Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 14 (1995), No. 4, 779–828

$\mathcal{L}^2$-Perturbations of Space-Periodic Equilibria of Navier-Stokes

B. Scarpellini

Abstract. We assume that a smooth equilibrium solution u_0, p_0 of Navier-Stokes on an infinite plate $\Omega = \mathbb{R}^2 \times (-\frac{1}{2}, +\frac{1}{2})$ is given, which is L-periodic with respect to the unbounded variables $x, y \in \mathbb{R}$. We investigate the stability of u_0, p_0 with respect to perturbations which are not L-periodic but belong to $L^2(\Omega)$. To this end we study the $L^2(\Omega)$-spectrum of the linearization around u_0, p_0 and describe it in terms of so-called Θ-periodic spectra in a similar way as it is done for Schrödinger equations with periodic potentials.

Keywords: *Navier-Stokes equation, stability, instability, direct integrals, Floquet periodicity*

AMS subject classification: 35 B 10, 35 B 20, 35 B 35, 35 Q 30

0. Introduction

In the present paper we treat a stability problem which has been invoked by D. Sattinger and K. Kirchgässner at different places [4, 10, 11] and which will be described in what follows in non-technical terms. Let a vector function $u_0(x,y,z)$ $(x, y \in \mathbb{R},\ z \in (-\frac{1}{2}, +\frac{1}{2}))$ be a smooth equilibrium of a nonlinear evolution equation $u_t = F(u)$ (typically $F(u) = \Delta u + f(u)$) and assume that besides satisfying some boundary conditions at $z = \pm\frac{1}{2}$ it is L-periodic in x and y for some $L > 0$. One can then discuss the stability of u_0 with respect to various classes of perturbations. An established way to proceed is to set $u = u_0 + v$ in $u_t = F(u)$ in order to find after some computational steps

$$v_t = (dF)(u_0)v + R(u_0, v) \tag{0.1}$$

with $(dF)(u_0)$ the derivative of F at u_0 and $R(u_0, v)$ a term such that $\|R(u_0, v)\| = o(\|v\|)$ in a suitable functional setting. A well known procedure amounts to test the stability of u_0 against L-periodic perturbations. To this end one considers (0.1) as an evolution equation in an appropriate space of functions which are L-periodic in x and y and satisfy the boundary conditions. The stability behaviour of u_0 is then reduced to a discussion of the stability of the equilibrium solution $v_0 \equiv 0$ of (0.1) in this particular functional setting. Under the proviso that the principle of linearized stability resp. instability has been justified, the stability behaviour of $v_0 \equiv 0$ is then essentially determined by the spectrum $\sigma_{\mathrm{per}}(dF(u_0))$ of $dF(u_0)$ considered as an unbounded operator on

B. Scarpellini: University Basel, Math. Institute, Rheinsprung 21, CH - 4051 Basel, Switzerland

ISSN 0232-2064 / \$ 2.50

the space of L-periodic functions in question. In [10], D. Sattinger suggests to investigate the stability of u_0 not against L-periodic perturbations, but against perturbations from some other function class. He also suggests to develop a Hill type theory for the linearization $dF(u_0)$. Such a program has been carried out in [12, 13] for the case where the basic equation $u_t = F(u)$ is a reaction-diffusion system $u_t = \Delta u + f(u)$, with u an m-vector $u = (u_1, \ldots, u_m)$ defined on $\mathbb{R}^n$ $(n \le 3)$ and assumed to be L-periodic in all its variables (no boundary conditions); $f(u)$ is a polynomial nonlinearity. As a model Hill type theory we have taken the theory of Schrödinger equations $i\psi_t = \Delta\psi + V\psi$ with L-periodic potential V, discussed in Reed and Simon [8]. This forced us to take as possible perturbations the class of vector functions which have components which are in $\mathcal{L}^2(\Omega)$ (with $\Omega = \mathbb{R}^n$ $(n \le 3)$ in [12, 13] while $\Omega = \mathbb{R}^2 \times (-\frac{1}{2}, +\frac{1}{2})$ here). In [12] the following problem is adressed (which is treated in the Schrödinger context in [8]):

(A) How is the spectrum $\sigma_{\text{per}}(dF(u_0))$ of $dF(u_0)$ as an operator on a space of L-periodic functions related to the spectrum $\sigma_{\mathcal{L}^2}(dF(u_0))$ of $dF(u_0)$ as an operator on a space of $\mathcal{L}^2(\Omega)$-functions?

This question cannot be treated directly, but, following the pattern set out in [8], requires a detour via a more general problem. To this end one needs the notion of "Floquet" or "Θ-periodic" function:

$$\begin{aligned} &f(x,y) \text{ is } \Theta = (\Theta_1, \Theta_2) - \text{periodic with respect to } x, y \text{ if} \\ &f(x+L, y) = e^{i\Theta_1} f(x,y) \text{ and } f(x, y+L) = e^{i\theta_2} f(x,y) \quad (\Theta \in [0, 2\pi]^2). \end{aligned} \tag{0.2}$$

The problem which is treated in [8] in the Schrödinger context and in [12] in the context of reaction diffusion systems is:

(A)* How are the spectra $\sigma_\Theta(dF(u_0))$ of $dF(u_0)$ as an operator on a space of Θ-periodic functions related to the spectrum $\sigma_{\mathcal{L}^2}(dF(u_0))$?

The answer given in [8, 12] is

(B) $\sigma_{\mathcal{L}^2}(dF(u_0)) = \bigcup_\Theta \sigma_\Theta(dF(u_0)) \qquad (\Theta \in [0, 2\pi]^2)$

what implies

(C) $\sigma_{\text{per}}(dF(u_0)) \subseteq \sigma_{\mathcal{L}^2}(dF(u_0))$.

Here we investigate as to what extent (B) and (C) or part of it remain true if $dF(u_0)$ is the linearization of Navier-Stokes around a periodic equilibrium solution restricted to the space of divergence free fields (see Subsection 1.1).

We now briefly describe content and results of this paper. In Section 1 the basic material regarding the $\mathcal{L}^2$-setting of Navier-Stokes on an infinite plate is compiled, mostly without proof; a glance at Section 1 suffices in a first reading. Section 2 contains the necessary prerequisites about Θ-periodic vector fields, Θ-periodic Stokes operators etc. A basic regularity result (see Theorems 2 and 2*) is stated without proof; a proof is given in [14]. In a first reading it suffices to take notice of the material in Section 2. Section 3 is crucial in that it contains the theory of direct integrals to the extent needed here. It is selfcontained to some extent but we can not avoid to borrow material from [12, 13]. The following basic result of independent interest is proved:

(D) $dF(u_0)_{\mathcal{L}^2}$ is unitarily equivalent to $\int_M dF(u_0)_\Theta d\Theta$

where $M = [0, 2\pi]^2$. Here $dF(u_0)_{\mathcal{L}^2}$ is $dF(u_0)$ acting on $\mathcal{L}^2$-vector fields, while $dF(u_0)_\Theta$ is $dF(u_0)$ acting on Θ-periodic vector fields. The second expression in (D), i.e. $\int_M$ is a direct integral in the sense of [8] and requires for its interpretation the vocabulary of Section 3 (resp. [8] or [12]). Relation (D) serves as starting point in Section 4 for the derivation of spectral relations similiar to (B) and (C). Here we assume the periodic equilibrium solution $u_0 = (u_1, u_2, u_3)$ to satisfy:

(E) u_1, u_2 are even in z and u_3 is odd in z.

This assumption, which could be dispensed with in principle, helps to simplify the presentation, gives finer and nicer results and gives more insight into the difficulties associated with the corners $(0,0)$, $(0,2\pi)$, $(2\pi,0)$ and $(2\pi,2\pi)$. The main result then is (Theorems 5 and 6 plus Corollaries)

(F) a complete description of $\sigma_{\mathcal{L}^2}(dF(u_0))$ in terms of $\sigma_\theta(dF(u_0))$ $(\Theta \in [0,2\pi]^2)$.

This is somewhat vague; a full interpretation of (F) requires a slight technical digression (see Subsection 4.4). However, two consequences of (F), close to (B) and (C) are the following: Let $\dot{M}$ be $[0,2\pi]^2$ minus the corners. Then (Corollary to Theorem 4)

(G) if $\Theta \in \dot{M}$, then $\sigma_\Theta(dF(u_0)) \subseteq \sigma_{\mathcal{L}^2}(dF(u0))$

and

(H) if $\lambda_0 \in \sigma_{\text{per}}(dF(u_0))$ is real, then $\lambda_0 \in \sigma_{\mathcal{L}^2}(dF(u_0))$.

What happens in the case of complex λ_0 is open; (F) does not contain (immediately at least) the necessary information. This leaves room for the possibility that u_0 is periodically unstable but $\mathcal{L}^2$-stable, a case excluded for reaction-diffusion systems by virtue of (C). The present paper is more difficult than [12] mainly because the four corners of $[0,2\pi]^2$ together with the divergence condition are a source of difficulties. For reasons of space we do not treat the principle of linearized instability, which can in fact be proved along the lines of [13], making thereby extensive use of Section 4 in the present paper. Likewise, we do not discuss the Benard problem; however, once one has mastered the difficulties of Navier-Stones alone, the Benard problem is easily accessible. This topic will be presented separately.

Notations. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. For $\mathcal{X}$ and $\mathcal{Y}$ Banach spaces, $\|\cdot\|_\mathcal{X}$ and $\|\cdot\|_\mathcal{Y}$ denote their respective norms, and for $\mathcal{U} \subseteq \mathcal{X}$ an open set, $C^p(\mathcal{U},\mathcal{Y})$ is the set of p-times continuously differentiable mappings from $\mathcal{U}$ to $\mathcal{Y}$. For $F \in C^1(\mathcal{U},\mathcal{Y})$, $dF(u)$ is the derivative of F at u. If the underlying space is fixed in a context, we write $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{X}}$. $L(\mathcal{X},\mathcal{Y})$ is the space of bounded linear operators T from $\mathcal{X}$ to $\mathcal{Y}$, with $\|T\|_\infty$ or even $\|T\|$ the operator norm. For T a bounded or unbounded operator on $\mathcal{X}$, having $E \subseteq \mathcal{X}$ as an invariant subspace, $\rho_E(T)$ and $\sigma_E(T)$ denote the resolvent set and spectrum of T restricted to E, respectively. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we set $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ where ∂_j is the derivative with respect to x_j and call $|\alpha| = \sum_j \alpha_j$ the order of D^α. For Ω having the segment property (see [1: p. 54]) an $f \in \mathcal{L}^2$ is in $H^p(\Omega)$ if and only if there is a sequence of $f_n \in C_0^p(\overline{\Omega})$ which is a Cauchy sequence with respect to the Sobolev norm $\|\cdot\|_{H^p(\Omega)}$ and such that $\lim_n \|f - f_n\|_{\mathcal{L}^2} = 0$. Here $C_0^p(\overline{\Omega})$ is the space of functions having compact support in $\overline{\Omega}$ and continuous derivatives up to order p; likewise with $C_0^p(\Omega)$. Finally, $(\cdot,\cdot)_p$ is the scalar product on $H^p(\Omega)$, given by

$$(u,v)_p = \sum_{|\alpha|\le p} (D^\alpha u, D^\alpha v)_0 \tag{0.3}$$

where $(u,v)_0 = \int_\Omega u(x)\overline{v}(x)\,dx$. We set $\mathcal{L}^2(\Omega) = H^0(\Omega)$ and write $\|\cdot\|_{H^p}$ instead of $\|\cdot\|_{H^p(\Omega)}$ if no confusion arises. We extend this notation to vectors and set

$$\|u\|_{\mathcal{L}^2}^2 = \|u_1\|_{\mathcal{L}^2}^2 + \|u_2\|_{\mathcal{L}^2}^2 + \|u_3\|_{\mathcal{L}^2}^2$$

whenever $u = (u_1,u_2,u_3) \in (\mathcal{L}^2)^3$, similarly with the Sobolev norms $\|u\|_{H^p}$. The scalar product in $(H^p)^3$ is $\langle\cdot,\cdot\rangle_p$, where

$$\langle u,v\rangle_p = (u_1,v_1)_p + (u_2,v_2)_p + (u_3,v_3)_p$$

where the components u_j and v_j of u and v are all in H^p. We write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_0$. We also use the following convention: if $A(x,y)$ depends only on x, y and $\zeta(z)$ only on z, then $A\zeta$ denotes the function $A(x,y)\zeta(z)$.

1. $\mathcal{L}^2$-setting of Navier-Stokes on an infinite plate

1.1 Navier-Stokes on an infinite plate. As a starting point we take the Navier-Stokes equations on an infinite plate $\Omega = \mathbb{R}^2 \times (-\frac{1}{2}, +\frac{1}{2})$:

$$\begin{aligned} u_t &= \nu\Delta u - (u\nabla)u - \nabla p - f \\ \operatorname{div} u &= 0 \end{aligned} \tag{1.1}$$

where $u = (u_1,u_2,u_3)$ satisfies Dirichlet boundary conditions at $z = \pm\frac{1}{2}$, p is the pressure and f a time independent outer force. We assume that a smooth equilibrium solution $u_0 = (u_1,u_2,u_3)$, p_0 of equations (1.1) is given, which is L-periodic in x and y, i.e. $u_0(x+\alpha L, y+\beta L, z) = u_0(x,y,z)$ $(\alpha,\beta \in \mathbb{Z})$ and likewise with p_0. Since the case of L_1-periodicity in x and L_2-periodicity in y leads to exactly the same technicalities as the case $L = L_1 = L_2$, we restrict us to the latter for simplicity of notation. Following the pattern set out in the introduction we insert $u = u_0 + v$ and $p = p_0 + \pi$ into (1.1). Using the assumption that u_0, p_0 is an equilibrium solution of (1.1) we obtain formally

$$v_t = \nu\Delta v + T_0 v - \nabla\pi - (v\nabla v) \tag{1.2}$$

$\operatorname{div} v = 0$ plus Dirichlet boundary conditions.

Here T_0 is the operator given formally by

$$T_0 v = -(u_0\nabla)v - (v\nabla)u_0. \tag{1.3}$$

Relations (0.2) have been obtained in a formal way but the intention is to let v be a member of the subspace

$$\begin{aligned} E = \mathcal{L}^2\text{-closure of } \; & f \in \left(H^2(\Omega)\cap H_0^1(\Omega)\right)^3 \\ \operatorname{div} f &= 0. \end{aligned} \tag{1.4}$$

We then must choose $\nabla\pi$ from the orthogonal complement $E_\perp$ of E, but contrary to the case of bounded Ω we can only assume

$$E_\perp = \mathcal{L}^2\text{-closure of } \quad \nabla p \qquad \left(p \in H^1(\Omega)\right). \tag{1.5}$$

In fact, examples $q \in E_\perp$ exist which are not of the form $q = \nabla p$, $p \in H^1(\Omega)$ (see Subsection 1.5). Letting P be the orthogonal projection onto E, we apply P to both sides of (1.3) in order to get

$$v_t = \nu P\Delta v + PT_0 v - P(v\nabla)v. \tag{1.6}$$

If one denotes by E_{per} and P_{per} the counterparts to E and P, respectively, in the periodic case, one obtains the corresponding equation

$$w_t = \nu P_{\text{per}}\Delta w + P_{\text{per}}T_0 w - P_{\text{per}}(w\nabla)w \tag{1.7}$$

with w an element of E_{per}. This is an evolution equation in the sense of Pazy [7: Chapter 6.3]; the principles of linearized stability and instability are known to hold (Kirchgässner [4, 5]). The situation is different for equation (1.6). It seems to be known among experts that (1.6) is indeed a well posed evolution equation although, as discussions indicate, it is difficult to find explicit references. This, and the fact that we have to handle space very economically forces us to state two auxiliary results on regularity properties of Stokes operators without proof. The proofs, which require some place, will be given in [14]. The central task of the paper is to study the relationship between the spectra

$$\sigma_{E_{\text{per}}}(\nu P_{\text{per}}\Delta + P_{\text{per}}T_0) \qquad \text{and} \qquad \sigma_E(\nu P\Delta + PT_0). \tag{1.8}$$

As pointed out in the introduction, this requires a new tool, that of Θ-periodic vector fields and related concepts, to be studied later.

1.2 Remarks on Sobolev spaces. First we fix some notations. Let $S_r = \{x \in \mathbb{R}^3 : |x| < r\}$, set $\partial'\Omega_r = (S_r \cap \partial\Omega) \cup (\Omega \cap \partial S_r)$ and $\Omega_r = S_r \cap \Omega$. Set also $\Gamma^+ = \mathbb{R}^2 \times \{+\frac{1}{2}\}$, $\Gamma^- = \mathbb{R}^2 \times \{-\frac{1}{2}\}$, $\Gamma_r^\pm = \Gamma^\pm \cap S_r$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, $\Gamma_r = \Gamma_r^+ \cup \Gamma_r^-$. Using well known results about traces on smooth bounded domains and the fact that Ω has the $(j,2)$-extension property (see Adams [1: pp. 83 - 94]) one can define for any $R > 0$ trace operators $\gamma_j^R \in L\big(H^{j+1}(\Omega), \mathcal{L}^2(\partial'\Omega_R)\big)$ $(j = 0, 1)$ such that:

(a) If $v \in H^1(\Omega) \cap C^1(\overline{\Omega})$, then $\gamma_0^R(v)(x) = v(x)$ for all $x \in \partial'\Omega_R$.

(b) If $u \in H^2(\Omega) \cap C^2(\overline{\Omega})$, then $\gamma_1^R(u)(x) = \left(\frac{\partial u}{\partial n}\right)(x)$ for all $x \in \partial'\Omega_R$, with $\left(\frac{\partial u}{\partial n}\right)(x)$ the outer unit normal at $x \in \partial'\Omega_R$.

(c) If $r \leq R$, them $\gamma_j^r(u_j)(x) = \gamma_j^R(u_j)(x)$ for a.e. $x \in \Gamma_r$, where $u_j \in H^{j+1}(\Omega)$ $(j = 0, 1)$.

