(\psi(\underline{y})D_{\underline{y}}^k\big)f(\underline{y})\,d(\underline{y})^n = \int_U \psi(\underline{y})A(\underline{y})f(\underline{y})\,d\underline{y}^n. \tag{12}$$

Now

$$\int_U \big(\psi(\underline{y})D_{\underline{y}}^k\big)f(\underline{y})\,d\underline{y}^n = \int_{\varphi^{-1}(U)} \Big(\psi(\varphi(\underline{x}))\,\tilde{J}_k(\varphi,\underline{x})D_{\underline{x}}^k\Big)\frac{J_{-k}(\varphi,\underline{x})}{\|c\underline{x}+d\|^{2n}}f(\varphi(\underline{x}))\,dx^n.$$

The right-hand side of the previous expression simplifies to

$$\int_{\varphi^{-1}(U)} \Big(\psi(\varphi(\underline{x}))\,\tilde{J}_k(\varphi,\underline{x})D_{\underline{x}}^k\Big)J_k(\varphi,\underline{x})f(\varphi(\underline{x}))\,dx^n.$$

The right-hand side of (12) can now be re-expressed as

$$-\int_{\varphi^{-1}(U)} \psi(\varphi(\underline{x}))\,\tilde{J}_k(\varphi,\underline{x})\tilde{J}_k(\varphi,\underline{x})^{-1}A(\varphi(\underline{x}))\,J_k(\varphi,\underline{x})^{-1}J_k(\varphi,\underline{x})\frac{f(\varphi(\underline{x}))}{\|c\underline{x}+d\|^{2n}}\,d\underline{x}^n$$

which simplifies to

$$-\int_{\varphi^{-1}(U)} \psi(\varphi(\underline{x}))\,\tilde{J}_k(\varphi,\underline{x})(\widetilde{c\underline{x}+d})\frac{A(\varphi(\underline{x}))}{\|c\underline{x}+d\|^{2k}}(\varphi,\underline{x})J_k(\varphi,\underline{x})f(\varphi(\underline{x}))\,d\underline{x}^n.$$

Consequently, $J_k(\varphi,\underline{x})f(\varphi(\underline{x}))$ is a generalized solution to the equation (11). A similar argument holds when k is even ∎

We now turn to look at solutions to the equation (10) on some bounded domain U. First, we consider the case $k = 1$, and with $A(\underline{y})$ a $\mathbb{C}^1$-function on a domain U_1, with $\mathrm{cl}U \subseteq U_1$. Suppose now that $g(\underline{y})$ is a $\mathbb{C}^1$-solution to the equation $D_{\underline{y}}g(\underline{y}) + A(\underline{y})g(\underline{y}) = 0$ on U_2, where $\mathrm{cl}U \subseteq U_2$ and $\mathrm{cl}U_2 \subseteq U_1$. Then, from Stokes' theorem we have that for each $\underline{y}_0 \in B(0,1)$

$$g(\underline{y}_0) - \int_{U_2} G_1(\underline{y} - \underline{y}_0)A(\underline{y})g(\underline{y})\, d\underline{y}^n = \int_{\partial \mathrm{cl} U_2} G_1(\underline{y} - \underline{y}_0)n(\underline{y})g(\underline{y})\, d\sigma(\underline{y}).$$

The term appearing on the right-hand side of the previous expression is a left-monogenic function.

Following [9: Chapter 4] and [10], we can go further than this. We shall first denote the right $A_n(\mathbb{C})$-module of bounded left-monogenic functions on U by $B_\infty(U, A_n(\mathbb{C}))$. This module is a generalization of the Bergman space of bounded analytic functions on a domain in the complex plane. We may now deduce

Theorem 9. *Suppose that $f(\underline{y}) \in B_\infty(U, A_n(\mathbb{C}))$ and $A(\underline{y})$ is L_p-integrable over $B(0,1)$, with $p > n$. Moreover, let $\sup_{\underline{y}\in U} \|\underline{y} - \underline{y}_0\|_q \|A(\underline{y})\|_p . < 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then the series*

$$\sum_{k=0}^{\infty} G_A *^k f \tag{13}$$

defines a bounded integrable function on U, where $G_A(\underline{y} - \underline{y}_0) = G_1(\underline{y} - \underline{y}_0)A(y)$, and $G_A *^k f$ denotes the k-fold convolutions of G_A over U acting on f. Moreover, the function (13) is a generalized solution to the equation $Dg(\underline{y}) + A(\underline{y})g(\underline{y}) = 0$.

Proof. That the series (13) defines a bounded function follows immediately from Hölder's inequality and the fact that f is a bounded left-monogenic function on U. On putting

$$g(\underline{y}) = \sum_{k=0}^{\infty} (G_A *^k f)(\underline{y})$$

we have from Hölder's inequality

$$\int_U |g(\underline{y})|\, d\underline{y}^n \leq C(U) \sum_{k=0}^{\infty} \sup_{\underline{y}_0 \in U} \|G_1(\underline{y} - \underline{y}_0)\|_q \|A(\underline{y})\|_p \|f\|_\infty < \infty \tag{14}$$

where $C(U)$ is the volume of U. Consequently, (13) defines a bounded, integrable function.

