Existence and Uniqueness of a Regular Solution of the Cauchy- Dirichlet Problem for Doubly Nonlinear Parabolic Equations

A. V. Ivanov

Abstract. Existence and uniqueness of some Holder continuous generalized solution of Cauchy-Dirichlet problem for a class of degenerate or singular quasilinear parabolic equations is established. Similar equations arise in the study of turbulent filtration of a gas or a fluid through porous media. **Figure 12**
 FIU : $\frac{1}{2}$
 FIU :

Keywords: *Quasilinear parabolic equations, generalized solutions, existence, uniqueness, Hölder estimates*

AMS subject classification: 35K55, 35K65, 76A05

1. Introduction

Let Ω be a bounded open set in \mathbb{R}^n $(n \geq 1), Q_T = \Omega \times (0, T], S_T = \partial \Omega \times (0, T],$ $\Gamma_T = S_T \cup (\overline{\Omega} \times \{t = 0\})$ the parabolic boundary of the cylinder Q_T . Consider in Q_T the equation Let Ω be a bounded open set in \mathbb{R}^n $(n \ge 1)$, $Q_T = \Omega \times (0, T]$, $S_T = \partial \Omega \times (0, T]$,
 $\Gamma_T = S_T \cup (\overline{\Omega} \times \{t = 0\})$ the parabolic boundary of the cylinder Q_T . Consider in Q_T

the equation
 $F[u] := \frac{\partial u}{\partial t} - \text{div } a(u, \nab$

$$
F[u] := \frac{\partial u}{\partial t} - \text{div}\, a(u, \nabla u) = f \tag{1.1}
$$

 σ
 $\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), f = f(x, t)$ is a given function and a

unction on $\mathbb{R} \times \mathbb{R}^n$ satisfying for all $(u, p) \in \mathbb{R} \times \mathbb{R}^n$ th
 $a(u, p) \cdot p \ge \nu_0 |u|^l |p|^m - \phi_0(u)$ ($\nu_0 > 0, \phi_0(u) \ge 0$))

$$
\overline{\Omega} \times \{t = 0\} \text{ the parabolic boundary of the cylinder } Q_T. \text{ Consider in } Q_T
$$
\n
$$
F[u] := \frac{\partial u}{\partial t} - \text{div } a(u, \nabla u) = f \qquad (1.1)
$$
\n
$$
\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right), f = f(x, t) \text{ is a given function and } a = (a^1, \dots, a^n) \text{ is a}
$$
\nfunction on $R \times R^n$ satisfying for all $(u, p) \in R \times R^n$ the inequalities\n
$$
a(u, p) \cdot p \ge v_0 |u|^l |p|^m - \phi_0(u) \qquad (\nu_0 > 0, \phi_0(u) \ge 0) \text{)}
$$
\n
$$
|a(u, p)| \le \mu_1 |u|^l |p|^{m-1} + \phi_1(u) \qquad (m > 1, l \ge 0, \phi_1(u) \ge 0).
$$
\n1.1), (1.2) are known as doubly nonlinear parabolic equations. Their proto-
\n
$$
F_0[u] := \frac{\partial u}{\partial t} - \text{div}(|u|^l |\nabla u|^{m-2} \nabla u) = 0. \qquad (1.3)
$$
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Equations (1.1), (1.2) are known as *doubly nonlinear parabolic equations.* Their prototype is

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F_0[u] := \frac{\partial u}{\partial t} - \text{div}\big(|u|^l |\nabla u|^{m-2} \nabla u\big) = 0. \tag{1.3}
$$

In this paper we consider a special case of doubly nonlinear parabolic equations. In particular we limit ourselves by consideration equations (1.1), (1.2) only for *m >* 1 and $l \geq 0$ (instead of more general conditions $m > 1$ and $l > 1 - m$.

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Equations (1.1) , (1.2) and in particular (1.3) arise in the study of turbulent filtration of a gas or of a fluid through porous media and non-Newtonian flows (see [13]).

Existence of generalized solutions of Cauchy- Dirichlet problem for doubly nonlinear parabolic equations were established first by Raviart [17] and J.-L. Lions [15] and then by many authors. In particular Bamberger stated in [1] his results on existence and uniqueness of some non-negative generalized solution of Cauchy- Dirichlet problem for a non-homogeneous equation $F_0[u] = f$ (see (1.3)).

Up to recent time there were no regularity results for doubly nonlinear parabolic equations. The simple modification of the Barenblatt explicit solutions lets to show that at least in the case $l > 1$ Hölderness is the best possible smoothness of generalized solutions of equation (1.3). Hence the key question of the regularity theory for doubly nonlinear parbolic equations is establishing Hölder estimates for their generalized solutions. At first such estimates were established in [4] for the case of, so-called, doubly degenerate parabolic equations, i.e. for eqautions (1.1), (1.2) in the case $m > 2$ and $l>0$.

This paper is devoted to the proof of existence and uniqueness of some Hölder continuous generalized solution of Cauchy- Dirichlet problem for equations of the type (1.1) , (1.2) . The crucial role is played by the Hölder estimates established by the author in [5 - 9].

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2. Statement of the main result

Assume that for any $u, v \in \mathbb{R}$ and $p, q \in \mathbb{R}^n$ we have

2. Statement of the main result
\nAssume that for any
$$
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$$
 and $p, q \in \mathbb{R}^n$ we have
\n(G) $|a(u, p)| \le \mu(|u|^l |p|^{m-1} + \overline{\mu}(|u|)) \qquad (\mu = \text{const} \ge 0, m > 1, l \ge 0)$

 $\overline{\mu}(s) \geq 0$ being non-decreasing.

Definition 2.1. We say that a non-negative function *u* bounded in Q_T is a *weak* solution of equation (1.1), (G) with $f \in L_1(Q_T)$ if

(b) for any $\phi \in C^1(\overline{Q}_T)$ with $\phi = 0$ on S_T and any $t_1, t_2 \in [0, T]$

*(a) ^u*E *C([0,T];L2(cl)),* Vu' E *Lm(QT) (* ⁼ *uO dxV + Jf (- U* + *a(u, u) . -* **i)** *dxdt* 0 (2.1)

where $u_x = (u_{x_1}, \ldots, u_{x_n})$ and u_{x_i} $(i = 1, \ldots, n)$ are defined by

decreasing.
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$$
Q_T
$$
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\nn (1.1), (G) with $f \in L_1(Q_T)$ if
\n T]; $L_2(\Omega)$, $\nabla u^{\sigma+1} \in L_m(Q_T)$ ($\sigma = \frac{l}{m-1}$)
\n $\in C^1(\overline{Q}_T)$ with $\phi = 0$ on S_T and any $t_1, t_2 \in [0, T]$
\n $\phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left(-u\phi_t + a(u, u_x) \cdot \nabla \phi - f\phi \right) dx dt = 0$ (2.1)
\n \dots, u_{x_n}) and u_{x_i} ($i = 1, ..., n$) are defined by
\n $u_{x_i} = \begin{cases} (1 + \sigma)^{-1} u^{-\sigma} \frac{\partial u^{\sigma+1}}{\partial x_i} & \text{in } \{Q_T : u > 0\} \\ 0 & \text{in } \{Q_T : u = 0\} \end{cases}$ (2.2)

Consider the Cauchy-Dirichiet problem

Existence and Uniqueness of a Regular Solution

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$$
\nchy-Dirichlet problem

\n
$$
F[u] := \frac{\partial u}{\partial t} - \text{div } a(u, \nabla u) = f \quad \text{in } Q_T
$$
\n
$$
u = \Psi \quad \text{on } \Gamma_T
$$
\n
$$
f \in L_1(Q_T) \qquad \text{and} \qquad 0 \le \Psi \in W_1^1(Q_T). \tag{2.4}
$$
\nWe say that a function u is a weak solution of the Cauchy-Dirichlet

\nif it is a weak solution of equation (1.1), (G) and $u = \Psi$ on Γ_T .

where

 $f \in L_1(Q_T)$ and $0 \le \Psi \in W_1^1(Q_T)$. (2.4)
Definition 2.2. We say that a function *u* is a *weak solution* of the Cauchy-Dirichlet problem (2.3), (2.4) if it is a weak solution of equation (1.1), (G) and $u = \Psi$ on Γ_T .

Remark 2.1. Every weak solution of equation (1.1), (G) and every function $\Psi \in$ $W_1^1(Q_T)$ have trace on Γ_T .

Definition 2.3. Let inf $(\Psi, \Gamma_T) > 0$. We say that a function *u* is a *strong solution* of the Cauchy-Dirichlet problem (2.3) if it is a weak solution of (2.3) and, moreover, $\inf (u, Q_T) > 0$ (and hence $u \in W_m^{1,0}(Q_T)$).

Definition 2.4. Let $\Psi \in \mathring{W}^1_1(Q_T)$. We say that a function *u* is a *quasistrong solution* of the Cauchy-Dirichiet problem (2.3) if it is a weak solution of (2.3) and, moreover, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of strong solutions of problems $W^{1,0}_{m}(Q_T)$.
 $\mathring{W}^{1}_{1}(Q_T)$. We s
 \pm problem (2.3)
 \pm $\{u_n\}_{n \in \mathbb{N}}$ of st
 $F[u_n] = f_n$
 $u_n = \Psi_n$

$$
F[u_n] = f_n \qquad \text{in} \quad Q_T
$$

$$
u_n = \Psi_n \qquad \text{on} \quad \Gamma_T
$$

such that

effnition 2.4. Let
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\nver, there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ of strong solutions of problems
\n
$$
F[u_n] = f_n \quad \text{in} \quad Q_T
$$
\n
$$
u_n = \Psi_n \quad \text{on} \quad \Gamma_T
$$
\nnat
\n
$$
u_n \longrightarrow u \quad \text{in} \ C([0,T]; L_1(\Omega)); \quad f_n \in L_1(Q_T), \quad f_n \longrightarrow f \quad \text{in} \quad L_1(Q_T)
$$
\n
$$
\Psi_n = \Psi + \varepsilon_n(x, t)
$$
\n
$$
\varepsilon_n \in W_1^1(Q_T) \cap C(\overline{Q}_T), \quad \inf(\varepsilon_n, \Gamma_T) > 0, \quad \sup(\varepsilon_n, \Gamma_T) \longrightarrow 0.
$$
\neffinition 2.5. Let $\Psi \in \mathring{W}_1^1(Q_T)$. We say that a function *u* is a *regular solution*

Definition 2.5. Let $\Psi \in \mathring{W}_1^1(Q_T)$. We say that a function u is a *regular solution* of the Cauchy-Dirichlet problem (2.3) if it is Hölder continuous in \overline{Q}_T and a quasistrong solution of equation (2.3).

Introduce the following assumptions:

Introduction of equation (2.6).

\nIntroduce the following assumptions:

\n
$$
(\Omega) \ |B_{\rho}(x) \cap \Omega| \leq (1 - \alpha_0) |B_{\rho}(x)| \ (x \in \partial \Omega, \rho \in (0, \rho_0)) \text{ for some } \rho_0 > 0, \ \alpha_0 \in (0, 1)
$$
\n(B1) $0 \leq \Psi \in \mathring{W}_2^1(Q_T) \cap C_{\beta, \beta/m}(\Gamma_T) \quad (\beta \in (0, 1))$

\n(RHS) $0 \leq f \in L_{\infty}(Q_T)$.

Moreover, assume that the following conditions are fulfilled for equation (1.1):

(RHS) $0 \le f \in L_{\infty}(Q_T)$.

Moreover, assume that the following conditions are fulfilled for equation (1.1):
 0) The functions $u^{-\alpha}a^i(u, u^{-\alpha}p)$ $(i = 1, ..., n; \alpha = \frac{l}{m})$ are continuous on $\overline{R}_+ \times R^n$.

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1) (Growth condition). For any $u \in \overline{R}_+$ and $p \in \overline{R}$
 $\left(\begin{array}{cc} 0 & 0 \end{array}\right)$ and $\left(\begin{array}{cc} 1 & 0 \end{array}\right)$

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\n(Growth condition). For any
$$
u \in \overline{I\!R}_+
$$
 and $p \in I\!R^n$
\n
$$
a(u, p) \cdot p \ge v_0 |u|^l |p|^m - \mu_0(|u|^{\delta} + 1) \quad \left(v_0 > 0; \begin{cases} 2 < \delta < m + l & \text{if } m + l > 2 \\ \delta = 2 & \text{if } m + l \le 2 \end{cases} \right)
$$
\n
$$
|a(u, p)| \le \mu_1 |u|^l |p|^{m-1} + \mu(|u|) |u|^{\alpha} \quad \left(\alpha = \frac{l}{m}, \ \mu(s) \ge 1 \text{ non-decreasing on } I\!R_+\right).
$$

2) (Strict monotonicity condition). There exists a constant $\nu_1 > 0$ and a continuous vector function $b: \mathbb{R} \to \mathbb{R}^n$ such that for any $u \in \mathbb{R}$ and $p, q \in \mathbb{R}$ t monotonicity condition). There exists a condition $b : \mathbb{R} \to \mathbb{R}^n$ such that for any $u \in \mathbb{R}$ and $(a(u,p) - a(u,q)) \cdot (p - q) \geq \nu_1 |u|^l |p - q|^{\kappa} (|p - q|)$

$$
(a(u,p) - a(u,q)) \cdot (p-q) \geq \nu_1 |u|^l |p-q|^{\kappa} (|p-b|^m + |q-b|^m)^{1-\kappa/m}
$$

where $\kappa = m$ if $m \geq 2$ and $\kappa = 2$ if $1 < m < 2$.

3) (Local Lipschitz condition). For any $u, v \in [\varepsilon, M]$ $(0 < \varepsilon < M)$ and any $p \in \mathbb{R}^n$
 $|a(u, p) - a(v, p)| \le \Lambda |u - v|(1 + |p|^{m-1})$

$$
|a(u, p) - a(v, p)| \le \Lambda |u - v|(1 + |p|^{m-1})
$$

where $\Lambda = \Lambda(\varepsilon, M) \geq 0$.

4) $(m, l) \in D \setminus \omega$ where

$$
|a(u, p) - a(v, p)| \le \Lambda |u - v|(1 + |p|^{m-1})
$$

\n1) ≥ 0 .
\nwhere
\n
$$
D = \{(m, l) : m > 1, l \ge 0\}
$$

\n
$$
\omega = \left\{(m, l) \in D : \frac{\sigma + 1}{\sigma + 2} \le \frac{1}{m} - \frac{1}{n}, \sigma = \frac{l}{m - 1}\right\}
$$

Theorem 2.1 (Existence and uniqueness of regular solution). Let conditions (Ω) , *(B!),* (RHS) *and* 0) - *4) hold. Then the Cauchy- Dirichlet problem (2.3) has exactly one regular solution.*

Remark 2.2. Conditions 0) - 3) are fulfilled for equation (1.3).

Remark 2.3. It is easy to see that $\Omega \subset F := \{(m, l) \in D : m + l < 2\}$. We constructed a counter-example (see [10]) showing that for every $(m, l) \in \omega$ the local boundedness of generalized solutions of equation (1.3) fails to be true.

Remark 2.4. Existence of Holder continuous weak solution of the Cauchy- Dirichlet problem for some class of equations of the type (1.1), (1.2) in the case $m \geq 2$ and $l \geq 0$ was proved in [11]. Existence and uniqueness of regular solution of the Cauchy-Dirichlet problem (2.3) under conditions (Ω), (BI), (RHS) and 0) - 3) and for $l \geq 0$, max $\left(1, \frac{2n}{n+2}\right) < m < 2, m+l > 2$ can be derived from results of [12]. The proofs of the results of [11] and [12] are based on using Hölder estimates established in [4] and [5 -9], respectively.

3. Uniqueness of quasistrong solution

In this section we state the uniqueness results of paper [12]. Assume at first that for any $u, v \in \mathbb{R}$ and any $p, q \in \mathbb{R}^n$ the function $a = (a^1, \ldots, a^n)$ satisfies the following conditions:

(G) $|a(u, p)| \le \mu(|p|^{m-1} + 1)$ $(\mu \ge 0)$ conditions:

any
$$
u, v \in \mathbb{R}
$$
 and any $p, q \in \mathbb{R}^n$ the function $a = (a^1, ..., a^n)$ satisfies the following
conditions:
\n $(\tilde{G}) |a(u, p)| \le \mu(|p|^{m-1} + 1)$ $(\mu \ge 0)$
\n(M) $(a(u, p) - a(u, q)) \cdot (p - q) \ge 0$
\n(L) $|a(u, p) - a(v, p)| \le \lambda |u - v|(|p|^{m-1} + 1)$ $(\Lambda = \text{const} \ge 0, m > 1)$.
\nDefinition 3.1. We say that a function u is a *generalized solution* of equation (1.1),
\n (\tilde{G}) if $u \in W_m^{1,0}(Q_T) \cap C([0, T]; L_1(\Omega))$ and for all $0 \le \phi \in W_m^1(Q_T) \cap L_\infty(Q_T)$ and any
\n $t_1, t_2 \in [0, T]$
\n
$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{\Omega} \Big(-u \phi_t + a(u, \nabla u) \cdot \nabla \phi - f \phi \Big) dx dt = 0.
$$
 (3.1)
\nAnalogously, the function u is called *subsolution* and *supersolution* if in (3.1) the sign

Definition 3.1. We say that a function *u* is a *generalized solution* of equation (1. 1), (\tilde{G}) if $u \in W^{1,0}_{m}(Q_T) \cap C([0,T]; L_1(\Omega))$ and for all $0 \leq \phi \in \mathring{W}^1_m(Q_T) \cap L_\infty(Q_T)$ and any $t_1, t_2 \in [0, T]$

$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{\Omega} \Big(-u \phi_t + a(u, \nabla u) \cdot \nabla \phi - f \phi \Big) dx dt = 0. \tag{3.1}
$$

Analogously, the function *u* is called *subsolution* and *supersolution* if in (3.1) the sign "=" is replacesd by " \leq " and " \geq ", respectively.