From the validity of Gauss' identity for domains such as Ω_r and elements $v \in C^1(\overline{\Omega}_r)$ and $\vec{u} \in (C^1(\overline{\Omega}_r))^3$ (see König [6]) we obtain via familiar approximation procedures the following extension to elements $v \in H^1(\Omega)$ and $\vec{u} = (u_1, u_2, u_3) \in (H^1(\Omega))^3$:

$$\int_{\partial'\Omega_R} \gamma_0^R(v)(\omega)\gamma_0^R(\vec{u})(\omega)\vec{n}(\omega)\, d\omega = \int_{\Omega_R} \left((\nabla v)\vec{u} + v \operatorname{div} \vec{u}\right) dx^3 \tag{1.9}$$

where

$$\gamma_0^R(\vec{u})(\omega) = \left(\gamma_0^R(u_1)(\omega), \gamma_0^R(u_2)(\omega), \gamma_0^R(u_3)(\omega)\right)$$

and with $\vec{n}(\omega)$ the outward unit normal at $\omega \in \partial'\Omega_R$. By noting that $\vec{n}(\omega)$ is $(0,0,+1)$ and $(0,0,-1)$ for $\omega \in \Gamma_R^+$ and $\omega \in \Gamma^+$, respectively, we have the decomposition

$$\begin{aligned}\int_{\partial'\Omega_R} \gamma_0^R(v)\gamma_0^R(\vec{u})\vec{n}\, d\omega &= \int_{\Omega\cap\partial S_R} \gamma_0^R(v)\gamma_0^R(\vec{u})\vec{n}\, d\omega \\ &\quad + \int_{\Gamma_R^+} \gamma_0^R(v)\gamma_0^R(u_3)\, d\omega - \int_{\Gamma_R^-} \gamma_0^R(v)\gamma_0^R(u_3)\, d\omega.\end{aligned} \tag{1.10}$$

We now define $H_0^1(\Omega)$ according to

$$f \in H_0^1(\Omega) \iff f \in H^1(\Omega) \text{ and } \gamma_0^R(f)(x) = 0 \text{ for a.e. } x \in \Gamma_R \text{ and } R > 0. \tag{1.11}$$

Thus for any sequence $f_n \in C_0^1(\overline{\Omega})$, $\|f_n - f\|_{H^1} \to 0$ and $R > 0$ we have that $\gamma_0^R(f_n) \to 0$ in $\mathcal{L}^2(\Gamma_R)$. Likewise we define

$$f \in \widehat{H}^2(\Omega) \iff f \in H^2(\Omega) \text{ and } \gamma_1^R(f)(x) = 0 \text{ for a.e. } x \in \Gamma_R \text{ and } R > 0. \tag{1.12}$$

Proposition 1.1. *The following assertions are true:*

(i) *If $f \in H^2(\Omega)$, then $\gamma_1^R(f) = \gamma_0^R(\frac{\partial f}{\partial z})$ a.e. on Γ_R^+ and $\gamma_1^R(f) = -\gamma_0^R(\frac{\partial f}{\partial z})$ a.e. on Γ_R^-.*

(ii) *$u \in H_0^1(\Omega)$ if and only if $u \in H^1(\Omega)$ and $\int_\Omega (\varphi \frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial z} u)\, dx = 0$ for all $\varphi \in H^1(\Omega)$.*

(iii) *$u \in \widehat{H}^2(\Omega)$ if and only if $u \in H^2(\Omega)$ and $\int_\Omega (\varphi \frac{\partial^2 u}{\partial z^2} + \frac{\partial \varphi}{\partial z}\frac{\partial u}{\partial z})\, dx = 0$ for all $\varphi \in H^1(\Omega)$.*

Proof. Assertion (i) follows via fundamental sequences by straightforward approximation arguments based on properties (a) - (c) of γ_j^R. With assertion (i) at disposal we have that $u \in \widehat{H}^2(\Omega)$ if and only if $u \in H^2(\Omega)$ and $\frac{\partial u}{\partial z} \in H_0^1(\Omega)$, yielding assertion (iii) as a consequence of assertion (ii).

Now assume first $u \in H_0^1(\Omega)$ and $\varphi \in H^1(\Omega)$. We apply (1.9) to φ and $\vec{v} = (0,0,u)$, taking (1.10) into account. We obtain

$$\int_{\Omega\cap\partial S_r} \gamma_0^r(\phi)\gamma_0^r(u)n_3\,d\omega = \int_{\Omega_r}\left(\varphi\frac{\partial u}{\partial z} + \frac{\partial\varphi}{\partial z}u\right)dx$$

where $\vec{n} = (n_1, n_2, n_3)$. Following the arguments in the proofs of Proposition 1.1 and Lemma 1 in [15] we infer that the function

$$f(r) = \int_{\Omega\cap\partial S_r} \gamma_0^r(\phi)\gamma_0^r(u)n_3\,d\omega$$

is in $\mathcal{L}^1(0,\infty)$. Since it is also continuous it follows that $\lim_n f(r_n) = 0$ for some sequence $r_n \uparrow \infty$. The "only if" part then immediately follows.

Now assume conversely that the right-hand side of the assumption in assertion (ii) is satisfied, and that $u \notin H_0^1(\Omega)$. Then there is an $R > 0$ such that $\gamma_0^R(u)(x) = 0$ for a.e. $x \in \Gamma_R$ fails. We thus may assume, e.g., that the set

$$\left\{x \in \Gamma_R^+ : \gamma_0^R(u)(x) \neq 0\right\}$$

has non-zero Γ-measure. Choose $\zeta \in C^\infty([-\frac{1}{2}, +\frac{1}{2}])$ as follows: $0 \le \zeta \le 1$, $\zeta = 0$ on $[-\frac{1}{2}, 0]$ and $\zeta = 1$ on $[\delta, +\frac{1}{2}]$ for some small $\delta > 0$. In accordance with our convention in 'Notations' we set $\varphi = \zeta u$. Clearly $\varphi \in H^1(\Omega)$. Moreover $\gamma_0^r(\varphi) = 0$ a.e. on Γ_r^- and $\gamma_0^r(\varphi) = \gamma_0^r(u)$ a.e. on Γ_r^+ for any r. With $\vec{v} = (0,0,u)$, the identity (1.9) then reduces to

$$\int_{\Omega\cap\partial S_r} \gamma_0^r(\phi)\gamma_0^r(u)n_3\,d\omega + \int_{\Gamma_r^+}\gamma_0^r(u)^2\,d\omega = \int_{\Omega_r}\left(\varphi\frac{\partial u}{\partial z} + \frac{\partial\varphi}{\partial z}u\right)dx.$$

For $r \ge R$ the second term on the left-hand side above remains greater or equal ε for some fixed $\varepsilon > 0$ while the first term tends to zero for a suitable chosen sequence $r_n \uparrow \infty$. This means that the right-hand side of the equality above remains greater or equal $\frac{\varepsilon}{2}$ as $r_n \uparrow \infty$, contradicting the assumption ∎

1.3 Fourier expansions. First we consider the eigenvalue problem $y'' + \lambda y = 0$ on $[-\frac{1}{2}, +\frac{1}{2}]$ both with Neumann and Dirichlet boundary conditions. A complete set $\{\varphi_p\}_{p\ge 0}$ of orthonormalized eigenfunctions for the Neumann case may be given as follows:

$$\begin{aligned} \varphi_0 &= 1 \\ \varphi_{2k} &= (-1)^k\sqrt{2}\cos 2\pi kx && \text{for } k \ge 1 \\ \varphi_{2k+1} &= (-1)^k\sqrt{2}\sin(2k+1)\pi x && \text{for } k \ge 0. \end{aligned} \tag{1.13}$$

Setting $\Lambda_p = p^2\pi^2$ we then have $\varphi_p'' + \Lambda_p\varphi_p = 0$ and, in addition,

$$\begin{aligned} \varphi_{2k}\left(+\frac{1}{2}\right) &= \varphi_{2k}\left(-\frac{1}{2}\right) = \sqrt{2} && \text{for } k \ge 1 \\ \varphi_{2k+1}\left(+\frac{1}{2}\right) &= -\varphi_{2k+1}\left(-\frac{1}{2}\right) = \sqrt{2} && \text{for } k \ge 0. \end{aligned} \tag{1.14}$$

A complete orthonormalized set $\{\psi_p\}_{p\geq 1}$ of eigenfunctions for the Dirichlet case is then obtained by setting

$$\Lambda_p^{1/2}\psi_p = \varphi_p' \qquad \text{whence } \psi_p' = -\Lambda_p^{1/2}\varphi_p \qquad (p \geq 1). \tag{1.15}$$

It will have great advantages to take the parity of the eigenfunctions into account. Therefore we set for later purposes

$$\begin{aligned} &\sigma_k = \varphi_{2k+1}, \quad \tau_k = \psi_{2k+1}, \quad \lambda_k = \Lambda_{2k+1} && (k \geq 0) \\ &\rho_0 = 1, \quad \rho_k = \varphi_{2k}, \quad \pi_k = \psi_{2k}, \quad \mu_k = \Lambda_{2k} && (k \geq 1). \end{aligned} \tag{1.16}$$

Next fix $f \in \mathcal{L}^2(\Omega)$ and set

$$A_k(x,y) = \int_{-1/2}^{+1/2} f(x,y,z)\varphi_k(z)\,dz \quad \text{and} \quad B_j(x,y) = \int_{-1/2}^{+1/2} f(x,y,z)\psi_j(z)\,dz.$$

Then $A_k, B_j \in \mathcal{L}^2(\mathbb{R}^2)$ $(k \geq 0,\ j \geq 1)$ and

$$f = \sum_{k=0}^{\infty} A_k(x,y)\varphi_k(z) = \sum_{j=1}^{\infty} B_j(x,y)\psi_j(z)$$

in the sense that $\|L_N - f\|_{\mathcal{L}^2}$ and $\|H_N - f\|_{\mathcal{L}^2}$ tend to zero where

$$L_N = \sum_{k=0}^{N} A_k\varphi_k \qquad \text{and} \qquad H_N = \sum_{j=1}^{N} B_j\psi_j.$$

Moreover we have that

$$\int_\Omega |f|^2 dx^3 = \sum_{k=0}^{\infty}\int_{\mathbb{R}^2} |A_k|^2 dx^2 = \sum_{j=1}^{\infty}\int_{\mathbb{R}^2} |B_j|^2 dx^2.$$

All this follows easily from Fubini's theorem and the completeness of the systems $\{\varphi_k\}_{k\geq 0}$ and $\{\psi_j\}_{j\geq 1}$, respectively. Important is the characterization of $H^1(\Omega)$ and $H_0^1(\Omega)$ in terms of Fourier series, provided by Proposition 1.2 below, in which A_k and B_j are as in (5).

Proposition 1.2. *Let $f \in \mathcal{L}^2(\Omega)$.*

(i) *$f \in H^1(\Omega)$ if and only if $A_k \in H^1(\mathbb{R}^2)$ $(k \geq 0)$ and if*

$$\sum_{k=0}^{\infty}\int_{\mathbb{R}^2} |\partial_x A_k|^2 dx^2, \qquad \sum_{k=0}^{\infty}\int_{\mathbb{R}^2} |\partial_y A_k|^2 dx^2, \qquad \sum_{k=1}^{\infty}\Lambda_k\int_{\mathbb{R}^2} |A_k|^2 dx^2$$

are all finite.

(ii) *$f \in H_0^1(\Omega)$ if and only if $B_j \in H^1(\mathbb{R}^2)$ $(j \geq 1)$ and if*

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}^2} |\partial_x B_j|^2 dx^2, \qquad \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} |\partial_y B_j|^2 dx^2, \qquad \sum_{j=1}^{\infty} \Lambda_j \int_{\mathbb{R}^2} |B_j|^2 dx^2$$

are all finite.

Proof. Consider assertion (i). If the right-hand side conditions on the A_k's are satisfied, then the $L_N = \sum_{k=0}^{N} A_k \varphi_k$ are all in $H^1(\Omega)$ and form a Cauchy sequence with respect to $\|\cdot\|_{H^1}$. Since $\lim_{N\to\infty} L_N = f$ in $\mathcal{L}^2(\Omega)$ we have that $f \in H^1(\Omega)$ and

$$\partial_x f = \sum_{k=0}^{\infty} (\partial_x A_k)\varphi_k, \qquad \partial_y f = \sum_{k=0}^{\infty} (\partial_y A_k)\varphi_k, \qquad \partial_z f = \sum_{k=1}^{\infty} \Lambda_k^{1/2} A_k \psi_k. \tag{1.17}$$

If conversely $f \in H^1(\Omega)$, then one easily verifies $A_k \in H^1(\mathbb{R}^2)$ $(k \geq 0)$ and in addition

$$\partial_x A_k = \int_{-1/2}^{+1/2} (\partial_x f)\varphi_k \, dz \qquad \text{and} \qquad \partial_y A_k = \int_{-1/2}^{+1/2} (\partial_y f)\varphi_k \, dz$$

a.e. on $\mathbb{R}^2$. From this, the relations

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^2} |\partial_x A_k|^2 dx^2 < \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} \int_{\mathbb{R}^2} |\partial_y A_k|^2 dx^2 < \infty$$

follow. In order to establish the third relation of (i), we use assertion (iii) of Proposition 1.1 replacing therein φ by f and setting $u = a\varphi_k$ $(k \geq 1)$ with $a \in C_0^\infty(\mathbb{R}^2)$ arbitrary but fixed; we note $u \in \widehat{H}^2(\Omega)$. From assertion (iii) of Proposition 1.1 and via the Fubini theorem we find

$$\int_{\mathbb{R}^2} dx^2 \cdot a \int_{-1/2}^{+1/2} \left(\frac{\partial f}{\partial z}\psi_k - \Lambda_k^{1/2} f\varphi_k \right) dz = 0 \qquad (k \geq 1).$$

By the arbitrariness of a we obtain

$$\int_{-1/2}^{+1/2} \frac{\partial f}{\partial z}\psi_k \, dz = \Lambda_k^{1/2} \int_{-1/2}^{+1/2} f\varphi_k \, dz = \Lambda_k^{1/2} A_k$$

a.e. on $\mathbb{R}^2$, from which the third relation on the right-hand side of assertion (i) follows.

The proof of assertion (ii) is essentially the same. In order to verify the third relation in the right-hand side of assertion (ii) under the assumption $f \in H_0^1(\Omega)$ we use Proposition 1.1/(ii) by setting $\varphi = a\varphi_k$ therein, with $a \in C_0^\infty(\mathbb{R}^2)$ arbitrary but fixed. By reasoning similar to the above one then finds

$$\Lambda_k^{1/2} \int_{-1/2}^{+1/2} f\psi_k \, dz = -\int_{-1/2}^{+1/2} \frac{\partial f}{\partial z}\varphi_k \, dz = \Lambda_k^{1/2} B_k$$

a.e. on $\mathbb{R}^2$, from which the required relation follows ∎

Remarks. Relations (1.17) in the above proof show that ∂_x, ∂_y and ∂_z commute with $\sum$, symbolically $\partial \sum = \sum \partial$. The proof of this fact applies verbatim to the case $f \in H_0^1(\Omega)$, $f = \sum_j B_j \psi_j$. As to higher Sobolev spaces all we need is

Proposition 1.3. *Let $f \in \mathcal{L}^2(\Omega)$, $f = \sum_k A_k \varphi_k$ and $A_k \in H^2(\mathbb{R}^2)$ $(k \geq 0)$. Assume that the expressions in Proposition 1.2/(i) and also all the expressions*

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^2} |\partial^2 A_k|^2 dx^2, \qquad \sum_{k=0}^{\infty} \Lambda_k \int_{\mathbb{R}^2} |\partial A_k|^2 dx^2, \qquad \sum_{k=0}^{\infty} \Lambda_k^2 \int_{\mathbb{R}^2} |A_k|^2 dx^2$$

are finite, where $\partial \in \{\partial_x, \partial_y\}$ and $\partial^2 \in \{\partial_x^2, \partial_{xy}^2, \partial_y^2\}$. Then $f \in H^2(\Omega)$ and $Df = \sum_k D(A_k \varphi_k)$ with D any derivative of order lesser or equal 2 in x, y and z. Likewise with $f = \sum_j B_j \psi_j$, $B_j \in H^2(\mathbb{R}^2)$ $(j \geq 1)$.

The straightforward proof, in which one recognizes $L_N = \sum_{k=0}^{N} A_k \varphi_k$ $(N \geq 0)$ as a Cauchy sequence in $H^2(\Omega)$, is omitted.

1.4 The Stokes operator. For the following, it helps to bring parity with respect to z into play: for $f \in \mathcal{L}^2(\Omega)$, $f \in \mathcal{L}_g^2(\Omega)$ if and only if f is even in z, and $f \in \mathcal{L}_u^2(\Omega)$ if and only if f is odd in z. Next set $L^2 = (\mathcal{L}^2)^3$, $L_g^2 = (\mathcal{L}_g^2(\Omega))^2 \times \mathcal{L}_u^2(\Omega)$ and $L_u^2 = (\mathcal{L}_u^2(\Omega))^2 \times \mathcal{L}_g^2(\Omega)$. Clearly $L^2 = L_g^2 \oplus L_u^2$. The scalar product on L^2 is given by $\langle u, v \rangle = \sum_{j=1}^{3} (u_j, v_j)_0$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Next we need the space E of divergence-free vector fields:

$$E \quad \text{is the } \mathcal{L}^2\text{-closure of all } f = (u, v, w) \in (H_0^1(\Omega))^3 \text{ such that } \operatorname{div} f = 0.$$

Evidently $E = E_g \oplus E_u$, where E_g is the $\mathcal{L}^2$-closure of all $f \in (H_0^1(\Omega))^3 \cap L_g^2$ with $\operatorname{div} f = 0$, and likewise with E_u. The orthogonal projections onto E, E_g and E_u are P, P_g and P_u, respectively. Next let Δ_d be the Laplacian on $\operatorname{dom}\Delta_d = H^2(\Omega) \cap H_0^1(\Omega)$. Then Δ_d is selfadjoint and $\Delta_d \leq -\varepsilon$ for some $\varepsilon > 0$. Moreover, it is easily seen that $L_g^2(\Omega)$ and $L_u^2(\Omega)$ reduce Δ_d, i.e. they are invariant under Δ_d. We also recall Gauss' formula

$$\int_{\Omega} (\nabla v \nabla u + v \Delta u)\, dx^3 = 0 \qquad (v \in H_0^1(\Omega),\ u \in H^2(\Omega))$$

(see, e.g., [6]). Δ_d induces a selfadjoint operator A on $\operatorname{dom} A = (H^2(\Omega) \cap H_0^1(\Omega))^3$ according to $A(u, v, w) = (\Delta u, \Delta v, \Delta w)$. The Stokes operator A_0 is now given as follows:

$$f \in \operatorname{dom} A_0 \quad \Longleftrightarrow \quad f \in \operatorname{dom} A,\ \operatorname{div} f = 0 \text{ and } \ A_0 f = PAf \ \text{ in this case.}$$

A_0 has some simple properties, provided by

Proposition 1.4. *A_0 is symmetric, densely defined in E and $A_0 \leq -\varepsilon$ for some $\varepsilon > 0$. The spaces E_g and E_u reduce A_0, i.e. $P_g A_0 \subseteq A_0 P_g$ and $P_u A_0 \subseteq A_0 P_u$.*

The straightforward proof is omitted.

