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Bounded Solutions and Oscillations of Concave Lagrangian Systems in Presence of a Discount Rate

J. Blot and P. Cartigny

Abstract. We study the bounded solutions with bounded derivative on $\mathbb{R}_+$ of Lagrangian systems in the form $\ell_x(x,\dot{x}) = \frac{d}{dt}\ell_{\dot{x}}(x,\dot{x}) - \delta\ell_{\dot{x}}(x,\dot{x})$, where $\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a concave differentiable function and δ a positive number. These systems are usual in the macroeconomic theory of growth. We formulate specific variational problems to study the bounded trajectories. We obtain results about the uniqueness of such solutions, and about the constant solutions and the almost-periodic solutions. We study the linear case and we describe a special non-local linearization method.

Keywords: *Bounded solutions, nonlinear oscillations, Lagrangian systems, optimal economic growth*

AMS subject classification: 34 C 11, 34 C 25, 34 C 27, 49 B 36, 90 A 16

0. Introduction

We study the bounded solutions, with bounded derivative, on $\mathbb{R}_+ = [0, \infty)$, of the systems

$$(\mathcal{E},\delta) \quad \ell_x(x,\dot{x}) = \frac{d}{dt}\ell_{\dot{x}}(x,\dot{x}) - \delta\ell_{\dot{x}}(x,\dot{x})$$

where $\delta > 0, \dot{x} = \frac{dx}{dt}$, and $\ell : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a concave differentiable function. Recall that a concave differentiable function automatically is of class C^1. When $\eta \in \mathbb{R}^n$,

Problem $\quad (\mathcal{E},\delta,\eta)$

stands for system $(\mathcal{E},\delta)$ with the initial condition $x(0) = \eta$. Taking $L(t,x,\dot{x}) = e^{-\delta t}\ell(x,\dot{x})$, system $(\mathcal{E},\delta)$ is equivalent to the Euler-Lagrange equation

$$L_x(t,x,\dot{x}) = \frac{d}{dt}L_{\dot{x}}(t,x,\dot{x}). \tag{1}$$

J. Blot: C.E.R.M.S.E.M., Université de Paris 1 Panthéon-Sorbonne, 90 rue de Tolbiac, 75634 Paris Cedex 13, France
P. Cartigny: G.R.E.Q.A.M., Université d'Aix-Marseille II, Centre de la Vieille Charité, 2 rue de la Charité, 13002 Marseille, France

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Under the classical Hilbert condition, $\ell_{\dot{x}\dot{x}}(x,\dot{x})$ invertible, the Lagrangian system $(\mathcal{E},\delta)$ is translatable in a normal second order system with $(x,\dot{x})$ as unknown function (see [12: p. 34]). In another hand, under some suitable assumptions, we can introduce the Hamiltonian functions

$$H(t,x,p) = \sup \left\{ p\cdot v + L(t,x,v) : v \in \mathbb{R}^n \right\}$$
$$h(x,p) = \sup \left\{ p\cdot v + \ell(x,v) : v \in \mathbb{R}^n \right\}$$

where $p \cdot v$ denotes the usual inner product of p and v in $\mathbb{R}^n$. We easily obtain the relation $H(t,x,p) = e^{-\delta t} h(x, e^{\delta t} p)$, and the Hamiltonian system associated to (1) is the following:

$$\begin{aligned} \dot{x} &= H_p(t,x,p) \\ \dot{p} &= -H_x(t,x,p) \end{aligned} \tag{2}$$

and the relation between the solutions of system (1) and those of system (2) is $p(t) = -L_{\dot{x}}(t,x(t),\dot{x}(t))$. Taking $P(t) = e^{\delta t} p(t)$, system (2) is equivalent to

$$\begin{aligned} \dot{x} &= h_p(x,P) \\ \dot{P} &= -h_x(x,P) + \delta P. \end{aligned} \tag{3}$$

For instance, assuming that h is of class C^1, if (x,P) is a solution of system (3), to say that (x,P) is bounded on $\mathbb{R}_+$ is equivalent to say that x is a solution of system $(\mathcal{E},\delta)$ which is bounded, with $\dot{x}$ bounded, on $\mathbb{R}_+$. These considerations show us that the natural concept of boundedness for a solution x of system $(\mathcal{E},\delta)$ is the boundedness of x and $\dot{x}$. The major motivation for this class of differential systems is the macroeconomic model of the optimal growth in infinite horizon (cf. [13, 16]). We consider the functional

$$J_\delta(x) = \int_0^\infty e^{-\delta t} \ell(x(t),\dot{x}(t))\,dt. \tag{4}$$

In this expression, δ is a discount rate, $\ell(x,\dot{x})$ is built from a utility function and some structural equations of the economic model, and then $J_\delta(x)$ appears as an intertemporal utility. The integral which defines J_δ does not ever exist and so, we ought to consider $\mathrm{dom} J_\delta$, the set of the functions $x : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ of class C^1 such that the above mentioned improper integral exists. The **Optimal Growth problem** is the following:

$(\mathcal{P},\delta,\eta)$ Maximize $J_\delta(x)$ when $x \in \mathrm{dom} J_\delta$, $x(0) = \eta$,

where η is a fixed initial value.

Among the numerous mathematical works about this subject, we quote of Rockafellar [15], Ekeland [11], and Medio [14]. The system (1), and consequently system $(\mathcal{E},\delta)$, is the Euler-Lagrange equation of problem $(\mathcal{P},\delta,\eta)$.

In this work we also consider the linear case. If A,B,C,D are real $(n\times n)$-matrices such that the $(2n \times 2n)$-matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symmetric negative semi-definite, we build the quadratic concave function

$$\Lambda(x,\dot{x}) = \frac{1}{2}(x^{\top}\,\dot{x}^{\top})\,S(x\,\dot{x}) = \frac{1}{2}\Big\{x^{\top}Ax + 2x^{\top}B\dot{x} + \dot{x}^{\top}D\dot{x}\Big\} \tag{5}$$

and the quadratic functional

$$Q_\delta(x) = \int_0^\infty e^{-\delta t}\Lambda\big(x(t),\dot{x}(t)\big)dt \tag{6}$$

which generates the linear Lagrangian system

$$(\mathcal{L},\delta) \quad Ax + B\dot{x} = \frac{d}{dt}(B^{\top}x + D\dot{x}) - \delta(B^{\top}x + D\dot{x}).$$

Further,

Problem $(\mathcal{L},\delta,\eta)$

stands for the system $(\mathcal{L},\delta)$ with the initial condition $x(0)=\eta$.

In preceding works [2 - 5, 7], which were concerned with the case $\delta = 0$, numerous structural results about the almost periodic solutions of system $(\mathcal{E},0)$ have been established in using original technics. We shall adapt some of these results to the case $\delta > 0$. The general viewpoint of our approach is the following : to study the bounded (specially the periodic or the almost-periodic) solutions of system $(\mathcal{E},\delta)$, we formulate some suitable variational problems on bounded functions spaces such that the bounded solutions of system $(\mathcal{E},\delta)$ are exactly the solutions of the variational problems.

Now we precisely describe the contents of the paper.

In Section 3 we introduce concave variational problems $(\mathcal{R},\delta,\eta)$ which are restrictions of problems $(\mathcal{P},\delta,\eta)$ to bounded differentiable functions spaces and which permit to characterize the bounded solutions of problem $(\mathcal{E},\delta,\eta)$ as optimizers of problem $(\mathcal{R},\delta,\eta)$. In Section 4, under second order conditions, we establish results about the uniqueness of the bounded solutions with bounded derivative of problem $(\mathcal{E},\delta,\eta)$ and we give some structural properties of these solutions. In Section 5 we establish that the presence of several periodic solutions with non-commensurable periods generates almost-periodic non-periodic solutions of system $(\mathcal{E},\delta)$. In Section 6 we study the constant solutions of system $(\mathcal{E},\delta)$ and the specific properties of problem $(\mathcal{E},\delta,\eta)$ when η is a constant solution of system $(\mathcal{E},\delta)$. In Section 7 we study the linear case $(\mathcal{L},\delta)$ and we generalize some results previously established in the case $\delta = 0$. In Section 8 we use a non-local linearization method special to the concave case in order to obtain properties of system $(\mathcal{E},\delta)$ from properties of system $(\mathcal{L},\delta)$.

1. Some notations

We precise our notations about certain subspaces of the space of bounded continuous functions.

• $BC^0(\mathbb{R}_+, \mathbb{R}^n)$ denotes the space of the bounded continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^n$. It is a Banach space when it is endowed with the supremum norm $\|x\|_\infty = \sup\{\|x(t)\| : t \in \mathbb{R}_+\}$.

• $BC^1 = BC^1(\mathbb{R}_+, \mathbb{R}^n) = \{x : x \in BC^0(\mathbb{R}_+, \mathbb{R}^n) \cap C^1(\mathbb{R}_+, \mathbb{R}^n),\ \dot{x} \in BC^0(\mathbb{R}_+, \mathbb{R}^n)\}$. It is a Banach space when it is endowed with the norm $\|x\|_1 = \|x\|_\infty + \|\dot{x}\|_\infty$.

• For $T > 0$, $C^1_T = C^1_T(\mathbb{R}_+, \mathbb{R}^n)$ is the space of the T-periodic continuously differentiable functions from $\mathbb{R}_+$ into $\mathbb{R}^n$.

• $AP^0 = AP^0(\mathbb{R}_+, \mathbb{R}^n)$ is the space of the Bohr almost-periodic functions from $\mathbb{R}_+$ into $\mathbb{R}^n$ (cf. [8]), and $AP^1 = AP^1(\mathbb{R}_+, \mathbb{R}^n) = \{x : x \in AP^0(\mathbb{R}_+, \mathbb{R}^n) \cap C^1(\mathbb{R}_+, \mathbb{R}^n),\ \dot{x} \in AP^0(\mathbb{R}_+, \mathbb{R}^n)\}$. $AP^0(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach subspace of $(BC^0(\mathbb{R}_+, \mathbb{R}^n), \|\cdot\|_\infty)$, and AP^1 is a Banach subspace of $(BC^1, \|\cdot\|_1)$.