Now suppose that $\psi : U \longrightarrow A_n(\mathbb{C})$ is a $\mathbb{C}^\infty$-function with compact support. As $g(\underline{y})$ is a bounded integrable function, the integral

$$\int_U (\psi(\underline{y})D_{\underline{y}}) \sum_{k=0}^{\infty} G_A *^k f(\underline{y})\, dy^n \tag{15}$$

is well-defined. Moreover, via the inequality (14), we have that the expression (15) is equal to

$$\sum_{k=0}^{\infty} \int_U (\psi(\underline{y}) D_{\underline{y}}) G_A *^k f(\underline{y})\, dy^n.$$

Via Fubini's theorem, we may observe that

$$\int_U (\psi(\underline{y}) D_{\underline{y}}) G_A *^k f(\underline{y})\, d_y^n = \int_U \int_U (\psi(\underline{y}) D_{\underline{y}}) G_1(\underline{y} - \underline{y}') A(\underline{y}') G_A *^{k-1} f(\underline{y}')\, dy^n d\underline{y}'.$$

As

$$\int_U (\psi(\underline{y}) D_{\underline{y}}) G_1(\underline{y} - \underline{y}')\, d\underline{y}^n = \psi(\underline{y}'),$$

then

$$\int_U (\psi(\underline{y}) D_{\underline{y}}) \sum_{k=0}^{\infty} G_A *^k f(\underline{y})\, d\underline{y}^n = \int_U \psi(\underline{y}) A(\underline{y}) \sum_{k=0}^{\infty} G_A *^k f(\underline{y})\, d\underline{y}^n.$$

Via a Möbius transform $\psi(\underline{x}) = \underline{y}$, it follows that if $F(\underline{x})$ is a bounded left-monogenic function on the domain $\varphi^{-1}(U)$, and $A(\underline{y})$ is an L^p-integrable function on U, with $p \geq n$, and $\sup_{\underline{y}_0 \in U} \|G_1(\underline{y} - \underline{y}_0)\|_q \|A(\underline{y})\|_p < 1$. Then $\sum_{k=0}^{\infty} G_B *^k F(\underline{x})$ is a generalized solution to the equation $(D_{\underline{x}} + B(\underline{x}))g(\underline{x}) = 0$ on $\varphi^{-1}(U)$ where

$$B(\underline{x}) = \frac{\widetilde{(c\underline{x} + d)} A(\varphi(\underline{x}))(c\underline{x} + d)}{\|c\underline{x} + d\|^2}.$$

Of particular importance here is the case where $\varphi^{-1}(U)$ is an unbounded domain. In greater generality we have

Theorem 10. *Suppose that $f(\underline{y})$ is a bounded solution to the equation $D^l g(\underline{y}) = 0$ on the bounded domain U, where $l = 1, \ldots, n-1$. Suppose also that $A(\underline{y})$ is an L^p-integrable function with $p > \frac{n}{l}$, and*

$$\sup_{\underline{y}_0 \in U} \|G_1(\underline{y} - \underline{y}_0)\|_q \|A(\underline{y})\|_p < 1$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{k=0}^{\infty} G_{l,B} *^k J_k(c\underline{x} + d) f(\varphi(\underline{x}))$$

is a generalized solution to the equation

$$(D_x^l + B_l(\underline{x}))g(\underline{x}) = 0$$

on $\varphi^{-1}(U)$, where

$$B_l(\underline{x}) = \begin{cases} \dfrac{\widetilde{(c\underline{x} + d)} A(\varphi(\underline{x}))(c\underline{x} + d)}{\|c\underline{x} + d\|^{xl}} & \text{when } l \text{ is odd} \\ \dfrac{A(\varphi(\underline{x}))}{\|c\underline{x} + d\|^{xl}} & \text{when } l \text{ is even} \end{cases}$$

and $G_{l,B}(\underline{x}-\underline{x}_0) = G_l(\underline{x}-\underline{x}_0)B_l(\underline{x})$.

Again, it may be observed that although U is a bounded domain, $\varphi^{-1}(U)$ need not be bounded. It may also be observed that if we simply assume that

$$\sup_{y_0\in U}\left\|\int_u G_g(\underline{y}-\underline{y}_0)A(\underline{y})\,d\underline{y}^n\right\|_1 < 1,$$

then the series $\sum_{l=0}^{\infty} G_{k,M} *^l f(\underline{y})$ gives a generalized solution to equation (10) on U for any bounded function satisfying $D^k f(\underline{y}) = 0$ on U. In this case, we do not need to assume that U is bounded, and we can allow k to be an arbitrary positive integer.

Let us now assume that $U = B(0,1)$, and that the series $\sum_{l=0}^{\infty} P_l(\underline{y})$ converges uniformly on $B(0,1)$ to the bounded function $f(\underline{y})$, where $D^k f(\underline{y}) = 0$ and $D^k P_l(\underline{y}) = 0$, and each $P_l(\underline{y})$ is a polynomial homogeneous of degree l. Letting $\varphi(x) = (e_1\underline{x}+1)(-\underline{x}-e_1)^{-1}$, the Cayley transform, we have

Proposition 5. *If* $k = 1,\ldots,n-1$ *and*

$$\sup_{\underline{y}_0\in B(0,1)} \|G_k(\underline{y}-\underline{y}_0)\|_q \|A(\underline{y})\|_p < 1$$

with $\frac{1}{p}+\frac{1}{q} = 1$ *and* $p > \frac{n}{k}$, *then the series*

$$Q_l(\underline{x}) = \sum_{m=0}^{\infty} G_{k,B} *^l J_k(\varphi,\underline{x})P_l(\varphi(\underline{x}))$$

is a generalized solution to the equation $\big(D^k + B(\underline{x})\big)h(\underline{x}) = 0$ *on the upper half-space* H^n, *where*

$$B(\underline{x}) = \begin{cases} \widetilde{(e_1\underline{x}+1)}A(\varphi(\underline{x}))(e_1\underline{x}+1)\|\underline{x}-e_1\|^{-2k} & \textit{when } k \textit{ is odd} \\ \dfrac{A(\varphi(\underline{x}))}{\|\underline{x}-e_1\|^{2k}} & \textit{when } k \textit{ is even.} \end{cases}$$

Moreover, the series $\sum_{l=0}^{\infty} Q_l(\underline{x})$ *converges uniformly on* H^n *to the function*

$$\sum_{m=0}^{\infty} G_{k,B} *^l J_k(\varphi,\underline{x})f(\varphi(\underline{x})).$$