In order to recognize A_0 as selfadjoint one introduces

Definition 1.1. $u \in (H_0^1(\Omega))^3$ with $\operatorname{div} u = 0$ is a weak solution of $A_0 u = f$ $(f \in E)$ if and only if

$$\sum_j \langle \nabla u_j, \nabla v_j \rangle + \langle f, v \rangle = 0 \quad \text{for all} \quad v \in (H_0^1(\Omega))^3$$

$$\operatorname{div} v = 0 \ \big(\text{with } u = (u_1, u_2, u_3) \ \text{ and } \ v = (v_1, v_2, v_3)\big).$$

It is readily established that given $f \in E$, there is at most one weak solution of equation $A_0 u = f$. By Proposition 1.4 and Lax-Milgram theory there is a selfadjoint (Friedrich's) extension $A_s \supseteq A_0$ given as follows:

$$u \in \operatorname{dom} A_s \iff u \text{ is the weak solution of}$$
$$A_0 u = f \ \text{ for some } \ f \in E \ \text{ and } \ A_s u = f \ \text{ in this case.}$$

Proposition 1.5. *$A_s \leq -\varepsilon$ with ε as in Proposition 1.4, and E_g, E_u reduce A_s, i.e. $P_u A_s \subseteq A_s P_u$ and $P_g A_s \subseteq A_s P_g$. If $u \in \operatorname{dom} A_0$ and $A_s u = f$, then $A_0 u = f$.*

Proof. The first part follows straightforwardly from the definitions. As to the second part, let $u \in \operatorname{dom} A_0$ satisfy $A_s u = f$, i.e. $\sum_j \langle \nabla u_j, \nabla v_j \rangle + \langle f, v \rangle = 0$ for all $v \in (H_0^1(\Omega))^3$ with $\operatorname{div} v = 0$. From Gauss' formula we then infer $\langle Au - f, v \rangle = 0$ for all v in a dense subset of E whence $PAu - Pf = 0$, i.e. $A_0 u = f$ since $Pf = f$ ∎

The main result about A_0 and A_s is the following

Theorem 1.1. *There exists $C > 0$ such that, for all $f \in E$ and $u \in \operatorname{dom} A_s$ satisfying $A_s u = f$, $u \in (H^2(\Omega))^3$ and $\|u\|_{H^2} \leq C \|f\|_{\mathcal{L}^2}$.*

In conjunction with Proposition 1.5 we get

Corollary. *$A_0 = A_s$, i.e. A_0 is selfadjoint.*

For reasons of space we cannot go into the proof of Theorem 1; the details for a comparable situation are in [14].

The main consequence from Theorem 1.1 concerns the operator T_0 given formally by (1.3) and now supplied by the setting $\operatorname{dom} T_0 = (H_0^1(\Omega))^3$.

Corollary. *Given $\varepsilon > 0$, there is a positive constant K_ε such that*

$$\|T_0 u\|_{\mathcal{L}^2} \leq \varepsilon \|A_s u\|_{\mathcal{L}^2} + K_\varepsilon \|u\|_{\mathcal{L}^2} \tag{$*$}$$

for all $u \in \operatorname{dom} A_s$.

Proof. We recall the operator A on $(\mathcal{L}^2(\Omega))^3$ such that $(Au)_j = \Delta u_j$ for $u \in \operatorname{dom} A = (H^2(\Omega) \cap H_0^1(\Omega))^3$. It is known that given $\varepsilon > 0$ there is a K_ε such that

$$\|(-A)^{1/2} u\|_{\mathcal{L}^2} \leq \varepsilon \|Au\|_{\mathcal{L}^2} + K_\varepsilon \|u\|_{\mathcal{L}^2} \qquad (u \in \operatorname{dom} A)$$

$$\|(-A)^{1/2} u\|_{\mathcal{L}^2}^2 = \sum_{j=1}^{3} \langle \nabla u_j, \nabla u_j \rangle$$

for $u \in \operatorname{dom}(-A)^{1/2} \equiv (H_0^1(\Omega))^3$. Moreover there is a positive constant C' such that

$$\|Au\|_{\mathcal{L}^2} \leq C'\|u\|_{H^2} \qquad (u \in \operatorname{dom} A).$$

On the other hand

$$\|u\|_{H^2} \leq C\|A_s u\|_{\mathcal{L}^2} \qquad (u \in \operatorname{dom} A_s)$$

by Theorem 1.1. Finally we have

$$\|T_0 u\|_{\mathcal{L}^2} \leq C''\left(\left(\sum_{j=1}^{3}\langle \nabla u_j, \nabla u_j\rangle\right)^{1/2} + \|u\|_{\mathcal{L}^2}\right)$$

for $u \in (H_0^1(\Omega))^3$, with a positive constant C'' depending only on the equilibrium solution u_0 which enters the definition of T_0 (see Subsection 1.1). The corollary now follows ∎

Remarks. By the Corollary $A_s + PT_0$ is the generator of a holomorphic semigroup on E (see Pazy [7]). This enables us to introduce fractional powers and to handle the nonlinearity in (1.6) in such a way that (1.6) becomes a nonlinear evolution equation in the sense of [7: Subsection 6.3]. These steps, necessary to establish the principles of linearized stability/instability are not needed here and not further discussed.

1.5 The projection operator. In order to discuss the projection operator P onto the subspace E (see (1.4)) we need the following

Lemma 1.1. *E is the $\mathcal{L}^2$-closure of the vector fields $f \in (H^1(\Omega))^2 \times H_0^1(\Omega)$, $\operatorname{div} f = 0$.*

By taking into account the results on Fourier expansions one recognizes Lemma 1.1 as consequence of

Proposition 1.6. *The following assertions are true:*

(i) *Fix $k \geq 1$, let $A, B \in H^1(\mathbb{R}^2)$ and $C = \Lambda_k^{-1/2}(\partial_x A + \partial_y B)$. Then $(A\varphi_k, B\varphi_k, C\psi_k) \in E$.*

(ii) *Let $A, B \in H^1(\mathbb{R}^2)$ satisfy $\partial_x A + \partial_y B = 0$. Then $(A\varphi_0, B\varphi_0, 0) \in E$.*

Proof. Since $k \geq 1$ we have $\int_{-1/2}^{+1/2} \varphi_k\, ds = 0$. Hence there is a sequence $\Phi_n \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$ such that $\lim_n \Phi_n = \varphi_k$ in $\mathcal{L}^2(-\frac{1}{2}, +\frac{1}{2})$ and $\int_{-1/2}^{+1/2} \Phi_n\, ds = 0$. Observe that $\int_{-1/2}^{z} \Phi_n\, ds \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$. Next let $A_n, B_n \in C_0^\infty(\mathbb{R}^2)$ be such that $\lim_n A_n = A$ and $\lim_n B_n = B$ in $H^1(\mathbb{R}^2)$. Recalling the convention in 'Notations' we set

$$f_n = \left(A_n\Phi_n, B_n\Phi_n, -(\partial_x A_n + \partial_y B_n)\int_{-1/2}^{z} \Phi_n\, ds\right)$$

whence $f_n \in (H_0^1(\Omega))^3$, $\operatorname{div} f = 0$ by construction. By (1.15) we have

$$\lim_n \int_{-1/2}^{z} \Phi_n\, ds = \int_{-1/2}^{z} \phi_k\, ds = -\Lambda_k^{-1/2}\psi_k(z) \qquad \text{in } \mathcal{L}^2\left(-\frac{1}{2}, +\frac{1}{2}\right).$$

By these remarks it follows that

$$\lim_n f_n = (A\varphi_k, B\varphi_k, C\psi_k) \qquad \text{in the } \mathcal{L}^2 - \text{sense}$$

proving assertion (i). Next let $A, B \in H^1(\mathbb{R}^2)$ satisfy $\partial_x A + \partial_y B = 0$, let $\Phi_n \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$ $(n \geq 1)$ be even and such that $\lim_n \Phi_n = \varphi_0 = 1$ in $\mathcal{L}^2(-\frac{1}{2}, +\frac{1}{2})$. Let $f_n = (A\Phi_n, B\Phi_n, 0)$. Clearly $f_n \in (H_0^1(\Omega))^3$, $\operatorname{div} f = 0$ and $\lim_n f_n = (A\varphi_0, B\varphi_0, 0)$ in the $\mathcal{L}^2$-sense, whence $(A\varphi_0, B\varphi_0, 0) \in E$, proving assertion (ii) ∎

Remarks. It is clear that if $k \geq 1$ is even or odd, then the sequence $\{f_n\}_{n \geq 0}$ can be chosen in L_g^2 or L_u^2. Lemma 1 respectively Proposition 6 enable us to reduce the investigation of the projectors P and $Q = 1 - P$ to straightforward manipulations with Fourier series and transforms. We will therefore be brief. In order to investigate $E_\perp = L^2 \ominus E$ we note

Proposition 1.7. *If $p \in H^1(\Omega)$, then $\nabla p \perp E$.*

Proof. By Lemma 1.1, Proposition 1.6 and Subsection 1.3 it suffices to show that if $p = p_k\varphi_k$ with $p_k \in H^1(\mathbb{R}^2)$, then ∇p is orthogonal to all fields of the form $(A\varphi_k, B\varphi_k, C\psi_k)$ with $A, B \in H^1(\mathbb{R}^2)$ and $C = \Lambda_k^{1/2}(\partial_x A + \partial_y B)$ if $k \geq 1$, and to all fields of the form $(A\varphi_0, B\varphi_0, 0)$ with $A, B \in H^1(\mathbb{R}^2)$ and $\partial_x A + \partial_y B = 0$ if $k = 0$. Both cases follow immediately if we express the arising scalar products and the assumptions in terms of the Fourier transforms $\hat{A}, \ldots, \hat{p}_k$ of $A, \ldots, p_k$, a computational step which we omit ∎

Next we invoke the Neumann operator $\widetilde{A}$ such that $\widetilde{A} = \Delta$ on $\operatorname{dom} \widetilde{A} = \widehat{H}^2(\Omega)$ (see (1.12)). It is known that $\widetilde{A}$ is selfadjoint and lesser or equal 0. We note also the validity of Gauss' formula (1.9) now under the assumption $v \in H^1(\Omega)$ and $u \in \widehat{H}^2(\Omega)$. Moreover it is easily seen that the subspace

$$\widehat{\mathcal{L}}^2(\Omega) = \left\{ f \in \mathcal{L}^2(\Omega) \,\middle|\, \int_{-1/2}^{+1/2} f\varphi_0 \, ds = 0 \ \text{ a.e. on } \ \mathbb{R}^2 \right\}$$

is invariant under $\widetilde{A}$, i.e. reduces $\widetilde{A}$. An important property of $\widetilde{A}$ is given by

Proposition 1.8. *$\widetilde{A}$ is boundedly invertible on $\widehat{\mathcal{L}}^2(\Omega)$.*

Proof. Let $f = \sum_{k=1}^\infty f_k\varphi_k$ in $\widehat{\mathcal{L}}^2(\Omega)$ be given. The unique solution $p \in \widehat{H}^2(\Omega) \cap \widehat{\mathcal{L}}^2(\Omega)$ of equation $\Delta p = f$ is then given by $\sum_{k=1}^\infty p_k\varphi_k = p$ where

$$\widehat{p}_k(\alpha, \beta) = \mu_k(\alpha, \beta)^{-1} \widehat{f}_k(\alpha, \beta) \qquad (k \geq 1)$$

where $\alpha, \beta \in \mathbb{R}$, $\mu_k = -(\alpha^2 + \beta^2 + \Lambda_k)$ and $\widehat{f}_k$ and $\widehat{p}_k$ are the Fourier transforms of f_k and p_k, respectively ∎

For use below it is convenient to introduce

$$\text{the space } \ \widehat{L}^2 = (\widehat{\mathcal{L}}^2(\Omega))^2 \times \mathcal{L}^2(\Omega)$$
$$\text{the set } \ \widetilde{H} = \left(H^1(\Omega) \cap \widehat{\mathcal{L}}^2(\Omega)\right)^2 \times H_0^1(\Omega).$$

We note that if $p \in \widehat{H}^2(\Omega) \cap \widehat{\mathcal{L}}^2(\Omega)$ and $f \in \widetilde{H}$, then $\nabla p \in \widetilde{H}$ and $\operatorname{div} f \in \widehat{\mathcal{L}}^2(\Omega)$ (Proposition 1.1). The space $\widehat{L}^2 \subseteq L^2$ turns out to be invariant under P and $Q = 1 - P$. In order to study the action of Q on $\widehat{L}^2$ we pick $f \in \widetilde{H}$ and let $p \in \widehat{H}^2(\Omega) \cap \widehat{\mathcal{L}}^2(\Omega)$ be the solution of equation $\Delta p = \operatorname{div} f$ (Proposition 1.8). Since $\nabla p \in \widetilde{H}$ by the above remark and $\operatorname{div}(f - \nabla p) = 0$, we have that $f - \nabla p \in E$ (Lemma 1.1) and $\nabla p \perp E$ (Proposition 1.7). We thus conclude

$$Qf = \nabla p \qquad \text{and} \qquad Pf = f - \nabla p.$$

The action of the projection operators Q and P on $\widehat{L}^2$ is thus determined by their action on the dense subset $\widetilde{H} \subseteq \widehat{L}^2$ according to the above description; the action on all of $\widehat{L}^2$ then follows via approximation. It remains to determine Q and P on the complement $L^2 \ominus \widehat{L}^2$. To this end we not that an $f \in L^2 \ominus \widehat{L}^2$ has necessarily the form

$$f = (A\varphi_0, B\varphi_0, 0) \qquad \big(A, B \in \mathcal{L}^2(\mathbb{R}^2),\ \varphi_0 = 1\big). \tag{1.18}$$

Now assume first that the Fourier transforms $\widehat{A}$ and $\widehat{B}$ are smooth, with compact supports in $\mathbb{R}^2 \setminus \{0\}$, whence $f \in (H^1(\Omega))^2 \times H_0^1(\Omega)$. Setting $p = p_0\varphi_0$ with

$$\widehat{p}_0 = -(i\alpha\widehat{A} + i\beta\widehat{B})\nu^{-1} \qquad (\nu = \alpha^2 + \beta^2)$$

we have that $p \in H^2(\Omega)$, $\nabla p \in (H^1(\Omega))^2 \times H_0^1(\Omega)$ and $\Delta p = \operatorname{div} f$. As in the previous case we conclude

$$Qf = \nabla p \qquad \text{and} \qquad Pf = f - \nabla p.$$

By a straightforward approximation argument one shows that in the general case of an f of the form (1.17) we have $Qf = (C\varphi_0, D\varphi_0, 0)$ where

$$\widehat{C} = \alpha(\alpha\widehat{A} + \beta\widehat{B})\nu^{-1} \qquad \text{and} \qquad \widehat{D} = \beta(\alpha\widehat{A} + \beta\widehat{B})\nu^{-1}, \tag{1.19}$$

what completes the description of Q on $L^2 \ominus \widehat{L}^2$ and hence on $L^2 = (\mathcal{L}^2(\Omega))^3$. From formulas (1.19) one extracts examples of fields $f \perp E$ which have not the form ∇p, for some $p \in H^1(\Omega)$. On the other hand it is clear from the above that the set $\{\nabla p\}_{p \in H^1(\Omega)}$ is dense in $E_\perp$. From the above analysis one also easily deduces that the spaces L_g^2 and L_u^2 are invariant under Q and P.

This concludes the discussion of the $\mathcal{L}^2$-setting of Navier-Stokes on an infinite plate. It concerns merely the linear part, the only one which is relevant here, but which is also essential for the handling of the nonlinear part, a point to be discussed elsewhere.

2. The Θ-periodic counterparts

2.1 Θ-periodic functions. In order to discuss the relation between $\mathcal{L}^2$-spectrum and periodic spectrum (see (1.8)) one is forced to proceed via an extension of periodicity, i.e. we need the concept of Θ-periodic functions. Let $L > 0$ as in (0.2) be fixed. Set $Q_L = (0, L)^2$ and $Q = Q_L \times (-\frac{1}{2}, +\frac{1}{2})$. Set, for a fixed small $\varepsilon > 0$, $M_\varepsilon = (-\varepsilon, 2\pi + \varepsilon)^2$ and let $\dot{M}_\varepsilon$ be M_ε minus the four points $(0,0)$, $(0, 2\pi)$, $(2\pi, 0)$ and $(2\pi, 2\pi)$. Let finally $\dot{M} = \dot{M}_\varepsilon \cap [0, 2\pi]^2$. By $\Theta = (\Theta_1, \Theta_2)$ we denote a typical point in M_ε, calling Θ "generic" if $\Theta \in \dot{M}_\varepsilon$. As before we set $\Omega = \mathbb{R}^2 \times (-\frac{1}{2}, +\frac{1}{2})$ and $\overline{\Omega} = \mathbb{R}^2 \times [-\frac{1}{2}, +\frac{1}{2}]$.

Next we need spaces. By $C^p_\Theta(Q)$ we denote the set of $f \in C^p_0(\overline{\Omega})$ such that

$$f(x + jL, y + kL, z) = e^{i(\Theta_1 j + \Theta_2 k)} f(x, y, z) \tag{2.1}$$

for $j, k \in \mathbb{Z}$, where $H^p_\Theta(Q)$ now denotes the set of $f \in \mathcal{L}^2(Q)$ such that $\lim_n \|f_n - f\|_{H^p} = 0$ holds for some sequence $f_n \in C^p_\Theta(Q)$. By (2.1), f admits a unique extension $\widetilde{f} \in H^p_{\mathrm{loc}}(\Omega)$ such that $\lim_n \|\widetilde{f} - f_n\|_{H^p} = 0$ holds on any bounded subdomain $\Omega' \subseteq \Omega$ and satisfying (2.1) in the a.e. sense; we identify f with $\widetilde{f}$ henceforth. Clearly $H^0_\Theta(Q) = \mathcal{L}^2(Q)$. The spaces $H^p(Q)$ are the usual Sobolev spaces on Q; for simplicity we denote the scalar product on $H^p(Q)$ again by $(\cdot,\cdot)_p$. It is also convenient to introduce spaces $C^1_{\Theta,0}$ and $\widehat{C}^2_\Theta$ as follows. Let $f \in C^1_{\Theta,0}$ if and only if $f \in C^1_\Theta(Q)$ and $f(x, y, \pm\frac{1}{2}) = 0$ for all $x, y \in \mathbb{R}$. Further, let $f \in \widehat{C}^2_\Theta(Q)$ if and only if $f \in C^2_\Theta(Q)$ and $\partial_z f(x, y, \pm\frac{1}{2}) = 0$ for all $x, y \in \mathbb{R}$.