• When $x \in AP^0$, its mean value is $\mathcal{M}\{x\} = \mathcal{M}\{x(t)\}_t = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$, and its Fourier-Bohr coefficients are $\mathrm{a}(x, \lambda) := \mathcal{M}\{x(t)e^{-i\lambda t}\}_t$ when $\lambda \in \mathbb{R}$. We denote by $\mathrm{Mod}\, x$ the $\mathbb{Z}$-module in $\mathbb{R}$ generated by $\{\lambda \in \mathbb{R} : a(x, \lambda) \neq 0\}$.

• Let $\omega = (\omega_1, \omega_2, \ldots, \omega_s)$ be a family of s real numbers which are $\mathbb{Z}$-linearly independent. Then for $k = 0, 1$, $QP^k = QP^k(\omega, \mathbb{R}_+, \mathbb{R}^n)$ is the space of the $x \in AP^k(\mathbb{R}_+, \mathbb{R}^n)$ such that $\mathrm{Mod} x$ is generated by ω. QP are the initials of quasi periodicity.

• When $\eta \in \mathbb{R}^n$, we note $E_\eta = \{x \in BC^1 : x(0) = \eta\}$.

• We consider the following assumptions:

$$\ell \text{ is concave and of class } C^1 \text{ on } \mathbb{R}^n \times R^n \tag{7}$$

$$\ell \text{ is concave and of class } C^2 \text{ on } \mathbb{R}^n \times R^n. \tag{8}$$

2. Variational tools

We build the variational setting which will permit us the study of the bounded solutions of system $(\mathcal{E}, \delta)$.

Lemma 1. *Let p and q be two positive integers, $\varphi : \mathbb{R}^p \longrightarrow \mathbb{R}^q$ a mapping and $\mathcal{N}_\varphi$ the Nemytski operator defined by $\mathcal{N}_\varphi(u) = \varphi \circ u$, for $u \in BC^0(\mathbb{R}_+, \mathbb{R}^n)$. Then we have:*

(i) *For $k \in \{0, 1, 2\}$, if $\varphi \in C^k(\mathbb{R}^p, \mathbb{R}^q)$, then*

$$\mathcal{N}_\varphi \in C^k(BC^0(\mathbb{R}_+, \mathbb{R}^p), BC^0(\mathbb{R}_+, \mathbb{R}^q)).$$

(ii) *For $k = 1$, when $u, h \in BC^0(\mathbb{R}_+, \mathbb{R}^p)$, then*

$$(\mathcal{N}'_\varphi(u) \cdot h)(t) = \varphi'(u(t)) \cdot h(t) \qquad \textit{for every } t \in \mathbb{R}_+.$$

(iii) *For* $k = 2$, *when* $u, h, h_1 \in BC^0(\mathbb{R}_+, \mathbb{R}^p)$, *then*

$$(\mathcal{N}''_\varphi(u)(h, h_1))(t) = \varphi''(u(t))\,(h(t), h_1(t)) \qquad \textit{for every } t \in \mathbb{R}_+.$$

Proof. We fix $u \in BC^0(\mathbb{R}_+, \mathbb{R}^p)$. Since the image $u(\mathbb{R}_+)$ is bounded in $\mathbb{R}^q$, its closure K is compact. Therefore there exists $r > 0$ such that the set $V_r = \{\eta \in \mathbb{R}^q : d(\eta, K) \leq r\}$ also is compact (cf. [9: Chapter 3, §18]). First we assume that φ is continuous. Then it is clear that $\varphi \circ u$ is continuous and that $\varphi(K)$ is compact. The inclusion $\varphi \circ u(\mathbb{R}_+) \subset \varphi(K)$ ensures that $\varphi \circ u \in BC^0(\mathbb{R}_+, \mathbb{R}^q)$. And so $\mathcal{N}_\varphi$ is a well-defined mapping from $BC^0(\mathbb{R}_+, \mathbb{R}^p)$ into $BC^0(\mathbb{R}_+, \mathbb{R}^q)$. By the Heine theorem, φ is uniformly continuous on V_r. Given $\varepsilon > 0$, we denote by α_ε the module of uniform continuity of φ on V_r and we define $\beta_\varepsilon = \min\{\alpha_\varepsilon, r\} > 0$. When $v \in BC^0(\mathbb{R}_+, \mathbb{R}^p)$ is such that $\|u - v\|_\infty < \beta_\varepsilon$, we have, for every $t \in \mathbb{R}_+$, $v(t) \in V_r$ and $\|u(t) - v(t)\| < \alpha_\varepsilon$ that implies $\|\varphi(u(t)) - \varphi(v(t))\| \leq \varepsilon$, i.e. $\|\mathcal{N}_\varphi(u) - \mathcal{N}_\varphi(v)\|_\infty \leq \varepsilon$. The continuity of $\mathcal{N}_\varphi$ is proven.

Now we assume that φ is of class C^1. Given $\varepsilon > 0$, by a mean value theorem (cf. [1: p. 144]) we have, for every $h \in BC^0(\mathbb{R}_+, \mathbb{R}^p)$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}&\Big\|\varphi(u(t) + h(t)) - \varphi(u(t)) - \varphi'(u(t)) \cdot h(t)\Big\| \\ &\qquad \leq \|h(t)\| \sup\Big\{\|\varphi'(u(t)) - \varphi'(\eta)\| : \eta \in [u(t), u(t) + h(t)]\Big\}.\end{aligned}$$

By the Heine theorem, φ' is uniformly continuous on V_r. We denote by γ_ε the module of uniform continuity of φ' on V_r, and we define $\sigma_\varepsilon = \min\{\gamma_\varepsilon, r\} > 0$. If $\|h\|_\infty \leq \sigma_\varepsilon$, then, for every $t \in \mathbb{R}_+$ and for every $\eta \in [u(t), u(t) + h(t)]$, we have $\|u(t) - \eta\| \leq \|h(t)\| \leq \sigma_\varepsilon$, therefore $\eta \in V_r$, and $\|u(t) - \eta\| \leq \gamma_\varepsilon$ implies $\|\varphi'(u(t)) - \varphi(\eta)\| \leq \varepsilon$. And so we have

$$\Big\|\mathcal{N}_\varphi(u + h) - \mathcal{N}_\varphi(u) - \mathcal{N}'_\varphi(u) \cdot h\Big\|_\infty \leq \varepsilon \|h\|_\infty \qquad \text{when } \|h\|_\infty \leq \sigma_\varepsilon,$$

hence $\mathcal{N}_\varphi$ is Fréchet differentiable in u, with $\mathcal{N}'_\varphi(u) \cdot h = \varphi'(u) \cdot h$. After that we have established in the case $k = 0$, since φ' is continuous, $\mathcal{N}_{\varphi'}$ also is continuous and consequently we obtain the continuity of $\mathcal{N}'_\varphi$. And so $\mathcal{N}_\varphi$ is of class C^1.

Finally we assume that φ is of class C^2. Using the previous reasoning to φ' instead of φ, we obtain that $\mathcal{N}_{\varphi'}$ is of class C^1 and that $\mathcal{N}'_{\varphi'}(u) \cdot h_1 = \varphi''(u) \cdot h_1$. From this, we obtain that $\mathcal{N}_\varphi$ is of class C^2 and assertion (iii) holds ∎

Lemma 2. *For every* $\delta > 0$ *and* $k \in \{0, 1, 2\}$, *if* $\ell \in C^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, *then* $J_\delta \in C^k(BC^1, \mathbb{R})$. *Further:*

(i) *When* $k = 1$, *for* $x, h \in BC^1$, *we have*

$$J'_\delta(x) \cdot h = \int_0^\infty e^{-\delta t} \ell'(x(t), \dot{x}(t)) \cdot (h(t), \dot{h}(t))\,dt. \tag{9}$$

(ii) *When* $k = 2$, *for* $x, h, h_1 \in BC^1$, *we have*

$$J''_\delta(x) \cdot (h, h_1) = \int_0^\infty e^{-\delta t} \ell''(x(t), \dot{x}(t))\Big((h(t), \dot{h}(t)), (h_1(t), \dot{h}_1(t))\Big)\,dt. \tag{10}$$

Proof. We split the functional J_δ in the following way:

$$J_\delta = I \circ \mathcal{N}_\ell \circ \mathcal{T}$$

where

$$\mathcal{T} : BC^1(\mathbb{R}_+, \mathbb{R}^n) \longrightarrow BC^0(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^n), \quad \mathcal{T}(x) = (x, \dot{x})$$

$$\mathcal{N}_\ell : BC^0(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^n) \longrightarrow BC^0(\mathbb{R}_+, \mathbb{R}), \quad \mathcal{N}_\ell(u,v) = \ell o(u,v)$$

$$I : BC^0(\mathbb{R}_+, \mathbb{R}) \longrightarrow \mathbb{R}, \quad I(u) = \int_0^\infty e^{-\delta t} u(t)\, dt$$

Here $\mathcal{N}_\ell$ is the Nemytski operator built on ℓ. We note that $\mathcal{T}$ and I are linear continuous, hence of class C^2. By Lemma 1, if ℓ is of class C^k, $\mathcal{N}_\ell$ is of class C^k, and so J_δ is of class C^k as a composition of mappings of class C^k. Formulae (9) and (10) are straightforward consequences of the application of the Chain Rule to the previous splitting ∎

Proposition 1. *Let $\delta > 0$ and $\eta \in \mathbb{R}^n$. We assume that $\ell \in C^1(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Then, for every $x \in E_\eta$, the two following assertions are equivalent:*

(i) *For every $h \in E_0$, $J'_\delta(x) \cdot h = 0$.*

(ii) *The function $t \longmapsto \ell_{\dot{x}}(x(t), \dot{x}(t))$ is of class C^1 on $\mathbb{R}_+$, and x is a solution of problem $(\mathcal{E}, \delta, \eta)$.*

If in adition we assume that condition (7) is fulfilled, then J_δ is concave and the two first assertions are equivalent to the following:

iii) $J_\delta(x) = \sup J_\delta(E_\eta)$.