3. The T-operator on unbounded domains

In [12] and elsewhere, the T-operator over a bounded domain U is defined to be the convolution

$$\int_U G_1(\underline{y}-\underline{y}_0)\psi(\underline{y})\,d\underline{y}^n$$

where $\underline{y}_0 \in U$ and $\psi(\underline{y})$ belongs to some suitable function space over U. If $\psi(\underline{y})$ is assumed to be a bounded C^1-function on U, then Theorem 6 tells us that

$$J_1(\varphi,\underline{x}_0)^{-1}\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}_0)J_{-1}(\varphi,\underline{x})\psi\big(\varphi(\underline{x})\big)\,d\underline{x}^n = \int_U G_1(\underline{y}-\underline{y}_0)\psi(\underline{y})\,d\underline{y}^n$$

where $\varphi(\underline{x})=\underline{y}$ and $\varphi(\underline{x}_0)=\underline{y}_0$. As $D_{\underline{y}} = J_{-1}(\varphi,\underline{x})^{-1}D_x J_1(\varphi,x)$, then

$$\begin{aligned} &J_{-1}(\varphi,\underline{x}_0)^{-1}D_{\underline{x}}\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}_0)J_{-1}(\varphi,\underline{x})\psi\big(\varphi(\underline{x})\big)\,d\underline{x}^n \\ &\qquad = D_{\underline{y}_0}\int_U G_1(\underline{y}-\underline{y}_0)\psi(\underline{y})\,d\underline{y}^n. \end{aligned}$$

From Theorem 4 we now have that

$$D_{\underline{x}_0}\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}_0)J_{-1}(\varphi,\underline{x})\psi\big(\varphi(\underline{x})\big)\,d\underline{x}^n = J_{-1}(\varphi,\underline{x}_0)\psi\big(\varphi(\underline{x}_0)\big).$$

In greater generality we obtain, by similar arguments, the following

Theorem 11. *Suppose that $\psi(\underline{y})$ is a bounded C^k-function on the bounded domain U. Then*

$$D^k_{\underline{x}_0}\int_{\varphi^{-1}(U)} G_k(\underline{x}-\underline{x}_0)J_{-k}(\varphi,\underline{x})\psi\big(\varphi(\underline{x}_0)\big)\,d\underline{x}^n = J_{-k}(\varphi,\underline{x}_0)\psi\big(\varphi(\underline{x}_0)\big)$$

when $k=1,\dots,n-1$ for n even and $k=1,\dots$ for n odd.

Theorem 11 tells us that the function

$$\int_{\varphi^{-1}(U)} G_k(\underline{x}-\underline{x}_0)J_{-k}(\varphi,\underline{x})\psi\big(\varphi(\underline{x})\big)\,d\underline{x}^n$$

is a solution to the equation $D^k f(\underline{x}) = J_{-k}(\varphi,\underline{x})\psi\big(\varphi(\underline{x})\big)$ over the unbounded domain $\varphi^{-1}(U)$, where $\psi\big(\varphi(\underline{x})\big)$ is a bounded C^k-function on U. Previously, one has needed to use an approximation theorem in order to solve the equation $D^k f = g$ on an unbounded domain (see [5: p. 161]).

Suppose now that $\psi(\underline{y}) \in L^p\big(U, A_n(\mathbb{C})\big)$ with $p>n$. Then, as has been observed in the previous section, it follows from Hölder's inequality that

$$\int_U G_1(\underline{y}-\underline{y}_0)\psi(\underline{y})\,d\underline{y}^n$$

is a bounded, measurable function on U. Consequently, for any $\mathbb{C}^\infty$-function $\eta : U \longrightarrow A_n(\mathbb{C})$ with compact support, we have that

$$(\eta(\underline{y}_0)D\underline{y}_0)\int_U G_1(\underline{y}-\underline{y}_0)\psi(\underline{y})\,d\underline{y}^n$$

is an L^1-function on U. Using Fubini's theorem and changing variables, we now see that

$$\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}_0)J_{-1}(\varphi,\underline{x})\psi(\varphi(\underline{x}))\,d\underline{x}^n$$

is a generalized solution to the equation

$$D_{\underline{x}}f(\underline{x}) = J_{-1}(\varphi,\underline{x})\psi(\varphi(\underline{x}))$$

on the unbounded domain $\varphi^{-1}(U)$.

Definition 3. Suppose that V is a domain in $\mathbb{R}^n$ and $m : V \longrightarrow \mathbb{R}\cup\{\infty\}$ is a non-negative measurable function. Then we shall denote the *right $A_n(\mathbb{C})$-module*

$$\left\{f : V \longrightarrow A_n(\mathbb{C}) : \quad \int_V |f(\underline{x})|^p m(\underline{x})\,d\underline{x}^n < \infty\right\}$$

by $L^p(V, m(\underline{x}))$.

The module $L^p(V, m(\underline{x}))$ is an example of a weighted L^p-space. We now have by similar arguments to those used to establish Theorem 10 the following

Theorem 12. *Let $k = 1,\ldots,n-1, p > \frac{n}{k}$ and*

$$I(x) \in L^p\Big(\varphi^{-1}(U), \|c\underline{x}+d\|^{-2n+(n+k)p}\Big).$$

Then

$$\int_{\varphi^{-1}(U)} G_k(\underline{x}-\underline{x}_0)I(\underline{x})\,d\underline{x}^n$$

is a generalized solution to the equation $D^k f(\underline{x}) = I(\underline{x})$ on $\varphi^{-1}(U)$.

We now turn to show how a number of results obtained over a bounded domain U, using the T-operator (in [11]), have analogues over the unbounded domain $\varphi^{-1}(U)$. We shall begin by solving the equation $\Delta u = k$ on $\varphi^{-1}(U)$, where $k(\underline{x})$ is a bounded C^1-function and

$$\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|k(\underline{x})\| < \infty. \tag{16}$$

As U is bounded and $\varphi^{-1}(U)$ is unbounded, then there is a point $\underline{y}_0 \in \overline{U}$ such that $\varphi^{-1}(\underline{y}_0) = \infty$.