Proposition 3.1 (Comparison Principle, see [12]). *Assume that conditions (C),* (M) and (L) hold. Let u₁ and u₂ be a generalized subsolution and a supersolution, *repsectively, such that* $\int_{t_1}^{t_2} + \int_{\Omega} \left(-u\phi_t + \right)$

on *u* is called *subs*

" and " \geq ", respect

(Comparison Princ
 *t u*₁ and *u*₂ *be a*
 F[*u*₁] $\leq f_1$
 *If u*₁ $\leq u_2$ *on S*_T: $a(u, \nabla u) \cdot \nabla \phi - f\phi\big)$
 a
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 aolution and *supersolu*
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 *iple, see [12]). Assur

<i>generalized subsolutio*
 and $F[u_2] \ge f_2$
 $= \partial \Omega \times (0, T]$, then for *where* $f_1, f_2 \in L_1(Q_T)$ *. If* $u_1 \le u_2$ *on* $S_T = \partial\Omega \times (0,T]$ *, then for any* $\tau \in (0,T]$ *we have* $f_1, f_2 \in L_1(Q_T)$ *. If* $u_1 \le u_2$ *on* $S_T = \partial\Omega \times (0,T]$ *, then for any* $\tau \in (0,T]$ *we have*

$$
F[u_1] \le f_1 \qquad \qquad and \qquad \qquad F[u_2] \ge f_2
$$

$$
F[u_1] \le f_1 \qquad and \qquad F[u_2] \ge f_2
$$

\n
$$
\vdots L_1(Q_T). \quad If \ u_1 \le u_2 \text{ on } S_T = \partial\Omega \times (0, T], \text{ then for any } \tau \in (0, T] \text{ we have}
$$

\n
$$
\int_{\Omega} (u_1 - u_2)^+ dx \Big|^{t=\tau} \le \int_{\Omega} (u_1 - u_2)^+ dx \Big|^{t=0}
$$

\n
$$
+ \int_{0}^{T} (f_1 - f_2) \operatorname{sign}(u_1 - u_2)^+ dx dt.
$$
\n(3.2)

Proof. Let $0 \le \eta \in \mathring{W}_{m}^{1,0}(Q_T) \cap L_{\infty}(Q_T), 0 < h < t_0 < t_2 < T - h, Q_{t_1,t_2} := \Omega \times$ $[t_1, t_2]$. Then from the conditions of Proposition 3.1 it follows (see also $[14:p.167,477]$) that

$$
\iint_{Q_{t_1,t_2}} \left\{ (u_1 - u_2)_{\tilde{h}t} \eta + \left(\left(a(u_1, \nabla u_1) \right)_{\tilde{h}} - \left(a(u_2, \nabla u_2) \right)_{\tilde{h}} \right) \cdot \nabla \eta \right\} dx dt
$$
\n
$$
\leq \iint_{Q_{t_1,t_2}} (f_1 - f_2)_{\tilde{h}} \eta \, dx dt \tag{3.3}
$$

where

$$
g_{\bar{h}} = \frac{1}{h} \int_{t-h}^{t} g(x,\tau) d\tau.
$$

Denote

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\n
$$
H_{\delta}(s) = \begin{cases} 1 & \text{if } s \ge \delta \\ \frac{s}{\delta} & \text{if } 0 < s < \delta \\ 0 & \text{if } s \le 0 \end{cases} \quad \text{and} \quad G_{\delta}(s) = \begin{cases} s - \frac{\delta}{2} & \text{if } s \ge \delta \\ \frac{s^2}{2\delta} & \text{if } 0 < s < \delta \\ 0 & \text{if } s \ge 0 \end{cases}
$$
\n
$$
G'_{\delta}(s) = H_{\delta}(s) \text{ on } \mathbb{R}. \text{ Set in (3.3)}
$$
\n
$$
\eta = H_{\delta}(u_1 - u_2).
$$
\nsly that the test function (3.4) is admissible. In view of the concavity of G_{δ} we have

so that $G'_{\delta}(s) = H_{\delta}(s)$ on *R*. Set in (3.3)

$$
\eta = H_{\delta}(u_1 - u_2). \tag{3.4}
$$

Obviously that the test function (3.4) is admissible. In view of the concavity of the function G_{δ} we have

$$
(u_1-u_2)_{\bar{h}t}H_{\delta}(u_1-u_2)\geq (G_{\delta}(u_1-u_2))_{\bar{h}t}.
$$

Then from (3.3) it follows that

$$
\iint_{Q_{t_1,t_2}} (G_{\delta}(u_1 - u_2))_{\bar{h}t} dx dt \n+ \iint_{Q_{t_1,t_2}} \left(\left(a(u_1, \nabla u_1) \right)_{\bar{h}} - \left(a(u_2, \nabla u_2) \right)_{\bar{h}} \right) \cdot \nabla (u_1 - u_2) H'_{\delta}(u_1 - u_2) dx dt \quad (3.5)\n\leq \iint_{Q_{t_1,t_2}} (f_1 - f_2)_{\bar{h}} H_{\delta}(u_1 - u_2) dx dt.
$$

obtain for any $\tau \in (0, T]$

Using the Newton-Leibnitz formula for the first term in (3.5) and then letting
$$
h \to 0
$$
 we obtain for any $\tau \in (0, T]$
\n
$$
\int_{\Omega} G_{\delta}(u_1 - u_2) dx \Big|_{0}^{\tau}
$$
\n
$$
+ \frac{1}{\delta} \iint_{\{Q_{0,\tau}: 0 < u_1 - u_2 < \delta\}} \left(a(u_1, \nabla u_1) - a(u_2, \nabla u_2) \right) \cdot \nabla (u_1 - u_2) dx dt \qquad (3.6)
$$
\n
$$
\leq \iint_{Q_{0,\tau}} (f_1 - f_2) H_{\delta}(u_1 - u_2) dx dt.
$$
\nTaking into account that

Taking into account that

into account that
\n
$$
G_{\delta}(u_1 - u_2) \longrightarrow (u_1 - u_2)^+
$$
 and $H_{\delta}(u_1 - u_2) \longrightarrow \text{sign}(u_1 - u_2)^+$

as $\delta \to 0$ we derive from (3.6) and conditions *(M)* and *(L)* that inequality (3.2) holds

Consider now the Cauchy-Dirichlet problem (2.3) assuming that condition (\tilde{G}) holds *and* $f \in L_1(Q_T)$, $\Psi \in W_1^1(Q_T)$.

Definition 3.2. We say that a function *u* is a *generalized solution* of the Cauchy-Dirichlet problem (2.3) if it is a generalized solution of equation (1.1) and $u = \Psi$ on Γ_T .

From Proposition 3.1 we can derive directly the following

Proposition **3.2.** *Let conditions (G),* (M) *and* (L) *are fulfilled. Then there is at most one generalized solution of the Cauchy-Dirichlet problem (2.3).*

Replace now condition $(\check{\mathrm{G}})$ by condition (G) (see Section 2) and consider instead of assumption (L) the local Lipschitz condition

Replace now condition (G) (see Section 2) and assumption (L) the local Lipschitz condition
\n
$$
|\tilde{L}| = |a(u, p) - a(v, p)| \leq \Lambda |u - v| (1 + |p|^{m-1}) \quad (\Lambda = \Lambda(\varepsilon, M) \geq 0)
$$
\nfor any $u, v \in [\varepsilon, M]$ $(0 < \varepsilon < M)$ and any $p \in \mathbb{R}^n$.

From Proposition 3.2 we can derive the following

Proposition 3.3. Let $\inf(\Psi, \Gamma_T) > 0$ and let conditions (G), (M) and (L) hold. *Then there is at most one strong (in sense of Definition* 2.3) *solution of the Cauchy-Dirichlet problem (2.3).*

The main uniqueness result for doubly nonlinear parabolic equations is the following

Theorem 3.1 (Uniqueness of quasistrong solution, see [12]). Let $\Psi \in \mathring{W}_1^1(Q_T)$ and *let the conditions (G),* (M) *and* (L) *be fulfilled. Then there is at most one quasistrong (in sense of Definition* 2.4) *solution of the Cauchy- Dirichlet problem (2.3).*

Proof. Let *u* and \tilde{u} be two quasistrong solutions of problem (2.3). Let (u_n, f_n, Ψ_n) $\rightarrow (u, f, \Psi)$ and $(\tilde{u}_n, \tilde{f}_n, \tilde{\Psi}_n) \rightarrow (\tilde{u}, f, \Psi)$ in sense of (2.5). Obviously we can choose subsequences $\{\Psi_n\}$ and $\{\widetilde{\Psi}_n\}$ such that $\sup(\Psi_n, S_T) \leq \inf(\widetilde{\Psi}_n, S_T)$ ($n \in \mathbb{N}$). Then we can apply Proposition 3.1, i.e., for any $\tau \in (0, T]$ *Jem* (2.3).

uniqueness result for doubly nonlinear parabolic equations is
 13.1 (Uniqueness of quasistrong solution, see [12]). Let $\Psi \in$

ions (G), (M) and (L) be fulfilled. Then there is at most or

definition 2.4)

$$
\int_{\Omega} (u_n - \tilde{u}_n)^+ dx \Big|^{t=\tau} \leq \int_{\Omega} (\Psi_n - \tilde{\Psi}_n) dx + \int_{0}^{t} |f_n - \tilde{f}_n| dx dt.
$$

Letting $n \to \infty$ and using (2.5) we obtain that $(u - \tilde{u})^+ = 0$ a.e. in Q_T

Remark 3.1. In some sense Definition 2.4 of quasistrong solution and Theorem 3.1 are similar to the definition of "limit of strong solutions" and the corresponding uniqueness theorem given by Bamberger [1) for equation (1.3). However instead of our condition inf $(u, Q_T) > 0$ in the definition of strong solution Bamberger used condition $\frac{\partial u}{\partial t} \in L_1(Q_T)$. (b) we obtain that $(u - \tilde{u})^+ = 0$ a.e. in ζ
ense Definition 2.4 of quasistrong solution of "limit of strong solutions" and t
Bamberger [1] for equation (1.3). Howe
he definition of strong solution Bamberg
lowing
that

We introduce now the following

Definition **3.3.** We say that a function *u* is a *maximal weak solution* of the Cauchy-Dirichlet problem (2.3) if it is a weak solution of problem (2.3) and, moreover, for any weak solution *v* of this problem we have

$$
u(x,t) \ge v(x,t) \qquad \text{in} \quad Q_T. \tag{3.7}
$$

The reason of uniqueness of quasistrong solution of equation (2.3) can be found by means the following proposition that easily follows from the proof of Proposition 3.1 given in [12).

Proposition 3.4 (A. V. Ivanov, W. Jäger and P. Z. Mkrtychyan). *Every quasistrong solution of the Cauchy-Dirichlet problem (2.1) is a maximal weak solution of this problem.*

Proof. Inequality (3.3) remains valid for any weak solutions u_1 and u_2 of the Cauchy-Dirichlet problem if we change ∇u_1 and ∇u_2 in (3.3) by $(u_1)_x$ and $(u_2)_x$, respec-Example 1.1 The time the sequence $\{u_1\}$ and $\{u_2\}$ in (3.3) by $(u_1)_x$ and $(u_2)_x$, respectively (where u_x is defined by (2.2)). Let *u* be a quasistrong and *v* be a weak solution of (2.3). Consider the sequence of (2.3). Consider the sequence $\{u_{(n)}\}$ of strong solutions of problems ralid for a
 ∇u_1 and ∇
 *e*t *u* be a

of strong :
 $=f_n$
 $= \Psi_n$ *U(n)* $f(u_n) = f_n$ in Q_T
 $u(n) = \Psi_n$ on Γ_T
 u(n) $\geq \varepsilon(n)$ and $\varepsilon(n) = \inf(\varepsilon_n, Q_T) > 0$ ($n \in \mathbb{N}$).
 u(n) $\geq \varepsilon(n)$ and $\varepsilon(n) = \inf(\varepsilon_n, Q_T) > 0$ ($n \in \mathbb{N}$).

3) in the case $u_1 = v$ and $u_2 = u(n)$. Set in (3.3) *f* (*s* + U(n) iii) \int for strong solutions of problems
 $\begin{aligned}\n\eta_0 &= f_n \quad \text{in} \quad Q_T \\
\eta_1 &= \Psi_n \quad \text{on} \quad \Gamma_T\n\end{aligned}$
 $\begin{aligned}\n\eta_2 &= u_n, \quad \text{Set in (3.3)} \\
\eta_3 &= H_\delta(v - u_{(n)})\n\end{aligned} \qquad (3.8)$
 $\begin{aligned}\n\eta_4 &= \inf(v - u_{(n)})\n\end{aligned}$ (3.8)
 $\begin{aligned}\n\$

$$
\mathcal{F}[u_{(n)}] = f_n \quad \text{in} \quad Q_T
$$

$$
u_{(n)} = \Psi_n \quad \text{on} \quad \Gamma_T
$$

satisfying conditions (2.5). In particular,

Consider (3.3) in the case $u_1 = v$ and $u_2 = u_{(n)}$. Set in (3.3)

$$
\eta = H_{\delta}(v - u_{(n)}) \tag{3.8}
$$

where $H_{\delta}(s)$ is defined like above. Such a test function is admissible for (3.3) because

$$
H_{\delta}(v-u_{(n)})=H_{\delta}((v-u_{(n)})^{+})=H_{\delta}((v_{(n)}-u_{(n)}))^{+})
$$

where $v_{(n)} = \sup(v, \epsilon_{(n)})$. Therefore, function (3.8) belongs to $\mathring{W}_{m}^{1,0}(Q_T) \cap L_{\infty}(Q_T)$. Then repeating arguments of the proof of Proposition *3.1* we obtain

the case
$$
u_1 = v
$$
 and $u_2 = u_{(n)}$. Set in (3.3)
\n
$$
\eta = H_{\delta}(v - u_{(n)})
$$
\n(3.8)
\nneed like above. Such a test function is admissible for (3.3) because
\n
$$
(v - u_{(n)}) = H_{\delta}((v - u_{(n)})^+) = H_{\delta}((v_{(n)} - u_{(n)}))^+)
$$
\n
$$
v, \epsilon_{(n)})
$$
. Therefore, function (3.8) belongs to $\mathring{W}_{m}^{1,0}(Q_T) \cap L_{\infty}(Q_T)$.
\nnumbers of the proof of Proposition 3.1 we obtain
\n
$$
\int_{\Omega} (v - u_{(n)})^+ dx \Big|^{t=r} \leq 0 \quad \text{for any } \tau \in (0, T].
$$
\n(3.9)
\n
$$
v = \int_{\Omega} (v - u_{(n)})^+ dx \Big|^{t=r} \leq 0 \quad \text{for any } \tau \in (0, T].
$$
\n(3.9)
\n
$$
v = \int_{\Omega} (v - u_{(n)})^+ dx \Big|^{t=r} \leq 0 \quad \text{for any } \tau \in (0, T].
$$

Using (2.5) we derive from (3.9) that $v - u \leq 0$ a.e. in Ω , i.e., inequality (3.7) is established \blacksquare

4. Holder estimates for doubly nonlinear parabolic equations

Establishing Holder estimates is the key question of the regularity problem for doubly nonlinear parabolic equations not only in view of the fact that HOlderness is the best possible smoothness for a large class of such equations. In fact HOlder estimates for bounded generalized solutions are crucial and the best difficult step in proving of existence of regular solution of Cauchy- Dirichlet problem for doubly nonlinear parabolic equations. **Example 13 Example 10 Complement Parabolic equations**
ates is the key question of the regularity problem for doubly
ions not only in view of the fact that Hölderness is the best
large class of such equations. In fact

Directly from our results *[5 - 9]* for doubly nonlinear parabolic equations of the full type

$$
\frac{\partial u}{\partial t} - \operatorname{div} a(x, t, u, \nabla u) + a_0(x, t, u, \nabla u) = 0 \qquad (4.1)
$$

with the limit growth conditions we can derive the following estimates for equations of the type (1.1) , (1.2) . Introduce condition

(H)
$$
a = (a^1, ..., a^n)
$$
 is continuous on $\mathbb{R} \times \mathbb{R}^n$ $(i = 1, ..., n)$
\n $a(u, p) \cdot p \ge v_0 |u|^l |p|^m - \varphi_0 \quad (\nu_0 > 0)$
\n $|a(u, p)| \le \mu_1 |u|^l |p|^{m-1} + |u|^\alpha \varphi_1 \quad (\alpha = \frac{l}{m})$
\n $|f(x, t)| \le \varphi_2$

where $\varphi_i = \text{const} \geq 0$ $(i = 0, 1, 2)$. For the sake of brevity we state here only global Hölder estimates (i.e. Hölder estimates up to the boundary) for equations (1.1) , (1.2) .

Theorem 4.1 (see [5 - 8]). Assume that $m+l\geq 2$ and let conditions (H) and (Ω) *hold. Let u be a weak solution of equation* (1.1) (in sense of Definition 2.1) such that *its trace on the parabolic boundary* Γ_T *is Hölder continuous. Then function u belongs to the class* $C^{\lambda,\lambda/m}(\overline{Q}_T)$ for some $\lambda \in (0,1)$. Moreover *K (4.2) (^u) ^A sup*

$$
\langle u \rangle_{\lambda, \overline{Q}_T} := \sup_{(x,t), (x',t') \in Q_T} \frac{|u(x,t) - u(x',t')|}{(|x - x'|^m + |t - t'|)^{\lambda/m}} \le K \tag{4.2}
$$

where $\lambda \in (0,1)$ and $K > 0$ depend only on $\sup(u, Q_T)$, n, m, l, ν_0 , μ_0 , φ_0 , φ_1 , φ_2 , $|\Omega|$, T , α_0 , ρ_0 and the Holder constant and exponent of the trace of function u on Γ_T .

Theorem 4.2 (see [9]). Assume that $m + l < 2$ and let conditions (H), (M), (L) and (Ω) hold. Let $u \in W_m^{1,0}(Q_T)$ be a weak solution of equation (1.1) (in sense of *Definition* 2.1) *such that its trace on the parabolic boundary* Γ_T *is Hölder continuous. Then u belongs to* $C^{\lambda,\lambda/m}(\bar{Q}_T)$ for some $\lambda \in (0,1)$. Moreover estimate (4.2) holds with some constants $\lambda \in (0,1)$ and $K > 0$ depending on the same data as in the case of *Theorem* 4.1 *(in particular* λ *and K are independent of* $\|\nabla u\|_{L_m(Q_T)}$ *and the constant* Λ *from condition* (L)).

Remark 4.1. Theorems 4.1 and 4.2 remain valid if the inequalities in condition (H) are fulfilled only for values *u* from the range of weak solution under consideration.

Remark 4.2. The proofs of Theorems 4.1 and 4.2 (as well as Holder estimates for general equations (4.1) in $[5 - 9]$ are concerned with some development of the methods of papers by De Giorgi, Ladyzhenskaya and Ural'tseva (see [141), DiBenedetto [3], Chen and DiBenedetto [2], and the author [4].

Remark 4.3. Other results on Hölder estimates for some classes of doubly nonlinear parabolic equations are obtained in [16, 18].