Proof. If assertion (ii) is true, in using Lemma 2, we have, for every $h \in E_0$,

$$\begin{aligned}
J'_\delta(x) \cdot h &= \int_0^\infty e^{-\delta t} \ell'(x(t), \dot{x}(t)) \cdot (h(t), \dot{h}(t)) dt \\
&= \int_0^\infty e^{-\delta t} \Big(\ell_x(x(t)\ \dot{x}(t)) \cdot h(t) + \ell_{\dot{x}}(x(t), \dot{x}(t)) \cdot \dot{h}(t)\Big) dt \\
&= \int_0^\infty e^{-\delta t} \left(\left(\frac{d}{dt} \ell_{\dot{x}}(x,\dot{x}) - \delta \ell_{\dot{x}}(x,\dot{x}) \right) \cdot h + \ell_{\dot{x}}(x,\dot{x}) \cdot \dot{h} \right) dt \\
&= \int_0^\infty \left(\frac{d}{dt} \{ e^{-\delta t} \ell_{\dot{x}}(x,\dot{x}) \} \cdot h + e^{-\delta t} \ell_{\dot{x}}(x,\dot{x}) \cdot \dot{h} \right) dt \\
&= \int_0^\infty \frac{d}{dt} \{ e^{-\delta t} \ell_{\dot{x}}(x,\dot{x}) \cdot h \} \, dt \\
&= \lim_{T \to \infty} e^{-\delta T} \ell_{\dot{x}}(x(T), \dot{x}(T)) \cdot h(T) - \ell_{\dot{x}}(\eta, \dot{x}(0)) \cdot h(0) \\
&= 0
\end{aligned}$$

since $\ell_{\dot{x}}(x(T), \dot{x}(T)) \cdot h(T)$ is bounded and $h(0) = 0$.

Conversely we assume that assertion (i) is true. From condition (9), for every $h \in E_0$, we have

$$0 = J_\delta'(x) \cdot h = \int_0^\infty e^{-\delta t} \ell_x(x, \dot{x}) \cdot h \, dt + \int_0^\infty e^{-\delta t} \ell_{\dot{x}}(x, \dot{x}) \cdot \dot{h} \, dt,$$

hence

$$\int_0^\infty e^{-\delta t} \ell_x(x, \dot{x}) \cdot h \, dt = - \int_0^\infty e^{-\delta t} \ell_{\dot{x}}(x, \dot{x}) \cdot \dot{h} \, dt.$$

We work on the components of these functions. We fix $k \in \{0, 1, ..., n\}$ and we denote

$$f(t) = e^{-\delta t} \ell_{x_k}(x(t), \dot{x}(t)) \quad \text{and} \quad g(t) = e^{-\delta t} \ell_{\dot{x}_k}(x(t), \dot{x}(t))$$
$$f_0, g_0 \quad \text{the restrictions of } f, g \text{ to } (0, \infty), \text{ respectively.}$$

Following the current notation of the Distributions Theory, $\mathcal{D}(0, \infty)$ is the space of the C^∞-functions with compact support. If $\psi \in \mathcal{D}(0, \infty)$, we can extend ψ onto $\mathbb{R}_+ = [0, \infty)$ to a function ψ_1 equal 0 at 0. And so $\psi_1 \in BC^1(\mathbb{R}_+, \mathbb{R})$. Then when we take $h = (h_1, h_2, \ldots, h_n)$, with $h_k = \psi_1$ and $h_j = 0$ if $j \neq k$, from an above mentioned equality, we obtain

$$\int_0^\infty f_0(t) \cdot \psi(t) \, dt = \int_0^\infty f(t) \cdot \psi_1(t) \, dt = - \int_0^\infty g(t) \cdot \dot{\psi}_1(t) \, dt = - \int_0^\infty g_0(t) \cdot \dot{\psi}(t) \, dt.$$

This relation means that f_0 is the distributional derivative of g_0 (cf. [17: Chapter II, §1]), and since f_0 is a continuous function, g_0 is of class C^1 and $f_0 = \dot{g}_0$. By the continuity of f, we obtain that g is of class C^1 on $\mathbb{R}_+$ and $f = \dot{g}$. We note that

$$e^{-\delta t} \ell_x = \frac{d}{dt}(e^{-\delta t} \ell_{\dot{x}}) = -\delta e^{-\delta t} \ell_{\dot{x}} + e^{-\delta t} \frac{d}{dt} \ell_{\dot{x}}$$

that implies $\ell_x = \frac{d}{dt} \ell_{\dot{x}} - \delta \ell_{\dot{x}}$ (this is system $(\mathcal{E}, \delta)$). Since k was arbitrarily choosen, we have proven assertion (ii).

Concerning the last assertion, the concavity of ℓ implies the concavity of J_δ, and since E_0 is the tangent space of E_η, assertion (i) is the first order necessary condition of (iii), and the concavity of J_δ ensures the equivalence between assertions (i) and (ii) ∎

We introduce the **Problem**

$(\mathcal{R}, \delta, \eta)$ Maximize $J_\delta(x)$ when $x \in E_\eta$.

Then Proposition 1 establishes that problem $(\mathcal{E}, \delta, \eta)$ is the Euler-Lagrange equation of the variational problem $(\mathcal{R}, \delta, \eta)$. And in the concave case, when $x \in BC^1$, x solves problem $(\mathcal{E}, \delta, \eta)$ if and only if x is an optimal solution of problem $(\mathcal{R}, \delta, \eta)$. In the following proposition we have collected some properties about the time translations.

Proposition 2. *Let $\ell \in C^1(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $\delta > 0$ and $x \in BC^1$. Then we have:*

(i) *For every $r \geq 0$,* $J_\delta(x(\cdot + r)) = e^{\delta r}\Big(J_\delta(x) - \int_0^r e^{-\delta t}\ell(x(t), \dot{x}(t))dt\Big)$.

(ii) *For every $r \geq 0$,* $\frac{d}{dr}J_\delta\big(x(\cdot + r)\big) = \delta J_\delta\big(x(\cdot + r)\big) - \ell\big(x(r), \dot{x}(r)\big)$.

(iii) $J_\delta\big(x(\cdot + r)\big) = J_\delta(x)$ *for every $r > 0$ if and only if $\ell\big(x(t), \dot{x}(t)\big) = \ell\big(x(0), \dot{x}(0)\big)$ for every $t \geq 0$.*

(iv) *If x is a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$, then, for every $r \geq 0$, $x(\cdot + r)$ is a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$, and if condition (7) is fulfilled, we have $J_\delta\big(x(\cdot + r)\big) = \sup J_\delta\big(E_{x(r)}\big)$.*

Proof. Assertion (i) is obtained in using the change of variable $s = t + r$. Differentiating the formula of (i) we obtain (ii). Assertion (iii) is a straightforward consequence of (ii). To prove assertion (iv) we note that the time translations of a solution of system $(\mathcal{E}, \delta)$ remains a solution of system $(\mathcal{E}, \delta)$ since system $(\mathcal{E}, \delta)$ is autonomous, and the last assertion is a consequence of Proposition 1 ∎

Proposition 3. *Let $\delta > 0$ and $\eta \in \mathbb{R}^n$, and let x and y be two solutions of problem $(\mathcal{E}, \delta, \eta)$ on $\mathbb{R}_+$ which belong to BC^1. We assume condition (8) fulfilled. Then the three following assertions hold and are equivalent:*

(i) $J''_\delta(x)(y - x, y - x) = 0$.

(ii) *For all $t \in \mathbb{R}_+$,* $\ell''\big(x(t), \dot{x}(t)\big) \cdot \big(y(t) - x(t), \dot{y}(t) - \dot{x}(t)\big) = 0$.

(iii) *For all $t \in \mathbb{R}_+$,*

$$\begin{aligned} &\ell_{xx}\big(x(t), \dot{x}(t)\big)\big(y(t) - x(t)\big) + \ell_{x\dot{x}}\big(x(t), \dot{x}(t)\big)\big(\dot{y}(t) - \dot{x}(t)\big) = 0 \\ &\ell_{\dot{x}x}\big(x(t), \dot{x}(t)\big)\big(y(t) - x(t)\big) + \ell_{\dot{x}\dot{x}}\big(x(t), \dot{x}(t)\big)\big(\dot{y}(t) - \dot{x}(t)\big) = 0. \end{aligned} \tag{11}$$

Proof. First we prove that assertion (i) holds. Since x and y are solutions of problem $(\mathcal{E}, \delta, \eta)$, by Proposition 1, $J_\delta(x) = J_\delta(y) = \sup J_\delta(E_\eta)$. Since E_η is convex and J_δ is concave, for every $\lambda \in [0, 1]$, we have $J_\delta\big((1 - \lambda)x + \lambda y\big) = \sup J_\delta(E_\eta)$. Hence $J'_\delta\big(x + \lambda(y - x)\big) \cdot h = 0$ for every $h \in E_0$. Therefore, $0 = \frac{d}{d\lambda}J'_\delta\big(x + \lambda(y - x)\big) \cdot h$, that implies $J''_\delta(x)(y - x, y - x)$, taking $h = y - x$ and $\lambda = 0$. Writing $\ell''(x, \dot{x})$ as a Hessian matrix

$$\begin{pmatrix} \ell_{xx}(x, \dot{x}) & \ell_{x\dot{x}}(x, \dot{x}) \\ \ell_{\dot{x}x}(x, \dot{x}) & \ell_{\dot{x}\dot{x}}(x, \dot{x}) \end{pmatrix}$$

we immediately see the equivalence of assertions (ii) and (iii). From assertion (ii) we have $\ell''(x, \dot{x})\big((y - x, \dot{y} - \dot{x}), (y - x, \dot{y} - \dot{x})\big) = 0$, and using formula (10) we obtain assertion (i). Conversely, since $\ell''(x, \dot{x})$ is negative semi-definite, by an argument of continuity, we can show that assertion (i) implies (ii) ∎

3. On the uniqueness

We give sufficient conditions to ensure the uniqueness of bounded solutions of system $(\mathcal{E},\delta)$, and we give some consequences of this uniqueness situation.