Now consider the function $h(\underline{x}) = J_{-1}(c\underline{x}+d)^{-1}k(\underline{x})$. It follows from (16) that $\|\underline{y}-\underline{y}_0\|^2\|h(\varphi^{-1}(\underline{y}))\|$ is a bounded function on $U\setminus\{\underline{y}_0\}$. We now look at the T-transform on U of $h(\varphi^{-1}(\underline{y}))$. First we have

Lemma 1. *The integral*

$$\int_U G_1(\underline{y}-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n$$

gives a well-defined function on $U \setminus \{\underline{y}_0\}$.

Proof. For each point $\underline{y}' \in U \setminus \{\underline{y}_0\}$ we have that

$$\int_U G_1(\underline{y}-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n = \int_{B\left(\underline{y}_0,\frac{1}{2}\|\underline{y}_0-\underline{y}'\|\right)\cap U} G_1(\underline{y}_0-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n + \int_{U\setminus B\left(\underline{y}_0,\frac{1}{2}\|\underline{y}_0-\underline{y}'\|\right)} G_1(\underline{y}-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n.$$

As $g(\underline{x})$ is a bounded function satisfying (16), it follows that $h(\varphi^{-1}(\underline{y}))$ is a bounded function on $U \setminus B(\underline{y}_0, \frac{1}{2}\|\underline{y}_0-\underline{y}'\|)$. Consequently, the integral

$$\int_{U\setminus B\left(\underline{y}_0,\frac{1}{2}\|\underline{y}_0-\underline{y}'\|\right)} G_1(\underline{y}-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n$$

is well-defined. Now

$$\left\| \int_{B\left(\underline{y}_0,\frac{1}{2}\|\underline{y}_0-\underline{y}'\|\right)\cap U} G_1(\underline{y}-\underline{y}')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n \right\| \le C\,\frac{2^{n-1}}{\|\underline{y}_0-\underline{y}'\|^{n-1}} \sup_{\underline{y}\in U}\|\underline{y}-\underline{y}_0\|^2\left\|h(\varphi^{-1}(\underline{y}))\right\| \int_0^{\frac{1}{2}\|\underline{y}_0-\underline{y}'\|} R^{n-3}dR$$

where C is some constant depending on the dimension n. As $\int_0^{\frac{1}{2}\|\underline{y}_0-\underline{y}'\|} R^{n-3}dR < \infty$, it follows that the integral is well-defined for each $\underline{y}' \in U \setminus \{\underline{y}_0\}$ ∎

We also have

Lemma 2. *For each* $\underline{y}' \in U \setminus \{\underline{y}_0\}$,

$$\int_{U\cap B\left(\underline{y}_0,\frac{1}{2}\|\underline{y}_0-\underline{y}'\|\right)} G_1(\underline{y}-\underline{y}'')h(\varphi^{-1}(\underline{y}))\,d\underline{y}^n \tag{17}$$

is a left-regular function on $U \cap B(\underline{y}', \frac{1}{4}\|\underline{y}_0-\underline{y}'\|)$.

Proof. Given $\varepsilon > 0$, we can find $h_i \in \mathbb{R}$ $(i = 1, ..., n)$ such that

$$\sup_{\underline{y} \in B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \; \sup_{\underline{y}'' \in B\left(\underline{y}', \frac{1}{4}\|\underline{y}_0 - \underline{y}'\|\right)} \left\| \frac{G_1(\underline{y} - \underline{y}'' + h_i e_i) - G_1(\underline{y} - \underline{y}'')}{h_i} - \frac{\partial}{\partial y_i'} G_1(\underline{y} - \underline{y}'') \right\| < \varepsilon.$$

Consequently,

$$\sup_{\underline{y} \in B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \; \sup_{\underline{y}'' \in B\left(\underline{y}', \frac{1}{4}\|\underline{y}_0 - \underline{y}'\|\right)} \int_{U \cap B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \left(\frac{G_1(\underline{y} - \underline{y}'' + h_i e_i) - G_1(\underline{y} - \underline{y}'')}{h_i} - \frac{\partial}{\partial y_i'} G_1(\underline{y} - \underline{y}'') \right) d\underline{y}^n$$

$$\leq \varepsilon C \sup_{\underline{y} \in U} \|\underline{y} - \underline{y}_0\|^2 \left\| h(\varphi^{-1}(\underline{y})) \right\| \frac{1}{n-3} \frac{\|\underline{y}_0 - \underline{y}'\|^{n-2}}{2^{n-2}}.$$

It follows from this that

$$\lim_{h_i \to 0} \frac{1}{h_i} \int_{U \cap B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \Big(G_1(\underline{y} - \underline{y}'' + h_i e_i) - G(\underline{y} - \underline{y}'') \Big) h(\varphi^{-1}(\underline{y})) dy^n$$

$$= \int_{U \cap B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \frac{\partial G_1}{\partial y_i''}(\underline{y} - \underline{y}'') h(\varphi^{-1}(\underline{y})) d\underline{y}^n$$

for each $\underline{y}'' \in B(y', \frac{1}{4}\|\underline{y}_0 - \underline{y}'\|)$. Thus, the integral (17) defines a left-monogenic function on $U \cap B(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|)$ ∎

Thus, we have

Lemma 3. *The identity*

$$D_{\underline{y}''} \int_U G_1(\underline{y} - \underline{y}'') h(\varphi^{-1}(\underline{y}))\, dy^n = h(\varphi^{-1}(\underline{y}''))$$

holds.