In order to handle boundary conditions we use again the notion of trace. To this end, let $\partial' Q$ be ∂Q minus the edges, more precisely $\partial' Q = \partial Q \setminus \cup \overline{pq}$ where $\overline{pq}$ runs through the closed edges connecting adjacent corners p and q of $\overline{Q}$. We also set $\Gamma'_\pm = \Gamma_\pm \cap \partial' Q$, $\Gamma' = \Gamma \cap \partial' Q = \Gamma'_+ \cup \Gamma'_-$ with Γ and $\Gamma_\pm$ as in Subsection 1.2.

Similarly one introduces boundary operators $\gamma_j \in L(H^{j+1}(Q), \mathcal{L}^2(\partial' Q))$ $(j = 0, 1)$ which satisfy (a) and (b) in Subsection 1.2 (with Q in place of Ω). Gauss formula (1.9) remains valid for $v \in H^1(Q)$ and $u \in (H^1(Q))^3$. In case where $v \in H^1_\Theta(Q)$ and $u = (u_1, u_2, u_3) \in (H^1_\Theta(Q))^3$ it assumes the form

$$\int_{\Gamma'_+} \gamma_0(\overline{v})\gamma_0(u_3)\, d\omega - \int_{\Gamma'_-} \gamma_0(\overline{v})\gamma_0(u_3)\, d\omega = \int_Q \big((\nabla \overline{v})u + \overline{v}\Delta u\big)\, dx^3. \tag{2.2}$$

The other boundary terms due to $\Omega \cap \partial' \Omega$ cancel since u is Θ-periodic while the complex conjugate $\overline{v}$ is $(-\Theta)$-periodic. We now define $H^1_{\Theta,0}(Q)$ and $\widehat{H}^2_\Theta(Q)$ according to

$$\begin{aligned} f \in H^1_{\Theta,0}(Q) \quad &\Longleftrightarrow \quad f \in H^1_\Theta(Q) \text{ and } \gamma_0(f) = 0 \text{ a.e. on } \Gamma' \\ f \in \widehat{H}^2_\Theta(Q) \quad &\Longleftrightarrow \quad f \in H^2_\Theta(Q) \text{ and } \gamma_1(f) = 0 \text{ a.e. on } \Gamma'. \end{aligned} \tag{2.3}$$

The periodic case arises for $\Theta = (0,0), (0, 2\pi), (2\pi, 0)$ and $(2\pi, 2\pi)$. We emphasize it by writing $H^1_{\mathrm{per}}(Q)$, $H^1_{\mathrm{per},0}(Q)$, ... instead of $H^1_0(Q)$, $H^1_{0,0}(Q)$, ..., respectively; likewise with $C^p_{\text{per}}(Q)$, $C^1_{\text{per},0}(Q)$, ... There is a simple connection between the periodic and the Θ-periodic case, expressed by the proposition below, in which we set

$$m(\Theta,\vec{x}) = e^{-i(\Theta_1 x+\Theta_2 y)L^{-1}} \qquad (\vec{x}=(x,y)) \tag{2.4}$$

Proposition 2.1. *The following assertions are true.*

(i) $f \in H^p_\Theta(Q)$ *if and only if* $m(\Theta,\vec{x})f \in H^p_{\text{per}}(Q)$

(ii) $f \in H^1_{\Theta,0}$ *if and only if* $m(\Theta,\vec{x})f \in H^1_{\text{per},0}(Q)$

(iii) *Likewise with* $\widehat{H}^2_\Theta(Q)$, $C^1_{\Theta,0}(Q)$ *etc.*

We omit the straightforward proof. Proposition 2.1 permits us to reduce statements on Θ-periodic functions to known statements on periodic functions. For simplicity we write henceforth H^2_Θ, $H^1_{\Theta,0}$, ... instead of $H^2_\Theta(Q)$, $H^1_{\Theta,0}(Q)$, ...

2.2 Fourier series. Fourier series are again the most useful tool in connection with pressure, divergence-free fields etc. In order to handle them in an economic way it is advisable to use some space-saving notations. We fix some $\Theta = (\Theta_1,\Theta_2) \in M_\varepsilon$ and let α,β,j,k,l range over $\mathbb{Z}$. We then set

$$\widehat{\alpha} = L^{-1}(2\pi\alpha+\Theta_1) \qquad\qquad e_{\alpha\beta} = L^{-1}e^{i\widehat{\alpha}x+i\widehat{\beta}y}$$
$$\text{and} \tag{2.5}$$
$$\widehat{\beta} = L^{-1}(2\pi\beta+\Theta_2) \qquad\qquad \widetilde{e}_{\alpha\beta} = L^{-1}e^{i\alpha x+i\beta y}.$$

Here and below we suppress the dependence on Θ or x and y if no ambiguity arises; in cases where this is not so we write more explicitely $e_{\alpha\beta}(\Theta)$ and $e_{\alpha\beta}(\Theta,\vec{x})$ or similar. We also let $\mathcal{L}^2_g(Q)$ and $\mathcal{L}^2_u(Q)$ be the subspaces of those f in $\mathcal{L}^2(Q)$ which are even and odd, respectively, in z. Finally we recall the eigenfunctions $\phi_k,\psi_k,\sigma_k,\tau_k,\ldots$ of (1.16). Complete orthonormal systems in $\mathcal{L}^2(Q)$ are given by $\{e_{\alpha\beta}\varphi_l\}_{l\geq 0}$ and $\{e_{\alpha\beta}\psi_l\}_{l\geq 1}$ and, similarly, by $\{e_{\alpha\beta}\sigma_l\}_{l\geq 0}$ and $\{e_{\alpha\beta}\tau_l\}_{l\geq 0}$ in $\mathcal{L}^2_u$ and $\mathcal{L}^2_g$, respectively. An $f \in \mathcal{L}^2(Q)$ thus admits the Fourier series

$$f = \sum_{l\geq 0} f_{\alpha\beta l}e_{\alpha\beta}\varphi_l = \sum_{l\geq 1} \widetilde{f}_{\alpha\beta l}e_{\alpha\beta}\psi_l \tag{2.6}$$

where $f_{\alpha\beta l} = (f, e_{\alpha\beta}\varphi_l)_0$ and $\widetilde{f}_{\alpha\beta l} = (f, e_{\alpha\beta}\psi_l)_0$. For use below and later we set

$$\widehat{\Lambda}_j(\alpha,\beta) = \begin{cases} \widehat{\alpha}^2+\widehat{\beta}^2+1 & \text{for } j=0 \\ \widehat{\alpha}^2+\widehat{\beta}^2+\Lambda_j & \text{for } j\geq 1. \end{cases} \tag{2.7}$$

Proposition 2.2. *Let* $f \in \mathcal{L}^2(Q)$. *Then the following assertions are true.*

(i) $f \in H^1_\Theta$ *if and only if* $\sum_j \widehat{\Lambda}_j(\alpha,\beta)|f_{\alpha\beta j}|^2 < \infty$

(ii) $f \in H^1_{\Theta,0}$ *if and only if* $\sum_j \widehat{\Lambda}_j(\alpha,\beta)|\widetilde{f}_{\alpha\beta j}|^2 < \infty$.

The proof proceeds along similar lines as that of Proposition 1.2.

Proposition 2.3. *Let $f \in \mathcal{L}^2(Q)$. Assume that $\sum_{j\ge 0} \widehat{\Lambda}_j(\alpha,\beta)^2 |f_{\alpha\beta j}|^2 < \infty$. Then $f \in H^2_\Theta$ and*

$$\|f\|^2_{H^2} \le C \left(\sum_{j\ge 0} \widehat{\Lambda}_j(\alpha,\beta)^2 |f_{\alpha\beta j}|^2 \right) \qquad (\alpha,\beta \in \mathbb{Z})$$

for some constant $C > 0$ independent of $\Theta \in M_\varepsilon$. Likewise with $\sum_{j\ge 1} \widehat{\Lambda}_j(\alpha,\beta)|\widetilde{f}_{\alpha\beta j}|^2$.

Proof (sketch). Let $L_N = \sum_j f_{\alpha\beta j} e_{\alpha\beta} \varphi_j \quad (|\alpha|,|\beta|, j \le N)$. By virtue of the assumption, $\{L_N\}_{N\ge 1}$ then is a Cauchy sequence in H^2, whence $f \in H^2_\Theta$. This implies that summation $\sum$ and differentiation d commute for any derivative d of order lesser or equal 2. We thus can express $\|f\|^2_{H^2}$ in terms of Fourier series. The statement then follows by observing that there is a positive constant C independent of $\alpha,\beta \in \mathbb{Z}$, $j \ge 0$ and $\Theta \in M_\varepsilon$ such that $\widehat{\Lambda}_j(\alpha,\beta) + \left(|\widehat{\alpha}| + |\widehat{\beta}| + \Lambda_j^{1/2}\right)^4 \le C\,\widehat{\Lambda}_j(\alpha,\beta)^2$ ∎

2.3 The Stokes operator. Next we come to the Θ-periodic version of the Stokes operator. For simplicity we write $\mathcal{L}^2, \mathcal{L}^2_g, \mathcal{L}^2_u$ instead of $\mathcal{L}^2(Q), \ldots$. We then set $L^2 = L^2(Q) = (\mathcal{L}^2)^3$, $L^2_g = (\mathcal{L}^2_g)^2 \times \mathcal{L}^2_u$ and $L^2_u = (\mathcal{L}^2_u)^2 \times \mathcal{L}^2_g$. We recall the notations $\|u\|_{\mathcal{L}^2}$, $\langle u, v\rangle$ and $\|u\|_{H^p}$, $\langle u,v\rangle_p$ for vector fields $u,v \in L^2$ and $u,v \in (H^p_\Theta)^3$, respectively. For $\Theta \in M_\varepsilon$ we define

$$E_\Theta \quad \text{is the } \mathcal{L}^2\text{-closure of all } f \in (H^1_{\Theta,0})^3 \text{ with } \operatorname{div} f = 0. \tag{2.8}$$

Clearly $E_\Theta = E^g_\Theta \oplus E^u_\Theta$; here E^g_Θ is the $\mathcal{L}^2$-closure of all $f \in (H^1_{\Theta,0})^3 \cap L^2_g$ such that $\operatorname{div} f = 0$, and likewise with E^g_Θ. We also let P_Θ, P^g_Θ and P^u_Θ be the orthogonal projections onto E_Θ, E^g_Θ and E^u_Θ, respectively. Next we let Δ_Θ be the Laplacian Δ on $\operatorname{dom}\Delta_\Theta = H^2_\Theta \cap H^1_{\Theta,0}$. The operator Δ_Θ so defined is selfadjoint and $\Delta_\Theta \le -\varepsilon_0$ for some ε_0 independent of $\Theta \in M_\varepsilon$; the subspaces $\mathcal{L}^2_g$ and $\mathcal{L}^2_u$ are easily recognized as invariant under Δ_Θ. From (2.2) one infers

$$\int_Q (\nabla v \nabla \overline{u} + v \Delta \overline{u})\, dx^3 = 0 \qquad (v \in H^1_{\Theta,0},\ u \in H^2_\Theta). \tag{2.9}$$

Δ_Θ induces a selfadjoint operator A_Θ on L^2 according to $\operatorname{dom} A_\Theta = \operatorname{dom}\Delta^3_\Theta$ and $(A_\Theta u)_j = \Delta_\Theta u_j$ for $u = (u_1,u_2,u_3) \in \operatorname{dom} A_\Theta$. The Stokes operator $A_0(\Theta)$ is then defined as follows:

$$\begin{aligned} &f \in \operatorname{dom} A_0(\Theta) \iff f \in \operatorname{dom} A_\Theta \text{ and } \operatorname{div} f = 0 \\ &\text{and } A_0(\Theta) f = P_\Theta A_\Theta f \text{ in this case.} \end{aligned} \tag{2.10}$$

$A_0(\Theta)$ has the following four simple properties. The proof is easy and therefore omitted.

Proposition 2.4. *$A_0(\Theta)$ is symmetric, densely defined (in E_Θ) and $A_0(\Theta) \le -\varepsilon_1$ for some Θ-independent $\varepsilon_1 > 0$. The spaces E^g_Θ and E^u_Θ reduce $A_0(\Theta)$, i.e. $P^s_\Theta A_0(\Theta) \subseteq A_0(\Theta) P^s_\Theta$ for $s \in \{g,u\}$.*

Of importance is

Definition 2.1. $u = (u_1, u_2, u_3) \in (H^1_{\Theta,0})^3$ with $\operatorname{div} u = 0$ is called a *weak solution* of equation $\Delta u = f$, for $f \in E_\Theta$, if

$$\sum_{j=1}^{3} \langle \nabla u_j, \nabla v_j \rangle + \langle f, v \rangle = 0 \quad \text{for all } v \in (H^1_{\Theta,0})^3 \text{ with } \operatorname{div} v = 0$$

where $v = (v_1, v_2, v_3)$.

Given $f \in E_\Theta$ there is at most one weak solution u of equation $\Delta u = f$. By Proposition 2.4 and Lax-Milgram theory there is a selfadjoint extension $A_s(\Theta) \supseteq A_0(\Theta)$ such that

$$u \in \operatorname{dom} A_s(\Theta) \iff u \text{ is the weak solution of } \Delta u = f$$
$$\text{for some } f \in E_\Theta \text{ and } A_s(\Theta)u = f \text{ in this case.}$$

From Proposition 2.4 and clause (2.9) we infer

Proposition 2.5. *$A_s(\Theta) \leq -\varepsilon_1$ with ε_1 as in Proposition 2.4 and E^g_Θ, E^u_Θ reduce $A_s(\Theta)$. If $u \in \operatorname{dom} A_0(\Theta)$ and $A_s(\Theta)u = f$, then $A_0(\Theta)u = f$.*

We now come to the Θ-periodic counterpart of Theorem 1.1. In its simplest form it states that $\operatorname{dom} A_s(\Theta) \subseteq (H^2_\Theta)^3$ what according to Proposition 2.5 yields $A_s(\Theta) = A_0(\Theta)$. However more is required for the needs of Section 4. In fact a more refined version of Theorem 1.1 is available. It is stated below in two parts without proof; a proof is given in [14].

First we fix some notations. With $\mathbb{C}$ the set of complex numbers we recall $\varepsilon > 0$ in $M_\varepsilon = (-\varepsilon, 2\pi + \varepsilon)^2$. We then set

$$\mathcal{M}_\varepsilon = \Big\{\Theta \in \mathbb{C}^2 : \operatorname{dist}(\Theta, [0, 2\pi]^2) < \varepsilon\Big\}.$$

We also set

$$S = \Big\{a_{\alpha\beta j} : \alpha, \beta \in \mathbb{Z}, \quad j \geq 0, \quad \sum |a_{\alpha\beta j}|^2 < \infty\Big\}$$

$$S' = \Big\{a_{\alpha\beta j} : \alpha, \beta \in \mathbb{Z}, \quad j \geq 1, \quad \sum |a_{\alpha\beta j}|^2 < \infty\Big\}.$$

These are Hilbert spaces under the norm $\|\underline{a}\|^2 = \sum_j |a_{\alpha\beta j}|^2$, where $\underline{a} = \{a_{\alpha\beta j}\}$. We adopt the following notations. If, e.g., $a \in \mathcal{L}^2_g$ and $a = \sum_j a_{\alpha\beta j} e_{\alpha\beta} \tau_j$ is the expansion with respect to the system $\{e_{\alpha\beta}\tau_j\}$, then clearly $\{a_{\alpha\beta j}\} \in S$; we then set $\underline{a} = \{a_{\alpha\beta j}\}$. Likewise in the case of expansions with respect to $\{e_{\alpha\beta}\sigma_j\}, \{e_{\alpha\beta}\varphi_j\}$ etc. Finally, in order to shorten expressions we set

$$F(x, y, p^k, k \leq 3) = (x^2 + y^2)^{-1}(x^2 p^1 + xyp^2) + p^3. \tag{2.11}$$

Definition 2.2 (Property (P)).

(i) A family $\{F_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z}, j\geq 0}$ of mappings $\mathcal{M}_\varepsilon \times S^3 \to \mathbb{C}$ has *property* (P) if
(1) for fixed $\Theta \in \mathcal{M}_\varepsilon$, $F_{\alpha\beta j}$ is linear in S^3

(2) there is C such such that for all $\Theta \in \mathcal{M}_\varepsilon$ and $\underline{a}, \underline{b}, \underline{c} \in S$, $\sum |F_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})|^2 \leq C(\|\underline{a}\|^2 + \|\underline{b}\|^2 + \|\underline{c}\|^2)$

(3) for fixed $\underline{a}, \underline{b}, \underline{c} \in S$, $F_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})$ is holomorphic in $\Theta \in \mathcal{M}_\varepsilon$.

(ii) A family $\{F_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 1}$ of mappings $\mathcal{M}_\varepsilon \times (S')^2 \times S \to \mathbb{C}$ has *property* (P) if conditions (1), (2) and (3) above with $(S')^2 \times S$ in place of S^3 hold.