Theorem 1. *We assume that condition* (8) *is fulfilled. Then we have:*

(i) *Given $\delta > 0$ and $\eta \in \mathbb{R}^n$. Let $x \in BC^1$ a solution of problem $(\mathcal{E},\delta,\eta)$ on $\mathbb{R}_+$. If, for every $t \in \mathbb{R}_+$, $\ell_{\dot{x}\dot{x}}(x(t),\dot{x}(t))$ is negative definite, or if, for every $t \in \mathbb{R}_+$, $\ell_{x\dot{x}}(x(t),\dot{x}(t))$ is invertible, then x is the unique solution of problem $(\mathcal{E},\delta,\eta)$ on $\mathbb{R}_+$ which belong to the space BC^1.*

(ii) *If, for every $(\eta,v) \in \mathbb{R}^n \times \mathbb{R}^n$, $\ell_{\dot{x}\dot{x}}(\eta,v)$ is negative definite, or if, for every $(\eta,v) \in \mathbb{R}^n \times \mathbb{R}^n$, $\ell_{\dot{x}\dot{x}}(\eta,v)$ is invertible, then, for each $\delta > 0$ and $\eta \in \mathbb{R}^n$, problem $(\mathcal{E},\delta,\eta)$ cannot possess more than one solution in the space BC^1.*

Proof. Assertion (ii) is a consequence of assertion (i), and so it is sufficient to prove the last. To simplify the writing we denote

$$A(t) = \ell_{xx}(x(t),\dot{x}(t)), \qquad B(t) = \ell_{x\dot{x}}(x(t),\dot{x}(t)), \qquad D(t) = \ell_{\dot{x}\dot{x}}(x(t),\dot{x}(t)).$$

Knowning that $D(t)$ is negative definite (resp. $B(t)$ is invertible) for every $t \in \mathbb{R}_+$, if $y \in BC^1$ is a solution of problem $(\mathcal{E},\delta,\eta)$ on $\mathbb{R}_+$, then by the second (resp. first) equation of system (11) we have

$$\begin{aligned} B(t)^{\top}\big(y(t)-x(t)\big) + D(t)\big(\dot{y}(t)-\dot{x}(t)\big) &= 0 \\ A(t)\big(y(t)-x(t)\big) + B(t)\big(\dot{y}(t)-\dot{x}(t)\big) &= 0, \end{aligned}$$

respectively, that implies

$$\begin{aligned} \frac{d}{dt}(y-x)(t) &= -D(t)^{-1}B(t)^{\top}(y-x)(t) \\ \frac{d}{dt}(y-x)(t) &= -B(t)^{-1}A(t)(y-x)(t), \end{aligned}$$

respectively. Moreover we have $(y-x)(0) = \eta - \eta = 0$, and the matrix-valued function $t \longmapsto -D(t)^{-1}B(t)$ (resp. $t \longmapsto -B(t)^{-1}A(t)$) is continuous on $\mathbb{R}_+$. By the theorem about the uniqueness of the solutions of the linear normal Cauchy problems (cf. [9: Chapter I, §5]) we obtain that $y - x = 0$, i.e. $y = x$ ∎

Remarks. Via the Hilbert theorem (cf. [12: p. 34]) the hypothesis

$$\ell_{\dot{x}\dot{x}}(\eta,v) \quad \text{negative definite for every } (\eta,v) \in \mathbb{R}^n \times \mathbb{R}^n$$

ensures that problem $(\mathcal{E},\delta,\eta)$ can be translatable as a normal second order equation and consequently as a normal first order system with $(x,\dot{x})$ as unknown function. Under some additional assumption (for instance ℓ to be of class C^3) we can use the uniqueness theorem about the normal Cauchy problems. And so a solution x of problem $(\mathcal{E},\delta,\eta)$ is completely determined by the initial value $\dot{x}(0)$ since $x(0)$ is fixed equal to η. Theorem 1 asserts that there is at most one initial value $\dot{x}(0)$ which can furnish a bounded

BC^1-solution of problem $(\mathcal{E}, \delta, \eta)$ on $\mathbb{R}_+$. When η is a constant solution of system $(\mathcal{E}, \delta)$, assuming that $\ell_{\dot{x}\dot{x}}$ in negative definite or that $\ell_{x\dot{x}}$ is invertible, by assertion (i) of Theorem 1, problem $(\mathcal{E}, \delta, \eta)$ cannot possess any other solution in BC^1, and consequently cannot possess any non-constant periodic solution.

Theorem 2. *We assume that condition* (8) *is fulfilled. Let $\delta > 0$ and $x \in BC^1$ a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$. We assume that, for every $t \in \mathbb{R}_+$, $\ell_{\dot{x}\dot{x}}(x(t), \dot{x}(t))$ is negative definite or that, for every $t \in \mathbb{R}_+$, $\ell_{x\dot{x}}(x(t), \dot{x}(t))$ is invertible. Then we have:*

(i) *For every $r \geq 0$, $x(\cdot + r)$ is the unique solution of problem $(\mathcal{E}, \delta, x(r))$ on $\mathbb{R}_+$ in the space BC^1.*

(ii) *If $y \in BC^1$ is a solution of system $(\mathcal{E}, \delta)$ such that $\{y(\cdot + r) : r \in \mathbb{R}_+\} \cap \{x(\cdot + r) : r \in \mathbb{R}_+\} = \emptyset$, then $x(\mathbb{R}_+) \cap y(\mathbb{R}_+) = \emptyset$.*

(iii) *If there exist $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ and $x(t_1) = x(t_2)$, then x is $(t_2 - t_1)$-periodic on $[t_1, \infty)$.*

(iv) *If there exist $t_1, t_2, t_3 \in \mathbb{R}_+$ such that $t_1 < t_2 < t_3$, $x(t_1) = x(t_2) = x(t_3)$ and $\frac{t_3 - t_2}{t_2 - t_1} \notin \mathbb{Q}$ or $\frac{t_3 - t_1}{t_2 - t_1} \notin \mathbb{Q}$, then there exists a $t_0 \in \mathbb{R}_+$ such that x is constant on $[t_0, \infty)$.*

(v) *If x is not periodic on any interval $[t_0, \infty)$, $t_0 \geq 0$, then x is one-to-one on $\mathbb{R}_+$.*

(vi) *When $n = 1$, either there exists a $t_0 \in \mathbb{R}_+$ such that x is constant on $[t_0, \infty)$ or x is strictly monotonic on $\mathbb{R}_+$.*

Proof. (i) By Proposition 2/(iv), $x(\cdot + r)$ is a solution of problem $(\mathcal{E}, \delta, x(r))$, and its uniqueness results of Theorem 1/(i).

(ii) If $x(\mathbb{R}_+) \cap y(\mathbb{R}_+) \neq \emptyset$, then there exist $s_0, s_1 \in \mathbb{R}_+$ such that $x(s_0) = y(s_1) = \eta$. By Proposition 2/(iv), $x(\cdot + s_0)$ and $y(\cdot + s_1)$ are two solutions of problem $(\mathcal{E}, \delta, \eta)$ in BC^1. Then, by Theorem 1/(i), $x(\cdot + s_0) = y(\cdot + s_1)$, therefore $\{x(\cdot + r) : r \in \mathbb{R}_+\} \cap \{y(\cdot + r) : r \in \mathbb{R}_+\} \neq \emptyset$.

(iii) Denoting $\eta = x(t_1) = x(t_2)$, by Proposition 2/(iv), $x(\cdot + t_1)$ and $x(\cdot + t_2)$ are two solutions of problem $(\mathcal{E}, \delta, \eta)$ on $\mathbb{R}_+$ in BC^1. By Theorem 1/(i), $x(t + t_1) = x(t + t_2)$ for every $t \in \mathbb{R}_+$. Hence, for every $s \geq t_1$, we have $x(s + (t_2 - t_1)) = x((s - t_1) + t_2) = x((s - t_1) + t_1) = x(s)$.

(iv) From assertion (iii), $t_3 - t_2, t_2 - t_1$ and $t_3 - t_2$ are periods of x at least on $[t_2, \infty)$. The hypothesis of non-commensurability and the continuity of x imply that x is constant on $[t_2, \infty)$.

(v) It is a consequence of assertion (iii).

(vi) We assume that x is not strictly monotonic on $\mathbb{R}_+$. Therefore there exist $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ and $x(t_1) = x(t_2)$. Hence, by assertion (iii), x is T-periodic, with $T = t_2 - t_1$, on $\mathbb{R}_+$. Denoting $y = x(\cdot + t_1)$, y is a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$ in BC^1 and is T-periodic. There exist $t_*, t^* \in [0, T]$ such that

$$y(t_*) = \min y([0, T]) = \min(\mathbb{R}_+) \quad \text{and} \quad y(t^*) = \max y([0, T]) = \max y(\mathbb{R}_+).$$

For each $r \in (0, T)$, we define $\Delta_r(t) = y(t) - y(t + r)$. We have $\Delta_r(t^*) \geq 0$ and $\Delta_r(t_*) \leq 0$. Therefore, by the Bolzano theorem, there exists $t_r \in [0, T]$ such that $\Delta_r(t_r) = 0$, i.e. $y(t_r) = y(t_r + r)$. Then, by assertion (iii), y is r-periodic. Since y is continuous and r-periodic for each $r \in (0, T)$, y is constant, hence x is constant on $[t_1, \infty)$ ∎

Remarks. The assertions (ii) and (iii) look like classical results about first order autonomous ordinary differential equations. But system $(\mathcal{E}, \delta)$ is a second order ordinary differential equation. In view of this, (ii) and (iii) appear as surprising results.