Proof. As $g(\underline{x})$ is a bounded C^1-function on $\varphi^{-1}(U)$, then $h(\varphi^{-1}(\underline{y}))$ is a bounded C^1-function on $U \setminus B(\underline{y}_0, \frac{1}{2}\|\underline{y}' - \underline{y}_0\|)$. From [11, 12] it now follows that

$$D_{\underline{y}''} \int_{U \setminus B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}' - \underline{y}_0\|\right)} G_1(\underline{y} - \underline{y}'') h(\varphi^{-1}(\underline{y}))\, d\underline{y}^n = h(\varphi^{-1}(\underline{y}'')).$$

The result now follows from Lemma 2 ∎

Using the change of variable arguments that we have employed throughout this paper, it may be observed on combining Lemmas 1 - 3 that we have deduced

Proposition 6. *Suppose that $g(\underline{x})$ is a bounded C^1-function defined on the bounded domain $\varphi^{-1}(U)$. Suppose also that $lim_{\underline{x}\to\infty}\|c\underline{x}+d\|^{n-1}\|g(\underline{x})\| < \infty$. Then the integral*

$$\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x})\,d\underline{x}^n$$

is a well-defined C^1-function on $\varphi^{-1}(U)$ and

$$D_{\underline{x}}\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x})\,d\underline{x}^n = g(\underline{x}').$$

It should be noted that Proposition 6 remains valid if we replace the condition $\lim_{\underline{x}\to\infty}\|c\underline{x}+d\|^{n-1}\|g(\underline{x})\| < \infty$ by the condition $\lim_{\underline{x}\to\infty}\|c\underline{x}+d\|^{\alpha}\|g(\underline{x})\| < \infty$, where $\alpha < n+3$. However, the reason for choosing $\alpha = n-1$ becomes apparent in

Lemma 4. *Suppose $f : \varphi^{-1}(U) \longrightarrow A_n(\mathbb{C})$ is a bounded C^1-function and $\lim_{\underline{x}\to\infty}$ $\|c\underline{x}+d\|^{n+1}\|f(\underline{x})\| < \infty$. Then*

$$\lim_{\underline{x}\to\infty}\|c\underline{x}+d\|^{n+1}\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')f(\underline{x}')\,d\underline{x}'^n < \infty.$$

Proof. Place $k(\underline{x}) = (c\underline{x}+d)^{n+1}f(\underline{x})$ and $l(\underline{y}) = k(\varphi^{-1}(\underline{y}))$. As $f(\underline{x})$ is a bounded C^1-function and $lim_{x\to\infty}\|c\underline{x}+d\|^{n-1}\|f(\underline{x})\| < +\infty$, it follows that $l(\underline{y})$ is a bounded C^1-function on U. So (see [12])

$$h(\underline{y}') = \int_U G_1(\underline{y}'-\underline{y})l(\underline{y})\,d\underline{y}^n$$

is a bounded C^1-function on U. But

$$\int_U G_1(\underline{y}'-\underline{y})l(\underline{y})\,d\underline{y}^n = J_1(\varphi,\underline{x}')^{-1}\int_{\varphi^{-1}(U)} G_1(\underline{x}'-\underline{x})J_{-1}(\varphi,x)k(\underline{x})\,d\underline{x}^n.$$

So

$$h(\varphi(\underline{x}')) = J_1(\varphi,\underline{x}')^{-1}\int_{\varphi^{-1}(U)} G_1(\underline{x}'-\underline{x})f(\underline{x})\,dx^n.$$

As $h(\varphi(\underline{x}))$ is a bounded C^1-function, it follows that

$$\lim_{\underline{x}\to\infty}\|c\underline{x}+d\|^{n+1}\|J_{-1}(\varphi,\underline{x})h(\varphi(\underline{x}))\| < +\infty$$

and the statement is proved ∎

Combining Lemmas 1 - 4 and Proposition 6 we obtain

Theorem 13. *Suppose that $k(\underline{x})$ is a bounded C^1-function on the unbounded domain $\varphi^{-1}(U)$. Suppose also that $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|k(\underline{x})\| < +\infty$. Then*

$$\int_{\varphi^{-1}(U)} G_1(\underline{x}'-\underline{x})\left(\int_{\varphi^{-1}(U)} G_1(\underline{x}''-\underline{x}')k(\underline{x}'')\,dx''^n\right)dx'^n$$

is a solution to the equation $\Delta u(\underline{x}) = k(\underline{x})$ on $\varphi^{-1}(U)$.

Via analogues of Lemmas 1 - 4 and Proposition 6 we have the following generalization of Theorem 13.

Theorem 14. *Suppose that $k(x)$ is a bounded C^l-function on the unbounded domain $\varphi^{-1}(U)$, where $l = 1,\ldots,\frac{n}{2}-1$ when n is even, and $l = 1,\ldots,\frac{n-1}{2}$ when n is odd. Suppose also that $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+l}\|(k\underline{x})\| < +\infty$. Then*

$$\int_{\varphi^{-1}(U)} G_l(\underline{x}'-\underline{x})\left(\int_{\varphi^{-1}(U)} G_l(\underline{x}''-\underline{x}')k(\underline{x}'')\,d\underline{x}''^n\right)d\underline{x}'^n$$

is a solution to the equation $\Delta^l u(\underline{x}) = k(\underline{x})$ on $\varphi^{-1}(U)$.

From now on we shall assume that the domain U has a Lipschitz continuous boundary. We shall now attempt to solve the equation

$$\Delta u(\underline{x}) = g(\underline{x})$$

on $\varphi^{-1}(U)$, subject to the condition $u(\underline{x}) = 0$ on $\partial\psi^{-1}(U)$. Moreover, $g(\underline{x})$ is a bounded C^1-function with $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|g(\underline{x})\| < \infty$. We first have

Lemma 5. *Suppose that $g : \varphi^{-1}(U) \longrightarrow A_n(\mathbb{C})$ is a bounded C^1-function and $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|g(\underline{x})\| < \infty$. Then*

$$\int_{\varphi_{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x}')\,d\underline{x}'^n$$

is an L^2-integrable function on $\varphi^{-1}(U)$.