5. The auxiliary Cauchy- Dirichiet problem

This section has an auxiliary character. At first we prove some generalization of the well-known Friedrieks inequality (cf. [14: pp. 529 - 530]) which will be used not only in this section. 22 cm^2
 21
 22
 22
 2
 2
 2¹
 1^{*1*}</sup>/*(* β +*1*)

Lemma 5.1. Let $\{\Psi_{\kappa}\}\$ be an orthonormal basis in $L_2(\Omega)$ and let $\beta \geq 0$ be fixed. Then for any $\varepsilon > 0$ there exists a number $\mathcal{N}_{\varepsilon}$ such that for any function u satisfying the *condition* hty (cf. [14: pp. 329 - 330]) which
be an orthonormal basis in $L_2(\Omega)$ a
ts a number \mathcal{N}_e such that for any fu
 $\dot{V}_m^1(\Omega)$ $\qquad (m>1, \frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2})$

$$
|u|^{\beta}u \in \mathring{W}_m^1(\Omega) \qquad \left(m > 1, \, \frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}\right) \tag{5.1}
$$

we have

$$
|u|^{\beta}u \in \mathring{W}_m^1(\Omega) \qquad \left(m > 1, \frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}\right) \tag{5.1}
$$

$$
||u||_{L_2(\Omega)} \le \left(\sum_{k=1}^{\mathcal{N}_{\epsilon}} (u, \Psi_{\kappa})^2\right)^{1/2} + \varepsilon ||\nabla (|u|^{\beta}u)||_{L_m(\Omega)}^{1/(\beta+1)} \tag{5.2}
$$

where $(u, \Psi_{\kappa}) := \int_{\Omega} u \Psi_{\kappa} dx$ and \mathcal{N}_{ϵ} does not depend on u.

$$
||u||_{L_2(\Omega)} \le \left(\sum_{k=1}^{\infty} (u, \Psi_{\kappa})^2\right) + \varepsilon ||\nabla(|u|^{\beta}u)||_{L_m(\Omega)}^{1/(\beta+1)}
$$
(5.2)
re $(u, \Psi_{\kappa}) := \int_{\Omega} u \Psi_{\kappa} dx$ and \mathcal{N}_{ϵ} does not depend on u.
Proof. It is sufficient to prove that for any $\delta > 0$ and $\varepsilon > 0$

$$
||u||_{L_2(\Omega)} \le (1+\delta) \left(\sum_{k=1}^{\mathcal{N}_{\epsilon,\delta}} (u, \Psi_{\kappa})^2\right)^{1/2} + \varepsilon ||\nabla(|u|^{\beta}u)||_{L_m(\Omega)}^{1/(\beta+1)}.
$$
(5.3)
lly, for the function $v = |u|^{\beta}u$ we have the well-known Sobolev inequality

$$
||v||_{L_r(\Omega)} \le c ||\nabla v||_{L_m(\Omega)} \qquad \left(r = \frac{2}{1+\beta} > 0\right)
$$
(5.4)
use from condition $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ it follows that $\frac{1}{r} > \frac{1}{m} - \frac{1}{n}$. Rewrite (5.4) as

Really, for the function $v = |u|^\beta u$ we have the well-known Sobolev inequality

$$
0 \leq (1 + \delta) \left(\sum_{k=1} (u, \Psi_{\kappa})^2 \right) + \varepsilon ||V(|u|^{p} u)||_{L_{m}(\Omega)} \tag{5.3}
$$

on $v = |u|^{\beta} u$ we have the well-known Sobolev inequality

$$
||v||_{L_{r}(\Omega)} \leq c ||\nabla v||_{L_{m}(\Omega)} \qquad \left(r = \frac{2}{1 + \beta} > 0\right) \tag{5.4}
$$

because from condition $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ it follows that $\frac{1}{r} > \frac{1}{m} - \frac{1}{n}$. Rewrite (5.4) as

$$
||u||_{L_2(\Omega)} \le c_1 ||\nabla (|u|^{\beta} u)||_{L_m(\Omega)}^{1/(\beta+1)}.
$$
\n(5.5)

Then from (5.3) and (5.5) it follows that

 $\ddot{}$

$$
||u||_{L_2(\Omega)} \le \left(\sum_{k=1}^{\mathcal{N}_{\epsilon,\delta}} (u, \Psi_\kappa)^2\right)^{1/2} + (c_1 \delta + \varepsilon) ||\nabla (|u|^\beta u)||_{L_m(\Omega)}^{1/(\beta+1)}, \tag{5.6}
$$

i.e., the result of Lemma 5.1 is true. So prove that (5.3) holds.

satisfying condition (5.1) such that for some fixed $\delta > 0$ and any $\nu \in \mathbb{N}$

If (5.3) is violated, then there exist an
$$
\varepsilon_0 > 0
$$
 and a sequence of functions $\{u_{\nu}\}$
sfying condition (5.1) such that for some fixed $\delta > 0$ and any $\nu \in \mathbb{N}$

$$
\|u_{\nu}\|_{L_2(\Omega)} > (1+\delta) \left(\sum_{k=1}^{\nu} (u_{\nu}, \Psi_k)^2\right)^{1/2} + \varepsilon_0 \left\|\nabla(|u_{\nu}|^{\beta} u_{\nu})\right\|_{L_m(\Omega)}^{1/(\beta+1)}.
$$
 (5.7)

Then for functions $\hat{u}_{\nu} = u_{\nu}/||u_{\nu}||_{L_2(\Omega)}$ we have

Existence and Uniqueness of a Regular Solution
\nfor functions
$$
\hat{u}_{\nu} = u_{\nu}/\|u_{\nu}\|_{L_2(\Omega)}
$$
 we have
\n
$$
1 = \|\hat{u}_{\nu}\|_{L_2(\Omega)} > (1+\delta) \left(\sum_{k=1}^{\nu} (\hat{u}_{\nu}, \Psi_k)^2\right)^{1/2} + \varepsilon_0 \left\|\nabla\left(|\hat{u}_{\nu}|^{\beta} \hat{u}_{\nu}\right)\right\|_{L_m(\Omega)}^{1/(\beta+1)}.
$$
\n(5.8)

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Then for functions $\hat{u}_{\nu} = u_{\nu}/||u_{\nu}||_{L_2(\Omega)}$ we have
 $1 = ||\hat{u}_{\nu}||_{L_2(\Omega)} > (1 + \delta) \left(\sum_{k=1}^{\nu} (\hat{u}_{\nu}, \Psi_k)^2\right)^{1/2} + \epsilon_0 ||\nabla (|\hat{u}_{\nu}|^{\beta} \hat{u}_{\nu})||_{L_m(\Omega)}^{1/(\beta+1)}$ (Denote $v_{\nu} = |\hat{u}_{\nu}|^{\beta} \hat{u}_{\nu}$. In view of (5.8) the norms $\|\nabla v_{\nu}\|_{L_m(\Omega)}$ are uniformly bounded and
hence (taking into account that $\frac{1}{r} > \frac{1}{m} - \frac{1}{n}$ for $r = \frac{2}{1+\beta}$) there exists some subsequence
 $\{v$ hence (taking into account that $\frac{1}{r} > \frac{1}{m} - \frac{1}{n}$ for $r = \frac{2}{1+\beta}$) there exists some subsequence $\{v_{\nu_s}\}$ converging strongly in $L_r(\Omega)$. It is easy to see then that the subsequence $\{u_{\nu_s}\}$ converges strongly in $L_2(\Omega)$ to some function $\hat{u} \in L_2(\Omega)$. Really, in view of the strict monotonicity of the function $x \to |x|^\beta x$ ($\beta > 0$) we have functions $\hat{u}_{\nu} = u_{\nu}$ /
= $\|\hat{u}_{\nu}\|_{L_2(\Omega)} > (1 + \nu)$
= $|\hat{u}_{\nu}|^{\beta} \hat{u}_{\nu}$. In view
king into account the account the strongly in $L_2(\Omega)$
icity of the function
 $c^{-1}|\hat{u}_{\nu} - \hat{u}_{\mu}|^{2+\beta} \leq$
ne constant $c > 0$ ar

$$
c^{-1}|\hat{u}_{\nu} - \hat{u}_{\mu}|^{2+\beta} \leq (|\hat{u}_{\nu}|^{\beta}\hat{u}_{\nu} - |\hat{u}^{\mu}|^{\beta}\hat{u}_{\mu})(\hat{u}_{\nu} - \hat{u}_{\mu}) \leq |v_{\nu} - v_{\mu}| |\hat{u}_{\nu} - \hat{u}_{\mu}|
$$

with some constant $c > 0$ and hence

 $\ddot{}$

$$
\leq (|\omega_{\nu}| \omega_{\nu} - |\omega| + \omega_{\mu})(\omega_{\nu} - \omega_{\mu}) \leq |\omega_{\nu} - \omega_{\mu}||\omega_{\nu}|
$$

> 0 and hence

$$
|\hat{u}_{\nu} - \hat{u}_{\mu}|^2 \leq c|\nu_{\nu} - \nu_{\mu}|^r \qquad \left(r = \frac{2}{1+\beta}\right).
$$

Moreover, it is obvious that $\|\hat{u}\|_{L_2(\Omega)} = 1$. The functions $P_{\nu_\mu}\hat{u}_{\nu_\mu} = \sum_{k=1}^{\nu_\mu} (\hat{u}_{\nu_\mu}, \Psi_k) \Psi_k$ also converge strongly in $L_2(\Omega)$ to \hat{u} because

$$
\begin{aligned} \left\| \hat{u} - P_{\nu_{\star}} \hat{u}_{\nu_{\star}} \right\|_{L_2(\Omega)} &= \left\| P_{\nu_{\star}} (\hat{u} - \hat{u}_{\nu_{\star}}) + (E - P_{\nu_{\star}}) \hat{u} \right\|_{L_2(\Omega)} \\ &\leq \left\| \hat{u} - \hat{u}_{\nu_{\star}} \right\|_{L_2(\Omega)} + \left\| (E - P_{\nu_{\star}}) \hat{u} \right\|_{L_2(\Omega)} \\ &\to 0 \quad \text{as } s \to \infty. \end{aligned}
$$

Then

$$
\|\hat{u} - P_{\nu_{\bullet}}\hat{u}_{\nu_{\bullet}}\|_{L_2(\Omega)} = \|P_{\nu_{\bullet}}(\hat{u} - \hat{u}_{\nu_{\bullet}}) + (E - P_{\nu_{\bullet}})\hat{u}\|_{L_2(\Omega)}
$$

\n
$$
\leq \|\hat{u} - \hat{u}_{\nu_{\bullet}}\|_{L_2(\Omega)} + \|(E - P_{\nu_{\bullet}})\hat{u}\|_{L_2(\Omega)}
$$

\n
$$
\to 0 \quad \text{as } s \to \infty.
$$

\n
$$
\left(\sum_{k=1}^{\nu_{\bullet}} (\hat{u}_{\nu_{\bullet}}, \Psi_k)^2\right)^{1/2} = \|P_{\nu_{\bullet}}\hat{u}_{\nu_{\bullet}}\|_{L_2(\Omega)} \longrightarrow \|\hat{u}\|_{L_2(\Omega)} = 1 \quad \text{as } s \to \infty.
$$
 (5.9)
\nwe of (5.8), (5.9) we obtain then the impossible inequality $1 \geq 1 + \delta$ **••**
\now we consider the Cauchy-Dirichlet problem
\n
$$
F[u] = f \quad \text{in } Q_T
$$

\n
$$
u = \Psi \quad \text{on } \Gamma_T
$$
 (5.10)

In view of (5.8), (5.9) we obtain then the impossible inequality $1 \geq 1 + \delta$

Now we consider the Cauchy-Dirichiet problem

then the impossible inequality
$$
1 \ge 1 + \delta
$$

\n-Dirichlet problem
\n
$$
F[u] = f \quad \text{in} \quad Q_T
$$
\n
$$
u = \Psi \quad \text{on} \quad \Gamma_T \tag{5.10}
$$

assuming the following:

 $(0')$ $a = (a^1, \ldots, a^n)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ 1') $a(u, p) \cdot p \ge v_0 |p|^m - \mu_0 \ (\nu_0 > 0), \ |a(u, p)| \le \mu_1(|p|^{m-1} + 1)$ for any $u \in \mathbb{R}, p \in \mathbb{R}^m$ *2'*) $(a(u,p)-a(u,q)) \cdot (p-q) \ge \nu_1|p-q|^m$ $(\nu_1>0)$ for any $u \in \mathbb{R}$, $p,q \in \mathbb{R}^m$ 3') $|a(u, p) - a(v, p)| \leq \Lambda |u - v| (|p|^{m-1} + 1)$ ($\Lambda \geq 0$) for any $u, v \in \mathbb{R}$ and $p, q \in \mathbb{R}^n$ *4'*) $m > max(1, \frac{2n}{n+2})$.

Proposition 5.1. Let f be measurable and bounded in Q_T , $\Psi \in W_2^1(Q_T)$ and let *conditions* $0'$) - 4') *hold. Then the Cauchy-Dirichlet problem* (5.10) *has exactly one generalized (in sense of Definition* 3.1) *solution u. Moreover this solution belongs to* $C([0, T]; L_2(\Omega)).$

Proof. Uniqueness of the generalized solution of problem (5.10) follows from Proposition 3.2. So we have to prove only existence of solution cited. The forthcoming proof is a suitable adaptation of the proof of Theorem 6.7 of [14: Chapter 51. miqueness of the generalized solution of problem (5.10) follow
we have to prove only existence of solution cited. The fort
daptation of the proof of Theorem 6.7 of [14: Chapter 5].
 $\in \mathbb{N}$ be a basis in $\tilde{W}_m^1(\Omega)$ on 3.1) solution u. Moreover this solution belongs to

neralized solution of problem (5.10) follows from Propo-

nly existence of solution cited. The forthcoming proof

coof of Theorem 6.7 of [14: Chapter 5].
 $\mathring{W}_m^1(\$

Let $\{\Psi_k\}_{k\in\mathbb{N}}$ be a basis in $\mathring{W}_m^1(\Omega)$ such that $\int_{\Omega}\Psi_k\Psi_l dx = \delta_k^l$ $(k, l \in \mathbb{N})$, where δ_k^l is the Kronecker delta, and

$$
\sup\left(|\Psi_k|,\Omega\right)+\sup\left(|\nabla\Psi_k|,\Omega\right)\leq c_k=\text{ const}\qquad (k\in I\!\!N).
$$

Set

$$
u^{\mathcal{N}} = \sum_{k=1}^{\mathcal{N}} c_k^{\mathcal{N}}(t) \Psi_k(x) \tag{5.11}
$$

where $\{c_k^N\}_{k=1,...,N}$ is the solution of the system of ordinary differential equations

$$
\sup(|\Psi_k|, \Omega) + \sup(|\Psi_k|, \Omega) \le c_k = \text{const} \qquad (\kappa \in \mathbb{N}).
$$
\n
$$
u^{\mathcal{N}} = \sum_{k=1}^{N} c_k^{\mathcal{N}}(t) \Psi_k(x) \qquad (5.11)
$$
\n
$$
u^{\mathcal{N}} = \sum_{k=1}^{N} c_k^{\mathcal{N}}(t) \Psi_k(x) \qquad (5.12)
$$
\n
$$
(u_i^{\mathcal{N}}, \Psi_k) + \left(a^i(u^{\mathcal{N}}, \nabla u^{\mathcal{N}}), \frac{\partial \Psi_k}{\partial x_i}\right) = (f, \Psi_k) \qquad (k = 1, \dots, N) \qquad (5.12)
$$
\n
$$
\text{conditions}
$$
\n
$$
c_k^{\mathcal{N}}(0) = (\Psi(x, 0), \Psi_k) \qquad (k = 1, \dots, N). \qquad (5.13)
$$
\nanditions of Proposition 5.1 it follows that the second and third terms in
\nounded and measurable functions of the variables $t, c_k^{\mathcal{N}}$ on any set $[0, T] \times$

with initial conditions

$$
c_k^N(0) = (\Psi(x,0), \Psi_k) \qquad (k = 1, ..., N). \qquad (5.13)
$$

From the conditions of Proposition 5.1 it follows that the second and third terms in (5.12) are bounded and measurable functions of the variables t, c_k^N on any set $[0, T] \times$ ${c_1, b_1 \leq c_2 \atop c \in \mathbb{R}^N}$ ($c \in \{k = 1, ..., N\}$); moreover these functions are continuous in c_k^N . There fore existence at least of one solution of (5.12), (5.13) will be established if we could show that all possible fore existence at least of one solution of (5.12), (5.13) will be established if we could show that all possible solutions of this problem are uniformly bounded on [0, *T].* Exactly in the same way as in [14: pp. 533 - 535] we can prove that the a priori estimate be functions of the variables t, c_k^N on any set $[0, T] \times$