The assertion (iv) indicates us that, when $n = 1$, under (8), a non-constant periodic solution cannot arise in system $(\mathcal{E}, \delta)$.

4. On the almost-periodic solutions

We show how periodic solutions of system $(\mathcal{E}, \delta)$ can generate almost-periodic solutions of this system.

Theorem 3. *We assume that* (7) *is fulfilled. Given $\delta > 0$, we assume that system $(\mathcal{E}, \delta)$ possesses two non-constant periodic solutions on $\mathbb{R}_+$, x_1 which is T_1-periodic and x_2 which is T_2-periodic. We also assume that $\frac{T_1}{T_2} \notin \mathbb{Q}$ and that $x_1(\mathbb{R}_+) \cap x_2(\mathbb{R}_+) \neq \emptyset$. Then system $(\mathcal{E}, \delta)$ possesses a uncountable family of quasi-periodic non-periodic solutions on $\mathbb{R}_+$ which are in the space $QP^1\left(\frac{2\pi}{T_1}, \frac{2\pi}{T_2}\right)$.*

Proof. By hypothesis, there exist $t_1, t_2 \in \mathbb{R}_+$ such that $x(t_1) = x(t_2) = \eta$. Then, by Proposition 2/(iv), $y_1 = x(\cdot + t_1)$ and $y_2 = x(\cdot + t_2)$ are solutions of problem $(\mathcal{E}, \delta, \eta)$ on $\mathbb{R}_+$, and $y_i \in C^1_{T_i}$, for $i = 1, 2$. For each $\lambda \in (0, 1)$,

$$z_\lambda = (1 - \lambda)y_1 + \lambda y_2 \in QP^1\left(\tfrac{2\pi}{T_1}, \tfrac{2\pi}{T_2}\right).$$

Since $\frac{T_1}{T_2} \notin \mathbb{Q}$, we have $y_1 \neq y_2$ and hence $\lambda \longmapsto z_\lambda$ is one-to-one. Therefore $\{z_\lambda : \lambda \in (0, 1)\}$ is a uncountable set. Since E_η is affine, $z_\lambda \in E_\eta$ for each $\lambda \in (0, 1)$. From the concavity of J_δ and Proposition 2/(iv),

$$J_\delta(z_\lambda) \geq (1 - \lambda)J_\delta(y_1) + \lambda J_\delta(y_2) = (1 - \lambda + \lambda) \sup J_\delta(E_\eta) = \sup J_\delta(E_\eta).$$

Therefore, by Proposition 1, z_λ is a solution of problem $(\mathcal{E}, \delta, \eta)$ on $\mathbb{R}_+$ ∎

Remarks. Theorem 3 is an extension of Theorem 3 of [2] to the case $\delta > 0$. Let us observe that in the previous case, $\delta = 0$, we did not need the hypothesis $x_1(\mathbb{R}_+) \cap x_2(\mathbb{R}_+) \neq \emptyset$. Here, with this additional hypothesis, for $\eta = x_1(t_1) = x_2(t_2)$, the problem $(\mathcal{E}, \delta, \eta)$ possesses a uncountable set of solutions in BC^1. Hence, taking account of Theorem 2/(i), we see that there necessarily exist $s_1, s_2, s'_1, s'_2 \in \mathbb{R}_+$ such that, for $i = 1, 2$,

$$\det \ell_{\dot{x}\dot{x}}\big(x_i(t_i + s_i), \dot{x}_i(t_i + s_i)\big) = 0 \quad \text{and} \quad \det \ell_{\dot{x}\dot{x}}\big(x_i(t_i + s'_i), \dot{x}_i(t_i + s'_i)\big) = 0,$$

We can say that the framework of Theorem 3 needs some degeneration on the second differential of ℓ.

5. On constant solutions

A constant solution of system $(\mathcal{E}, \delta)$ is a vector $c \in \mathbb{R}^n$ which verifies the equation

$$(\mathcal{C}, \delta) \quad \ell_x(c,0) + \delta \ell_{\dot{x}}(c,0) = 0.$$

In [3] one finds a classification of the constant solutions of this equation in the case $\delta = 0$. When $\delta > 0$, we distinguish between two cases: $\ell_x(c,0) = 0$ and $\ell_x(c,0) \neq 0$. In all the results of this section we shall assume that condition (7) is fulfilled.

Proposition 4. *Let $c \in \mathbb{R}^n$. Then all the following assertions are equivalent:*

(i) *There exists $\delta > 0$ such that c is a solution of equation $(\mathcal{C}, \delta)$ which verifies $\ell_x(c,0) = 0$.*

(ii) *There exist $\delta_1 > 0$ and $\delta_2 > 0, \delta_1 \neq \delta_2$, such that c is a solution of equations $(\mathcal{C}, \delta_1)$ and $(\mathcal{C}, \delta_2)$.*

(iii) *For every $\delta > 0$, c is a solution of equation $(\mathcal{C}, \delta)$.*

(iv) $\ell'(c,0) = 0$.

(v) $\ell(c,0) = \sup \ell(\mathbb{R}^n \times 0) = \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$.

(vi) *For every $\delta > 0$, $\frac{1}{\delta}\ell(c,0) = J_\delta(c) = \sup J_\delta(E_\eta) = \sup J_\delta(BC^1)$.*

Proof. From (i) we obtain $\ell_{\dot{x}}(c,0) = 0 = \ell_x(c,0)$ and assertion (ii) is satisfied. If (ii) holds, then $(\delta_2 - \delta_1)\ell_{\dot{x}}(c,0) = 0$ implies $\ell_{\dot{x}}(c,0) = 0$. Therefore $\ell_x(c,0) = 0$ by equation $(\mathcal{C}, \delta_1)$, and consequently, for every $\delta > 0$, we have equation $(\mathcal{C}, \delta)$, and assertion (iii) is verified. From (iii) we deduce that $\ell_x(c,0) = \ell_{\dot{x}}(c,0) = 0$, hence assertion (iv) holds. Under (iv), because of the concavity of ℓ, $\ell(c,0) = \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, $\ell(c,0) \leq \sup \ell(\mathbb{R}^n \times 0) \leq \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$ imply the two equalities of assertion (v). The first equality of (vi) results of a simple calculation. By (v), for every $x \in BC^1$, we have $\ell(c,0) \geq \ell(x(t), \dot{x}(t))$ that implies $J_\delta(c) \geq J_\delta(x)$. Hence $J_\delta(c) = \sup J_\delta(BC^1)$. Moreover, $J_\delta(c) \leq \sup J_\delta(E_c) \leq \sup J_\delta(BC^1) = J_\delta(c)$ imply the two last equalities of assertion (iv). Finally, if (vi) is true, then, for every $\eta \in \mathbb{R}^n$, $\eta \in BC^1$. Therefore $J_\delta(c) \geq J_\delta(\eta)$, i.e. $\frac{1}{\delta}\ell(c,0) \geq \frac{1}{\delta}\ell(\eta,0)$, i.e. $\ell(c,0) \geq \ell(\eta,0)$, and by the first order necessary condition of minimality, we have $\ell_x(c,0) = 0$. Moreover, $\sup J_\delta(E_c) \geq J_\delta(c) = \sup J_\delta(BC^1) \geq \sup J_\delta(E_c)$. Therefore $J_\delta(c) = \sup J_\delta(E_c)$ and, by Proposition 1, we can assert that c is a solution of system $(\mathcal{E}, \delta)$. Hence c is a solution of equation $(\mathcal{C}, \delta)$, and assertion (i) holds ∎

Proposition 5. *If there exists $\eta \in \mathbb{R}^n$ such that $\ell_x(\eta,0) = 0$ and $\ell_{\dot{x}}(\eta,0) \neq 0$, then, for every $\delta > 0$, when $c \in \mathbb{R}^n$ is a solution of system $(\mathcal{E}, \delta)$, we necessarily have $\ell_x(c,0) \neq 0$.*

Proof. Let $\delta > 0$, and $c \in \mathbb{R}^n$ a solution of system $(\mathcal{E}, \delta)$ such that $\ell_x(c,0) = 0$. By the lemma of [3: p. 461], we necessarily have $\ell_{\dot{x}}(c,0) \neq 0$ and consequently c cannot solve system $(\mathcal{E}, \delta)$. This is a contradiction ∎

Proposition 6. *Let $c \in \mathbb{R}^n$ such that $\ell'(c,0) = 0$ and let $x \in E_c$. Then all the following assertions are equivalent:*

(i) *There exists $\delta > 0$ such that x is a solution of problem $(\mathcal{E}, \delta, c)$ on $\mathbb{R}_+$.*

(ii) *For every $\delta > 0$, x is a solution of problem $(\mathcal{E}, \delta, c)$ on $\mathbb{R}_+$.*

(iii) *For every $t \in \mathbb{R}_+$, $\ell(x(t), \dot{x}(t)) = \ell(c, 0)$.*

(iv) *For every $t \in \mathbb{R}_+$, $\ell'(x(t), \dot{x}(t)) = 0$.*

Proof. The implications (iv) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious. Since ℓ is concave, $\ell'(c,0) = 0$ is equivalent to $\ell(c,0) = \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$. From (iii) we obtain, for every $t \in \mathbb{R}_+$, $\ell(x(t), \dot{x}(t)) = \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore $\ell'(x(t), \dot{x}(t)) = 0$. That proves the implication (iii) $\Rightarrow$ (iv). It remains us to prove the implication (i) $\Rightarrow$ (iii). From (i) and Proposition 1, we have $J_\delta(x) = \sup J_\delta(E_c)$, and since c is a solution of problem $(\mathcal{E}, \delta, c)$, we have $J_\delta(c) = \sup J_\delta(E_c)$. Therefore $J_\delta(x) - J_\delta(c) = 0$. From Proposition 4, we have $\ell(c,0) = \sup \ell(\mathbb{R}^n \times \mathbb{R}^n)$, therefore, for every $t \in \mathbb{R}_+$, we obtain $\ell(c,o) - \ell(x(t), \dot{x}(t)) \geq 0$, hence $e^{-\delta t}(\ell(c,0) - \ell(x(t), \dot{x}(t))) \geq 0$. Since $0 = \int_0^\infty e^{-\delta t}(\ell(c,0) - \ell(x(t), \dot{x}(t)))dt$, the usual properties of the integral permits us to assert that $\ell(c,0) - \ell(x(t), \dot{x}(t)) = 0$, for every $t \in \mathbb{R}_+$ ∎

Remarks. An assertion equivalent to the previous is the inclusion $\{(x(t), \dot{x}(t)) : t \in \mathbb{R}_+\} \subset \operatorname{Arg\,max} \ell$.