Proof. From Lemma 4 we have that

$$\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\left\|\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x}')\,d\underline{x}'^n\right\| < \infty.$$

Consequently, upon setting

$$k(\underline{x}) = \int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x}')\,d\underline{x}'^n,$$

we have

$$\int_{\varphi^{-1}(U)} \mathrm{Re}\left(k(\underline{x})\bar{k}(\underline{x})\right)d\underline{x}^n \le \sup_{\underline{x}\in\varphi^{-1}(U)} \|k(\underline{x})\|^2\left(\int_{U_1} d\underline{x}^n + c_n\int_1^\infty R^{-n+1}dR\right)$$

where $\mathrm{Re}(k(\underline{x})\bar{k}(\underline{x}))$ denotes the real part of $k(\underline{x})\bar{k}(\underline{x})$, c_n is a positive constant which depends on dimension, and U is a bounded subdomain of $\varphi^{-1}(U)$.

The proof of Lemma 15 also tells us that any bounded C^1-function $h : \varphi^{-1}(U) \longrightarrow A_n(\mathbb{C})$ which satisfies $\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-1}\|h(\underline{x})\| < \infty$ is an L^2-integrable function on $\varphi^{-1}(U)$. Let $\mathcal{A}^2(\varphi^{-1}(U), -n+1)$ denote the right-$A_n(\mathbb{C})$ pre-Hilbert module of bounded solutions to the Dirac equation $Df(\underline{x}) = 0$ on a neighbourhood of $\varphi^{-1}(U)$, satisfying the asymptotic condition

$$\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-1}\|f(\underline{x})\| < \infty.$$

For each point $\underline{x}_0 \in \mathbb{R}^n \setminus \varphi^{-1}(U)$, we have that $G_1(\underline{x} - \underline{x}') \in \mathcal{A}^2(\varphi^{-1}(U), -n+1)$. Let $\mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$ denote the right-$A_n(\mathbb{C})$ pre-Hilbert module of bounded C^1-functions defined on a neighbourhood of $\varphi^{-1}(U)$, $g : \varphi^{-1}(U) \longrightarrow A_n(\mathbb{C})$, such that $\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-1}\|g(\underline{x})\| < \infty$ and

$$\int_{\varphi^{-1}(U)} \bar{g}(\underline{x})f(\underline{x})\, d\underline{x}^n = 0 \qquad \text{for all } f, g \in \mathcal{A}^2(\varphi^{-1}(U), -n+1).$$

If we allow $L^2(\varphi^{-1}(U), -n+1)$ to denote the right-$A_n(\mathbb{C})$ pre-Hilbert module of bounded C^1-functions $g : \varphi^{-1}(U) \longrightarrow A_n(\mathbb{C})$ such that $\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-1}\|g(\underline{x})\| < \infty$, then we have

Lemma 6. *The identity*

$$L^2(\varphi^{-1}(U), -n+1) = \mathcal{A}^2(\varphi^{-1}(U), -n+1) \oplus \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$$

holds.

We now try to give a better characterization of $\mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$. First, we need

Lemma 7. *Suppose that $g(\underline{x})$ is a bounded C^1-function on $\varphi^{-1}(U)$ and $\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-1}\|g(\underline{x})\| < \infty$. Then*

$$\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-3} \left\| \int_{\varphi^{-1}(U)} G_1(\underline{x} - \underline{x}')g(\underline{x}')\, d\underline{x}'^n \right\| < \infty.$$

Proof. Suppose $\underline{y}_0 \in \overline{U}$ is such that $\varphi(\underline{y}_0) = \infty$. Then, on putting $k(\underline{x}) = \|c\underline{x} + d\|^{n-1}\|g(\underline{x})\|$ we get that

$$\lim_{\underline{y}\to\underline{y}_0} \|\underline{y} - \underline{y}_0\|^2 \|h(\varphi^{-1}(y))\| < +\infty.$$

Now

$$\int_U G_1(\underline{y} - \underline{y}')h(\varphi^{-1}(\underline{y}))\, dy^n = \int_{U\cap B(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|)} G_1(\underline{y} - \underline{y}')h(\varphi^{-1}(\underline{y}))\, d\underline{y}^n + \int_{U\setminus B(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|)} G_1(\underline{y} - \underline{y}')h(\varphi^{-1}(\underline{y}))\, d\underline{y}^n.$$

Furthermore,

$$\left\| \int_{U \cap B\left(\underline{y}_0, \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} G_1(\underline{y} - \underline{y}') h(\varphi^{-1}(\underline{y}))\, d\underline{y}^n \right\|$$
$$\leq C_1 \left(\sup_{\underline{y} \in U} \|\underline{y} - \underline{y}_0\|^2 \left\| h(\varphi^{-1}(y)) \right\| \right) \frac{1}{\|y_0 - \underline{y}'\|}$$

for some positive number C_1. Also,

$$\left\| \int_{U \setminus B\left(\underline{y}', \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} G_1(\underline{y} - \underline{y}') h(\varphi^{-1}(\underline{y}))\, d\underline{y}^n \right\|$$
$$\leq \frac{C_2}{\|\underline{y}' - \underline{y}_0\|^2} \int_{U \setminus B\left(\underline{y}', \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \|G_1(\underline{y} - \underline{y}')\| \, \|\underline{y}_0 - \underline{y}\|^2 \|h(\varphi^{-1}(y))\| \, d\underline{y}^n$$

for some positive number C_2. Now

$$\int_{U \setminus B\left(\underline{y}', \frac{1}{2}\|\underline{y}_0 - \underline{y}'\|\right)} \|G_1(\underline{y} - \underline{y}')\| \, \|\underline{y}_0 - \underline{y}\|^2 \|h(\varphi^{-1}(y))\| \, d\underline{y}^n$$
$$\leq C_3 \sup_{\underline{y} \in U} \|\underline{y}_0 - \underline{y}\|^2 \left\| h(\varphi^{-1}(y)) \right\| \int_U \|G_1(\underline{y} - \underline{y}')\| \, d\underline{y}^n$$

for some positive number C_3. As U is a bounded domain, it now follows that

$$\lim_{\underline{y} \to \underline{y}_0} \|\underline{y}_0 - \underline{y}\|^2 \left\| \int_U G_1(\underline{y} - \underline{y}') h(\varphi^{-1}(y))\, d\underline{y}^n \right\| < +\infty.$$