}; moreover these functions are continuous in c_k^N . There-

solution of (5.12), (5.13) will be established if we could

sof this problem are uniformly bounded **2 (v)** = ($\mathbf{Y}(t, 0)$, $\mathbf{Y}(t)$ = 1
 Proposition 5.1 it follows that

measurable functions of the va
 \ldots , N)}; moreover these function

f one solution of (5.12), (5.13)

utions of this problem are unifo

4: pp the second and third terms in

riables t, c_k^N on any set $[0, T] \times$

ms are continuous in c_k^N . There-

will be established if we could

rmly bounded on $[0, T]$. Exactly

that the a priori estimate
 $2r$ $\geq c$ (5.14)
 Froposition 5.1

I measurable function 5.1
 \ldots , N) ; moreover

of one solution of

olutions of this proved
 14 : pp. 533 - 535

sup $||u^N||_{L_2(\Omega)}^2$
 $u \in [0,T]$

ant c independent

sup $\sum_{t \in [0,T]}^N |c_k^N(t)|^2$

least of

$$
\sup_{t \in [0,T]} \|u^N\|_{L_2(\Omega)}^2 + \|\nabla u^N\|_{L_m(Q_T)}^m \le c \tag{5.14}
$$

holds with some constant c independent of *N.* Then from (5.14) it follows that

int c independent of N. Then from (3.14) it follows that
\n
$$
\sup_{t \in [0,T]} \sum_{k=1}^{N} |c_k^N(t)|^2 = \sup_{t \in [0,T]} ||u^N||^2_{L_2(\Omega)} \le c \qquad (5.15)
$$
\nleast of one solution (5.12), (5.13) is established. From (5.14)
\n34]) that
\n
$$
||u^N||_{L_{m(n+2)/n}(Q_T)} \le c \qquad (5.16)
$$
\nindependent of N. Moreover, for any fixed k the functions

and hence existence at least of one solution (5.12), (5.13) is established. $_i$ From (5.14) it follows (see [14: p. 534]) that
 $||u^N||_{L_{m(n+2)/n}(Q_T)} \le c$ (5.16) it follows (see [14: p. 534]) that $\sup_{t \in [0,T]} \sum_{k=1} |c_k^N(t)|^2 = \sup_{t \in [0,T]} ||u^N||_{L_2(\Omega)}^2 \le c$ (5.15)
 1 least of one solution (5.12), (5.13) is established. *i* From (5.14)
 534]) that
 $||u^N||_{L_{m(n+2)/n}(Q_T)} \le c$ (5.16)
 s independent of *N*. Moreover, fo

$$
||u^N||_{L_{m(n+2)/n}(Q_T)} \leq c \tag{5.16}
$$

where the constant *c* is independent of *N.* Moreover, for any fixed *k* the functions

$$
l_{N,k}(t) = (u^N(x,t), \Psi_k(x)) \qquad (N, k \in I\!\!N). \tag{5.17}
$$

are equicontinuous (with respect to *N*) in t on $[0, T]$. Together with (5.14) it gives the possibility (see [14: p. 535]) to choose some subsequence $\{u^N\}$ that converges weakly in $L_2(\Omega)$ uniformly with respect to *t* on $[0, T]$ to some function *u* such that Existence and Uniqueness of a Regular Solution 763

ct to N) in t on [0, T]. Together with (5.14) it gives the

choose some subsequence $\{u^N\}$ that converges weakly

tt to t on [0, T] to some function u such that

sup Existence and Uniqueness of a Regular Solution 763

pect to *N*) in *t* on [0, *T*]. Together with (5.14) it gives the

to choose some subsequence $\{u^N\}$ that converges weakly

sect to *t* on [0, *T*] to some function *IIII* I on $[0, T]$. Together with (5.14) it gives the
ose some subsequence $\{u^N\}$ that converges weakly
t on $[0, T]$ to some function u such that
 $||u||_{L_2(\Omega)}, [0, T]) \leq c.$ (5.18)
1 count that
weakly in $L_m(Q_T)$ as

$$
\sup\left(\|u\|_{L_2(\Omega)}, [0, T]\right) \le c. \tag{5.18}
$$

Moreover, using again (5.14) we can count that

\n- 4: p. 535]) to choose some subsequence
$$
\{u''\}
$$
 that converges weakly y with respect to t on $[0, T]$ to some function u such that\n
$$
\sup\left(\|u\|_{L_2(\Omega)}, [0, T]\right) \leq c.
$$
\n
\n- 5.18)
\n- gain (5.14) we can count that\n
$$
\frac{\partial u^N}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \qquad \text{weakly in } L_m(Q_T) \text{ as } N \to \infty
$$
\n
\n- 5.19)
\n- 5.19
\n

and hence $u \in \mathring{W}_{m}^{1,0}(Q_T)$ and

$$
\|\nabla u\|_{L_m(Q_T)} \leq c \tag{5.20}
$$

with some constant c depending only on the data (see [14: p. 535]).

Obviously, from (5.12) it follows that the integral identity

hence
$$
u \in \mathring{W}_m^{1,0}(Q_T)
$$
 and
\n
$$
\|\nabla u\|_{L_m(Q_T)} \leq c
$$
\n(5.20)
\nsome constant c depending only on the data (see [14: p. 535]).
\nObviously, from (5.12) it follows that the integral identity
\n
$$
\int_{\Omega} u^N \varphi \, dx \Big|_0^r + \iint_{Q_T} \Big(-u^N \varphi_t + a(u^N, \nabla u^N) \cdot \nabla \varphi \Big) dx dt = \iint_{Q_T} f \varphi \, dx dt
$$
\n(5.21)
\nso for any $\tau \in (0, T]$ and $\iota = \sum_{i=1}^N d_i(\tau) \Psi(\tau)$ where d, are arbitrary functions

holds for any $\tau \in (0,T]$ and $\varphi = \sum_{k=1}^{N} d_k(t) \Psi_k(x)$ where d_k are arbitrary functions continuous in *t* on [0, *T*] and $\gamma = \sum_{k=1}^{\infty} a_k(x) x_k(x)$ where a_k are distintives *d_k*. Denote the class of such functions φ as \mathcal{P}_N . Obviously, u^N belong to \mathcal{P}_N . Denote $A_N^i =$ $a^{i}(u^{N}, \nabla u^{N})$ $(i = 1, ..., N)$. In view of the second inequality in condition 1') and estimate (5.14) we have the uniform (with respect to N) estimate *III.* $\left(\int_{Q_r}^{\infty} \left(1 + \frac{1}{2} \ln \frac{1}{2} \ln \left(1 + \frac{1}{2} \ln \frac{1}{2} \ln \left(1 + \frac{1}{2} \ln$ $\left(-u^N\varphi_t + a(u^N, \nabla u^N) \cdot \nabla \varphi\right) dx dt = \iint_{Q_\tau} f\varphi dx$

and $\varphi = \sum_{k=1}^N d_k(t)\Psi_k(x)$ where d_k are arbitr

and having bounded on $[0, T]$ generalized derivative

ns φ as \mathcal{P}_N . Obviously, u^N belong to \mathcal{P}_N .

$$
||A_i^N||_{L_{m'}(Q_T)} \le c \qquad (i = 1, ..., N; N \in \mathbb{N}).
$$
 (5.22)

Therefore we can count that there exist functions $A_i \in L_{m'}(Q_T)$ such that

$$
A_i^N \longrightarrow A_i \qquad \text{weakly in } L_{m'}(Q_T). \tag{5.23}
$$

Using estimate (5.14) and taking into account that $u^N \to u$ weakly in $L_2(\Omega)$ (uniformly with respect to *t* on $[0, T]$) we derive from inequality (5.2) in the case $\beta = 0$ for the difference $u^N - u^{N_1}$ that $\text{Uern (with respect to } N)$ estimate
 $y \leq c$ ($i = 1, ..., N; N \in \mathbb{N}$).
 e exist functions $A_i \in L_{m'}(Q_T)$ such th
 A_i weakly in $L_{m'}(Q_T)$.

into account that $u^N \to u$ weakly in L_2

erive from inequality (5.2) in the case
 u str *U* count that there exist functions $A_i \in L_{m'}(Q_T)$ such that
 $A_i^N \longrightarrow A_i$ weakly in $L_{m'}(Q_T)$. (5.23)

14) and taking into account that $u^N \rightarrow u$ weakly in $L_2(\Omega)$ (uniformly

on [0, *T*]) we derive from inequality (5. $\rightarrow A_i$ weakly in $L_{m'}(Q_T)$.

Ing into account that $u^N \rightarrow u$ weakly in
 e derive from inequality (5.2) in the c
 $\rightarrow u$ strongly in $L_{2,m}(Q_T)$

strongly in $L_2(\Omega)$ for a.e. $t \in [0, T]$
 $u^N \rightarrow u$ a.e. in Q_T .

d conditio we derive from inequality (5.2) in the case
 $N \longrightarrow u$ strongly in $L_{2,m}(Q_T)$

at
 ι strongly in $L_2(\Omega)$ for a.e. $t \in [0, T]$
 $u^N \longrightarrow u$ a.e. in Q_T .

and condition 4')
 $u^N \longrightarrow u$ weakly in $L_2(Q_T)$.

$$
u^N \longrightarrow u \qquad \text{strongly in} \ \ L_{2,m}(Q_T) \tag{5.24}
$$

and hence we can count that

$$
u^N \longrightarrow u \qquad \text{strongly in} \ \ L_2(\Omega) \text{ for a.e.} \quad t \in [0, T] \tag{5.25}
$$

and

$$
u^N \longrightarrow u \qquad \text{a.e. in } Q_T. \tag{5.26}
$$

Moreover, in view of (5.16) and condition *41)*

$$
u^N \longrightarrow u \qquad \text{weakly in} \quad L_2(Q_T). \tag{5.27}
$$

Then from (5.21) and (5.23) - (5.27) we can conclude that for a.e. $\tau \in (0,T]$ and $\varphi \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$

vanov
\n21) and (5.23) - (5.27) we can conclude that for a.e.
$$
\tau \in (0, T]
$$
 and
\n
$$
\int_{\Omega} u \varphi \, dx \Big|_{0}^{\tau} + \iint_{Q_{\tau}} \left(-u \varphi_{t} + A_{i} \varphi_{x_{i}} \right) dx dt = \iint_{Q_{\tau}} f \varphi \, dx dt.
$$
\n(5.28)
\nwe way as in [14: p. 538] we can derive from (5.28) and (5.18) that
\n $u \in C([0, T]; L_{2}(\Omega))$ \n(5.29)
\nhat identity (5.28) holds for any $\tau \in (0, T]$. Moreover, we establish that
\n0, T]
\n
$$
\frac{1}{2} \int_{\Omega} u^{2} dx \Big|_{0}^{\tau} + \iint_{Q_{\tau}} A_{i} u_{x_{i}} dx dt = \iint_{Q_{\tau}} fu \, dx dt.
$$
\n(5.30)
\nu is a generalized solution of (5.10) it is sufficient to establish that

In the same way as in $[14: p. 538]$ we can derive from (5.28) and (5.18) that

$$
u \in C([0, T]; L_2(\Omega))
$$
\n(5.29)

and to prove that identity (5.28) holds for any $\tau \in (0, T]$. Moreover, we establish that for every $\tau \in (0, T]$

z as in [14: p. 538] we can derive from (5.28) and (5.18) that
\n
$$
u \in C([0, T]; L_2(\Omega))
$$
 (5.29)
\n
$$
\begin{aligned}\n&\text{density (5.28) holds for any } \tau \in (0, T]. \text{ Moreover, we establish that} \\
&\frac{1}{2} \int_{\Omega} u^2 dx \Big|_0^{\tau} + \iint_{Q_{\tau}} A_i u_{z_i} dx dt = \iint_{Q_{\tau}} f u \, dx dt. \qquad (5.30) \\
&\text{generalized solution of (5.10) it is sufficient to establish that} \\
&\iint_{Q_{\tau}} A_i \varphi_{z_i} dx dt = \iint_{Q_{\tau}} a^i(u, \nabla u) \varphi_{z_i} dx dt \qquad (5.31) \\
&\mathcal{P}_k \text{ because } \bigcup_{k=1}^{\infty} \mathcal{P}_k \text{ is dense in } \mathring{W}^1_m(Q_T). \text{ To prove (5.31) it is that} \\
&\frac{\partial u^N}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \quad (i = 1, ..., N) \qquad \text{a.e. in } Q_{\tau} \qquad (5.32) \\
&\text{5.32) and (5.26), the continuity of the functions } a^i(u, p), \text{condition and the Vitali theorem we obtain that for any } \omega \in \mathbb{H} \rvert_{\infty}^{\infty} \rvert_{\infty}^{\infty}.\n\end{aligned}
$$

To prove that u is a generalized solution of (5.10) it is sufficient to establish that

$$
\iint_{Q_r} A_i \varphi_{x_i} dx dt = \iint_{Q_r} a^i(u, \nabla u) \varphi_{x_i} dx dt
$$
\n(5.31)
\n
$$
k \text{ because } \bigcup_{k=1}^{\infty} P_k \text{ is dense in } \mathring{W}_m^1(Q_T). \text{ To prove (5.31) it is that}
$$
\n
$$
\frac{\partial u^N}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \quad (i = 1, ..., N) \qquad \text{a.e. in } Q_r \qquad (5.32)
$$
\n(5.32) and (5.26), the continuity of the functions $a^i(u, p)$, condition

for any $\varphi \in \bigcup_{k=1}^{\infty} P_k$ because $\bigcup_{k=1}^{\infty} P_k$ is dense in $\mathring{W}_m^1(Q_T)$. To prove (5.31) it is $\begin{aligned} \text{suffix} \text{a} & \text{and} \text{b} & \text{or} \text{b} \ \text{or} \text{b} & \text{or} \text{c} \end{aligned}$

$$
\frac{\partial u^N}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \quad (i = 1, ..., N) \qquad \text{a.e. in } Q_\tau \tag{5.32}
$$

because in view of (5.32) and (5.26), the continuity of the functions $a^{i}(u, p)$, condition 1'), estimate (5.14) and the Vitali theorem we obtain that for any $\varphi \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$

$$
\begin{aligned}\n\text{ew of (5.32) and (5.26), the continuity of the functions } a^i(u) \\
\text{5.14) and the Vitali theorem we obtain that for any } \varphi \in \bigcup_{N \to \infty} \iint_{Q_{\tau}} a^i(u^N, \nabla u^N) \varphi_{x_i} \, dxdt = \iint_{Q_{\tau}} a^i(u, \nabla u) \varphi_{x_i} \, dxdt.\n\end{aligned}
$$

On the other hand in view of (5.23)

d in view of (5.23)
\n
$$
\lim_{N \to \infty} \iint_{Q_{\tau}} a^i(u^N, \nabla u^N) \varphi_{x_i} dx dt = \iint_{Q_{\tau}} A_i \varphi_{x_i} dx dt.
$$

Hence (5.32) implies (5.31) . The remainder of this section is devoted to proving of (5.32).

Choosing $\varphi = u^N$ in (5.21) we obtain

$$
\lim_{N \to \infty} \iint_{Q_{\tau}} a^{i} (u^{N}, \nabla u^{N}) \varphi_{x_{i}} dx dt = \iint_{Q_{\tau}} A_{i} \varphi_{x_{i}} dx dt.
$$

5.32) implies (5.31). The remainder of this section is devoted to proving of
osing $\varphi = u^{N}$ in (5.21) we obtain

$$
\frac{1}{2} \int_{\Omega} (u^{N})^{2} dx \Big|_{0}^{r} + \iint_{Q_{\tau}} a(u^{N}, \nabla u^{N}) \cdot \nabla u^{N} dx dt = \iint_{Q_{\tau}} f u^{N} dx dt.
$$
 (5.33)
5.25) and (5.27) we derive from (5.33) and (5.30) that for any $\tau \in (0, T]$

Using (5.25) and (5.27) we derive from (5.33) and (5.30) that for any $\tau \in (0, T]$

$$
\lim_{N \to \infty} \iint_{Q_r} a^i(u^N, \nabla u^N) \varphi_{x_i} dx dt = \iint_{Q_r} A_i \varphi_{x_i} dx dt.
$$

plies (5.31). The remainder of this section is devoted to proving of

$$
= u^N \text{ in (5.21) we obtain}
$$

$$
\int^2 dx \Big|_0^r + \iint_{Q_r} a(u^N, \nabla u^N) \cdot \nabla u^N dx dt = \iint_{Q_r} f u^N dx dt. \qquad (5.33)
$$

$$
\iota
$$
(5.27) we derive from (5.33) and (5.30) that for any $\tau \in (0, T]$
$$
\lim_{N \to \infty} \iint_{Q_r} a(u^N, \nabla u^N) \cdot \nabla u^N dx dt = \iint_{Q_r} A_i u_{x_i} dx dt. \qquad (5.34)
$$

Using now condition 2') we have

Existence and Uniqueness of a Regular Solution

\n
$$
V_{1} \iint_{Q_{r}} |\nabla u^{N} - \nabla u|^{m} dx dt
$$
\n
$$
\leq \iint_{Q_{r}} \left(a(u^{N}, \nabla u^{N}) - a(u^{N}, \nabla u) \right) \cdot (\nabla u^{N} - \nabla u) dx dt.
$$
\n(5.35)

\n(5.23) and (5.34) and taking into account (in view of 1'), (5.20) and (5.26)

\n
$$
a^{i}(u^{N}, \nabla u) \longrightarrow a^{i}(u, \nabla u)
$$
 strongly in $L_{m'}(Q_{T})$ as $N \to \infty$ (5.36)

\n(5.37)

\n
$$
\lim_{N \to \infty} \iint_{Q_{r}} |\nabla u^{N} - \nabla u|^{m} dx dt = 0.
$$
 (5.37)

\nn (5.37) it follows that (5.32) holds for some subsequence $\{u^{N}\}$.

Using (5.19) , (5.23) and (5.34) and taking into account (in view of 1'), (5.20) and (5.26)) that

$$
a^{i}(u^{N}, \nabla u) \longrightarrow a^{i}(u, \nabla u) \qquad \text{strongly in } L_{m'}(Q_{T}) \text{ as } N \to \infty \tag{5.36}
$$

we derive from (5.35)

$$
\lim_{N \to \infty} \iint_{Q_r} |\nabla u^N - \nabla u|^m \, dx dt = 0. \tag{5.37}
$$

But from (5.37) it follows that (5.32) holds for some subsequence $\{u^N\}$

6. A priori estimates for solutions of regularized Cauchy-Dirichlet problems

In view of Theorem 3.1 to prove Theorem 1.1 it is sufficient to establish the following

Theorem 6.1. *Let conditions* (Ω), (BI), (RHS) *and* 0) - 4) *hold. Then the Cauchy-*
 n) $F[u] = f$ *in* Q_T
 $u = \Psi$ *on* Γ_T *Dirichlet problem* prove Theorem 1.1 it is sufficient to establish the following
 mditions (Ω), (BH), (RHS) and 0) - 4) hold. Then the Cauchy-
 n sense of Definition 2.5) solution.

16.1 correspondent to the case
 $m \ge 2$

(CD) F[u] = f in QT

has at least one regular (in sense of Definition 2.5) *solution.*

The result of Theorem 6.1 correspondent to the case

$$
m \ge 2 \qquad \text{and} \qquad l \ge 0 \tag{6.1}
$$

can be derived from the proof of the main theorem of the paper [11] if to use Theorem *4.1* of the present paper. Therefore we shall prove Theorem 6.1 only in the case when (in sense of Definition 2.5) solution.

em 6.1 correspondent to the case
 $m \ge 2$ and $l \ge 0$ (6.1)

proof of the main theorem of the paper [11] if to use Theorem

. Therefore we shall prove Theorem 6.1 only in the case w

It is easy to see that

$$
\omega\subset \{(m,l):1
$$

The proof of Theorem 6.1 correspondent to the case (6.2) can be easily transformed in one applicable in the case (6.1).

In the remainder of this paper we assume that all conditions (Ω) , (BI) , (RHS) and 0) - 4) of Theorem 6.1 and also condition (6.2) are fulfilled.