Let $c \in \mathbb{R}^n$ such that $\ell'(c,0) = 0$. Let $x \in BC^1$ a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$ such that $c \in x(\mathbb{R}_+)$. Then there exists $t_0 \in \mathbb{R}_+$ such that, for every $t \geq t_0, \ell(x(t), \dot{x}(t)) = \ell(c,0)$. This fact is easy to verify: there exists $t_0 \in \mathbb{R}_+$ such that $x(t_0) = c$. Then $x(\cdot + t_0)$ is a solution on $\mathbb{R}_+$ of problem $(\mathcal{E}, \delta, c)$, and we conclude in using Proposition 6.

Proposition 7. *Let $c \in \mathbb{R}^n$ such that $\ell'(c,0) = 0$ and let $x \in AP^1$. Then all the following assertions are equivalent:*

(i) *x is a solution of system $(\mathcal{E}, 0)$ on $\mathbb{R}_+$.*

(ii) *For every $t \in \mathbb{R}_+$, $\ell(x(t), \dot{x}(t)) = \ell(c,0)$.*

(iii) *For every $t \in \mathbb{R}_+$, $\ell'(x(t), \dot{x}(t)) = 0$.*

(iv) *There exist $\delta_1 > 0$ and $\delta_2 > 0$, $\delta_1 \neq \delta_2$, such that x is a solution on $\mathbb{R}_+$ of systems $(\mathcal{E}, \delta_1)$ and $(\mathcal{E}, \delta_2)$.*

(v) *For every $\delta \geq 0$, x is a solution on $\mathbb{R}_+$ of system $(\mathcal{E}, \delta)$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is established in Proposition 4 of [3]. The equivalence (ii) $\Leftrightarrow$ (iii) results of the concavity of ℓ. The implications (iii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iv) are obvious. From (iv) we have $\ell_x(x,\dot{x}) - \frac{d}{dt}\ell_{\dot{x}}(x,\dot{x}) = -\delta_1 \ell_{\dot{x}}(x,\dot{x}) = -\delta_2 \ell_{\dot{x}}(x,\dot{x})$, and since $\delta_1 \neq \delta_2$ we obtain $\ell_{\dot{x}}(x,\dot{x}) = 0$ and consequently $\ell_x(x,\dot{x}) = 0$, that implies $\ell'(x,\dot{x}) = 0$. This proves the implication (iv) $\Rightarrow$ (iii) ∎

Remarks. Proposition 7 is an extension of Proposition 4 of [3] to the case $\delta > 0$. In Proposition 7, we do not assume that $x(0) = c$. It is important to note that the existence of a $\delta > 0$ such that $x \in AP^1$ is a solution of system $(\mathcal{E}, \delta)$ is not an assertion equivalent to the assertions of Proposition 7.

Proposition 8. *Let $c \in \mathbb{R}^n$ such that $\ell'(c,0) = 0$, and let $\delta > 0$. Then we have:*

(i) *If there exists two non-constant periodic solutions of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$, x_1 which is T_1-periodic and x_2 which is T_2-periodic such that $\frac{T_1}{T_2} \notin \mathbb{Q}$ and $\ell(x_i(t), \dot{x}_i(t)) = \ell(c,0)$, for $i = 1,2$ and every $t \in \mathbb{R}_+$, then there exists a uncountable family of quasi-periodic non-periodic solutions of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$. Moreover each element z of this family belongs to $QP^1(\frac{2\pi}{T_1}, \frac{2\pi}{T_2})$ and satisfies, for every $t \in \mathbb{R}_+$, $\ell(z(t), \dot{z}(t)) = \ell(c,0)$.*

(ii) *If $z \in AP^1$ is a solution of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$ such that, for every $t \in \mathbb{R}_+$, $\ell(z(t), \dot{z}(t)) = \ell(c,0)$, then there exist periodic solutions x of system $(\mathcal{E}, \delta)$ on $\mathbb{R}_+$ which verify, for every $t \in \mathbb{R}_+$, $\ell(x(t), \dot{x}(t)) = \ell(c,0)$. Moreover, the mean value $\mathcal{M}\{z\}$ of z is a constant solution of system $(\mathcal{E}, \delta)$ and verifies $\ell(\mathcal{M}\{z\}, 0) = \ell(c,0)$.*

Proof. To obtain assertion (i), we use the results of [2] and the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 7. For assertion (ii), we use [3] ∎

Proposition 9. *We assume that (8) is fulfilled. Let $c \in \mathbb{R}^n$ a solution of system $(\mathcal{E}, \delta)$ with $\delta > 0$. If $\dim \operatorname{Ker} \ell''(c,0) \leq 1$, then c is the unique solution of problem $(\mathcal{E}, \delta, c)$ in AP^1.*

Proof. It is the same as that of Proposition 3 of [7] ∎

6. The linear case

In this section we discuss the linear systems $(\mathcal{L}, \delta)$. We consider the condition

$$S = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \quad \text{is symmetric negative definite.} \tag{12}$$

This hypothesis ensures the concavity of Λ defined in (5). System $(\mathcal{L}, \delta)$ is a particular case of system $(\mathcal{E}, \delta)$, hence all the results of the preceding sections are relevant for system $(\mathcal{L}, \delta)$. For instance, using the functional Q_δ defined in (6), given $\delta > 0$ and $\eta \in \mathbb{R}^n$, the application of Proposition 1 to system $(\mathcal{L}, \delta)$ gives us, for $h \in E_\eta$, the following equivalences:

$$\begin{aligned} Q_\delta(h) = \sup Q_\delta(E_\eta) \quad &\Longleftrightarrow \quad Q'_\delta(h) \cdot k = 0 \text{ for all } k \in E_0 \\ &\Longleftrightarrow \quad h \text{ is a solution of } (\mathcal{L}, \delta, \eta) \text{ on } \mathbb{R}_+. \end{aligned}$$

And when $\eta = 0$, these assertions are also equivalent to the following one:

$$Q_\delta(h) = 0.$$

In using Proposition 3, when $\eta = 0$, these assertions are also equivalent to

$$\left.\begin{aligned} Ah(t) + B\dot{h}(t) = 0 \\ B^T h(t) + D\dot{h}(t) = 0 \end{aligned}\right\} \quad \text{for every } t \in \mathbb{R}_+. \tag{13}$$

The application of Theorem 1 on system $(\mathcal{L}, \delta)$ furnishes the following result:

When D is negative definite or when B is invertible, problem $(\mathcal{L}, \delta, \eta)$ cannot possess more than one solution in BC^1, for each $\delta > 0$. In particular, 0 is the unique solution of problem $(\mathcal{L}, \delta, 0)$ in BC^1.

We note that $\Lambda_x(x, \dot{x}) = Ax + B\dot{x}$ and $\Lambda_{\dot{x}} = B^\top x + D\dot{x}$ (Λ is defined in formula (5)), and so the constant solutions of system $(\mathcal{L}, \delta)$ are the vectors $c \in \mathbb{R}^n$ which solve problem

$$(\mathcal{CL}, \delta) \quad (A + \delta B^\top)c = 0.$$

The condition $\Lambda_x(c, 0) = 0$ used in Section 6 becomes $Ac = 0$, i.e. $c \in \operatorname{Ker} A$.

Proposition 10. *We assume that condition (12) is fulfilled. Then we have:*

(i) *If $\det A = 0$, then $\operatorname{Ker} A = \bigcap_{\delta > 0}\{c \in \mathbb{R}^n : c$ solves equation $(\mathcal{CL}, \delta)\}$.*

(ii) *If $\det A \neq 0$ (for instance A is negative definite), then there are at most n distinct positive values of δ for which system $(\mathcal{L}, \delta)$ can possess non-zero constant solutions.*

Proof. (i) From Lemma 1 of [7] there exists a real $(n \times n)$-matrix M such that $B^\top = MA$. If $Ac = 0$, then $B^\top c = 0$ and therefore $(A + \delta B^\top)c = 0$ for every $\delta > 0$. Taking $\delta \to 0+$, we obtain $Ac = 0$.