In the theorem below, $u = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ is in $\operatorname{dom} A_s(\Theta) \cap E^g_\Theta$ and satisfies $A_s(\Theta)u = f$ for some $f = (a, b, c) \in E^g_\Theta$ and $\Theta \in M_\varepsilon$. The components $\mathcal{A}, \ldots, c$ then have Fourier expansions with respect to the complete orthonormal systems $\{e_{\alpha\beta\tau_j}\}$ in $\mathcal{L}^2_g$ and $\{e_{\alpha\beta\sigma_j}\}$ in $\mathcal{L}^2_u$, i.e.

$$\begin{aligned} a &= \sum a_{\alpha\beta j} e_{\alpha\beta\tau_j}, & b &= \sum b_{\alpha\beta j} e_{\alpha\beta\tau_j}, & c &= \sum c_{\alpha\beta j} e_{\alpha\beta\sigma_j}, \\ \mathcal{A} &= \sum A_{\alpha\beta j} e_{\alpha\beta\tau_j}, & \mathcal{B} &= \sum B_{\alpha\beta j} e_{\alpha\beta\tau_j}, & \mathcal{C} &= \sum C_{\alpha\beta j} e_{\alpha\beta\sigma_j} \end{aligned} \tag{2.12}$$

where $\alpha, \beta \in \mathbb{Z}$ and $j \geq 0$.

Theorem 2.1. *There are families*

$$\{\mathcal{A}^k_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 0}, \ \{\mathcal{B}^k_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 0} \quad (k = 1, 2, 3) \qquad \textit{and} \qquad \{\mathcal{C}_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 0}$$

having property (P), *as follows. Let* $u \in \operatorname{dom} A_s(\Theta) \cap E^g_\Theta$ *satisfy* $A_s(\Theta)u = f$ *for some* $f \in E^g_\Theta$ *and* $\Theta \in M_\varepsilon$. *The Fourier expansions of their components* a, b, c *and* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *in* (2.12) *then satisfy*

(i) $\widehat{\lambda}_j A_{\alpha\beta j} = a_{\alpha\beta j} + F(\widehat{\alpha}, \widehat{\beta}, \mathcal{A}^k_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c}), k \geq 3)$

(ii) $\widehat{\lambda}_j B_{\alpha\beta j} = b_{\alpha\beta j} + F(\widehat{\beta}, \widehat{\alpha}, \mathcal{B}^k_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c}), k \geq 3)$

(iii) $\widehat{\lambda}_j C_{\alpha\beta j} = c_{\alpha\beta j} + \mathcal{C}_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})$

where $\widehat{\lambda}_j = \widehat{\alpha}^2 + \widehat{\beta}^2 + \lambda_j$ $(\alpha, \beta \in \mathbb{Z},\ j \geq 0)$.

There is a variant for the case where $f = (a, b, c)$ and $u = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ are in E^u_Θ and $\operatorname{dom} A_s(\Theta) \cap E^u_\Theta$, respectively. Recalling (1.16) we now have the expansions

$$\begin{aligned} a &= \sum a_{\alpha\beta j} e_{\alpha\beta\pi_j} & b &= \sum b_{\alpha\beta j} e_{\alpha\beta\pi_j} & c &= \sum c_{\alpha\beta \dot{k}} e_{\alpha\beta\rho_{\dot{k}}} \\ \mathcal{A} &= \sum A_{\alpha\beta j} e_{\alpha\beta\pi_j} & \mathcal{B} &= \sum B_{\alpha\beta j} e_{\alpha\beta\pi_j} & \mathcal{C} &= \sum C_{\alpha\beta \dot{k}} e_{\alpha\beta\rho_{\dot{k}}}. \end{aligned} \tag{2.13}$$

Theorem 2.1*. *There are families*

$$\{\widehat{\mathcal{A}}_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 1}, \ \{\widehat{\mathcal{B}}_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 1} \qquad \textit{and} \qquad \{\widehat{\mathcal{C}}_{\alpha\beta j}\}_{\alpha,\beta\in\mathbb{Z},j\geq 0}$$

having property (P), *as follows. Let* u *in* $\operatorname{dom} A_s(\Theta) \cap E^u_\Theta$ *satisfy* $A_s(\Theta)u = f$ *for* $f \in E^u_\Theta$ *and some* $\Theta \in M_\varepsilon$. *The Fourier expansions of their components* a, b, c *and* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *in* (2.13) *then satisfy*

(i) $\widehat{\mu}_j A_{\alpha\beta j} = \widehat{\mathcal{A}}_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})$

(ii) $\widehat{\mu}_j B_{\alpha\beta j} = \widehat{\mathcal{B}}_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})$

(iii) $(1+\widehat{\mu}_k) C_{\alpha\beta k} = \widehat{\mathcal{C}}_{\alpha\beta k}(\Theta, \underline{a}, \underline{b}, \underline{c})$

where $\alpha, \beta \in \mathbb{Z}$, $j \geq 1$, $k \geq 0$ *and* $\widehat{\mu}_k = (\widehat{\alpha}^2 + \widehat{\beta}^2 + \mu_k)$.

Corollary 2.1. *The following assertions are true.*

(i) *There exists a positive constant* C_1 *as follows. If* $u \in \operatorname{dom} A_s(\Theta) \cap E_g^{\Theta}$ *and* $A_s(\Theta)u = f$ *for some* $\Theta \in \dot{M}_\varepsilon$ *and* $f \in E_g^{\Theta}$, *then* $u \in (H_\Theta^2)^3$ *and* $\|u\|_{H^2} \leq C_1 \|f\|_{\mathcal{L}^2}$.

(ii) *Likewise with* $u \in \operatorname{dom} A_s(\Theta) \cap E_u^{\Theta}$ *and* $f \in E_u^{\Theta}$, *and likewise with* $u \in \operatorname{dom} A_s(\Theta)$ *and* $f \in E^{\Theta}$.

Proof. We prove assertion (i) via Theorem 2; the first part of assertion (ii) follows from Theorem 2* in the same way, while the second then follows via Proposition 2.4. First note that since $\Theta \in \dot{M}_\varepsilon$ we have that $\widehat{\alpha}^2 + \widehat{\beta}^2 > 0$ whence $\widehat{\alpha}^2(\widehat{\alpha}^2 + \widehat{\beta}^2)^{-1} \leq 1$ and $|\widehat{\alpha}\widehat{\beta}|(\widehat{\alpha}^2 + \widehat{\beta}^2)^{-1} \leq 1$. From Theorem 2/(i) and the definition of F in (2.11) we infer

$$|\widehat{\lambda}_j|^2 |A_{\alpha\beta j}|^2 \leq 2|a_{\alpha\beta j}|^2 + 2\sum_{k=1}^{3} |\mathcal{A}^k_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})|^2.$$

By summing over $\alpha, \beta \in \mathbb{Z}$ and $j \geq 0$ and then using (2) in Definition 2.2/(i) we infer

$$\sum_{\alpha,\beta,j} |\widehat{\lambda}_j|^2 |A_{\alpha\beta j}|^2 \leq 2\|\underline{a}\|^2 + 2C(\|\underline{a}\|^2 + \|\underline{b}\|^2 + \|\underline{c}\|^2)$$

with the Θ-independent constant C of Definition 2.2. Since $\|f\|^2_{\mathcal{L}^2} = \|\underline{a}\|^2 + \|\underline{b}\|^2 + \|\underline{c}\|^2$ we infer

$$\sum |\widehat{\lambda}_j|^2 |A_{\alpha\beta j}|^2 \leq C_1 \|f\|^2_{\mathcal{L}^2} \tag{2.14}$$

for some positive Θ-independent constant C_1. We now invoke Proposition 2.3 according to which $\mathcal{A} = \sum A_{\alpha\beta j} e_{\alpha\beta} \tau_j$ is in H_Θ^2, satisfying

$$\|\mathcal{A}\|^2_{H^2} \leq C_2 \sum |\widehat{\lambda}_j|^2 |A_{\alpha\beta j}|^2 \tag{2.15}$$

for some positive Θ-independent constant C_2. By (2.14) and (2.15) we have

$$\|\mathcal{A}\|_{H^2} \leq C_3 \|f\|_{\mathcal{L}^2} \tag{2.16}$$

for some positive Θ-independent constant C_3. By a similar reasoning we infer $\mathcal{B}, \mathcal{C} \in H_\Theta^2$ and find a positive Θ-independent constant C_4 such that

$$\|\mathcal{B}\|_{H^2}, \|\mathcal{C}\|_{H^2} \leq C_4 \|f\|_{\mathcal{L}^2} \tag{2.17}$$

proving assertion (i) and thus the corollary ∎

Remark. The $\mathcal{A}^k_{\alpha\beta j}, \ldots$ in Theorems 2 and 2* are constructed explicitely in [14]. We note the disparity between the E_g-case (Theorem 2) and the E_u-case (Theorem 2*). In contrast to the second case, singular factors such as $\widehat{\alpha}^2(\widehat{\alpha}^2 + \widehat{\beta}^2)^{-1}$ appear in the first case, which force us to restrict Θ to $\dot{M}_\epsilon$. These singular factors are a source of complications in the perturbation-theoretic Section 4.

For what follows we recall the operator T_0 in (1.3) and the stipulation $\operatorname{dom} T_0 = (H^1(Q))^3$. We note the further

Corollary 2.2. *Given δ there is a positive constant K_δ such that*

(i) $\|T_0 u\|_{\mathcal{L}^2} \leq \delta \|A_s(\Theta)u\|_{\mathcal{L}^2} + K_\delta \|u\|_{\mathcal{L}^2}$

(ii) $\|T_0 u\|^2_{\mathcal{L}^2} \leq \delta \|A_s(\Theta)u\|^2_{\mathcal{L}^2} + K_\delta \|u\|^2_{\mathcal{L}^2}$

for any $\Theta \in \dot{M}_\epsilon$ and $u \in \operatorname{dom} A_s(\Theta)$.

Proof. It suffices to prove assertion (ii); assertion (i) then follows by rescaling K_δ. Thus fix $\Theta \in \dot{M}_\epsilon$ and recall the operator Δ_Θ at the beginning of this subsection. Δ_Θ is selfadjoint with pure point spectrum $\widehat{\Lambda}_j(\alpha,\beta)$ (see (2.7)) and associated eigenfunctions $e_{\alpha\beta}\psi_j$ $(\alpha,\beta \in \mathbb{Z},\ j \geq 1)$ (see (1.16)) giving rise to the expansion

$$u = \sum a_{\alpha\beta j} e_{\alpha\beta}\psi_j \qquad \text{for } u \in \mathcal{L}^2(Q).$$

From the spectral theorems for selfadjoint operators we infer

$$\|\Delta u\|^2_{\mathcal{L}^2} = \sum \widehat{\Lambda}_j^2 |a_{\alpha\beta j}|^2$$

if $u \in \operatorname{dom} \Delta_\Theta$. Since $-\Delta_\Theta \geq \varepsilon_0 > 0$ it follows from the theory of quadratic forms that $H^1_{\Theta,0} = \operatorname{dom}(-\Delta_\Theta)^{1/2}$ and

$$\|\nabla u\|^2_{\mathcal{L}^2} = \sum \widehat{\Lambda}_j |a_{\alpha\beta j}|^2 \qquad (u \in H^1_{\Theta,0}).$$

By elementary arguments we infer given δ there is a positive constant K_δ such that

$$\widehat{\Lambda}_j(\alpha,\beta) \leq \delta \widehat{\Lambda}_j(\alpha,\beta)^2 + K_\delta$$

for $\alpha,\beta \in \mathbb{Z},\ j \geq 1$ and $\Theta \in M_\epsilon$. From all these relations we infer that given δ there is a positive constant K_δ such that

$$\sum_{j=1}^{3} \|\nabla u_j\|^2_{\mathcal{L}^2} \leq \delta \sum_{j=1}^{3} \|\Delta u_j\|^2_{\mathcal{L}^2} + K_\delta \|u\|^2_{\mathcal{L}^2}$$

for $\Theta \in M_\epsilon$ and $u = (u_1, u_2, u_3) \in \operatorname{dom} \Delta^3_\Theta$. On the other hand there is a positive Θ-independent constant C_0 such that

$$\|\Delta v\|^2_{\mathcal{L}^2} \leq C_0 \|v\|^2_{H^2} \qquad (v \in H^2(Q)).$$

According to the last corollary there is a positive Θ-independent constant C_1 such that

$$\|u\|^2_{H^2} \leq C_1 \|A_s(\Theta)u\|^2_{\mathcal{L}^2} \qquad (u \in \operatorname{dom} A_s(\Theta),\ \Theta \in \dot{M}_\epsilon).$$

Finally we find a positive constant C_2 depending only on T_0 such that

$$\|T_0 u\|^2_{\mathcal{L}^2} \leq C_2 \left(\sum_{j=1}^{3} \|\nabla u_j\|^2_{\mathcal{L}^2} + \|u\|^2_{\mathcal{L}^2} \right)$$

for $u \in (H^1(Q))^3$. Assertion (ii) now follows from the last relations upon rescaling K_δ ∎

2.4 The projection operators. We now investigate the projection operators P_Θ and Q_Θ more closely. To this end we assume that Θ is generic, i.e. $\Theta \in \dot{M}_\varepsilon$. Under this assumption, essentially all relevant properties of P_Θ and Q_Θ can be obtained via Fourier series.

First we stress an alternative definition of E_Θ.

Lemma 2.1. *E_Θ is the $\mathcal{L}^2$-closure of all f in $(H^1_\Theta)^2 \times H^1_{\Theta,0}$ such that* $\operatorname{div} f = 0$.

Proof. We proceed as in the proof of Propositon 1.6, i.e. for fixed $j \geq 1$ we consider the field

$$f = \left(ae_{\alpha\beta}\varphi_j, be_{\alpha\beta}\varphi_j, \Lambda_j^{-1/2}(i\widehat{\alpha}a + i\widehat{\beta}b)\psi_j\right). \tag{2.18}$$

We then pick a sequence of functions $\Phi_n \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$ such that $\int_{-1/2}^{+1/2} \Phi_n \, ds = 0$ and $\lim_n \Phi_n = \psi_j$ in $\mathcal{L}^2$ and define

$$f_n = \left(ae_{\alpha\beta}\Phi_n, be_{\alpha\beta}\Phi_n, -(i\widehat{\alpha}a + i\widehat{\beta}b)\int_{-1/2}^{z} \Phi_n \, ds\right).$$

By arguing as in the proof of Proposition 1.6 we infer that $\lim_n f_n = f$ in $\mathcal{L}^2$ and $f_n \in (H^1_{\Theta,0})^3$ with $\operatorname{div} f_n = 0$, whence $f \in E_\Theta$. In case $j = 0$ we consider a field

$$f = \left(ae_{\alpha\beta}\varphi_0, be_{\alpha\beta}\varphi_0, 0\right) \tag{2.19}$$

with $i\widehat{\alpha}a + i\widehat{\beta}b = 0$. As sequence of approximating fields we take

$$f_n = (ae_{\alpha\beta}\Phi_n, be_{\alpha\beta}\Phi_n, 0)$$

where $\Phi_n \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$ and $\lim_n \Phi_n = \varphi_0 = 1$ in $\mathcal{L}^2$. Again $f_n \in (H^1_{\Theta,0})^3$ with $\operatorname{div} f_n = 0$ and $\lim_n f_n = f$ in $\mathcal{L}^2$. Since the linear hull of fields of type (2.18) and (2.19) is H^1-dense in $(H^1_\Theta)^2 \times H^1_{\Theta,0}$, the statement follows ∎

Proposition 2.6. *If $p \in H^1_\Theta$, then $\nabla p \perp E_\Theta$.*

Proof. It suffices to prove the statement for p's of the form $p = e_{\alpha\beta}\varphi_j$ since the linear hull of such p's is H^1-dense in H^1_Θ. For such a p we have

$$\nabla p = \begin{cases} (i\widehat{\alpha}e_{\alpha\beta}\varphi_0, i\widehat{\beta}e_{\alpha\beta}\varphi_0, 0) & \text{if } j = 0 \\ (i\widehat{\alpha}e_{\alpha\beta}\varphi_j, i\widehat{\beta}e_{\alpha\beta}\varphi_j, \Lambda_j^{1/2}e_{\alpha\beta}\psi_j) & \text{if } j \geq 1. \end{cases}$$

It now suffices to show that any of these fields is orthogonal to any of the fields (2.18) and (2.19). But this follows immediately by computation ∎

Next we introduce the Neumann operator $\widetilde{\Delta}_\Theta$ which acts like Δ on $\operatorname{dom}\widetilde{\Delta}_\Theta = \widehat{H}^2_\Theta$, with $\widehat{H}^2_\Theta$ as in (2.3). It is easy to see that $\widetilde{\Delta}_\Theta$ is selfadjoint with pure point spectrum consisting of the eigenvalues $\widetilde{\Lambda}_j(\alpha,\beta)$ $(j \geq 0;\ \alpha, \beta \in \mathbb{Z})$ where

$$\widetilde{\Lambda}_j = \widetilde{\Lambda}_j(\alpha,\beta) = -(\widehat{\alpha}^2 + \widehat{\beta}^2 + \Lambda_j) \tag{2.20}$$

with associated normalized eigenfunctions $e_{\alpha\beta}\varphi_j$. Note that $\widetilde{\Lambda}_j \le -\varepsilon_\Theta$ for some $\varepsilon_\Theta > 0$ due to our assumption $\Theta \in \dot{M}_\epsilon$. By familiar spectral theorems we have that an $f \in \mathcal{L}^2$ is in $\widehat{H}^2_\Theta$ if and only if

$$\sum \widetilde{\Lambda}_j^2 |a_{\alpha\beta j}|^2 < \infty \qquad \text{where } f = \sum a_{\alpha\beta j} e_{\alpha\beta}\varphi_j. \tag{2.21}$$

Since $-\widetilde{\Delta}_\Theta \ge \varepsilon_\Theta$ it follows that given $g \in \mathcal{L}^2$, there is a unique solution $p \in \widehat{H}^2_\Theta$ of $\widetilde{\Delta}_\Theta p = g$. With $p = \sum p_{\alpha\beta j} e_{\alpha\beta}\varphi_j$ and with the series subject to (2.21) one infers by termwise differentiation that $\nabla p \in (H^1_\Theta)^2 \times H^1_{\Theta,0}$. Now fix $f \in (H^1_\Theta)^2 \times H^1_{\Theta,0}$ and let $p \in \widehat{H}^2_\Theta$ be the solution of $\widetilde{\Delta}_\Theta p = \operatorname{div} f$. By Lemma 2.1, Proposition 2.6 and the above remarks we have that $f - \nabla p$ and ∇p are in $(H^1_\Theta)^2 \times H^1_{\Theta,0}$, that $\operatorname{div}(f - \nabla p) = 0$ and $\nabla p \perp E_\Theta$, whence

$$Q_\Theta f = \nabla p \qquad \text{and} \qquad P_\Theta f = f - \nabla p.$$

It is usefull to express ∇p in terms of Fourier series. Thus let $f = (a, b, c)$ with

$$a = \sum_{j\ge 0} a_{\alpha\beta j} e_{\alpha\beta}\varphi_j, \qquad b = \sum_{j\ge 0} b_{\alpha\beta j} e_{\alpha\beta}\varphi_j, \qquad c = \sum_{j\ge 1} c_{\alpha\beta j} e_{\alpha\beta}\psi_j$$

with α, β ranging over $\mathbb{Z}$. Set also

$$\gamma_{\alpha\beta j} = \begin{cases} i\widehat{\alpha} a_{\alpha\beta 0} + i\widehat{\beta} b_{\alpha\beta 0} & \text{for } j = 0 \\ i\widehat{\alpha} a_{\alpha\beta j} + i\widehat{\beta} b_{\alpha\beta j} - \lambda_j^{1/2} c_{\alpha\beta j} & \text{for } j \ge 1. \end{cases} \tag{2.22}$$

Then $\operatorname{div} f = \sum \gamma_{\alpha\beta j} e_{\alpha\beta}\varphi_j$ and with p above,

$$p = \sum \widetilde{\Lambda}_j(\alpha,\beta)^{-1} \gamma_{\alpha\beta j} e_{\alpha\beta}\varphi_j. \tag{2.23}$$

The gradient ∇p is now given by:

$$\begin{aligned} \partial_x p &= \sum_{j\ge 0} i\widehat{\alpha}\widetilde{\Lambda}_j^{-1} \gamma_{\alpha\beta j} e_{\alpha\beta}\varphi_j \\ \partial_y p &= \sum_{j\ge 0} i\widehat{\beta}\widetilde{\Lambda}_j^{-1} \gamma_{\alpha\beta j} e_{\alpha\beta}\varphi_j \qquad\qquad (\alpha, \beta \in \mathbb{Z}) \\ \partial_z p &= \sum_{j\ge 1} \widetilde{\Lambda}_j^{-1} \Lambda_j^{1/2} \gamma_{\alpha\beta j} e_{\alpha\beta}\psi_i. \end{aligned} \tag{2.24}$$

A simple approximation argument shows that (2.24) remains valid for arbitrary $f = (a, b, c) \in L^2$, i.e. if $Q_\Theta f = (u, v, w)$, then the Fourier expansions of u, v and w are given by those for $\partial_x p, \partial_y p$ and $\partial_z p$ above. From this remark one reads off from (2.24) that the subspaces L^2_g and L^2_u are invariant under Q_Θ.