Upon changing variables, it now follows that

$$\lim_{\underline{x} \to \infty} \|c\underline{x} + d\|^{n-3} \left\| \int_{\varphi^{-1}(U)} G_1(\underline{x} - \underline{x}') g(\underline{x}')\, d\underline{x}'^n \right\| < \infty$$

and the statement is proved ∎

We now have

Proposition 7. *Suppose that $n \geq 4$ and $\partial \mathrm{cl} \varphi^{-1}(U)$ is bounded. Then for each $g \in \mathcal{A}^2(\varphi^{-1}(U), -n+1))^{\perp}$ there is a bounded C^2-function h on $\varphi^{-1}(U)$ such that $Dh = g$ and $h(\underline{x}) = 0$ on $\partial \mathrm{cl} \varphi^{-1}(U)$.*

Proof. Suppose that $g \in \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$. Then from Proposition 6 we have

$$g(\underline{x}) = D_{\underline{x}} \int_{\varphi^{-1}(U)} G_1(\underline{x} - \underline{x}')g(\underline{x}')\, d\underline{x}'^n.$$

Put

$$k(\underline{x}) = \int_{\varphi^{-1}(U)} G_1(\underline{x} - \underline{x}')g(\underline{x}')\, d\underline{x}'^n.$$

If the positive number R is chosen to be sufficiently large, we now have that

$$\int_{\varphi^{-1}(U)\cap, B(\underline{o},R)} (k(\underline{x})D)f(\underline{x})\, d\underline{x}^n$$
$$= \int_{\partial \mathrm{cl}\varphi^{-1}(U)} \bar{k}(\underline{x})f(\underline{x})\, d\sigma(\underline{x}) - \int_{S^{n-1}(\underline{o},R)} \bar{k}(\underline{x})n(\underline{x})f(\underline{x})\, d\sigma(\underline{x})$$

for each $f \in \mathcal{A}^2(\varphi^{-1}(U), -n+1)$, where $S^{n-1}(\underline{o}, R)$ is the sphere in $\mathbb{R}^n$ of radius R and centered at $\underline{o}$. As $lim_{\underline{x}\to\infty}\|c\underline{x} + d\|^{n-1}\|f(\underline{x})\| < +\infty$, it follows from Lemma 7 that, when $n \geq 4$, we have

$$\lim_{R\to\infty} \left\| \int_{S^{n-1}(\underline{o},R)} \bar{k}(\underline{x})n(\underline{x})f(\underline{x})\, dS^{n-1}(\underline{o}, R) \right\| = 0.$$

Consequently,

$$\int_{\varphi^{-1}(U)} (k(\underline{x})D)f(\underline{x})\, d\underline{x}^n = \int_{\partial \mathrm{cl}\varphi^{-1}(U)} \bar{k}(\underline{x})n(\underline{x})f(\underline{x})\, d\sigma(\underline{x}).$$

So

$$\int_{\partial \mathrm{cl}\varphi^{-1}(U)} \bar{k}(\underline{x})n(\underline{x})f(\underline{x})\, d\sigma(\underline{x}) = 0.$$

Upon placing $f(\underline{x}) = G_1(\underline{x} - \underline{x}_1)$, where $\underline{x}_1 \in \mathbb{R}^n \setminus \varphi^{-1}(U)$ and letting $\underline{x}_1$ approach the boundary of $\varphi^{-1}(U)$, we obtain

$$\text{P.V.} \int_{\partial \mathrm{cl}\varphi^{-1}(U)} \bar{k}(\underline{x})n(\underline{x})G_1(\underline{x} - \underline{x}_1)\, d\sigma(\underline{x}) = \frac{1}{2}\bar{k}(\underline{x}')$$

for almost all $\underline{x}' \in \partial \mathrm{cl}\varphi^{-1}(U)$. Upon setting

$$h(\underline{x}) = k(\underline{x}) - \int_{\partial \mathrm{cl}\varphi^{-1}(U)} G_1(\underline{x} - \underline{x}')n(\underline{x}')k(\underline{x}')\, d\sigma(\underline{x}')$$

we obtain the result ∎

Using Proposition 7 and the projection operator

$$P: \mathcal{A}^2(\varphi^{-1}(U), -n+1) \oplus \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp} \longrightarrow \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$$

we obtain

Theorem 15. *Suppose that $n \geq 4$ and $\partial\mathrm{cl}\varphi^{-1}(U)$ is bounded. Suppose also that $g(\underline{x})$ is a bounded C^1-function on $\varphi^{-1}(U)$ and $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|g(\underline{x})\| < \infty$. Then there is a unique C^2-function $u(\underline{x})$ on $\varphi^{-1}(U)$ satisfying $\Delta u = g$ and $u(\underline{x}) = 0$ on $\partial\mathrm{cl}\varphi^{-1}(U)$. Moreover,*

$$Du(\underline{x}) = P\int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x}')\,dx'^n.$$