(ii) We have $(A + \delta B^\top) = -\delta A(-A^{-1}B^\top - \frac{1}{\delta}I)$. Let $c \neq 0$ in $\mathbb{R}^n$. Then we have the following equivalences:

$$
\begin{aligned}
&(A + \delta B^\top)c = 0 \\
&\iff \quad (-A^{-1}B^\top)c = \frac{1}{\delta}c \\
&\iff \quad c \text{ is an eigenvector associated to the eigenvalue } \frac{1}{\delta} \text{ of } -A^{-1}B^\top.
\end{aligned}
$$

Therefore system $(\mathcal{L}, \delta)$ possesses a non-zero constant solution if and only if $\frac{1}{\delta}$ is an eigenvalue of $-A^{-1}B^\top$. Since $-A^{-1}B^\top$ possesses at most n real distinct positive eigenvalues, we obtain the announced result ∎

Proposition 11. *We assume that condition (12) is fulfilled. Let $h \in C^1(\mathbb{R}_+, \mathbb{R}^n)$. If $h(\mathbb{R}_+) \subset \operatorname{Ker} A \cap \operatorname{Ker} D$, then h is a solution of system $(\mathcal{L}, \delta)$ on $\mathbb{R}_+$, for each $\delta > 0$.*

Proof. From Lemma 1 of [7], there exist M and N, two $n \times n$ real matrices such that $B^\top = MA$ and $B = ND$. And so $B^\top h(t) = 0 = Bh(t)$, hence $B\dot{h}(t) = 0 = D\dot{h}(t)$. Therefore h is a solution of system $(\mathcal{L}, \delta)$ on $\mathbb{R}_+$, for every $\delta > 0$.

Proposition 12. *We assume that condition (12) is fulfilled. Let $h \in E_0$ a solution of problem $(\mathcal{L}, \delta, 0)$ on $\mathbb{R}_+$, with $\delta > 0$. Then we have:*

(i) *For every $t \in \mathbb{R}_+$, $Ah(t) \cdot h(t) = D\dot{h}(t) \cdot \dot{h}(t) = -B\dot{h}(t) \cdot h(t)$.*

(ii) *If B is symmetric, then $h(\mathbb{R}_+) \subset \operatorname{Ker} A \cap \operatorname{Ker} D$.*

Proof. (i) In the system (13), it is sufficient to calculate the inner product of the first equation and of $h(t)$, and the inner product of the second equation and of $\dot{h}(t)$.

(ii) By the second equation of (13), we have $B^\top h(t) + D\dot{h}(t) = 0$. We know that $B^\top = B = ND$, and taking $k(t) = Dh(t)$, we obtain $Nk(t) + \dot{k}(t) = 0$. Therefore $k(t) = e^{-Nt}k(0) = 0$ (cf. [9: Chapter III, §4]). And so $h(t) \in \operatorname{Ker} D$, that implies $D\dot{h} = 0$, and by assertion (i), $Ah(t) \cdot h(t) = 0$. Since A is symmetric negative semi-definite, we have $Ah(t) = 0$, i.e. $h(t) \in \operatorname{Ker} A$ ∎

Proposition 13. We assume that condition (12) is fulfilled. Let $h \in BC^1$ a solution of problem $(\mathcal{L}, \delta, 0)$ on $\mathbb{R}_+$ such that $0 \in h(\mathbb{R}_+)$. Then there exists $t_0 \in \mathbb{R}_+$ such that $h(\cdot + t_0)$ is a solution of system (13) on $\mathbb{R}_+$.

Proof. Let $t_0 \in \mathbb{R}_+$ such that $h(t_0) = 0$. By Proposition 2, $h(\cdot + t_0)$ is a solution of problem $(\mathcal{L}, \delta, 0)$ on $\mathbb{R}_+$, therefore $h(\cdot + t_0)$ is a solution of system (13) on $\mathbb{R}_+$ ∎

In order to study the almost-periodic solutions of system $(\mathcal{L}, \delta)$, we extend to the case $\delta > 0$ the method used in [7] to the case $\delta = 0$. For each $\lambda \in \mathbb{R}$, we introduce the $n \times n$ complex matrix

$$U_{\delta,\lambda} = \lambda^2 D + i\lambda(B - B^\top) + A + \delta(B^\top + i\lambda D) \tag{14}$$

where $i = \sqrt{-1}$, and the polynomial

$$P_\delta(\lambda) = \det U_{\delta,\lambda}. \tag{15}$$

We have:

(i) $\deg P_\delta \le 2n$

(ii) for every $\delta > 0$ and $\lambda \in \mathbb{R}$, $\overline{U_{\delta,\lambda}} = U_{\delta,-\lambda}$ where the upper bar denotes the complex conjugation

(iii) $\overline{P_\delta(\lambda)} = P_\delta(-\lambda)$

and so, if λ is a real root of P_δ, $-\lambda$ also is a real root of P_δ.

Lemma 3. *Given $\delta > 0$, if λ is a non-zero real root of P_δ and if $\xi \in \operatorname{Ker} U_{\delta,\lambda}$ (in $\mathbb{C}^n$), then $t \mapsto x(t) = \xi e^{i\lambda t} + \overline{\xi} e^{-i\lambda t}$ is a real $\frac{2\pi}{\lambda}$-periodic solution of system $(\mathcal{L}, \delta)$ on $\mathbb{R}_+$. Conversely, if system $(\mathcal{L}, \delta)$ possesses a non-constant T-periodic solution $(T > 0)$, then there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $\frac{2\pi k}{T}$ is a non-zero real root of P_δ.*

Proof. For every $t \in \mathbb{R}_+$, taking $z(t) = \xi e^{i\lambda t} \in \operatorname{Ker} U_{\delta,\lambda}$ in $\mathbb{C}^n$, we have $\dot{z}(t) = i\lambda z(t)$ and $\ddot{z}(t) = -\lambda^2 z(t)$. From $U_{\delta,\lambda} z(t) = 0$ we obtain

$$-D\ddot{z}(t) + (B - B^\top)\dot{z}(t) + A + \delta B^\top z(t) + \delta D\dot{z}(t) = 0,$$

i.e.

$$Az(t) + B\dot{z}(t) = \frac{d}{dt}\big(B^\top z(t) + D\dot{z}(t)\big) - \delta\big(B^\top z(t) + D\dot{z}(t)\big).$$

And so, z is a complex solution of system $(\mathcal{L}, \delta)$. Since the coefficients of system $(\mathcal{L}, \delta)$ are real, $\overline{z}$ also is a solution of system $(\mathcal{L}, \delta)$, and since system $(\mathcal{L}, \delta)$ is linear, $x = z + \overline{z}$ is a real solution of system $(\mathcal{L}, \delta)$.

Conversely, if x is a real solution non-constant T-periodic of system $(\mathcal{L},\delta)$, then there exists $k \in \mathbb{Z}\setminus\{0\}$ such that the Fourier coefficient $\mathrm{a}(x, \frac{2\pi k}{T}) \neq 0$. From the relation $\mathrm{a}(\dot{\varphi},\lambda) = i\lambda \mathrm{a}(\varphi,\lambda)$, taking $\lambda = \frac{2\pi k}{T}$, we obtain

$$(A + i\lambda B)\mathrm{a}(x,\lambda) = i\lambda(B^\top + i\lambda D)\mathrm{a}(x,\lambda) - \delta(B^\top + i\lambda D)\mathrm{a}(x,\lambda),$$

i.e. $U_{\delta,\lambda}\mathrm{a}(x,\lambda) = 0$, hence we have $\det U_{\delta,\lambda} = P_\delta(\lambda) = 0$ since $\mathrm{a}(x,\lambda) \neq 0$ ∎

Proposition 14. *Let $h \in AP^1$ a solution of system $(\mathcal{L},\delta)$ on $\mathbb{R}_+$, with $\delta > 0$. Then we have:*

(i) *$\mathcal{M}\{h\}$ is a constant solution of $(\mathcal{L},\delta)$, i.e. $\mathcal{M}\{h\} \in \mathrm{Ker}(A+\delta B^\top)$.*

(ii) *For every $\lambda \in \mathbb{R}, \mathrm{a}(x,\lambda) \in \mathrm{Ker}\, U_{\delta,\lambda}$.*

(iii) *System $(\mathcal{L},\delta)$ possesses periodic solutions defined on $\mathbb{R}_+$ which are non-constant when h is non-constant.*

Proof. Taking $\lambda = 0$ in (ii) we obtain assertion (i). With the same argument as in the proof of Lemma 3, converse part, we obtain that, for every $\lambda \in \mathbb{R}$, $U_{\delta,\lambda}\mathrm{a}(h,\lambda) = 0$ and assertion (ii) is proven.

To prove assertion (iii) we use a theorem of Besicovitch (cf. [6]) which asserts that, for $\varphi \in AP^1$ and each $T < 0$ and denoting $c_\nu(\varphi) = \frac{1}{\nu}\sum_{\alpha=1}^{\nu}\varphi(\cdot + \alpha T)$, the sequence $(c_\nu(\varphi))_\nu$ uniformly converges on towards a T-periodic C^1-function, and the sequence of the derivatives $\left(\frac{d}{dt}c_\nu(\varphi)\right)_\nu$ also is uniformly convergent on $\mathbb{R}_+$. We denote by x the limit of $(c_\nu(\varphi))_\nu$, $x \in C^1_T$. Since h solves system $(\mathcal{L},\delta)$ and since the functional operators c_ν are linear, we have, for every $\nu \in \mathbb{N}_*$,

$$c_\nu\left(\frac{d}{dt}(B^\top h + D\dot{h})\right) = Ac_\nu(h) + Bc_\nu(\dot{h}) + \delta\big(B^\top c_\nu(h) + Dc_\nu(\dot{h})\big).$$

Moreover,

$$c_\nu\left(\frac{d}{dt}(B^T h + D\dot{h})\right) = \frac{d}{dt}c_\nu(B^\top h + D\dot{h}).$$

When $\nu \to \infty$, we obtain

$$\frac{d}{dt}(B^\top x + D\dot{x}) = Ax + B\dot{x} + \delta(B^\top x + D\dot{x}),$$

i.e. x solves system $(\mathcal{L},\delta)$. When h is non-constant, there exists $\lambda \neq 0$ such that $\mathrm{a}(h,\lambda) \neq 0$. Taking $T = \frac{2\pi}{\lambda}$, the function x provided by the Besicovitch theorem satisfies $\mathrm{a}(x,\lambda) = \mathrm{a}(h,\lambda)$. Hence $\mathrm{a}(h,\lambda) \neq 0$ with $\lambda \neq 0$ that implies the non-constancy of x ∎

Remark. Since system $(\mathcal{L},\delta)$ is linear, if system $(\mathcal{L},\delta)$ possesses periodic solutions with non-commensurable periods, then system $(\mathcal{L},\delta)$ possesses non-periodic quasi-periodic solutions built as sum of the periodic solutions.