In order to stress a further invariance, let $f \in \widehat{\mathcal{L}}^2$ if and only if $f \in \mathcal{L}^2$ and $\int_{-1/2}^{+1/2} f\varphi_0 \, ds = 0$; let $\check{\mathcal{L}}^2 = \mathcal{L}^2 \ominus \widehat{\mathcal{L}}^2$ and denote by $\{0\}$ the null space. According to the formulas (2.24) we read off that $\widehat{L}^2 = (\widehat{\mathcal{L}}^2)^2 \times \mathcal{L}^2$ and $\check{L}^2 = (\check{\mathcal{L}}^2)^2 \times \{0\}$ are invariant under Q_Θ. On $\check{L}^2$, Q_Θ acts as follows. With $f = (a, b, 0)$ in $\check{L}^2$ and $a = \sum a_{\alpha\beta j} e_{\alpha\beta}\varphi_0$ and $b = \sum b_{\alpha\beta j} e_{\alpha\beta}\varphi_0$ we have that $Q_\Theta f = (u, v, 0)$ where

$$\begin{aligned} u &= -\sum \left(\widehat{\alpha}^2 a_{\alpha\beta} + \widehat{\alpha}\widehat{\beta} b_{\alpha\beta}\right) \widetilde{\Lambda}_0(\alpha,\beta)^{-1} e_{\alpha\beta}\varphi_0 \\ v &= -\sum \left(\widehat{\alpha}\widehat{\beta} a_{\alpha\beta} + \widehat{\beta}^2 b_{\alpha\beta}\right) \widetilde{\Lambda}_0(\alpha,\beta)^{-1} e_{\alpha\beta}\varphi_0. \end{aligned} \tag{2.25}$$

That is, the same singular factors as in Theorem 2 reappear, a source of concern in Section 4.

3. Direct integrals

3.1 The language of direct integrals. Below we describe the method of direct integrals which connects the $A_s P + PT_0 P$ and $A_s(\Theta)P_\Theta + P_\Theta T_0 P_\Theta$. The description is selfcontained but as to results we rely on [2, 8, 12, 13].

Let $\mathcal{H}'$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}'}$ and norm $\| \cdot \|_{\mathcal{H}'}$. Recall $M = [0, 2\pi]^2$ and Lebesgue measure μ on M. A mapping $\Phi : M \to \mathcal{H}'$ is measurable if it is defined for a.e. $\Theta \in M$ and if $\langle g, \Phi(\cdot) \rangle$ is measurable for all $g \in \mathcal{H}'$; $\|\Phi(\cdot)\|_{\mathcal{H}'}$ is then also measurable. A Hilbert space $\mathcal{H} = \int_M \mathcal{H}' d\mu$ is then defined as follows. It consists of the set of (equivalence classes) of measurable mappings Φ such that

$$\begin{aligned} &\int_M \|\Phi(\Theta)\|^2_{\mathcal{H}'} d\mu < \infty \\ &\int_M \langle \Phi_1(\Theta), \Phi_2(\Theta) \rangle_{\mathcal{H}'} d\mu = \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} \quad \text{for} \quad \Phi_1, \Phi_2 \in \mathcal{H}. \end{aligned} \tag{3.1}$$

The basic example is given by $\mathcal{H}' = \mathcal{L}^2(Q)$ and $\mathcal{H} = \mathcal{L}^2(M \times Q)$. Given $\Phi \in \mathcal{H}$ we have by the Fubini theorem that $\Phi(\Theta, \cdot) \in \mathcal{H}'$ for a.e. $\Theta \in M$ and

$$\int_M \|\Phi(\Theta, \cdot)\|^2_{\mathcal{H}'} d\mu = \int_{M \times Q} |\Phi(\Theta, x, y, z)|^2 d\Theta^2 dx dy dz. \tag{3.2}$$

The measurability is evident. Next we describe a unitary mapping V from $\mathcal{L}^2(\Omega)$ onto $\mathcal{H}$. Thus set $\underline{x} = (x, y)$, $n = (n_1, n_2)$, $|n| = \max(|n_1|, |n_2|)$ and let $\Theta n = \Theta_1 n_1 + \Theta_2 n_2$ for $\Theta = (\Theta_1, \Theta_2) \in M$. Now pick $f \in C_0^p(\overline{\Omega})$. Since supp$f \subseteq \overline{\Omega}$ is compact, there is for every $\underline{x}' \in \mathbb{R}^2$ a neighbourhood $U_\epsilon \ni \underline{x}'$ and an $N > 0$ such that

$$f(\underline{x} + nL, z) = f(x + n_1 L, y + n_2 L, z) = 0$$

for $\underline{x} \in U_\epsilon$, $|n| \geq N$ and $|z| \leq \frac{1}{2}$. We then have that

$$\psi_f(\Theta, \underline{x}, z) = \frac{1}{2\pi} \sum_n e^{in\Theta} f(\underline{x} + nL, z) = \frac{1}{2\pi} \sum_{|n| \leq N} e^{in\Theta} f(\underline{x} + nL, z) \tag{3.3}$$

for all $\underline{x} \in U_\varepsilon$ and $|z| \leq \frac{1}{2}$. From (3.3) we read off that $\psi_f(\Theta, \cdot) \in C^p_\Theta(\overline{\Omega})$. By the Heine-Borel theorem and the above remarks there is an $N > 0$ such that $f(\underline{x} + nL, z) = 0$ for $\underline{x} \in \overline{Q}_L = [0, L]^2$, $|n| \geq N$ and $|z| \leq \frac{1}{2}$. We thus have that

$$\int_{M \times Q} |\psi_f|^2 d\Theta d\underline{x} dz = \sum_{|n| \leq N} \int_Q |f(\underline{x} + nL, z)|^2 d\underline{x} dz \tag{3.4}$$

for $\underline{x} \in \overline{Q}_L$ and $|z| \leq \frac{1}{2}$. But due to our choice of N the right-hand side of (3.4) is just $\|f\|^2_{\mathcal{L}^2(\Omega)}$. Thus the mapping $Vf = \psi_f$ given by (3.3) for $f \in C^p_0(\overline{\Omega})$ is a Hilbert space isometry from $C^p_0(\overline{\Omega}) \subseteq \mathcal{L}^2(\Omega)$ into $\mathcal{L}^2(M \times Q)$, which extends into a Hilbert space isometry from $\mathcal{L}^2(\Omega)$ into $\mathcal{L}^2(M \times Q)$. In order to recognize V as unitary, i.e. as "onto", we let V' be the isometry from $\mathcal{L}^2(\mathbb{R}^2)$ into $\mathcal{L}^2(M \times Q_L)$ which we obtain if we restrict (3.3) to functions $f \in C^p_0(\overline{\Omega})$ which do not depend on z. It is shown in [12] (see also [8]) that V' is onto $\mathcal{L}^2(M \times Q_L)$, i.e. unitary. Now it is easily established that if $f = g\Phi$ with $g \in \mathcal{L}^2(\mathbb{R}^2)$ and $\Phi \in \mathcal{L}^2(-\frac{1}{2}, +\frac{1}{2})$, then $Vf = (V'g)\Phi$, whence $V(\sum g_j \Phi_j) = \sum (V'g_j)\Phi_j$ for any finite sum $\sum g_j \Phi_j$ such that $g_j \in \mathcal{L}^2(\mathbb{R}^2)$ and $\Phi_j \in \mathcal{L}^2(-\frac{1}{2}, +\frac{1}{2})$. Since the finite sums $\sum_j h_j \Phi_j$ with $h_j \in \mathcal{L}^2(M \times Q_L)$ and $\Phi_j \in \mathcal{L}^2(-\frac{1}{2}, +\frac{1}{2})$ are $\mathcal{L}^2$-dense in $\mathcal{L}^2(M \times Q)$, the unitarity of V follows from that of V'.

In the lemma below, $x_1 = x$, $x_2 = y$, $x_3 = z$ and $\partial_j = \partial_{x_j}$, $\partial_{jk} = \partial^2_{x_j x_k}$.

Lemma 3.1. *The following assertions are true.*

(i) *Let f in $\widehat{H}^2(\Omega)$ or in $H^2(\Omega)$. Then there exists $E \subseteq M$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies:*

(1) *$(Vf)(\Theta, \cdot)$ is in $\hat{H}^2_\Theta$ or H^2_Θ, respectively.*

(2) *$(V\partial_j f)(\Theta, \cdot) \in H^1_\Theta$ and $(V\partial_j f)(\Theta, \cdot) = \partial_j(Vf)(\Theta, \cdot)$.*

(3) *$(V\partial_{jk} f)(\Theta, \cdot) \in \mathcal{L}^2(Q)$ and $(V\partial_{jk} f)(\Theta, \cdot) = \partial_{jk}(Vf)(\Theta, \cdot)$ $(j, k = 1, 2, 3)$.*

(ii) *Let f in $H^1_0(\Omega)$ or $H^1(\Omega)$. Then there is $E \subseteq M$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies:*

(1) *$(Vf(\Theta, \cdot)$ is in $H^1_{\Theta,0}$ or H^1_Θ, repectively.*

(2) *$(V\partial_j f)(\Theta, \cdot) \in L^2(Q)$ and $(V\partial_j f)(\Theta, \cdot) = \partial_j(Vf)(\Theta, \cdot)$.*

Proof. Since it is essentially the same as that of Lemma 1 in [12] we only stress the main points. First note that by virtue of (3.3) we have that

$$\partial_{jk}(V\Phi)(\Theta, \cdot) = (V\partial_{jk}\Phi)(\Theta, \cdot) \quad \text{or} \quad \partial_j(V\Phi)(\Theta, \cdot) = (V\partial_j\Phi)(\Theta, \cdot),$$

respectively, for $j, k = 1, 2, 3$ and any $\Phi \in C^2_0(\overline{\Omega})$ and

$$(V\Phi)(\Theta, \cdot) \in \hat{C}^2_\Theta(\overline{\Omega}) \qquad \text{if } \Phi \in \hat{C}^2_0(\overline{\Omega}).$$

Now let $f \in \widehat{H}^2(\Omega)$. Based on our remarks concerning the Neumann operator $\widetilde{A}$ in Subsection 1.5 and on Proposition 1.8 one infers that the Fourier series $\sum f_j \varphi_j$ of f satisfies the assumptions of Proposition 1.3, whence $\lim_N \|f - L_N\|_{H^2} = 0$ where

$L_N = \sum_{j=0}^{N} f_j \varphi_j$. Since $L_N \in \widehat{H}^2(\Omega)$, and by a simple approximation argument one finds $\Phi_n \in \widehat{C}_0^2(\overline{\Omega})$ such that $\lim \|f - \Phi_n\|_{H^2} = 0$. By the unitarity of V and (3.3) we then have that

$$\|Vf - V\Phi_n\|_{\mathcal{L}^2}, \qquad \|Vg_j - \partial_j V\Phi_n\|_{\mathcal{L}^2}, \qquad \|Vh_{jk} - \partial_{jk} V\Phi_n\|_{\mathcal{L}^2}$$

all tend to zero as $n \uparrow \infty$ where $\|\cdot\|_{\mathcal{L}^2}$ is the norm in $\mathcal{L}^2(M \times Q)$ and where $g_j = \partial_j f$ and $h_{jk} = \partial_{jk} f$. Via the Fubini theorem one then finds a set $E \subseteq M$ with $\mu(E) = \mu(M)$ and a subsequence $\{\Phi_{n_k}\}$ such that $\Theta \in E$ implies that

$$(Vf)(\Theta,\cdot), \qquad (V\partial_j f)(\Theta,\cdot), \qquad (V\partial_{jk} f)(\Theta,\cdot)$$

are all in $\mathcal{L}^2(Q)$ and such that

$$\|(Vf)(\Theta,\cdot) - (V\Phi_{n_k})(\Theta,\cdot)\|_{\mathcal{L}^2}$$

$$\|(Vg_j)(\Theta,\cdot) - \partial_j(V\Phi_{n_k})(\Theta,\cdot)\|_{\mathcal{L}^2}$$

$$\|(Vh_{jk})(\Theta,\cdot) - \partial_{jk}(V\Phi_{n_k})(\Theta,\cdot)\|_{\mathcal{L}^2}$$

all tend to zero as $k \uparrow \infty$, where now $\|\cdot\|_{\mathcal{L}^2}$ is the norm in $\mathcal{L}^2(Q)$. But this is exactly what is claimed in assertion (i).

If merely $f \in H^2(\Omega)$, then the sequence $\Phi_n \in C_0^2(\overline{\Omega})$ exists by definition of $H^2(\Omega)$. The proof of assertion (ii) is quite the same and omitted ∎

Corollary. *For $f \in H^2(\Omega) \cap H_0^1(\Omega)$ there is $E \subseteq M$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies that $(Vf)(\Theta,\cdot) \in H_\Theta^2 \cap H_{\Theta,0}^1$ and that assertions* (2) *and* (3) *of Lemma* 3.1/(i) *hold.*

Lemma 3.2. *Let $h \in C^0(\overline{\Omega})$ be L-periodic in x and y, and let $f \in \mathcal{L}^2(\Omega)$. Then there is $E \subseteq M$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies that $(Vf)(\Theta,\cdot)$, $(Vhf)(\Theta,\cdot)$ $\in \mathcal{L}^2(Q)$ and that $h(Vf)(\Theta,\cdot) = (V'hf)(\Theta,\cdot)$.*

Proof. It is based on $(V\Phi h) = h(V\Phi)$ for $\Phi \in C_0^p(\overline{\Omega})$ and similar to the above but simpler and omitted ∎

The above setting extends straightforwardly to the vector-valued case. As "fiber" space we take $\mathcal{H}'' = (\mathcal{L}^2(Q))^3$. The direct integral $\mathcal{H}^* = \int_M \mathcal{H}'' d\mu$ is now the set of measurable mappings $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ which map M into $\mathcal{H}''$ according to $\Phi(\Theta) = (\Phi_1(\Theta), \Phi_2(\Theta), \Phi_3(\Theta)) \in \mathcal{H}''$ for a.e. $\Theta \in M$ such that

$$\int_M \|\Phi(\Theta)\|_{\mathcal{H}''}^2 d\mu = \sum_{j=1}^{3} \int_M \|\Phi_j(\Theta)\|_{\mathcal{H}'}^2 d\mu < \infty$$

and with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}^*}$ defined in terms of $\langle\cdot,\cdot\rangle_{\mathcal{H}''}$ in the obvious way. A unitary mapping U from $(\mathcal{L}^2(\Omega))^3$ onto $\mathcal{H}^*$ is then given by

$$U\Phi = (V\Phi_1, V\Phi_2, V\Phi_3) \tag{3.5}$$

with V as above. There are obvious extensions of Lemmas 3 and 4 and the Corollary to the vector-valued case, the most important being the lemma below in which A, T_0 and A_Θ are the operators in Subsections 1.4 and 2.3, respectively.

Lemma 3.3. *Let f in $(H^2(\Omega) \cap H_0^1(\Omega))^3$ or in $(H^1(\Omega))^3$. Then there is $E \subseteq M$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies:*

(i) $(Uf)(\Theta, \cdot) \in (H^2_\Theta \cap H^1_{\Theta,0})^3$
$(UAf)(\Theta, \cdot) \in (\mathcal{L}^2(Q))^3$ *and* $(UAf)(\Theta, \cdot) = A_\Theta(Uf)(\Theta, \cdot)$.

(ii) $(UT_0 f)(\Theta, \cdot) \in (\mathcal{L}^2(Q))^3$ *and* $T_0(Uf)(\Theta, \cdot) = (UT_0 f)(\Theta, \cdot)$.