Consider the regularized Cauchy-Dirichiet problems

³¹¹(RCD) S,e,N *Fs,,N[u] a-t -* 6Vu - div a((u), Vu) = *I* in QT *u=P+e* on r7, 6>0, *(u) min { max (u,e),N}, c >0, N> E. (6.3)*

where

Without loss of generality we can and shall count that $\delta \leq 1$ and $\varepsilon \leq 1$. It is easy to see that in view of conditions 0) - *4)* and (6.2) and the structure of the left-hand side of equation in $(RCD)_{\delta,\epsilon,N}$ assumptions 0') - 3') of Proposition 5.1 are fulfilled with $m=2$ Without loss of generality we can and shave that in view of conditions 0) - 4) and equation in $(RCD)_{\delta,\epsilon,N}$ assumptions 0') - because $\epsilon \leq \chi(u) \leq N$ and $|p|^{m-1} + 1 \leq$ $|p| + 1$ for any $m \in (1, 2)$. in $N > \varepsilon$.
 i and $\varepsilon \leq 1$. It is

ire of the left-han

in are fulfilled with

in Q_T

on Γ_T

Denote $v = u - \varepsilon$ and consider the Cauchy-Dirichlet problem

conditions 0) - 4) and (6.2) and the structure of the left-hand side of
\n5,
$$
\varepsilon, N
$$
 assumptions 0') - 3') of Proposition 5.1 are fulfilled with $m = 2$
\n ε and $|p|^{m-1} + 1 \le |p| + 1$ for any $m \in (1, 2)$.
\n ε and consider the Cauchy-Dirichlet problem
\n
$$
\frac{\partial v}{\partial t} - \delta \nabla v - \text{div } a(\chi(v + \varepsilon), \nabla v) = f \qquad \text{in} \quad Q_T
$$
\n
$$
v = \Psi \qquad \text{on} \quad \Gamma_T
$$
\n(6.4)

where $\Psi \in \mathring{W}_2^1(Q_T)$. In view of previous conclusions it follows obviously that for the problem (6.4) all conditions of Proposition 5.1 are fulfilled with $m = 2$. Hence there exists exactly one generalized solution *v* of this problem (such that $v \in C([0,T]; L_2(\Omega)) \cap$ $\mathring{W}_2^1(Q_T)$). But then the Cauchy-Dirichlet problem $(RCD)_{\delta,\epsilon,N}$ has exactly one generalized solution *u* such that $u \in C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$, i.e., we proved the following

Lemma 6.1. *For any* $\delta > 0, \epsilon > 0, N > \epsilon$ the Cauchy-Dirichlet problem $(RCD)_{\delta,\epsilon,N}$ *has exactly one generalized solution* $u \in C([0, T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$ *.*

In the remainder of this section we consider problem $(RCD)_{\delta,\epsilon,N}$ for $\delta \geq 0, \epsilon > 0$ and $N > \epsilon$. Now the term "generalized solution u" means in particular that $u \in$ $C\big([0,T];L_2(\Omega)\big)\cap W_2^{1,0}(Q_T)$ in the case $\delta>0$ and $u\in C\big([0,T];L_2(\Omega)\big)\cap W_{m}^{1,0}(Q_T)$ in the case $\delta = 0$. *i* $0, N > \varepsilon$ *the Cauchy-Dirichlet problem* $(\text{RCD})_{\delta,\varepsilon,N}$
 $u \in C([0,T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T)$.

we consider problem $(\text{RCD})_{\delta,\varepsilon,N}$ for $\delta \geq 0, \varepsilon > 0$

ralized solution u" means in particular that $u \in$

case $\delta > 0$

Lemma 6.2. Let u be a generalized solution of problem $(RCD)_{\delta,\epsilon,N}$ for any fixed $\delta \geq 0, \varepsilon > 0$ and $N > \varepsilon$. Then

$$
\inf(u, Q_T) \ge \varepsilon. \tag{6.5}
$$

Proof. Obviously that the conditions of Theorem 6.1 imply validity of the as- $\delta \geq 0, \varepsilon > 0$ and $N > \varepsilon$. Then
 $\inf(u, Q_T) \geq \varepsilon$. (6.5)
 Proof. Obviously that the conditions of Theorem 6.1 imply validity of the as-

sumptions (G), (M) and (L) of Proposition 3.1 for the operator $F_{\delta,\varepsilon,N}[u]$ if $\delta > 0$. Then taking into account that $F_{\delta, \epsilon, N}[u] = f$, $F_{\delta, \epsilon, N}[\epsilon] = 0$ and $u = \epsilon$ on S_T , we can apply Proposition 3.1 for $u_1 = \epsilon, u_2 = u$ and $f_1 = 0, f_2 = f$. Using that $u_1 = \varepsilon \leq \Psi + \varepsilon = u_2$ on $\Omega \times \{t = 0\}$ (because $\Psi \geq 0$) we derive from (3.2) that $(\varepsilon - u)^+ \leq 0$ a.e. in Q_T , i.e., $u \geq \varepsilon$ a.e. in $Q_T \blacksquare$

Lemma 6.3. There exist constants c_1 and c_2 depending on n, m, l , the parameters *from conditions 1) - 3), and* $\sup(\Psi,\overline{Q}_T)$ *such that for any generalized solution u of* Existence and Uniqueness of
 lants c_1 and c_2 depending
 Ψ, \overline{Q}_T *such that for any*
 $\delta \geq 0, \varepsilon > 0$ and $N \geq c_1$ *i*
 $sup(u, Q_T) \leq c_1$ \mathbf{L} Existence and Uniqueness of a Regular Solution
 There exist constants c_1 and c_2 depending on n, m, l , the paral l) - 3), and $\sup(\Psi, \overline{Q}_T)$ such that for any generalized solution
 c, N with any fixed $\delta \geq 0,$

$$
sup(u, Q_T) \le c_1 \tag{6.6}
$$

and

problem (RCD)_{6, \epsilon, N} with any fixed
$$
\delta \ge 0, \epsilon > 0
$$
 and $N \ge c_1$ we have
\n
$$
sup(u, Q_T) \le c_1
$$
\n(6.6)
\nand
\n
$$
\epsilon^l \iint_{Q_T} |\nabla u|^m dx dt \le \iint_{Q_T} |\nabla u^{\alpha+1}|^m dx dt \le c_2 \qquad (\alpha = \frac{l}{m}).
$$
\n(6.7)
\nProof. The proof of validity of estimates (6.6) and (6.7) in the case $m + l \ge 2$ is
\n $lim_{n \to \infty} \ln |U_n|$. The proof of validity of semicial to find a sum (6.7) in the case $m + l \ge 2$ is

Proof. The proof of validity of estimates (6.6) and (6.7) in the case $m + l \geq 2$ is given in [11]. The case $m + l < 2$ required to find new (a more difficult) version of the Moser method of establishing L_{∞} -estimates. It was made in paper [10]. In the case $m + l < 2$ the lemma follows from Theorems 1.1 and 1.2 of [10]

Remark 6.1. In the remainder of this paper we consider problem $(RCD)_{\delta,\epsilon,N}$ with $N = c_1$ where the constant c_1 is defined by Lemma 6.3. In view of estimates (6.5) and (6.6) we can rewrite problem $(RCD)_{\delta,\epsilon,N}$ as **Remark 6.1.** In the remainder of this paper we conside:
 $N = c_1$ where the constant c_1 is defined by Lemma 6.3. In v

(6.6) we can rewrite problem $(RCD)_{\delta,\epsilon,N}$ as

(**RCD**) $_{\delta,\epsilon}$, $F_{\delta,\epsilon}[u] := \frac{\partial u}{\partial t} - \delta \nabla u - \text{div } a$

$$
(\text{RCD})_{\delta,\epsilon} \quad F_{\delta,\epsilon}[u] := \frac{\partial u}{\partial t} - \delta \nabla u - \text{div } a(u, \nabla u) = f \quad \text{in } Q_T
$$

$$
u = \Psi + \epsilon \quad \text{on } \Gamma_T
$$

where $\delta > 0$ and $\varepsilon > 0$.

Lemma 6.4. *Let u be a generalized solution of the Cauchy- Dirichlet problem* $(\text{RCD})_{\delta,\varepsilon}$ for $\delta = 0$ and $\varepsilon > 0$. Then there exist constants $\lambda \in (0,1)$ and $K > 0$ *independent of* ε *such that (see (4.21)) Example 1 (a) (d) (d)*

$$
\langle u \rangle_{\lambda, \overline{Q}_T} \le K. \tag{6.8}
$$

Proof. In view of conditions 1) -3), Remark *6.1,* estimates (6.5)- (6.7) and Remark 4.1 we can apply either Theorem 4.1 or Theorem 4.2 and hence establish (6.8) with some $\lambda \in (0,1)$ and $K > 0$ independent of $\varepsilon \blacksquare$

7. The passing to the limit as $\delta \rightarrow 0$

In this section we show that generalized solutions u_{δ} of the Cauchy-Dirichlet problems $(RCD)_{\delta,\epsilon}$ (for any fixed $\epsilon > 0$) tend to a generalized solution of the Cauchy-Dirichlet problem

$$
(\text{RCD})_{\epsilon} \ \ F_{\epsilon}[u] := \frac{\partial u}{\partial t} - \text{div}\, a(u, \nabla u) = f \quad \text{in} \ \ Q_T
$$

$$
u = \Psi + \epsilon \quad \text{on} \ \Gamma_T
$$

as $\delta \rightarrow 0$. For proving this we use estimates (6.5) - (6.7) and Lemma 5.1 with appropriate $\beta > 0$.

Obviously, the functions u_{δ} satisfy for any $\tau \in (0, T]$ and every function $\phi \in \mathring{W}^1_2(Q_T)$ the integral identity

$$
\int_{\Omega} u_{\delta} \phi \, dx \Big|_{0}^{r} + \iint_{Q_{T}} \Big(-u_{\delta} \psi + \delta \nabla u_{\delta} \cdot \nabla \phi + a(u_{\delta}, \nabla u_{\delta}) \cdot \nabla \phi - f \phi \Big) dx dt = 0. \tag{7.1}
$$

Set here $\phi = \Psi \in C_0^1(\Omega)$. Then from condition 1) and estimates (6.6) and (6.7) it follows that for any $t_1, t_2 \in [0, T]$ we have

byology, the functions
$$
u_{\delta}
$$
 satisfy for any $\tau \in (0, T]$ and every function $\phi \in W_2(Q_T)$
\ntegral identity

\n
$$
u_{\delta}\phi \, dx \Big|_{0}^{\tau} + \iint_{Q_T} \Big(-u_{\delta}\psi + \delta \nabla u_{\delta} \cdot \nabla \phi + a(u_{\delta}, \nabla u_{\delta}) \cdot \nabla \phi - f\phi \Big) dx dt = 0. \tag{7.1}
$$

\nare $\phi = \Psi \in C_0^1(\Omega)$. Then from condition 1) and estimates (6.6) and (6.7) it is that for any $t_1, t_2 \in [0, T]$ we have

\n
$$
\Big| \int_{\Omega} u_{\delta} \Psi \, dx \Big|_{t_1}^{t_2} \Big| \leq c \int_{t_1}^{t_2} \int_{\Omega} \Big(|\nabla u_{\delta}| + |\nabla u_{\delta}^{\alpha+1}|^{m-1} + 1 \Big) dx dt
$$

\n
$$
\leq c \Big(\Big(|t_2 - t_1| |\Omega| \Big)^{1/2} + \Big(|t_2 - t_1| |\Omega| \Big)^{1/m} + |t_2 - t_1| |\Omega| \Big).
$$

\nhere it follows that the integrals $\int_{\Omega} u_{\delta} \Psi \, dx$ $(\delta \in (0, 1))$ are equicontinuous (with $t \to \delta$) in t on $[0, T]$ for any fixed $\Psi \in C^1(\overline{\Omega})$. Using the density of $C^1(\overline{\Omega})$ in I .

From here it follows that the integrals $\int_{\Omega} u_{\delta} \Psi dx$ ($\delta \in (0,1)$) are equicontinuous (with respect to δ) in t on $[0, T]$ for any fixed $\Psi \in C_0^1(\overline{\Omega})$. Using the density of $C_0^1(\overline{\Omega})$ in $L_2(\Omega)$ and the uniform boundedness of the sequence $\{u_\delta\}$ in Q_T (see (6.6)) we can derive from here that there exists a subsequence $\{u_{\delta}\}\$ which converges weakly in $L_2(\Omega)$, uniformly with respect to t on $[0, T]$, to some function u satisfying inequality (5.18) with a constant c independent of δ (see also [14: pp. 182-183]). Moreover, in view of (6.5) - (6.7) we can count that ²¹
 $V_1 \overline{N}$ (7.2)
 $\leq c \Big(\Big(|t_2 - t_1| |\Omega| \Big)^{1/2} + \Big(|t_2 - t_1| |\Omega| \Big)^{1/m} + |t_2 - t_1| |\Omega| \Big)$.

ws that the integrals $\int_{\Omega} u_{\delta} \Psi dx$ ($\delta \in (0, 1)$) are equicontinuous (with

on $[0, T]$ for any fixed $\Psi \in C_0^1(\overline{\Omega})$ $\leq c \Big(\big(|t_2 - t_1| |\Omega| \big)^{1/2} + \big(|t_2 - t_1| |\Omega| \big)^{1/m} + |t_2 - t_1| |\Omega| \Big).$

is that the integrals $\int_{\Omega} u_{\delta} \Psi dx$ ($\delta \in (0, 1)$) are equicontinuous (with $[0, T]$ for any fixed $\Psi \in C_0^1(\overline{\Omega})$. Using the density of $C_0^1(\$ vow that the integrals $\int_{\Omega} u_{\delta} \Psi dx$ ($\delta \in (0,1)$) are equicontinuous (with

on $[0, T]$ for any fixed $\Psi \in C_0^1(\overline{\Omega})$. Using the density of $C_0^1(\overline{\Omega})$ in $L_2(\Omega)$

boundedness of the sequence $\{u_{\delta}\}$ in Q_T (s *Superformals the* $\int_{\Omega} u_{\theta} dV = C_0^1(\overline{\Omega})$ *. Using the density of* $C_0^1(\overline{\Omega})$ *in* $L_2(\Omega)$ *as of the sequence* $\{u_{\theta}\}\}$ *in* Q_T *(see (6.6)) we can derive from sequence* $\{u_{\theta}\}\}$ *which converges weakly in* $L_2(\Omega)$ *,* respect to *t* on [0, *T*], to so
ependent of δ (see also [1
ount that
 $\nabla u_{\delta}^{\alpha+1} \longrightarrow \nabla$
 $\nabla u_{\delta} \longrightarrow \nabla$
 $\nabla \overline{\delta} \nabla u_{\delta} \longrightarrow 0$
sur
 $\varepsilon^{l} \iint_{Q_T} |\nabla u_{\delta}|^m c$
 $+ \iint_{Q_T} |\nabla$

$$
\nabla u_{\delta}^{\alpha+1} \longrightarrow \nabla u^{\alpha+1} \qquad \text{weakly in } L_m(Q_T) \text{ as } \delta \to 0 \tag{7.3}
$$

$$
\nabla u_{\delta} \longrightarrow \nabla u \qquad \text{ weakly in } L_m(Q_T) \text{ as } \delta \to 0 \tag{7.4}
$$

$$
\sqrt{\delta} \nabla u_{\delta} \longrightarrow 0 \qquad \text{ weakly in } L_2(Q_T) \quad \text{as } \delta \to 0 \tag{7.5}
$$

$$
\sup(u_{\delta}, Q_T) + \sup(u, Q_T) \le c_1 \tag{7.6}
$$

and

Unit that

\n
$$
\nabla u_{\delta}^{\alpha+1} \longrightarrow \nabla u^{\alpha+1} \qquad \text{weakly in } L_{m}(Q_{T}) \text{ as } \delta \to 0 \qquad (7.3)
$$
\n
$$
\nabla u_{\delta} \longrightarrow \nabla u \qquad \text{weakly in } L_{m}(Q_{T}) \text{ as } \delta \to 0 \qquad (7.4)
$$
\n
$$
\sqrt{\delta} \nabla u_{\delta} \longrightarrow 0 \qquad \text{weakly in } L_{2}(Q_{T}) \text{ as } \delta \to 0 \qquad (7.5)
$$
\n
$$
\sup(u_{\delta}, Q_{T}) + \sup(u_{\delta}, Q_{T}) \leq c_{1} \qquad (7.6)
$$
\n
$$
\varepsilon^{l} \iint_{Q_{T}} |\nabla u_{\delta}|^{m} dx dt + \varepsilon^{l} \iint_{Q_{T}} |\nabla u|^{m} dx dt + \int_{Q_{T}} |\nabla u^{\alpha+1}|^{m} dx dt \leq c_{2}
$$
\n
$$
\varepsilon \alpha = \frac{l}{m}.
$$
 Denote $A_{\delta}^{i} = a^{i}(u_{\delta}, \nabla u_{\delta}) \quad (i = 1, ..., n).$ In view of condition 1) and a
lities (7.6) and (7.7) we have the estimate uniform with respect to δ \n
$$
||A_{\delta}^{i}||_{L_{m'}(Q_{T})} \leq c \qquad (i = 1, ..., n; \delta > 0).
$$
\n
$$
(7.8)
$$
\nwe can count that there exist functions $A^{i} \in L_{m'}(Q_{T}) \quad (i = 1, ..., n)$ such that

\n
$$
A_{\delta}^{i} \longrightarrow A^{i} \quad \text{weakly in } L_{m'}(Q_{T}) \text{ as } \delta \to 0 \qquad (i = 1, ..., n).
$$
\n(7.9)

\nne other hand, from inequalities (7.6) and (7.7) it follows that for any $\delta, \delta' > 0$

where $\alpha = \frac{l}{n}$. Denote $A^i_\delta = a^i(u_\delta, \nabla u_\delta)$ $(i = 1, ..., n)$. In view of condition 1) and inequalities (7.6) and (7.7) we have the estimate uniform with respect to δ $A_{\delta}^{i} = a^{i}(u_{\delta}, \nabla^{i})$
 *A*_{$_{\delta}$} $||_{L_{m'}(Q_T)} \leq$

Then we can count that there exist functions $A^i \in L_{m'}(Q_T)$ $(i = 1, \ldots, n)$ such that

 $A^i_{\delta} \longrightarrow A^i$ weakly in $L_{m'}(Q_T)$ as $\delta \rightarrow 0$ $(i=1,\ldots,n)$.