Theorem 4. *We fix $\delta > 0$. Then we have:*

(i) *$P_\delta = 0$ if and only if, for each finite or countable subset F of $\mathbb{R}$, there exists a solution $x \in AP^1$ of system $(\mathcal{L}, \delta)$ such that $\{\lambda \in \mathbb{R} : \mathbf{a}(x, \lambda) \neq 0\} \supset F$.*

(ii) *If $P_\delta \neq 0$ and if P_δ possesses non-zero real roots, denoting by $\lambda_1, ..., \lambda_\nu$ the distinct real positive roots of P_δ, each function of the form*

$$h(t) = c + \sum_{k=1}^{\nu} \left(\xi_k e^{i\lambda_k t} + \overline{\xi} e^{-i\lambda_k t}\right)$$

with $c \in \operatorname{Ker}(A + \delta B^{\mathsf{T}})$ and $\xi_k \in \operatorname{Ker} U_{\delta,\lambda_k}$ is an almost-periodic solution of system $(\mathcal{L}, \delta)$. Conversely each $h \in AP^1$ which is a solution of system $(\mathcal{L}, \delta)$ necessarily has this form.

(iii) *P_δ does not possess any non-zero real root if and only if system $(\mathcal{L}, \delta)$ does not possess any non-zero solution in AP^1.*

Proof. (i) First we assume that $P_\delta = 0$ and we fix a countable subset F of $\mathbb{R}$. We build a bijective mapping $j \longmapsto \lambda_j$, from $\mathbb{N}$ into F, and so $F = \{\lambda_j : j \in \mathbb{N}\}$. Since $P_\delta = 0$, for each $j \in \mathbb{N}$, we have $P_\delta(\lambda_j) = 0$. Therefore there exists $\xi_j \in \operatorname{Ker} U_{\delta,\lambda_j}$, $\xi_j \neq 0$. For each $j \in \mathbb{N}$, we choose $w_j \in \mathbb{C} \setminus \{0\}$ such that, denoting $a_j = w_j \xi_j$, we have $\|a_j\| < 2^{-j}$, $\|\lambda_j a_j\| < 2^{-j}$ and $\|\lambda_j^2 a_j\| < 2^{-j}$. We take $x_j(t) = a_j e^{i\lambda_j t} + \overline{a_j} e^{-i\lambda_j t}$, and so by Lemma 3, x_j is a solution of system $(\mathcal{L}, \delta)$. Since system $(\mathcal{L}, \delta)$ is linear, for every $m \in \mathbb{N}$, $\sum_{j=0}^{m} x_j$ also is a solution of system $(\mathcal{L}, \delta)$. With our choice of the a_j, the three sequences $\left(\sum_{j=0}^{m} x_j\right)_m$, $\left(\sum_{j=0}^{m} \dot{x}_j\right)_m$ and $\left(\sum_{j=0}^{m} \ddot{x}_j\right)_m$ uniformly converge on $\mathbb{R}_+$ towards $x, \dot{x}$ and $\ddot{x}$, respectively, which are almost-periodic, and we easily verify that x is a solution of system $(\mathcal{L}, \delta)$ in AP^1. Moreover, for each $j \in \mathbb{N}$, $\mathbf{a}(x, \lambda) = a_j \neq 0$, hence $F \subset \{\lambda \in \mathbb{R} : \mathbf{a}(x, \lambda) \neq 0\}$. When F is finite, we proceed in a more simple similar way.

Conversely, we consider an one-to-one mapping $j \longmapsto \lambda_j$, from $\mathbb{N}$ into $\mathbb{R}$. By hypothesis, taking $F = \{\lambda_j : j \in \mathbb{N}\}$, there exists a solution $x \in AP^1$ of system $(\mathcal{L}, \delta)$ such that, for every $j \in \mathbb{N}$, $\mathbf{a}(x, \lambda_j) \neq 0$. By Proposition 14, for every $j \in \mathbb{N}$, $\mathbf{a}(x, \lambda_j) \in \operatorname{Ker} U_{\delta,\lambda_j}$. Therefore $\operatorname{Ker} U_{\delta,\lambda_j} \neq \{0\}$, hence $P_\delta(\lambda_j) = 0$. And so P_δ possesses an infinite set of roots, that implies $P_\delta = 0$.

(ii) By Lemma 3 and by the linearity of system $(\mathcal{L}, \delta)$, h is a solution of system $(\mathcal{L}, \delta)$. Conversely, if h is a solution of system $(\mathcal{L}, \delta)$ in AP^1, then, by Proposition 14, $\mathbf{a}(x, \lambda) \neq 0$ implies $\operatorname{Ker} U_{\delta,\lambda} \neq \{0\}$. Hence $P_\delta(\lambda) = 0$, and so $\lambda \in \{\pm\lambda_j : j = 1, ..., \nu\}$. Therefore, the only non-zero Fourier-Bohr coefficients of h are among the $\mathbf{a}(x, \pm\lambda_j)$, and consequently, h is in the announced form.

(iii) By Proposition 14/(ii), if P_δ does not possess any real root, then the Fourier-Bohr coefficients of a solution of system $(\mathcal{L}, \delta)$ in AP^1 necessarily are zero. Therefore, from [8], this solution is equal to zero. Conversely, if system $(\mathcal{L}, \delta)$ does not possess any solution in AP^1, then, by Proposition 14/(ii) and Lemma 3, P_δ does not possess any real root ∎

Remarks. Theorem 4 is an extension of Theorem 2 of [7] to the case $\delta > 0$.

7. A non-local linearization

Because of the concavity of problem $(\mathcal{E},\delta,\eta)$, we can built a linearization technique which provides non-local results about $(\mathcal{E},\delta)$.

Proposition 15. *We assume that condition* (8) *is fulfilled. Let $\delta > 0$ and $c \in \mathbb{R}^n$ a constant solution of system $(\mathcal{E},\delta)$. We consider system $(\mathcal{L},\delta)$ with $A = \ell_{xx}(c,0), B = \ell_{x\dot{x}}(c,0)$ and $D = \ell_{\dot{x}\dot{x}}(c,0)$. Then system $(\mathcal{L},\delta)$ satisfies condition* (12), *and when $x \in BC^1$ is a solution of problem $(\mathcal{E},\delta,c)$ on $\mathbb{R}_+$, $x-c$ is a solution of problem $(\mathcal{L},\delta,0)$ on $\mathbb{R}_+$.*

Proof. In Proposition 3, we take $x = c$ and $y = x$. Then system (11) becomes system (13) and as we remark it at the beginning of Section 6, system (13) is equivalent to problem $(\mathcal{L},\delta,0)$ ∎

Theorem 5. *We assume that condition* (8) *is fulfilled. Let $\delta > 0$ and $c \in \mathbb{R}^n$ a constant solution of system $(\mathcal{E},\delta)$ such that $\ell_{x\dot{x}}(c,0) = \ell_{\dot{x}x}(c,0)$. Then we have:*

(i) *If $x \in BC^1$ is a solution of problem $(\mathcal{E},\delta,c)$ on $\mathbb{R}_+$, then $x(\mathbb{R}_+)-c \subset \mathrm{Ker}\,\ell_{xx}(c,0) \cap \mathrm{Ker}\,\ell_{\dot{x}\dot{x}}(c,0)$.*

(ii) *If $y \in BC^1$ is a solution of system $(\mathcal{E},\delta)$ on $\mathbb{R}_+$ such that $c \in y(\mathbb{R}_+)$, then there exists $t_0 \in \mathbb{R}_+$ such that $y([t_0,\infty)) - c \subset \mathrm{Ker}\,\ell_{xx}(c,0) \cap \mathrm{Ker}\,\ell_{\dot{x}\dot{x}}(c,0)$.*

Proof. Using Propositions 15 and 13 we obtain assertion (i). We consider $t_0 \in \mathbb{R}_+$ such that $y(t_0) = c$ and take $x = y(\cdot + t_0)$. Then assertion (ii) is a consequence of (i) ∎

Theorem 6. *Under the assumptions of Proposition 15, we have:*

(i) *If P_δ does not possess any real root, then problem $(\mathcal{E},\delta,c)$ does not possess any non-constant almost-periodic solution.*

(ii) *If $P_\delta \neq 0$ and if $\lambda_1,\dots,\lambda_\nu$ are the distinct real positive roots of P_δ, then a solution of problem $(\mathcal{E},\delta,c)$ in AP^1 necessarily has the form*

$$x(t) = c + m + 2\mathcal{R}e\left(\sum_{k=1}^{\nu} \xi_k e^{i\lambda_k t}\right),$$

where

$$\begin{aligned} m &\in \mathrm{Ker}\Big(\ell_{xx}(c,0) + \delta\ell_{\dot{x}x}(c,0)\Big) \\ \xi_k &\in \mathrm{Ker}\Big(\lambda_k^2\ell_{\dot{x}\dot{x}}(c,0) + i\lambda_k\big(\ell_{x\dot{x}}(c,0) - \ell_{\dot{x}x}(c,0)\big) \\ &\qquad + \ell_{xx}(c,0) + \delta\big(\ell_{\dot{x}x}(c,0) + i\lambda_k\ell_{\dot{x}\dot{x}}(c,0)\big)\Big) \subset \mathbb{C}^n. \end{aligned}$$

Proof. This is a straightforward consequence of Proposition 15 and Theorem 4 ∎

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