Proof. It follows immediately from Lemmas 3 and 4 and the Corollary ■

Next we come to the direct integrals of bounded and unbounded operators, notions discussed at length in [2] for the bounded selfadjoint case, in [8] for the unbounded case and in [12, 13] for semigroup generators. Here we stress the definitions, but rely on [2, 8, 12, 13] as for as results are concerned. A family $\{B(\Theta)\}_{\Theta \in M} \subset L(\mathcal{H}'', \mathcal{H}'')$ of bounded operators is measurable, if $\langle g, B(\cdot)f\rangle_{\mathcal{H}''}$ is measurable for all $f, g \in \mathcal{H}''$. If we have that $\|B(\Theta)\|_\infty \leq C$ for a.e. $\Theta \in M$ for some positive constant C, then a bounded operator $\widetilde{B} \in L(\mathcal{H}^*, \mathcal{H}^*)$ exists according to

$$(\widetilde{B}\phi)(\Theta) = B(\Theta)\varphi(\Theta) \qquad \text{for a.e } \Theta \in M \;\text{ and }\; \varphi \in \mathcal{H}^*. \tag{3.6}$$

That $\widetilde{B}\varphi \in \mathcal{H}^*$ is shown in [2, 8]. We write $\widetilde{B} = \int_M B(\Theta)\, d\mu$ (for details see [8: p. 281] and [2: Subsection II.2]). Next let $\{A(\Theta)\}_{\Theta \in M}$ be a family of linear operators on $\mathcal{H}''$. An unbounded operator $\widetilde{A} = \int_M A(\Theta)\, d\mu$ on $\mathcal{H}^*$ is defined according to

Definition 3.1. $\varphi \in \operatorname{dom} \widetilde{A}$ if and only if:

(i) $\varphi(\Theta) \in \operatorname{dom} A(\Theta)$ for a.e. $\Theta \in M$

(ii) $\Theta \to A(\Theta)\varphi(\Theta)$ is measurable

(iii) $\int_M \|A(\Theta)\varphi(\Theta)\|^2_{\mathcal{H}''}\, d\mu < \infty$.

For such φ we set $(\widetilde{A}\varphi)(\Theta) = A(\Theta)\varphi(\Theta)$ for a.e. Θ.

Remarks. There are cases in which in Definition 3.1 condition (ii) is a consequence of condition (i). Operator families for which this is the case will be said to have *property* (M). Such is, e.g., the case if the $A(\Theta)$ are semigroup generators such that $(\lambda_0, \infty) \subseteq \rho(A(\Theta))$ $(\Theta \in M)$ for some λ_0 and if there are $\lambda \in (\lambda_0, \infty)$ and a positive constant C such that

(i) $(A(\cdot) - \lambda)^{-1}$ is measurable on M

(ii) $\|(A(\Theta) - \lambda)^{-1}\|_\infty \leq C$ for all $\Theta \in M$.

For a proof see [12: Lemma 4] and [13: Appendix]. If in addition the $A(\Theta)$ are selfadjoint, then $\widetilde{A}$ is selfadjoint (see [8: Theorem XIII.85]). It may be usefull to take parity into account by setting

$$\begin{aligned} &\mathcal{H}''_g = (\mathcal{L}^2_g(Q))^2 \times \mathcal{L}^2_u(Q), \qquad \mathcal{H}''_u = (\mathcal{L}^2_u(Q))^2 \times \mathcal{L}^2_g(Q) \\ &\mathcal{H}^*_g = \int_M \mathcal{H}''_g d\mu, \qquad \mathcal{H}^*_u = \int_M \mathcal{H}''_u d\mu. \end{aligned} \tag{3.7}$$

One then has the decomposition $\mathcal{H}^* = \mathcal{H}_g^* \oplus \mathcal{H}_u^*$. Recalling $L^2(\Omega)^3 = L_g^2 \oplus L_u^2$ in Subsection 1.4 one then has that U maps L_g^2 and L_u^2 unitarily onto $\mathcal{H}_g^*$ and $\mathcal{H}_u^*$, respectively. Likewise, recalling $\widehat{\mathcal{L}}^2(\Omega)$ and $\check{\mathcal{L}}^2(\Omega)$ in Subsection 1.5, and $\widehat{\mathcal{L}}^2(Q)$ and $\check{\mathcal{L}}^2(Q)$ in Subsection 2.4 we set

$$\begin{aligned} \widehat{\mathcal{H}}'' &= (\widehat{\mathcal{L}}^2(Q)) \times \mathcal{L}^2(Q), & \check{\mathcal{H}}'' &= (\check{\mathcal{L}}^2(Q))^2 \times \{0\} \\ \widehat{\mathcal{H}}^* &= \int_M \widehat{\mathcal{H}}''\, d\mu, & \check{\mathcal{H}}^* &= \int_M \check{\mathcal{H}}''\, d\mu. \end{aligned} \tag{3.8}$$

We then have the decomposition $\mathcal{H}^* = \widehat{\mathcal{H}}^* \oplus \check{\mathcal{H}}^*$, and U maps $\widehat{L}^2$ and $\check{L}^2$ unitarily onto $\widehat{\mathcal{H}}^*$ and $\check{\mathcal{H}}^*$, respectively.

Now we come to the main result of this section, which serves as basis for the next section, but which has independent interest. First we note that every linear operator L on $(\mathcal{L}^2(\Omega))^3$ has a unitary transplant $\widehat{L} = ULU^{-1}$ on $\mathcal{H}^*$. Next we recall the operators A, T_0 and P in Subsections 1.4 and 1.5, and A_Θ, P_Θ in Subsections 2.3 and 2.4. Let $\{T_\Theta\}_\Theta$ be the family of unbounded operators on $(\mathcal{L}^2(Q))^3$, which are formally given by (1.3) but supplied with the stipulation $\operatorname{dom} T_\Theta = (H^1_\Theta)^3$. The result in question then is

Theorem 3.1. *The equation* $\widehat{P}(\widehat{A} + \widehat{T}_0)\widehat{P} = \int_M P_\Theta(A_\Theta + T_\Theta)P_\Theta\, d\mu$ *holds.*

The proof of this theorem will be given in Subsection 3.2, assuming the following lemma, whoose proof is relegated to Subsection 3.3.

Lemma 3.4. *The equations* $\widehat{P} = \widetilde{P} = \int_M P_\Theta\, d\mu$ *are valid.*

3.2 Proof of Theorem 3.1. For simplicity we write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for $\langle \cdot, \cdot \rangle_{\mathcal{H}''}$ and $\| \cdot \|_{\mathcal{H}^*}$, respectively, if it is clear to which space the symbols refer; likewise with $\int$ instead of $\int_M$. Next we stress that the operators $A(\Theta)$ and $B(\Theta)$ above need only be defined for a.e. $\Theta \in M$. In fact all families below are defined at least on $\dot{M}$, i.e. M minus the corners. For Θ a corner we might set, e.g., $A(\Theta) = 0$ or $A(\Theta) = 1$; we leave this open.

We start with some remarks and consider a family $\{A(\Theta)\}_{\Theta \in M}$ of operators on $\mathcal{H}''$ which in all cases of interest has property (M), although the statements below hold without this property. The operator $\widetilde{A} = \int A(\Theta)\, d\mu$ is then defined via Definition 3.1. We also let $\{B(\Theta)\}_{\Theta \in M}$ and $\{C(\Theta)\}_{\Theta \in M}$ be two measurable families of bounded operators on $\mathcal{H}''$ such that, for some constant c, $\|B(\Theta)\|_\infty \le c$ and $\|C(\Theta)\|_\infty \le c$ for a.e. Θ, giving rise to the bounded operators $\widetilde{B} = \int B(\Theta)\, d\mu$ and $\widetilde{C} = \int C(\Theta)\, d\mu$; property (M) is then automatically satisfied (see [12, 13]). We also set

$$\widetilde{G} = \int A(\Theta)B(\Theta)\, d\mu, \quad \widetilde{H} = \int B(\Theta)A(\Theta)\, d\mu, \quad \widetilde{L} = \int C(\Theta)A(\Theta)B(\Theta)\, d\mu.$$

Straightforwardly from the definitions we infer

$$\widetilde{A}\widetilde{B} \subseteq \widetilde{G}, \qquad \widetilde{H} \subseteq \widetilde{B}\widetilde{A}, \qquad \widetilde{C}\widetilde{A}\widetilde{B} \subseteq \widetilde{L}. \tag{3.9}$$

We also note the implication

$$A(\Theta) \text{ are symmetric} \quad \Longrightarrow \quad \widetilde{A} \text{ is symmetric.} \tag{3.10}$$

Next let A be a semigroup generator on a Banach space X and T an operator which is A-bounded with relative bound zero (see [3: p. 190]). Then

$$A + T \qquad \text{is a semigroup generator on } X$$

(see [7: p. 80] and [3: p. 190]). We also have the following maximality property.

Proposition 3.1. *Let A and B be semigroup generators such that $A \subseteq B$. Then $A = B$.*

Proof. We have to prove $\operatorname{dom} B \subseteq \operatorname{dom} A$. Fix $\lambda \in \rho(A) \cap \rho(B)$, pick $x \in \operatorname{dom} B$ and set $(B-\lambda)x = y$. Then there is $z \in \operatorname{dom} B$ with $(A-\lambda)z = y$. Since $A \subseteq B$, $(B-\lambda)z = y$ holds. By unicity $x = z \in \operatorname{dom} A$, proving the claim ∎

Similarly, if A is selfadjoint and $B \supseteq A$ a symmetric extension, then $A = B$ since A is maximal, i.e. has no proper symmetric extension. We now apply these remarks to the "Dirichlet"-operators A and A_Θ. First fix $f \in \mathcal{H}''$. Then $A_\Theta^{-1} f$ can be expressed componentwise by Fourier series with respect to the eigenfunctions $e_{\alpha\beta}\psi_j$ of Δ_Θ, from which the measurability of A_Θ^{-1} for $\Theta \in M$ is easily deduced. Since $\sigma(A_\Theta) \subseteq (-\infty, 0]$ $(\Theta \in M)$ we have that the family $\{A_\Theta\}_{\Theta \in M}$ has property (M) by the remarks in Subsection 3.1. Also, since the A_Θ are selfadjoint, $\widetilde{A} = \int A_\Theta \, d\mu$ is symmetric by (3.10).

Next pick $\varphi \in \operatorname{dom} \widehat{A}$, i.e. $\varphi = Uf$ for some $f \in \operatorname{dom} A$, and set $\varphi_A = UAf$. By Lemma 3.3, $A_\Theta \varphi(\Theta) = \varphi_A(\Theta)$ for a.e. Θ whence $\varphi \in \operatorname{dom} \widetilde{A}$ and $\widehat{A}\varphi = \widetilde{A}\varphi$. That is, $\widehat{A} \subseteq \widetilde{A}$ and thus by (3.10)

$$\widetilde{A} = \int A_\Theta \, d\mu \qquad \text{is selfadjoint and } \widehat{A} = \widetilde{A}. \tag{3.11}$$

Similar remarks apply to T_0 (see Subsection 1.4) and T_Θ, given by (1.3) but with $\operatorname{dom} T_\Theta = (H_\Theta^1)^3$. Property (M) of the family $\{T_\Theta\}_{\Theta \in M}$ reduces via components to property (M) of the families $\{\partial_j(\Theta)\}_\Theta$ on $\mathcal{H}' = \mathcal{L}^2(Q)$, where $\partial_j(\Theta) = \partial_j$ on its domain H_Θ^1. This in turn is easily proved by Fourier series arguments. Next, let $\varphi \in \operatorname{dom} \widehat{T}_0$, i.e. $\varphi = Uf$ for some $f \in \operatorname{dom} T_0 = (H^1(\Omega))^3$ and put $\varphi_T = U(T_0 f)$. By Lemma 5 we have that $\varphi_T(\Theta) = T_\Theta \varphi(\Theta)$ for a.e. Θ, whence $\varphi \in \operatorname{dom} \widetilde{T}$ and $\widetilde{T}\varphi = \widehat{T}_0\varphi$, i.e.

$$\widehat{T}_0 \subseteq \widetilde{T}_0, \qquad \text{where } \widetilde{T} = \int T_\Theta \, d\mu. \tag{3.12}$$

The last remark is provided by

Proposition 3.2. *The following assertions are true.*

(i) *If $f \in (H^1(\Omega))^2 \times H_0^1(\Omega)$ and $f \in E$, then $\operatorname{div} f = 0$.*

(ii) *If $f \in (H_\Theta^1)^2 \times H_{\Theta,0}^1$ and $f \in E_\Theta$, then $\operatorname{div} f = 0$, $\Theta \in M$.*

Proof. In case of (ii), $f = P_\Theta f = f - \nabla p$ for $p \in \widehat{H}^2_\Theta$ with $\Delta p = \operatorname{div} f$, whence $\operatorname{div} f = 0$. In case of (i), we have that $f = (a, b, c)$, with $a = \sum_{j\geq 0} A_j\varphi_j$, $b = \sum_{j\geq 0} B_j\varphi_j$ and $c = \sum_{j\geq 1} C_j\psi_j$ subject to Proposition 1.2. By Proposition 1.7, $\langle f, \nabla p\rangle = 0$ for all $p \in H^1(\Omega)$. Testing this relation for all p's of the form $p_j\varphi_j$ with $p_j \in H^1(\mathbb{R}^2)$ $(j \geq 0)$ we infer straightforwardly via Fourier transforms that $\partial_x A_j + \partial_y B_j - \Lambda_j^{1/2} C_j = 0$ $(j \geq 1)$ and $\partial_x A_0 + \partial_y B_0 = 0$, i.e. $\operatorname{div} f = 0$ ∎

As a consequence we have that $\operatorname{dom} A_s = \operatorname{dom} A \cap E$ and $\operatorname{dom} A_s(\Theta) = \operatorname{dom} A_\Theta \cap E_\Theta$ whence

$$A_s P = PAP \qquad \text{and} \qquad A_s(\Theta)P_\Theta = P_\Theta A_\Theta P_\Theta.$$

We now come to the proof of Theorem 3.1 proper which splits into propositions.

Proposition 3.3. *The equation* $\widehat{P}\widehat{A}\widehat{P} = \int P_\Theta A_\Theta P_\Theta \, d\mu$ *is valid.*

Proof. The operator PAP is selfadjoint: $PAP = A_s$ on E and $PAP = 0$ on $E_\perp$. By the same reason, $P_\Theta A_\Theta P_\Theta$ is selfadjoint. From Lemma 3.4 and clauses (3.9) and (3.11) we infer

$$\widehat{P}\widehat{A}\widehat{P} = \widetilde{P}\widetilde{A}\widetilde{P} \subseteq \int P_\Theta A_\Theta P_\Theta \, d\mu.$$

Since $P_\Theta A_\Theta P_\Theta$ is selfadjoint and by a remark above, $\int P_\Theta A_\Theta P_\Theta \, d\mu$ is a symmetric extension of the selfadjoint $\widetilde{P}\widetilde{A}\widetilde{P}$, whence by maximality they coincide ∎

We now combine Proposition 3.3 with Lemma 3.4, and clauses (3.9) and (3.12) in order to infer

$$\widehat{P}\widehat{A}\widehat{P} + \widehat{P}\widehat{T}_0\widehat{P} \subseteq \int P_\Theta A_\Theta P_\Theta \, d\mu + \int P_\Theta T_\Theta P_\Theta \, d\mu. \tag{3.13}$$

We can replace "$\subseteq$" by "$=$" if we recognize the right-hand side of (3.13) as a semigroup generator. In fact

Proposition 3.4. $\int P_\Theta T_\Theta P_\Theta \, d\mu$ *is* $(\int P_\Theta A_\Theta P_\Theta \, d\mu)$*-bounded with relative bound zero.*

Proof. Set provisionally

$$L_\Theta = P_\Theta A_\Theta P_\Theta, \qquad H_\Theta = P_\Theta T_\Theta P_\Theta$$
$$\widetilde{L} = \int L_\Theta d\mu, \qquad \widetilde{H} = \int H_\Theta d\mu.$$

Pick $\varphi \in \operatorname{dom} \widetilde{L}$. By this assumption we have that

$$P_\Theta\varphi(\Theta) \in (H^2_\Theta \cap H^1_{\Theta,0})^3 \text{ for a.e. } \Theta \qquad \text{and} \qquad \int \|L_\Theta\varphi(\Theta)\|^2 d\mu < \infty. \tag{3.14}$$

Next recall the second corollary to Theorems 2 and 2* and fix $\varepsilon > 0$. By assertion (ii) in this corollary and clause (3.14) we have for a.e. Θ

$$\|H_\Theta\varphi(\Theta)\|^2 \leq \varepsilon\|L_\Theta\varphi(\Theta)\|^2 + K_\varepsilon\|\varphi(\Theta)\|^2 \tag{3.15}$$

for a positive Θ-independent constant K_ε. By integrating (3.15) we get

$$\|\widetilde{H}\varphi\|^2 \leq \varepsilon\|\widetilde{L}\phi\|^2 + K_\varepsilon\|\phi\|^2$$

with $\operatorname{dom} \widetilde{L} \subseteq \operatorname{dom} \widetilde{H}$ included by virtue of (3.14) ∎

By this proposition and the remarks prior to Proposition 3.1 we have that $\widetilde{L}+\widetilde{H}$ is a semigroup generator. By the second corollary to Theorem 1.1 we recognize $PAP+PT_0P$ and hence $\widehat{P}\widehat{A}\widehat{P}+\widehat{P}\widehat{T}_0\widehat{P}$ as a semigroup generator having $\widetilde{L}+\widetilde{H}$ as an extension. By Proposition 3.1 they coincide:

$$\widehat{P}\widehat{A}\widehat{P}+\widehat{P}\widehat{T}_0\widehat{P} = \int P_\Theta A_\Theta P_\Theta \, d\mu + \int P_\Theta T_\Theta P_\Theta \, d\mu. \tag{3.16}$$

Thus Theorem 3.1 reduces to

Proposition 3.5. *The equation*

$$\int P_\Theta A_\Theta P_\Theta \, d\mu + \int P_\Theta T_\Theta P_\Theta \, d\mu = \int P_\Theta (A_\Theta + T_\Theta) P_\Theta \, d\mu$$

holds.