On the other hand, from inequalities (7.6) and (7.7) it follows that for any $\delta, \delta' > 0$

\n- Denote
$$
A_{\delta}^{i} = a^{i}(u_{\delta}, \nabla u_{\delta})
$$
 $(i = 1, \ldots, n)$. In view of condition 1) and (5) and (7.7) we have the estimate uniform with respect to δ $||A_{\delta}^{i}||_{L_{m'}(Q_T)} \leq c$ $(i = 1, \ldots, n; \delta > 0).$ (7.8) count that there exist functions $A^{i} \in L_{m'}(Q_T)$ $(i = 1, \ldots, n)$ such that $A_{\delta}^{i} \longrightarrow A^{i}$ weakly in $L_{m'}(Q_T)$ as $\delta \rightarrow 0$ $(i = 1, \ldots, n).$ (7.9) hand, from inequalities (7.6) and (7.7) it follows that for any $\delta, \delta' > 0$ $\iint_{Q_T} |\nabla (|u_{\delta} - u_{\delta'}|^{\beta}(u_{\delta} - u_{\delta'}))|^{m} dx dt \leq c$ $\left(\beta = \frac{\sigma}{\sigma + 2}\right)$ (7.10)
\n

(7.12)

with some constant *c* independent of δ . Really, in view of the definition of β we have $\frac{\beta+1}{2} = \frac{\sigma+1}{\sigma+2}$ and hence the conditions $m > 1$, $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ of Lemma 5.3 are fulfilled for ith some $\frac{+1}{2} = \frac{\sigma + 1}{\sigma + 2}$
 $\frac{\sigma}{2} = \frac{\sigma}{\sigma + 2}$ in Existence and Uniqueness of a Regular Solution 769

constant c independent of δ . Really, in view of the definition of β we have

and hence the conditions $m > 1$, $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ of Lemma 5.3 are fulfille $\frac{a}{\sqrt{2}}$ – $\frac{b}{\sigma+2}$ and nence the conditions $m > 1$, $\frac{c}{m} < \frac{c}{n} + \frac{c}{2}$ or Lemma 5.3 are fulfilled for $\beta = \frac{b}{\sigma+2}$ in view of condition 4). It is easy to see that from inequalities (7.6) and (7.7) it follows that the constant c in (7.10) is independent of δ . Using (7.10) and taking into account that $u_{\delta} \to u$ weakly in $L_2(\Omega)$, uniformly with respect to *t* on [0, *T*], we derive with some constant c independent of δ . F
 $\frac{\beta+1}{2} = \frac{\sigma+1}{\sigma+2}$ and hence the conditions $m > 0$
 $\beta = \frac{\sigma}{\sigma+2}$ in view of condition 4). It is easy

it follows that the constant c in (7.10) is in

account that $u_{\$ from inequality (5.2) in the case $\beta = \frac{\sigma}{\sigma+2}$ for the difference $u_{\delta} - u_{\delta'}$ that Existence and Uniqueness of a l

c independent of δ . Really, in view of the

ce the conditions $m > 1$, $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ of Le

condition 4). It is easy to see that from ine

onstant c in (7.10) is independen Existence and Uni

dent of δ . Really, in

itions $m > 1$, $\frac{1}{m} < \frac{1}{n}$
 i). It is easy to see tl

1(7.10) is independen

1 $L_2(\Omega)$, uniformly w

se $\beta = \frac{\sigma}{\sigma+2}$ for the dis

strongly in $L_{2,m}(Q_T)$

a.e. in Q_T *c* independent of δ . Really, in view of the definition of β we have
 u ce the conditions $m > 1$, $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ of Lemma 5.3 are fulfilled for

condition 4). It is easy to see that from inequalities *c* independent of δ . Really, in view of the definition of β we have
 uce the conditions $m > 1$, $\frac{1}{m} < \frac{1}{n} + \frac{1+\beta}{2}$ of Lemma 5.3 are fulfilled for

condition 4). It is easy to see that from inequalities (*u* is sy to see that from inequalities (7.6) and (7.7)

independent of δ . Using (7.10) and taking into

niformly with respect to t on [0, T], we derive

for the difference $u_{\delta} - u_{\delta'}$ that
 $L_{2,m}(Q_T)$ (7.12)
 L_2

$$
u_{\delta} \longrightarrow u \quad \text{strongly in} \quad L_{2,m}(Q_T) \tag{7.11}
$$

$$
u_{\delta} \longrightarrow u \quad \text{weakly in} \quad L_2(Q_T) \tag{7.13}
$$

$$
u_{\delta} \longrightarrow u \quad \text{strongly in} \quad L_2(\Omega) \text{ for a.e. } t \in [0, T]. \tag{7.14}
$$

Then from (7.2) , (7.5) , and (7.12) \cdot (7.14) we can derive that for a.e. $\tau \in (0, T]$ and any $\phi\in \mathring{W}^1_2(Q_T)$

$$
\int_{\Omega} u \phi \, dx \Big|_{0}^{r} + \iint_{Q_{r}} \Big(-u \phi_{t} + A^{i} \phi_{x_{i}} - f \phi \Big) dx dt = 0. \tag{7.15}
$$

The following proposition is well-known (see, for example, [111).

Proposition 7.1. *Let the function g satisfy a Lipschitz condition uniformly on JR and its derivative g' be continuous everywhere on JR with possible exception of finitely many points at which g' has a discontinuity of the first order. Further, let* **Proposition 7.1.** Let the function g satisfy a Lipschitz condition uniformly on \mathbb{R} and its derivative g' be continuous everywhere on \mathbb{R} with possible exception of finitely many points at which g' has a di $\begin{aligned} \textit{tive} \;\; g' \;\; b \\ \textit{b} \;\; \textit{at} \;\; \textit{wh} \\ \textit{b)} \;\; \cap \; & W^{1,1}_{m} \\ \textit{At} \;\; \textit{least} \\ \int_{\Omega} \; & u \phi \; dx \end{aligned}$

$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u \phi_t + f_i \phi_{x_i} + f_0 \phi \Big) dx dt = 0
$$

and let $u = \varphi$ on S_T . Then for any $t_1, t_2 \in [0, T]$ we have

$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u \phi_t + f_i \phi_{x_i} + f_0 \phi \Big) dx dt = 0
$$

at $u = \varphi$ on S_T . Then for any $t_1, t_2 \in [0, T]$ we have

$$
\int_{\Omega} (G(u) - ug(\varphi)) \, dx \Big|_{t_1}^{t_2}
$$

$$
+ \int_{t_1}^{t_2} \int_{\Omega} \Big(u g'(\varphi) \varphi_t + f_i \big(g'(u) u_{x_i} - g'(\varphi) \varphi_{x_i} \big) + f_0 \big(g(u) - g(\varphi) \big) \Big) dx dt \qquad (7.16)
$$

$$
= 0
$$

$$
G(u) = \int_0^u g(\xi) \, d\xi.
$$

sing Proposition 7.1 we can conclude (in the same way as in [14: p. 538]) that in
of (7.15) and (5.18) or (7.6) condition (5.29) holds for the function u . Moreover,
Proposition 7.1 we can derive from (7.15) that for any $\tau \in (0, T]$ we have

$$
\int_{\Omega} \Big(\frac{1}{2} u^2 - u \varepsilon \Big) dx \Big|_0^{\tau} + \iint_{Q_{\tau}} \Big(A^i u_{x_i} - f(u - \varepsilon) \Big) dx dt = 0. \qquad (7.17)
$$

where $G(u) = \int_0^u g(\xi) d\xi$.

Using Proposition 7.1 we can conclude (in the same way as in [14: p. 538]) that in view of (7.15) and (5.18) or (7.6) condition (5.29) holds for the function *u.* Moreover, using Proposition 7.1 we can derive from (7.15) that for any $\tau \in (0, T]$ we have

$$
\int_{\Omega} \left(\frac{1}{2} u^2 - u \varepsilon \right) dx \Big|_{0}^{r} + \iint_{Q_r} \left(A^{i} u_{x_i} - f(u - \varepsilon) \right) dx dt = 0. \tag{7.17}
$$

In view of (5.29) the integral identity (7.15) holds for any $\tau \in (0, T]$.

To prove that *u* is a generalized solution of problem $(RCD)_\epsilon$ it is sufficient to establish that

\n Integral identity (7.15) holds for any
$$
\tau \in (0, T]
$$
.\n is a generalized solution of problem $(RCD)_{\epsilon}$ it is sufficient to\n
$$
\iint_{Q_{\tau}} A^i \phi_{x_i} dx dt = \iint_{Q_{\tau}} a^i(u, \nabla u) \phi_{x_i} dx dt
$$
\n (7.18)\n \n because $C_0^1(\Omega)$ is dense in $\mathring{W}_m^1(Q_T)$). To prove this it is sufficient\n
$$
\frac{\partial u_{\delta}}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i}
$$
\n a.e. in Q_T \n (i = 1, ..., n)\n (7.19)\n and (7.12), the continuity of the functions a^i , condition 1'),\n (7.19)\n and the Vitali theorem we obtain that for any $\phi \in \mathring{W}_m^1(Q_T)$ \n

for any $\phi \in C_0^1(\Omega)$ (because $C_0^1(\Omega)$ is dense in $\mathring{W}_m^1(Q_T)$). To prove this it is sufficient to establish that

$$
\frac{\partial u_{\delta}}{\partial x_{i}} \longrightarrow \frac{\partial u}{\partial x_{i}} \quad \text{a.e. in } Q_{T} \qquad (i = 1, ..., n)
$$
 (7.19)

because in view of (7.19) and (7.12), the continuity of the functions a^i , condition 1'), estimates (7.6) and (7.7) and the Vitali theorem we obtain that for any $\phi \in \mathring{W}^1_m(Q_T)$ and any $\tau \in (0, T]$ $\iint_{Q_{\tau}} A^i \phi_{x_i} dx dt = \iint_{Q_{\tau}} a^i(u, \nabla u) \phi_{x_i} dx dt$ (7.18)

(2) (because $C_0^1(\Omega)$ is dense in $\mathring{W}_m^1(Q_T)$). To prove this it is sufficient
 $\frac{\partial u_{\delta}}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i}$ a.e. in Q_T ($i = 1, ..., n$) (7.19)

of (7.19) and

$$
\lim_{\delta \to 0} \iint_{Q_{\tau}} a^i(u_{\delta}, \nabla u_{\delta}) \phi_{x_i} dx dt = \iint_{Q_{\tau}} a^i(u, \nabla u) \phi_{x_i} dx dt.
$$
 (7.20)

On the other hand, in view of (7.9) the left-hand side here is equal to that of (7.18). Hence (7.19) implies (7.18).

nates (7.6) and (7.7) and the Vitali theorem we obtain that for any
$$
\phi \in \check{W}_m^1(Q_T)
$$

\nany $\tau \in (0, T]$
\n
$$
\lim_{\delta \to 0} \iint_{Q_{\tau}} a^i(u_{\delta}, \nabla u_{\delta}) \phi_{x_i} dx dt = \iint_{Q_{\tau}} a^i(u, \nabla u) \phi_{x_i} dx dt.
$$
\n(7.20)
\nthe other hand, in view of (7.9) the left-hand side here is equal to that of (7.18).
\nChoosing $\phi = u_{\delta} - \varepsilon$ in (7.1) we obtain with the aid of Proposition 7.1 that
\n
$$
\int_{\Omega} \left(\frac{1}{2} u_{\delta}^2 - u_{\delta} \varepsilon \right) dx \Big|_{0}^{\tau} + \iint_{Q_{\tau}} \left(a^i(u_{\delta}, \nabla u_{\delta}) \frac{\partial u_{\delta}}{\partial x_i} - f(u_{\delta} - \varepsilon) \right) dx dt = 0.
$$
\n(7.21)
\nug (7.13) and (7.14) we derive from (7.21) and (7.17) that for any $\tau \in (0, T]$
\n
$$
\lim_{\delta \to 0} \iint_{Q_{\tau}} a^i(u_{\delta}, \nabla u_{\delta}) \frac{\partial u_{\delta}}{\partial x_i} dx dt = \iint_{Q_{\tau}} A^i \frac{\partial u}{\partial x_i} dx dt.
$$
\n(7.22)
\nng now condition 2) we have

Using (7.13) and (7.14) we derive from (7.21) and (7.17) that for any
$$
\tau \in (0, T]
$$

\n
$$
\lim_{\delta \to 0} \iint_{Q_{\tau}} a^i(u_{\delta}, \nabla u_{\delta}) \frac{\partial u_{\delta}}{\partial x_i} dx dt = \iint_{Q_{\tau}} A^i \frac{\partial u}{\partial x_i} dx dt.
$$
\n(7.22)

Using now condition 2) we have

$$
\nu_1 J_{\delta} := \nu_1 \iint_{Q_{\tau}} \frac{|\nabla u_{\delta} - \nabla u|^2}{\left(|\nabla u_{\delta}|^m + |\nabla u|^m\right)^{2/m - 1}} dx dt
$$

\n
$$
\leq \iint_{Q_{\tau}} \left(a(u_{\delta}, \nabla u_{\delta}) - a(u, \nabla u) \right) \cdot (\nabla u_{\delta} - \nabla u) dx dt
$$
(7.23)
\n
$$
=:\mathcal{H}_{\delta}.
$$

\n7.5), (7.9) and (7.22) and taking into account that in view of (7.12), (7.6)
\ndition 1) and the Vitali theorem
\n
$$
a^i(u_{\delta}, \nabla u) \longrightarrow a^i(u, \nabla u) \text{ strongly in } L_{m'}(Q_T) \text{ as } \delta \to 0
$$
(7.24)
\n1. (7.23) that
\n
$$
\lim_{\delta \to 0} \mathcal{H}_{\delta} = 0.
$$
(7.25)

Using (7.3) - (7.5) , (7.9) and (7.22) and taking into account that in view of (7.12) , (7.6) and (7.7), condition 1) and the Vitali theorem

$$
a^{i}(u_{\delta}, \nabla u) \longrightarrow a^{i}(u, \nabla u) \quad \text{strongly in} \quad L_{m'}(Q_T) \quad \text{as} \quad \delta \to 0 \tag{7.24}
$$

we derive from (7.23) that

$$
\lim_{\delta \to 0} \mathcal{H}_{\delta} = 0. \tag{7.25}
$$

Using this limit and inequalities $0 \leq J_{\delta} \leq \nu_1^{-1} \mathcal{H}_{\delta}$ we obtain

$$
is \text{tence and Uniqueness of a Regular Solution} \qquad 771
$$
\n
$$
J_{\delta} \leq \nu_1^{-1} \mathcal{H}_{\delta} \text{ we obtain}
$$
\n
$$
\lim_{\delta \to 0} J_{\delta} = 0. \qquad (7.26)
$$
\n
$$
J_{\delta} = 0.
$$

Show that from here it follows that (7.19) is true. Denote

Existence and Uniqueness of a Regular Solution
\nqualities
$$
0 \leq J_{\delta} \leq \nu_1^{-1} \mathcal{H}_{\delta}
$$
 we obtain
\n
$$
\lim_{\delta \to 0} J_{\delta} = 0.
$$
\n(7.26)
\nallows that (7.19) is true. Denote
\n
$$
h_{\delta}(x, t) := \frac{|\nabla u_{\delta} - \nabla u|^2}{\left(|\nabla u_{\delta}|^m + |\nabla u|^m\right)^{2/m-1}}.
$$
\n(7.27)

From $\lim_{\delta \to 0} J_{\delta} = 0$ it follows that there exist some subsequence $\{\delta\}$ and subset $\tilde{Q} \subset$ Q_r , $|\overline{Q}| = |Q_r|$, such that Existence and Uniqueness of a Regular S

Using this limit and inequalities $0 \le J_{\delta} \le \nu_1^{-1} \mathcal{H}_{\delta}$ we obtain
 $\lim_{\delta \to 0} J_{\delta} = 0$.

Show that from here it follows that (7.19) is true. Denote
 $h_{\delta}(x, t) := \frac{|\nabla u_{\$

$$
\lim_{\delta \to 0} h_{\delta}(x,t) = 0 \qquad \text{on } \ \widetilde{Q}.
$$
\n(7.28)

Without loss of generality we can count that $\frac{\partial u}{\partial x_i}$ are finite on \tilde{Q} , i.e., $|\nabla u|$ is bounded (non-uniformly) at any point $(x, t) \in \tilde{Q}$. In view of (7.27) we have for any $(x, t) \in \tilde{Q}$ $V + |Vu|^m$
ist some subsequence
0 on \tilde{Q} .
t $\frac{\partial u}{\partial z_i}$ are finite on \tilde{Q}
view of (7.27) we hav
 $\nabla u_\delta | -c)^2$
 $u_\delta | +c)^{2-m}$

$$
h_{\delta}(x,t) \ge \frac{(|\nabla u_{\delta}| - c)^2}{(|\nabla u_{\delta}| + c)^{2-m}} \tag{7.29}
$$

with a constant *c* depending on $(x,t) \in \tilde{Q}$. Suppose now that $|\nabla u_{\delta}|$ is unbounded in some point $(x,t) \in \tilde{Q}$. Then $|\nabla u_{\delta}| \to \infty$ for some subsequence $\{\delta\}$ and hence in view of (7.29) we obtain that for this subsequence $\lim_{\delta\to\infty} h_{\delta}(x,t) = \infty$, i.e., we obtain a contradiction with (7.28). Hence t c depending on $(x, t) \in \tilde{Q}$. Suppose now that $|\nabla u_{\delta}|$ is unbounce $t \in \tilde{Q}$. Then $|\nabla u_{\delta}| \to \infty$ for some subsequence $\{\delta\}$ and hence in that for this subsequence $\lim_{\delta \to \infty} h_{\delta}(x, t) = \infty$, i.e., we obtain

$$
|\nabla u_{\delta}|
$$
 are bounded (non-uniformly) at any point of \tilde{Q} . (7.30)

Then from (7.27), (7.28) and (7.30) it follows that the numerators of h_{δ} tend to zero on \widetilde{Q} as $\delta \to 0$, i.e. (7.19) is true. Therefore the function $u \in C([0,T]; L_2(\Omega)) \cap W^{1,0}_m(Q_T)$ is a generalized solution of problem $(RCD)_\epsilon$. From Lemmas 6.2 and 6.3 it follows that this function satisfies estimates (6.5) - (6.8) . In view of (6.5) and Proposition 3.3 the function u is a unique strong solution of problem $(RCD)_e$. So we proved the following

Lemma 7.1. For any fixed $\epsilon > 0$ there exist exactly one strong solution (in sense *of Definition* 2.3) *of problem* (RCD)e *satisfying estimates (6.5) - (6.8) with constants* $c_1, c_2, \lambda \in (0,1)$ and K independent of ε .