Adaptations of our previous arguments, and arguments presented in [11], gives us

Theorem 16. *Suppose that $n \geq 4$ and $\partial\mathrm{cl}\varphi^{-1}(U)$ is bounded. Suppose also that $\lambda(\underline{x})$ is a continuous function on $\partial\mathrm{cl}\varphi^{-1}(U)$ which extends to a bounded C^1-function λ' on $\varphi^{-1}(U)$, and $\lambda'(\underline{x})$ has compact support. If $g(\underline{x})$ is a bounded C^1-function on $\varphi^{-1}(U)$ satisfying $\lim_{\underline{x}\to\infty} \|c\underline{x}+d\|^{n+1}\|g(\underline{x})\| < \infty$, then there is a unique C^2-function $u(\underline{x})$ on $\varphi^{-1}(U)$ satisfying $\Delta u = g$ and $u(\underline{x}) = \lambda(\underline{x})$ on $\partial\mathrm{cl}\varphi^{-1}(U)$.*

We shall now show that we can drop the assumption that $\varphi^{-1}(U)$ has a bounded boundary. We begin with

Lemma 8. *Suppose that $n \geq 4$ and $g \in \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$. Then, upon setting*

$$k(\underline{x}) = \int_{\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')g(\underline{x}')\,d\underline{x}'^n$$

we have

$$\text{P.V.}\int_{\partial\mathrm{cl}\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')n(\underline{x}')k(\underline{x}')\,d\sigma(\underline{x}') = \frac{1}{2}k(\underline{x})$$

for almost all $\underline{x} \in \partial\mathrm{cl}\varphi^{-1}(U)$.

The proof follows similar lines to that of Proposition7.

Suppose now that $k(\underline{x})$ is as in Lemma 8. Then we have

Lemma 9. *The integral*

$$\int_{\partial\mathrm{cl}\varphi(U)} G_1(\underline{x}-\underline{x}_0)n(\underline{x})k(\underline{x})\,d\sigma(\underline{x})$$

gives a well-defined solution to the Dirac equation on $\varphi^{-1}(U)$.

Proof. Now,

$$\begin{aligned}
&\int_{\partial\mathrm{cl}\varphi^{-1}(U)} G_1(\underline{x}-\underline{x}')n(\underline{x}')k(\underline{x}')\,d\sigma(\underline{x}') \\
&\qquad = \int_{\partial\mathrm{cl}U} J_1(\tilde{c}\underline{y}-\tilde{a})^{-1}G_1(\underline{y}'-\underline{y})J_1(\tilde{c}\underline{y}'-\tilde{a})^{-1} \\
&\qquad\quad \times J_1(\tilde{c}\underline{y}-\tilde{a})n(\underline{y}')J_1(\tilde{c}\underline{y}'-\tilde{a})k(\varphi^{-1}(y))\,d\sigma(\underline{y}')
\end{aligned}$$

$\varphi^{-1}(\underline{x}') = \underline{y}'$ and $\underline{x} = (-\tilde{d}\underline{y} + \tilde{b})(\tilde{c}\underline{y} - \tilde{a})^{-1}$. Consequently,

$$\begin{aligned}\int_{\partial \mathrm{cl}\varphi^{-1}(U)} & G_1(\underline{x}' - \underline{x})n(\underline{x}')k(\underline{x}')\,d\sigma(\underline{x}') \\ &= J_1(\varphi^{-1}, \underline{y})^{-1} \int_{\partial \mathrm{cl} U} G_1(\underline{y}' - \underline{y})n(\underline{y}')J_1(\varphi^{-1}, \underline{y}')k(\varphi^{-1}(\underline{y}))\,d\sigma(\underline{y}').\end{aligned} \tag{18}$$

As $\lim_{\underline{x}\to\infty} \|c\underline{x} + d\|^{n-3}\|k(\underline{x})\| < +\infty$, it follows that

$$\lim_{\underline{y}\to\underline{y}_0} \|\underline{y} - \underline{y}_0\|^2 \left\| J_1(\varphi^{-1}, \underline{y}')k(\varphi^{-1}(\underline{y}))\right\| < +\infty$$

where $\varphi^{-1}(\underline{y}_0) = +\infty$. As $n \geq 4$, it follows that

$$\left\| \int_{\partial \mathrm{cl} U} G_1(\underline{y}' - \underline{y})n(\underline{y}')J_1(\varphi^{-1}, \underline{y}')k(\varphi^{-1}(\underline{y}'))\,d\sigma(\underline{y}') \right\| < \infty$$

for each $\underline{y} \in U$. Thus

$$\int_{\partial \mathrm{cl} U} G_1(\underline{y}' - \underline{y})n(\underline{y}')J_1(\varphi^{-1}, \underline{y}')k(\varphi^{-1}(\underline{y}'))\,d\sigma(\underline{y}')$$

defines a left-monogenic function on U. From (18) it follows that

$$\int_{\partial \mathrm{cl}\varphi^{-1}(U)} G_1(\underline{x}' - \underline{x})n(\underline{x}')k(\underline{x}')\,d\sigma(\underline{x}')$$

defines a left-monogenic function on $\varphi^{-1}(U)$ ∎

It may easily be checked that $k(\underline{x})$ is a locally Lipschitz continuous function on $\varphi^{-1}(U)$. From this observation, and Lemmas 8 and 9, we obtain

Proposition 8. *Suppose that $n \geq 4$. Then for each $g \in \mathcal{A}^2(\varphi^{-1}(U), -n+1)^{\perp}$ there is a C^2-function h on $\varphi^{-1}(U)$ such that $Dh = g$ and $h(\underline{x}) = 0$ on $\partial \mathrm{cl}\varphi^{-1}(U)$.*

The result follows upon placing

$$h(\underline{x}) = k(\underline{x}) - \int_{\partial \mathrm{cl}\varphi^{-1}(U)} G_1(\underline{x}' - \underline{x})n(\underline{x}')k(\underline{x}')\,d\sigma(\underline{x}').$$

It follows that the condition that $\partial \mathrm{cl}\varphi^{-1}(U)$ to be a bounded set, appearing in Theorems 15 and 16 can be dropped.

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