8. The passing to the limit as $\varepsilon \to 0$

Now we are ready to prove Theorem 6.1 and hence Theorem 1.1. In the remainder of this section we denote the solution of problem $(RCD)_e$ as u_e . We are going to realize the passing to the limit as $\varepsilon \to 0$ using the a priori estimates (6.5) - (6.8). This passing can be done in the same way as one in [12] where existence of regular solution of problem 8. The passing to the limit as $\varepsilon \to 0$
Now we are ready to prove Theorem 6.1 and hence Theorem 1.1. In
this section we denote the solution of problem (RCD)_{ϵ} as u_{ϵ} . We are go
passing to the limit as $\varepsilon \to 0$ u **as** $\varepsilon \rightarrow$
6.1 and h
problem (1
e a priori (2) where $\left(1, \frac{2n}{n+2}\right)$
e can con
kly in L_m Theorem 1.
 \sum_{ℓ} as u_{ℓ} . We

ates (6.5) - ℓ

ince of regul
 $n < 2$, $m + i$

that there ℓ
 $\ell = 1, ...$ are ready to prove Theorem 6.1 and hence T

on we denote the solution of problem $(RCD)_{\epsilon}$

o the limit as $\epsilon \to 0$ using the a priori estimat

in the same way as one in [12] where existen

s proved in the case $l \geq 0$,

In view of estimates (6.5) - (6.8) we can conclude that there exists a function *u* such that

$$
u_{\epsilon} \to u \qquad \text{uniformly in} \quad Q_T \tag{8.1}
$$

in the same way as one in [12] where existence of regular solution of problem
\ns proved in the case
$$
l \ge 0
$$
, max $(1, \frac{2n}{n+2}) < m < 2$, $m + l \ge 2$.
\new of estimates (6.5) - (6.8) we can conclude that there exists a function u such
\n $u_{\epsilon} \to u$ uniformly in Q_T (8.1)
\n
$$
\frac{\partial}{\partial x_i} u_{\epsilon}^{\alpha+1} \longrightarrow u^{\alpha} u_{x_i}
$$
 weakly in $L_m(Q_T)$ $\left(i = 1, ..., n; \alpha = \frac{l}{m}\right)$ (8.2)
\n $0 \le \inf(u, Q_T) \le \sup(u, Q_T) \le c_1$ (8.3)
\n
$$
\iint_{Q_T} u^l |u_x|^m dx dt \le c_2
$$
 (8.4)
\n(4.2))
\n $\langle u \rangle_{\lambda, \overline{Q}_T} \le K.$ (8.5)
\nand (8.4) we used the following notation similar to one from Definition 2.3:

$$
\leq \inf(u, Q_T) \leq \sup(u, Q_T) \leq c_1 \tag{8.3}
$$

$$
\iint_{Q_T} u^l |u_z|^m dx dt \le c_2 \tag{8.4}
$$

and (see (4.2))

$$
\langle u \rangle_{\lambda, \overline{Q}_T} \le K. \tag{8.5}
$$

In (8.2) and (8.4) we used the following notation similar to one from Definition 2.3:

ux = *(u i,,. . . ,u,,)* **U'j** *((a+1)u°* on{QT:u>0} (a= J_. (8.6) ⁼¹0 on *{QT* = 01 *in)*

Obviously, $u^{\alpha}u_{x_i} \in L_m(Q_T)$ $(i = 1, ..., n)$ (in view of (8.4)). In view of the boundedness of *u* and inequality $\sigma = \frac{l}{m-1} > \alpha$ the expressions for u_{x_i} in (8.6) and (2.2) coin $u_x = (u_{x_1}, \ldots, u_{x_n})$
 $u_{x_i} = \begin{cases} (\alpha + 1)^{-1} u^{-\alpha} & \text{on } \{Q_T : u > 0\} \\ 0 & \text{on } \{Q_T : u = 0\} \end{cases} \qquad (\alpha = \frac{l}{m})$.

Obviously, $u^{\alpha} u_{x_i} \in L_m(Q_T)$ $(i = 1, \ldots, n)$ (in view of (8.4)). In view of the bound

ness of *u* and inequality cide. Moreover, from condition $u^{\alpha}u_{x_i} \in L_m(Q_T)$ it follows that $u^{\sigma}u_{x_i} \in L_m(Q_T)$ $(i = 1, ..., n)$. We use below the following auxiliary propositions (see [11] or [12]). *i* (8.6)
 $\frac{1}{2}$ $\frac{1}{2}$

Proposition 8.1. Let the function u be bounded and non-negative in Q_T and such *that* $\nabla u^{\alpha+1} \in L_m(Q_T)$ *for some* $\alpha \geq 0$ *. Further, let the function* \overline{u} *be defined by*

$$
\overline{u} = \sup(u - \varepsilon_1, 0) \qquad (\varepsilon_1 = const > 0). \tag{8.7}
$$

Then \overline{u} *has generalized derivatives* $\frac{\partial \overline{u}}{\partial x_i} \in L_m(Q_T)$ $(i = 1, \ldots, n)$ *such that*

$$
r \text{ some } \alpha \ge 0. \text{ Further, let the function } \overline{u} \text{ be defined by}
$$
\n
$$
= \sup(u - \varepsilon_1, 0) \qquad (\varepsilon_1 = \text{const} > 0). \qquad (8.7)
$$
\n
$$
erivatives \frac{\partial \overline{u}}{\partial x_i} \in L_m(Q_T) \quad (i = 1, ..., n) \text{ such that}
$$
\n
$$
\frac{\partial \overline{u}}{\partial x_i} = \begin{cases} u_{x_i} & \text{in } \{Q_T : u > \varepsilon_1\} \\ 0 & \text{in } \{Q_T : 0 \le u \le \varepsilon_1\} \end{cases} \qquad (8.8)
$$
\n
$$
(8.6). \text{ Moreover,}
$$
\n
$$
\lim_{\varepsilon_1 \to 0} \left\| u^{\alpha} \frac{\partial \overline{u}}{\partial x_i} - u^{\alpha} u_{x_i} \right\|_{L_m(Q_T)} = 0. \qquad (8.9)
$$

where u_{x_i} are defined by (8.6). Moreover,

$$
\lim_{\epsilon_1 \to 0} \left\| u^{\alpha} \frac{\partial \overline{u}}{\partial x_i} - u^{\alpha} u_{x_i} \right\|_{L_m(Q_T)} = 0.
$$
\n(8.9)

Proposition 8.2. Let $A^i \in L_{m'}(Q_T)$ $(i = 1, ..., n)$ and $B \in L_{m'}(Q_T)$ $(\frac{1}{m} +$ $\frac{1}{m'} = 1$, $m > 1$), and let the function u be bounded and non-negative in Q_T and such $\phi \in \mathring{W}^1_m(Q_T)$ Existence and Uniqueness of a Regular Solution 773

8.2. Let $A^i \in L_{m'}(Q_T)$ $(i = 1, ..., n)$ and $B \in L_{m'}(Q_T)$ $\left(\frac{1}{m} +$

and let the function u be bounded and non-negative in Q_T and such
 (Q_T) for some $\alpha \geq 0$. Assume

$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u \phi_t + u^{\alpha} A^i \phi_{x_i} + B \phi \Big) dx dt = 0. \tag{8.10}
$$

Let $\varphi \in W_{m}^{1}(Q_{T})$ and $u = \varphi$ on S_{T} . Then for any $t_{1}, t_{2} \in [0, T]$

$$
\frac{1}{m'} = 1, m > 1, \text{ and let the function } u \text{ be bounded and non-negative in } Q_T \text{ and such} \nthat $\nabla u^{\alpha+1} \in L_m(Q_T)$ for some $\alpha \ge 0$. Assume that for any $t_1, t_2 \in [0, T]$ and any $\phi \in \mathring{W}_m^1(Q_T)$
\n
$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u \phi_t + u^{\alpha} A^i \phi_{x_i} + B \phi \Big) dx dt = 0. \tag{8.10}
$$
\n
$$
\text{Let } \varphi \in W_m^1(Q_T) \text{ and } u = \varphi \text{ on } S_T. \text{ Then for any } t_1, t_2 \in [0, T]
$$
\n
$$
\int_{\Omega} \left(\frac{1}{2} u^2 - u \varphi \right) dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(u \varphi_t + u^{\alpha} A^i (u_{x_i} - \varphi_{x_i}) + B(u - \varphi) \Big) dx dt = 0 \tag{8.11}
$$
$$

where u_{x_i} are defined by (8.8).

Returning to (8.1) - (8.5) we see that the function *u* is non-negative and bounded $\int \ln Q_T$, $u \in C^{\lambda,\lambda/m}(\overline{Q}_T)$, $\nabla u^{\alpha+1} \in L_m(Q_T)$ $(\alpha = \frac{l}{m})$ (so that $\nabla u^{\alpha+1} \in L_m(Q_T)$, $\sigma =$ $\frac{l}{m-1}$) and $u = \psi$ on Γ_T . Hence to prove Theorem 6.1 it is sufficient to show that for any $t_1, t_2 \in [0, T]$ and $\phi \in \mathring{W}^1_m(Q_T)$ *udd* $\left\{ \begin{array}{l} \int_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{t_1}^{t_2} (u\varphi_t + u^2 A(u_{x_i} - \varphi_{x_i}) + B(u - \varphi)) dx dt = 0 \quad (8.11) \end{array} \right\}$
 Und by (8.8).

(8.1) - (8.5) we see that the function *u* is non-negative and bounded
 $\int_{t_1}^{t_2} (\overline{Q}_T$

$$
I = \int_{\Omega} \int_{\Omega} \int_{\Omega} \left(\int_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left(-u\phi_t + a(u, u_x) \cdot \nabla \phi - f\phi \right) dx dt = 0 \qquad (8.12)
$$
\n
$$
\int_{\Omega} u\phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left(-u\phi_t + a(u, u_x) \cdot \nabla \phi - f\phi \right) dx dt = 0 \qquad (8.12)
$$
\nfind by (8.8) in the case $\alpha = \frac{l}{m}$. Really, in this case from the kind of $(CD)_{\epsilon}$ it will follow that u is a quasistrong and hence regular solution of $A_{\epsilon}^{i} = u_{\epsilon}^{-\alpha} a^{i}(u_{\epsilon}, \nabla u_{\epsilon}) \qquad \left(\alpha = \frac{l}{m}; i = 1, ..., n \right)$. (8.13)

\nsecond inequality in condition 1) and estimate (8.4) we have the uniform

where u_x is defined by (8.8) in the case $\alpha = \frac{l}{m}$. Really, in this case from the kind of the problem $(RCD)_\epsilon$ it will follow that u is a quasistrong and hence regular solution of problem (CD). (8) in the case $\alpha = \frac{l}{m}$. Really, in this case from the kind of
 III follow that *u* is a quasistrong and hence regular solution of

colds denote
 $\alpha^a (u_e, \nabla u_e)$ $\left(\alpha = \frac{l}{m}; i = 1, ..., n\right)$. (8.13)

uality in conditio

To prove that *(8.12)* holds denote

$$
A_{\epsilon}^{i} = u_{\epsilon}^{-\alpha} a^{i}(u_{\epsilon}, \nabla u_{\epsilon}) \qquad \left(\alpha = \frac{l}{m}; i = 1, ..., n\right).
$$
 (8.13)
of the second inequality in condition 1) and estimate (8.4) we have the uniform

$$
||A_{\epsilon}^{i}||_{L_{m'}(Q_{T})} \leq c \qquad (i = 1, ..., n).
$$
 (8.14)

$$
\therefore
$$
 can count that there exist functions $a^{i} \in L_{m'}(Q_{T})$ such that

$$
A_{\epsilon}^{i} \longrightarrow A^{i} \qquad
$$
 weakly in $L_{m'}(Q_{T})$ as $\epsilon \to 0$ $(i = 1, ..., n).$ (8.15)
 $\epsilon \to 0$ in the integral identity

In view of the second inequality in condition 1) and estimate (8.4) we have the uniform estimate

$$
|A_e^i| \Big|_{L_{m'}(Q_T)} \le c \qquad (i = 1, ..., n). \tag{8.14}
$$

Then we can count that there exist functions $a^i \in L_{m'}(Q_T)$ such that

$$
A_{\epsilon}^{i} \longrightarrow A^{i} \qquad \text{weakly in} \ \ L_{m'}(Q_{T}) \text{ as } \epsilon \to 0 \qquad \quad (i=1,\ldots,n). \tag{8.15}
$$

Letting $\varepsilon \to 0$ in the integral identity

count that there exist functions
$$
a^i \in L_{m'}(Q_T)
$$
 such that
\n→ Aⁱ weakly in $L_{m'}(Q_T)$ as $\epsilon \to 0$ $(i = 1,...,n)$. (8.15)
\n0 in the integral identity
\n
$$
\int_{\Omega} u_{\epsilon} \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u_{\epsilon} \phi_t + a(u_{\epsilon}, \nabla u_{\epsilon}) \cdot \nabla \phi - f \phi \Big) dx dt = 0
$$
 (8.16)

for $\phi \in \mathring{W}_m^1(Q_T)$ we obtain in view of (8.1) and (8.15) that for any $t_1, t_2 \in [0, T]$ and $\phi \in \mathring{W}_m^1(Q_T)$ for $\phi \in \mathring{W}^1_m(Q_T)$ we obtain in view of (8.1) and (8.15) that for any $t_1, t_2 \in [0, T]$ and $\phi \in \mathring{W}^1_m(Q_T)$

now

\nwe obtain in view of (8.1) and (8.15) that for any
$$
t_1, t_2 \in [0, T]
$$
 and

\n
$$
\int_{\Omega} u \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u \phi_t + u^{\alpha} A^i \phi_{x_i} - f \phi \Big) \, dx dt = 0.
$$
\n(8.17)

\nFrom 6.1 it is sufficient to show that

To prove Theorem 6.1 it is sufficient to show that

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\nfor
$$
\phi \in \mathring{W}^1_m(Q_T)
$$
 we obtain in view of (8.1) and (8.15) that for any $t_1, t_2 \in [0, T]$ and
\n $\phi \in \mathring{W}^1_m(Q_T)$
\n
$$
\int_{\Omega} u\phi \,dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \Big(-u\phi_t + u^\alpha A^i \phi_{x_i} - f\phi \Big) \,dxdt = 0. \qquad (8.17)
$$
\nTo prove Theorem 6.1 it is sufficient to show that
\n
$$
\int_{t_1}^{t_2} \int_{\Omega} u^\alpha A^i \phi_{x_i} dxdt = \int_{t_1}^{t_2} \int_{\Omega} a^i(u, \nabla u) \phi_{x_i} dxdt = 0 \qquad \text{for any } \phi \in \mathring{C}(\overline{Q}_T) \qquad (8.18)
$$
\nbecause A^i , $a^i(u, \nabla u) \in L_{m'}(Q_T)$ and $\mathring{C}^1(\overline{Q}_T)$ is dense in $\mathring{W}^1_m(Q_T)$. To prove equality
\n(8.18) it is sufficient to establish that for some subsequence $\{\epsilon\}$
\n $u_\epsilon^\alpha \frac{\partial u_\epsilon}{\partial x_i} \longrightarrow u^\alpha u_{x_i} \qquad \text{a.e. in } Q_T \qquad (i = 1, ..., n) \qquad (8.19)$
\nbecause in view of (8.19) and (8.1), the continuity of the functions $u^{-\alpha} a^i(u, u^{-\alpha} p)$ ($\alpha = \frac{1}{m}$) on $\overline{R}_+ \times R^n$, condition 1), the uniform estimate (6.7) for $u = u_\epsilon$ and the Vitali

because A^i , $a^i(u, \nabla u) \in L_{m'}(Q_T)$ and $\mathring{C}^1(\overline{Q}_T)$ is dense in $\mathring{W}^1_m(Q_T)$. To prove equality (8.18) it is sufficient to establish that for some subsequence $\{\epsilon\}$

$$
u_{\varepsilon}^{\alpha} \frac{\partial u_{\varepsilon}}{\partial x_{i}} \longrightarrow u^{\alpha} u_{x_{i}} \quad \text{a.e. in } Q_{T} \quad (i = 1, ..., n)
$$
 (8.19)

Thus *Aⁱ*, $a^i(u, \nabla u) \in L_{m'}(Q_T)$ and $\mathcal{C}^1(\overline{Q}_T)$ is dense in $\mathcal{W}_m^1(Q_T)$. To prove equality
 i) it is sufficient to establish that for some subsequence $\{\varepsilon\}$
 $u_{\varepsilon}^{\partial} \frac{\partial u_{\varepsilon}}{\partial x_i} \longrightarrow u^{\alpha} u_{x_i}$ theorem we obtain that for any $\phi \in \mathcal{C}^1(\overline{Q}_T)$ the integral

Let
$$
u_{\epsilon} \neq \frac{\partial u_{\epsilon}}{\partial x_{i}} \longrightarrow u^{\alpha} u_{\epsilon_{i}}
$$
 are in Q_{T} $(i = 1, \ldots, n)$ (8.19)

\nif (8.19) and (8.1), the continuity of the functions $u^{-\alpha} a^{i}(u, u^{-\alpha} p)$ ($\alpha = i$, condition 1), the uniform estimate (6.7) for $u = u_{\epsilon}$ and the Vitali in that for any $\phi \in \mathcal{C}^{1}(\overline{Q}_{T})$ the integral

\n
$$
\iint_{t_{1}}^{t_{2}} a(u_{\epsilon}, \nabla u_{\epsilon}) \cdot \nabla \phi \, dx dt
$$

\n
$$
= \iint_{t_{1}}^{t_{2}} u_{\epsilon}^{\alpha} A_{\epsilon}^{i} \phi_{\epsilon_{i}} dx dt
$$
 (8.20)

\n
$$
= \iint_{t_{1}}^{t_{2}} u_{\epsilon}^{\alpha} (u_{\epsilon}^{-\alpha} a^{i}(u_{\epsilon}, u_{\epsilon}e^{-\alpha}(u_{\epsilon}e^{\alpha} \nabla u_{\epsilon})) \phi_{\epsilon_{i}} dx dt
$$

tends to the integral $\int_{t_1}^{t_2} \int_{\Omega} a^i(u, u_x) \phi_{x_i} dx dt$. On the other hand, in view of (8.1) and (8.18).

The remainder of this section is devoted to the proof of (8.19). Applying Proposition 7.1 with $g(\xi) = \xi - \varepsilon$ we derive from (7.16) that

(8.15) the integral (8.20) tends to the integral
$$
\int_{t_1}^{t_2} \int_{\Omega} u^{\alpha} A^i \phi_{x_i} dx dt
$$
. Hence (8.19) implies
\n(8.18).
\nThe remainder of this section is devoted to the proof of (8.19). Applying Proposition
\n7.1 with $g(\xi) = \xi - \varepsilon$ we derive from (7.16) that
\n
$$
\int_{\Omega} \left(\frac{1}{2} u_{\varepsilon}^2 - \varepsilon u_{\varepsilon} \right) dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left(a^i(u_{\varepsilon}, \nabla u_{\varepsilon}) \frac{\partial u}{\partial x_i} - f(u - \varepsilon) \right) dx dt = 0.
$$
\n(8.21)

Applying Proposition 8.2 with $g(\xi) = \xi$ and using that $u = 0$ on S_T (in view of (8.1),

tends to the integral
$$
\int_{t_1}^{t_1} \int_{\Omega} a^t(u, u_x) \phi_{x_i} dx dt
$$
. On the other hand, in view of (8.1) and
\n(8.15) the integral (8.20) tends to the integral $\int_{t_1}^{t_2} \int_{\Omega} u^{\alpha} A^i \phi_{x_i} dx dt$. Hence (8.19) implies
\n(8.18).
\nThe remainder of this section is devoted to the proof of (8.19). Applying Proposition
\n7.1 with $g(\xi) = \xi - \varepsilon$ we derive from (7.16) that
\n
$$
\int_{\Omega} \left(\frac{1}{2} u_{\varepsilon}^2 - \varepsilon u_{\varepsilon} \right) dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left(a^t(u_{\varepsilon}, \nabla u_{\varepsilon}) \frac{\partial u}{\partial x_i} - f(u - \varepsilon) \right) dx dt = 0.
$$
\n(8.21)
\nApplying Proposition 8.2 with $g(\xi) = \xi$ and using that $u = 0$ on S_T (in view of (8.1),
\nbecause $u_{\varepsilon} = \varepsilon$ on S_T) we derive from (8.11)
\n
$$
\int_{\Omega} \frac{1}{2} u^2 dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left(u^{\alpha} A^i u_{x_i} - f u \right) dx dt = 0.
$$
\n(8.22)

Using (8.1) we derive from (8.21) and (8.22) that

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\nfive from (8.21) and (8.22) that

\n
$$
\lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\Omega} a^i(u_{\epsilon}, \nabla u_{\epsilon}) \frac{\partial u_{\epsilon}}{\partial x_i} dx dt = \int_{t_1}^{t_2} \int_{\Omega} u^{\alpha} A^i u_{x_i} dx dt.
$$
\n(8.23)

\n(8.7). Obviously that the following proposition holds (see also [11])

Let \bar{u} be defined by (8.7). Obviously that the following proposition holds (see also [11])

Proposition 8.3. We have the convergences

$$
t_1 \Omega
$$

defined by (8.7). Obviously that the following proposition holds (see also [11]).
osition 8.3. We have the convergence
 $u_{\epsilon}^{-\alpha} a^{i}(u_{\epsilon}, \nabla \bar{u}) \longrightarrow u^{-\alpha} a^{i}(u, \nabla \bar{u})$ strongly in $L_{m'}(Q_T)$ as $\epsilon \to 0$ (8.24)
 $u^{-\alpha} a^{i}(\omega, \nabla \bar{u}) \longrightarrow u^{-\alpha} a^{i}(\omega, \omega)$, then also in $L_{m'}(Q_T)$ also a (8.25).

position 8.3. We have the convergences

\n
$$
u_{\epsilon}^{-\alpha} a^{i}(u_{\epsilon}, \nabla \bar{u}) \longrightarrow u^{-\alpha} a^{i}(u, \nabla \bar{u}) \quad \text{strongly in } L_{m'}(Q_{T}) \text{ as } \epsilon \to 0 \qquad (8.24)
$$
\n
$$
u_{\epsilon}^{-\alpha} a^{i}(u, \nabla \bar{u}) \longrightarrow u^{-\alpha} a^{i}(u, u_{\epsilon}) \quad \text{strongly in } L_{m'}(Q_{T}) \text{ as } \epsilon_{1} \to 0. \qquad (8.25)
$$
\ncondition 2) we have

Using now condition 2) we have

et
$$
\vec{u}
$$
 be defined by (8.7). Obviously that the following proposition holds (see also [11]).
\nProposition 8.3. We have the convergences
\n $u_{\epsilon}^{-\alpha}a^{i}(u_{\epsilon},\nabla\vec{u}) \longrightarrow u^{-\alpha}a^{i}(u,\nabla\vec{u})$ strongly in $L_{m'}(Q_{T})$ as $\epsilon \to 0$ (8.24)
\n $u_{\epsilon}^{-\alpha}a^{i}(u,\nabla\vec{u}) \longrightarrow u^{-\alpha}a^{i}(u,\nabla\vec{u})$ strongly in $L_{m'}(Q_{T})$ as $\epsilon \to 0$ (8.25)
\nsing now condition 2) we have
\n $\nu_{1}\mathcal{H}_{\epsilon,\epsilon_{1}}$
\n $:= \nu_{1}\int_{t_{1}}^{t_{2}}u_{\epsilon}^{i}|\nabla u_{\epsilon}-\nabla\vec{u}|^{2}(|\nabla u_{\epsilon}-b(u_{\epsilon})|^{m}+|\nabla\vec{u}-b(u_{\epsilon})|^{m})^{1-2/m}dxdt$
\n $\leq \int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}}(a^{i}(u_{\epsilon},\nabla u_{\epsilon})-a^{i}(u_{\epsilon},\nabla\vec{u}))(\frac{\partial u}{\partial x_{i}}-\frac{\partial\vec{u}}{\partial x_{i}})dxdt$ (8.26)
\n $= \int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}}(a^{i}(u_{\epsilon},\nabla u_{\epsilon})\frac{\partial u_{\epsilon}}{\partial x_{i}}-A^{i}_{\epsilon}u_{\epsilon}^{\alpha}\frac{\partial\vec{u}}{\partial x_{i}}-u_{\epsilon}^{-\alpha}a^{i}(u_{\epsilon},\nabla\vec{u})\left(u_{\epsilon}^{\alpha}\frac{\partial u_{\epsilon}}{\partial x_{i}}-u_{\epsilon}^{\alpha}\frac{\partial\vec{u}}{\partial x_{i}}\right))dxdt$
\n $=:J_{\epsilon,\epsilon_{1}}.$
\nsing (8.1), (8.2), (8.15), (8.23), (8.24) and letting $\epsilon \to 0$ we obtain
\n
$$
\lim_{\epsilon \to 0} J_{\epsilon,\epsilon_{1}}.
$$

\n $= \int_{t_{1}}^{t$

Using (8.1), (8.2), (8.15), (8.23), (8.24) and letting $\varepsilon \to 0$ we obtain

$$
= \int_{t_1}^{t_1} \int_{\Omega} \left(a^i(u_{\epsilon}, \nabla u_{\epsilon}) \frac{\partial u_{\epsilon}}{\partial x_i} - A^i_{\epsilon} u_{\epsilon}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} - u_{\epsilon}^{-\alpha} a^i(u_{\epsilon}, \nabla \bar{u}) \left(u_{\epsilon}^{\alpha} \frac{\partial u_{\epsilon}}{\partial x_i} - u_{\epsilon}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) \right) dx dt
$$

\n
$$
= J_{\epsilon, \epsilon_1}.
$$

\n
$$
= \int_{t_{-1}}^{t_2} \int_{\Omega} \left(A^i \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) - u^{-\alpha} a^i(u, \nabla \bar{u}) \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) \right) dx dt \qquad (8.27)
$$

\n
$$
= \int_{t_1}^{t_2} \int_{\Omega} \left(A^i \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) - u^{-\alpha} a^i(u, \nabla \bar{u}) \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) \right) dx dt \qquad (8.27)
$$

\n
$$
=: \tilde{J}_{\epsilon_1}.
$$

\n
$$
= \int_{\epsilon_1}^{t_2} \int_{\epsilon_1}^{t_1} \left(A^i \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) - u^{-\alpha} a^i(u, \nabla \bar{u}) \left(u^{\alpha} u_{x_i} - u^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right) \right) dx dt \qquad (8.28)
$$

\n
$$
=: \tilde{J}_{\epsilon_1}.
$$

\n
$$
=:\tilde{J}_{\epsilon_1}.
$$

\n
$$
=:\tilde{J}_{\epsilon_1}.
$$

\n
$$
=:\tilde{J}_{\epsilon_1}.
$$

\n<math display="</math>

Using (8.25) and (8.9) we derive from (8.27)

$$
\lim_{\epsilon_1 \to 0} \tilde{J}_{\epsilon_1} = 0. \tag{8.28}
$$

From (8.27) and (8.28) it follows that there exist subsequences $\{\varepsilon_k\}$ and $\{\varepsilon_{1k}\}$ tending to zero such that $\lim_{k\to\infty} J_{\epsilon_k,\epsilon_{1k}} = 0$. Because $0 \leq \mathcal{H}_{\epsilon_k,\epsilon_{1k}} \leq J_{\epsilon_k,\epsilon_{1k}}$ we derive from here that

$$
\lim_{k \to \infty} \mathcal{H}_{\epsilon_k, \epsilon_{1k}} = 0. \tag{8.29}
$$

Rewrite $\mathcal{H}_{\varepsilon_k,\varepsilon_{1k}}$ as

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\nRewrite
$$
\mathcal{H}_{\epsilon_k, \epsilon_{1k}}
$$
 as
\n
$$
\mathcal{H}_{\epsilon_k, \epsilon_{1k}}
$$
\n
$$
= \int_{t_1}^{t_2} \int_{\Omega} \frac{|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k} - u_{\epsilon_k}^{\alpha} \nabla \overline{u}|^2}{(|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k} - u_{\epsilon_k}^{\alpha} b(u_{\epsilon_k})|^m + |u_{\epsilon_k}^{\alpha} \nabla \overline{u} - u_{\epsilon_k}^{\alpha} b(u_{\epsilon_k})|^m)^{(2-m)/m}} dx dt
$$
\n
$$
=:\int_{t_1}^{t_2} h_k(x, t) dx dt.
$$
\nIs should be recalled that $\overline{u} = \sup(u - \epsilon_{1k}, 0)$ herein. From (8.28) and (8.29) it follows that there exist a subsequence $\{k\}$ and a subset $\widetilde{Q} \subset Q_{t_1, t_2} = \Omega \times [t_1, t_2], |\widetilde{Q}| = |Q_{t_1, t_2}|,$ such that
\n
$$
\lim_{k \to \infty} h_k(x, t) = 0 \quad \text{on } \widetilde{Q}.
$$
\n
$$
=:\int_{t_1, t_2}^{t_1} h_k(x, t) dx dt.
$$
\n
$$
\lim_{k \to \infty} h_k(x, t) = 0 \quad \text{on } \widetilde{Q}. \tag{8.31}
$$
\nWithout loss of generality we can count that the derivatives $\frac{\partial u^{\alpha+1}}{\partial x_i}$ are finite on \widetilde{Q} ($i = 1, ..., n$). Then using (8.1) and (8.3), the definition of \overline{u} (see (8.7)), (8.8) and (8.6),

Is should be recalled that $\bar{u} = \sup(u - \varepsilon_{1k}, 0)$ herein. From (8.28) and (8.29) it follows that there exist a subsequence $\{k\}$ and a subset $\widetilde{Q} \subset Q_{t_1,t_2} = \Omega \times [t_1,t_2], |\widetilde{Q}| = |Q_{t_1,t_2}|,$ such that

$$
\lim_{k \to \infty} h_k(x,t) = 0 \qquad \text{on } \ \widetilde{Q}.\tag{8.31}
$$

1,...,n). Then using (8.1) and (8.3), the definition of *ü* (see (8.7)), (8.8) and (8.6), Ium $h_k(x,t) = 0$ on \tilde{Q} .

Interval is a subsequence $\{k\}$ and a subset $Q \subset Q_{t_1,t_2} = \frac{1}{2}$ $\{k\}$ $\{t_1,t_2\}$, $|Q| = |Q$

and $\lim_{k \to \infty} h_k(x,t) = 0$ on \tilde{Q} .

It loss of generality we can count that the derivat om (8.28) and (8.29) it follows
 $t_1, t_2 = \Omega \times [t_1, t_2], |\tilde{Q}| = |Q_{t_1, t_2}|,$
 $(\tilde{Q}, 31)$
 $(\tilde{Q}, 32)$
 $(\tilde{Q}, 32)$
 $(\tilde{Q}, 32)$
 $(\tilde{Q}, 33)$
 $(\tilde{Q}, t) \in \tilde{Q}$

$$
|u_{\epsilon_k}^{\alpha} \nabla \bar{u}|
$$
 and $|u_{\epsilon_k}^{\alpha} b(u_{\epsilon_k})|$ are bounded (non-uniformly) on \tilde{Q} . (8.32)

On the other hand, in view of the definition of the functions h_k we can estimate

and the continuity of the vector function
$$
b(u)
$$
 we can conclude that
\n
$$
|u_{\epsilon_k}^{\alpha} \nabla \bar{u}| \text{ and } |u_{\epsilon_k}^{\alpha} b(u_{\epsilon_k})| \text{ are bounded (non-uniformly) on } \tilde{Q}.
$$
\n(8.32)
\nOn the other hand, in view of the definition of the functions h_k we can estimate
\n
$$
h_k(x,t) \ge \frac{(|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}| - c)^2}{c(|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}| + c)^{2-m}} \qquad ((x,t) \in \tilde{Q})
$$
\n(8.33)

with some constant *c* depending on (x, t) . Assume that $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$ are unbounded at some point $(x, t) \in \tilde{Q}$. Then for some subsequence $\{k\}$ we have $|u_{\varepsilon_k}^{\alpha} \nabla u_{\varepsilon_k}| \to \infty$ as $k \to \infty$ and hence (using that $m \in (1,2)$) we derive from (8.33) that the definition of the functions h_k we
 $\frac{(|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}| - c)^2}{c(|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}| + c)^{2-m}}$ $((x, t) \in \tilde{Q})$

ing on (x, t) . Assume that $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon}$

for some subsequence $\{k\}$ we have
 $m \in (1, 2)$) Bond point $(x, t) \in Q$. Then for some subsequence $\{u_f\}$ we have $|u_{\epsilon_k} \vee u_{\epsilon_k}| \to \infty$ as $k \to \infty$ and hence (using that $m \in (1, 2)$) we derive from (8.33) that
 $\lim_{k \to \infty} h_k(x, t) = \infty$ on \tilde{Q} . (8.34)

But this give $\frac{u_{\epsilon_k} - u_{\epsilon_k}}{u_{\epsilon_k} - u_{\epsilon_k}} + c$ $(x, t) \in Q$

on (x, t) . Assume that $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$ are unborsome subsequence $\{k\}$ we have $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$
 $\in (1, 2)$ we derive from (8.33) that
 $h_k(x, t) = \infty$ on $\tilde{Q$

$$
\lim_{k \to \infty} h_k(x,t) = \infty \qquad \text{on } \ \widetilde{Q}.\tag{8.34}
$$

But this gives a contradiction with (8.31). Hence

$$
u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k} | \qquad \text{are bounded (non-uniformly) on } \ \widetilde{Q}. \tag{8.35}
$$

 \widetilde{Q} as $k \to \infty$, i.e., But this gives a contradiction with (8.31). Hence
 $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$ are bounded (non-uniformly) on \tilde{Q} . (8.35)

Then from (8.31), (8.30) and (8.35) it follows that the numerators of h_k tend to zero on
 $\tilde{$

$$
u(t) \in Q.
$$
 Then for some subsequence $\{k\}$ we have $|u_{\epsilon_k}^* \vee u_{\epsilon_k}| \to \infty$ as
since (using that $m \in (1, 2)$) we derive from (8.33) that

$$
\lim_{k \to \infty} h_k(x, t) = \infty \quad \text{on } \widetilde{Q}.
$$
 (8.34)
to contradiction with (8.31). Hence
 $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$ are bounded (non-uniformly) on $\widetilde{Q}.$ (8.35)
1), (8.30) and (8.35) it follows that the numerators of h_k tend to zero on
the.

$$
\lim_{k \to \infty} \left| u_{\epsilon_k}^{\alpha} \frac{\partial u_{\epsilon_k}}{\partial x_i} - u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right| = 0 \quad \text{on } \widetilde{Q} \quad (i = 1, ..., n).
$$
 (8.36)
at

$$
\lim_{k \to \infty} \left| u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} - u^{\alpha} \frac{\partial u}{\partial x_i} \right| = 0 \quad \text{on } \widetilde{Q} \quad (i = 1, ..., n).
$$
 (8.37)

Remark now that

$$
\lim_{k \to \infty} \left| u_{\varepsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} - u^{\alpha} \frac{\partial u}{\partial x_i} \right| = 0 \quad \text{on } \ \widetilde{Q} \qquad (i = 1, \dots, n). \tag{8.37}
$$

 $\lim_{k \to \infty} h_k(x,$ contradiction with (8.3
 $|u_{\epsilon_k}^{\alpha} \nabla u_{\epsilon_k}|$ are both
 $h(k,30)$ and (8.35) it for
 $\lim_{\epsilon \to \infty} u_{\epsilon_k}^{\alpha} \frac{\partial u_{\epsilon_k}}{\partial x_i} - u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i}$
 $\lim_{k \to \infty} u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} - u^{\alpha} \frac{\partial u}{\partial x_i}$ and hence (8.37) follows from (8.1). On the other hand, if $(x, t) \in \tilde{Q}$ and $u(x, t) = 0$, then $\frac{\partial \hat{u}(x, t)}{\partial x_i} = 0$ for any k and hence (8.37) follows from (8.1) and the definition of \tilde{u}_{x_i} . $\lim_{k \to \infty} \left| u_{\epsilon_k}^{\alpha} \frac{\partial u_{\epsilon_k}}{\partial x_i} - u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} \right| = 0 \quad \text{on } \widetilde{Q} \qquad (i = 1, ..., n).$ (8.36)

Remark now that
 $\lim_{k \to \infty} \left| u_{\epsilon_k}^{\alpha} \frac{\partial \bar{u}}{\partial x_i} - u^{\alpha} \frac{\partial u}{\partial x_i} \right| = 0 \quad \text{on } \widetilde{Q} \qquad (i = 1, ..., n).$ (8.37)

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Finally, from (8.36) and (8.37) it follows obviously that (8.19) holds. Theorem 6.1 (and hence Theorem 1.1) is proved.

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