Proof. We retain the notations L_Θ, H_Θ and $\widetilde{L}, \widetilde{H}$ in the proof of Proposition 3.3 and set also

$$B_\Theta = P_\Theta(A_\Theta + T_\Theta)P_\Theta \qquad \text{and} \qquad \widetilde{B} = \int B_\Theta d\mu.$$

In a first step one shows the implication

$$\varphi \in \operatorname{dom}\widetilde{L} \cap \operatorname{dom}\widetilde{B} \quad \Longrightarrow \quad (\widetilde{L}+\widetilde{H})\varphi = \widetilde{B}\varphi.$$

The proof of this amounts to evaluate the definitions straightforwardly; we may safely omit it. It remains to show $\operatorname{dom}\widetilde{L} = \operatorname{dom}\widetilde{B}$. One half of this is provided by

$$\operatorname{dom}\widetilde{L} \subseteq \operatorname{dom}\widetilde{B}.$$

But this is settled by (3.15) which permits us to infer $\varphi \in \operatorname{dom}\widetilde{B}$ from $\varphi \in \operatorname{dom}\widetilde{L}$. It remains to prove

$$\operatorname{dom}\widetilde{B} \subseteq \operatorname{dom}\widetilde{L}. \tag{3.17}$$

Thus fix $\varepsilon > 0$ small and let K_ε be such that (ii) in Corollary 2.2 holds. Next let $\varphi \in \operatorname{dom}\widetilde{B}$, whence $P_\Theta\varphi(\Theta) \in \operatorname{dom} A_\Theta$ for a.e. Θ and

$$\|\widetilde{B}\varphi\|^2 = \int \|P_\Theta(A_\Theta + T_\Theta)P_\Theta\varphi(\Theta)\|^2 d\mu < \infty. \tag{3.18}$$

By elementary reasons we have that

$$\|L_\Theta\varphi(\Theta)\|^2 \le 2\|B_\Theta\varphi(\Theta)\|^2 + 2\|H_\Theta\varphi(\Theta)\|^2. \tag{3.19}$$

By our choice of ε and K_ε, inequality (3.15) is available. Thus we can insert the right-hand side of this inequality for $\|H_\Theta\varphi(\Theta)\|^2$ into (3.19). Since $\varepsilon > 0$ is small, we get after a rearrangement of terms

$$(1-2\varepsilon)\|L_\Theta\varphi(\Theta)\|^2 \le 2\|B_\Theta\varphi(\Theta)\|^2 + 2K_\varepsilon\|\phi(\Theta)\|^2. \tag{3.20}$$

By (3.18), the integral over the right-hand side of (3.20) is finite, and so is the integral over the left-hand side, proving $\varphi \in \operatorname{dom}\widetilde{L}$. Thus (3.17) holds whence the proposition follows ∎

Theorem 3.1 has a variant, whose proof is virtually the same. We now assume that the equilibrium solution (u_1, u_2, u_3) which determines T_0 formally via (1.3) satisfies

$$u_1, u_2 \text{ are even in } z \qquad \text{and} \qquad u_3 \text{ is odd in } z. \tag{3.21}$$

Next recall the spaces $\mathcal{H}''_g = (\mathcal{L}^2_g(Q))^2 \times \mathcal{L}^2_u(Q)$ and $\mathcal{H}''_u = (\mathcal{L}^2_u(Q))^2 \times \mathcal{L}^2_g(Q)$, giving rise to the direct integrals $\mathcal{H}^*_u = \int \mathcal{H}''_u \, d\mu$ and $\mathcal{H}^*_g = \int \mathcal{H}''_g \, d\mu$. As noted earlier, A, A_s and P leave L^2_g and L^2_u invariant while $A_\Theta, A_s(\Theta)$ and P_Θ leave $\mathcal{H}''_g$ and $\mathcal{H}''_u$ invariant. Based on (3.21) it is easily checked that T_0 (with $\operatorname{dom} T_0 = (H^1(\Omega))^3$) leaves L^2_g and L^2_u invariant, while T_Θ (with $\operatorname{dom} T_\Theta = (H^1_\Theta)^3$) leaves $\mathcal{H}''_g$ and $\mathcal{H}''_u$ invariant. Since the unitary U maps L^2_g onto $\mathcal{H}^*_g$ and L^2_u onto $\mathcal{H}^*_u$ we can restrict the arguments leading to Theorem 3.1 to the pairs $L^2_g, \mathcal{H}^*_g$ and $L^2_u, \mathcal{H}^*_u$, respectively. In order to state the variant let D^g be the restriction of the linear operator D to L^2_g or $\mathcal{H}''_g$ according to the case; likewise with D^u.

Theorem 3.1*. *The equation* $U^g P^g (A^g + T^g_0) P^g (U^g)^{-1} = \int P^g_\Theta (A^g_\Theta + T^g_\Theta) P^g_\Theta \, d\mu$ *holds, likewise with* u *for* g.

Remark. Another way to express Theorem 3.1* is

$$(\widehat{P}(\widehat{A} + \widehat{T}_0)\widehat{P})^g = \int (P_\Theta(A_\Theta + T_\Theta)P_\Theta)^g d\mu. \tag{3.22}$$

The relationship expressed by Theorems 3.1 and 3.1* is in our view fundamental in that they relate, via unitarity U, the physical operator $P(A + T_0)P$ on $(\mathcal{L}^2(\Omega))^3$ with the Θ-periodic objects $P_\Theta(A_\Theta + T_\Theta)P_\Theta$ via the concept of direct integral. How to exploit this relationship will be seen in the next subsection.

Since the equations in Theorems 3.1 and 3.1* remain invariant under multiplication with a scalar $\nu > 0$, we can replace A by νA, since the factor ν may be put into the equilibrium solution (u_1, u_2, u_3) defining T_0.

3.3 Proof of Lemma 3.4. The proof of Lemma 3.4 is based on Lemma 3.1 and three remarks. First, since $P + Q = \mathrm{Id}$ and $P_\Theta + Q_\Theta = \mathrm{Id}$ it suffices to prove

$$\widehat{Q} = \widetilde{Q} = \int Q_\Theta \, d\mu. \tag{3.23}$$

The measurability of the family $\{Q_\Theta\}_{\Theta \in M}$ is an easy consequence of the Fourier series representation in Subsection 2.4, but it is also contained in the arguments below and is not further discussed. The third remark is given by

Proposition 3.6. *There is a set* $\mathcal{S} \subseteq (H^1(\Omega))^2 \times H^1_0(\Omega))$ *which is dense in* $(\mathcal{L}^2(Q))^3$ *and such that* $f \in \mathcal{S}$ *implies*

$$\Delta p = \operatorname{div} f \quad \textit{and} \quad Qf = \nabla p \qquad \textit{for some } \; p \in \widehat{H}^2(\Omega).$$

Proof. By the arguments in Subsection 1.5 $\mathcal{S}$ may be taken as the union $\mathcal{S}_1 \cup \mathcal{S}_2$ where

(a) $\mathcal{S}_1 \subseteq (H^1(\Omega) \cap \widehat{\mathcal{L}}^2(\Omega))^2 \times H^1_0(\Omega)$

(b) $\mathcal{S}_2$ is the set of $f = (A\varphi_0, B\varphi_0, 0)$ with $A, B \in H^2(\mathbb{R}^2)$

whose Fourier transforms $\widehat{A}$ and $\widehat{B}$ have compact support in $\mathbb{R}^2 \setminus \{0\}$ ∎

By Proposition 3.6, $\widehat{Q} = \widetilde{Q}$ if they coincide on the set US. Thus the proof of (3.23) and hence of Lemma 3.4 reduces to

Proposition 3.7. *If $f \in S$, then $\widehat{Q}Uf = \widetilde{Q}Uf$.*

Proof. Let $f \in S$. By Proposition 3.6, f is in $(H^1(\Omega))^2 \times H_0^1(\Omega)$, and there is $p \in \widehat{H}^2(\Omega)$ such that

$$\Delta p = \operatorname{div} f \qquad \text{and} \qquad Qf = \nabla p. \tag{3.24}$$

By (3.5) and a repeated application of Lemma 3.1 we find a set $E \subseteq \dot{M}$ with $\mu(E) = \mu(M)$ such that $\Theta \in E$ implies

$$\begin{matrix} (Uf)(\Theta,\cdot) \in (H^1_\Theta)^2 \times H^1_{\Theta,0} \\ \\ (Vp)(\Theta,\cdot) \in \widehat{H}^2_\Theta \end{matrix} \qquad \text{and} \qquad \begin{matrix} (V\operatorname{div} f)(\Theta,\cdot) = \operatorname{div}(Uf)(\Theta,\cdot) \\ \nabla(Vp)(\Theta,\cdot) = U(\nabla p)(\Theta,\cdot) \\ \Delta(Vp)(\Theta,\cdot) = V(\Delta p)(\Theta,\cdot). \end{matrix} \tag{3.25}$$

Exploiting the commutativity expressed by the right three equations one finds by straightforward computation for $\Theta \in E$ that

$$\operatorname{div}(Uf)(\Theta,\cdot) = \Delta(Vp)(\Theta,\cdot). \tag{3.26}$$

By the first and second relation in (3.25), by (3.26) and Subsection 2.5 we infer

$$Q_\Theta(Uf)(\Theta,\cdot) = \nabla(Vp)(\Theta,\cdot). \tag{3.27}$$

On the other hand, since $\widehat{Q}Uf = UQf$, and by exploiting once more the commutativity in (3.25) we find

$$(\widehat{Q}Uf)(\Theta,\cdot) = U(\nabla p)(\Theta,\cdot) = \nabla(Vp)(\Theta,\cdot). \tag{3.28}$$

Thus by (3.27) and (3.28)

$$Q_\Theta(Uf)(\Theta,\cdot) = (\widehat{Q}Uf)(\Theta,\cdot) \qquad (\Theta \in E)$$

whence $\widetilde{Q}Uf = \widehat{Q}Uf$ by definition ∎

4. Spectral relations

4.1 Holomorphic considerations. If we would be generous we would claim that, with Theorems 3.1 and 3.1* at disposal, we can proceed as in [12] in order to infer the validity of (B) and (C) in Section 1. While the situation is not so simple, we intend to restore it such that the results in [12] become applicable.

In order to simplify the presentation we assume that the equilibrium solution which defines T_0 via (1.3) is in L^2_g, i.e. satisfies (3.21). In this case, the spaces L^2_g and L^2_u are invariant under $P_\Theta(A_s(\Theta) + T_0)P_\Theta$, what allows us to treat them seperately, allowing some simplifications. We treat the difficult case L^2_g, contenting us with some remarks as to L^2_u. Our first aim is to extend P_Θ and $A_s(\Theta)^{-1}P_\Theta$ into the complex. To this end we set henceforth $\nu = \nu(\alpha,\beta) = \widehat{\alpha}^2 + \widehat{\beta}^2$ and replace the complex domain $\mathcal{M}_\varepsilon$ (see Subsection 2.3) by a smaller one as follows. With $\Theta_0 \in \dot{M}_\varepsilon$ we associate a complex spherical neighbourhood $\mathcal{U}_{\Theta_0}$ so small that

(i) $\mathcal{U}_{\Theta_0} \subseteq \mathcal{M}_\varepsilon$.

(ii) $|\widehat{\alpha}^2\nu^{-1}|, |\widehat{\alpha}\widehat{\beta}\nu^{-1}|, |\widehat{\beta}^2\nu^{-1}| \le \frac{3}{2}$ for $\Theta \in \mathcal{U}_{\Theta_0}$ and $\alpha,\beta \in \mathbb{Z}$.

Note that in case of (ii), only $\alpha,\beta \in \{0,-1\}$ have to be considered. We then set

$$\widetilde{\mathcal{M}}_\varepsilon = \cup_\Theta \mathcal{U}_\Theta \qquad (\Theta \in \dot{M}_\varepsilon). \tag{4.1}$$

The purpose of the shrinking $\mathcal{M}_\varepsilon \to \widetilde{\mathcal{M}}_\varepsilon$ is to keep the factor ν^{-1} under control.

Lemma 4.1. *There are holomorphic families $\{R_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$ and $\{P'_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$ of bounded linear operators on L^2_g such that $P'_\Theta = P^g_\Theta$ and $R_\Theta = A_s(\Theta)^{-1}P^g_\Theta$ if $\Theta \in \dot{M}_\varepsilon$.*

Remark. As to holomorphy we refer to [3: pp. 365 - 366] and the remarks in [12]. Since weak, strong and uniform holomorphy coincide, we simply speak of holomorphy. The proof, which yields more information than provided by the lemma, is routine but tedious in detail and may be skipped in first reading. We do not formalize every step but content us with an outline.

We first aim at P'_Θ and recall the factor $m(\Theta,\cdot)$ (see (2.4)), the $\widetilde{e}_{\alpha\beta}$ (see (2.5)) and S, S' (see Subsection 2.3). Next we fix $f \in \mathcal{L}^2_g$ and $\Theta \in \dot{M}_\varepsilon$. The Fourier coefficients with respect to $e_{\alpha\beta}\tau_j$ are given by

$$f_{\alpha\beta j} = (e_{\alpha\beta}\tau_j, f)_0 = (\widetilde{e}_{\alpha\beta}\tau_j, m(\Theta,\cdot)f)_0. \tag{4.2}$$

If we allow $\Theta \in \mathcal{M}_\varepsilon$, then (4.2) still makes sense and it follows that the family of mappings $\{M^g_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ such that $M^g_\Theta f = \{f_{\alpha\beta j}\}$ is a holomorphic family of bounded operators from $\mathcal{L}^2_g$ onto S, which has a bounded holomorphic inverse $(M^g_\Theta)^{-1}$. Likewise we introduce holomorphic families $\{N^g_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$, $\{M^u_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ and $\{N^u_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ which perform analoguous tasks but: N^g_Θ with respect to $e_{\alpha\beta}\rho_j$, and for $f \in \mathcal{L}^2_u$, M^u_Θ with respect to $e_{\alpha\beta}\sigma_j$ and N^u_Θ with respect to $e_{\alpha\beta}\pi_j$. These mappings extend to the vector case, i.e. for $f = (a,b,c)$ in L^2_g we set

$$\begin{aligned} M_\Theta f &= (M^g_\Theta a, M^g_\Theta b, M^u_\Theta c) \in S^3 \\ N_\Theta f &= (N^g_\Theta a, N^g_\Theta b, N^u_\Theta c) \in S^2 \times S'. \end{aligned} \tag{4.3}$$

$\{M_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ and $\{N_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ are holomorphic families of operators from L^2_g onto S^3 and $S^2 \times S'$, respectively, having holomorphic inverses M^{-1}_Θ and N^{-1}_Θ. We note that there is a Θ-independent unitary map W from $S^2 \times S'$ onto S^3 such that

$$WN_\Theta = M_\Theta \qquad (\Theta \in \mathcal{M}_\varepsilon). \tag{4.4}$$

Next we consider the projections P_Θ and $Q_\Theta = 1 - P_\Theta$ described by (2.24). The restrictions P^g_Θ and Q^g_Θ to L^2_g are obtained by omitting the φ_{2j+1} in the first two, and the ψ_{2j+1} in the last of the series in (2.24), resulting in expansions in terms of $e_{\alpha\beta\rho_j}$ and $e_{\alpha\beta\pi_j}$, respectively. From this remark and an analysis of (2.24) we extract sets of functionals $P^i_{\alpha\beta j}$ and $Q^i_{\alpha\beta j}$ $(j \geq 0,\ i = 1,2,3)$ and $S_{\alpha\beta j}$ $(j \geq 1)$ for $\alpha, \beta \in \mathbb{Z}$ having property (P) of Definition 2.2, which describe P^g_Θ as follows. Given $f = (a,b,c)$ in L^2_g and $\Theta \in \dot{M}_\varepsilon$, let

$$P^g_\Theta f = (\mathcal{A}, \mathcal{B}, \mathcal{C})$$

with

$$\{A_{\alpha\beta j}\} = N^g_\Theta \mathcal{A}, \qquad \{B_{\alpha\beta j}\} = N^g_\Theta \mathcal{B}, \qquad \{C_{\alpha\beta j}\} = N^u_\Theta \mathcal{C}.$$

Let also $\underline{a} = N^g_\Theta a$, $\underline{b} = N^g_\Theta b$ and $\underline{c} = N^u_\Theta c$. Then

$$A_{\alpha\beta j} = \frac{\widehat{\alpha}^2}{\nu} P^1_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c}) + \frac{\widehat{\alpha}\widehat{\beta}}{\nu} P^2_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c}) + P^3_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c}) \tag{4.5}$$

and likewise with $B_{\alpha\beta j}$, $Q^i_{\alpha\beta j}$ and $C_{\alpha\beta j} = S_{\alpha\beta j}(\Theta, \underline{a}, \underline{b}, \underline{c})$. That is we have a description of P^g_Θ in terms of functionals similar to that of $A_s(\Theta)^{-1}$ via Theorem 2. If we drop the condition $\Theta \in \dot{M}_\varepsilon$, i.e. admit $\Theta \in \widetilde{\mathcal{M}}_\varepsilon$ and let $\underline{a}, \underline{b}$ and $\underline{c}$ range over $S^2 \times S'$, then we see that the right-hand side of (4.5) defines a holomorphic family $\{\widetilde{P}_\Theta\}_\Theta$ of bounded linear operators from $S^2 \times S'$ into $S^2 \times S'$ which in case that $\Theta \in \dot{M}_\varepsilon$ is tied to P^g_Θ via

$$\widetilde{P}^g_\Theta N_\Theta f = N_\Theta P^g_\Theta f \qquad (\Theta \in \dot{M}_\varepsilon,\ f \in L^2_g) \tag{4.6}$$

from which we extract the holomorphy family $\{P'_\Theta\}_\Theta$ of Lemma 4.1 according to

$$P'_\Theta = N_\Theta^{-1} \widetilde{P}^g_\Theta N_\Theta \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon). \tag{4.7}$$

Next we come to $A_s(\Theta)^{-1}$ on E^g_Θ, i.e. to its description in Theorem 2.1. Here too we can look at the systems (i) - (iii) and of Theorem 2 as describing a holomorphic family $\{\widetilde{R}_\Theta\}_{\Theta \in \widetilde{\mathcal{M}}_\varepsilon}$ of bounded operators from S^3 to S^3, which is tied to $A_s(\Theta)^{-1}$ as follows: if $\Theta \in \dot{M}_\varepsilon$ and $f = (a,b,c) \in E^g_\Theta$, then

$$A_s(\Theta)^{-1} f = M_\Theta^{-1} \widetilde{R}_\Theta M_\Theta f. \tag{4.8}$$

The holomorphic extension from $\dot{M}_\varepsilon$ to $\widetilde{\mathcal{M}}_\varepsilon$ of $A_s(\Theta)^{-1} P^g_\Theta$ is then given by

$$R_\Theta = M_\Theta^{-1} \widetilde{R}_\Theta W \widetilde{P}^g_\Theta N_\Theta \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon) \tag{4.9}$$

where (4.4) and (4.7) has been used.

Next we discuss formal properties of the extensions R_Θ and P'_Θ, refraining thereby from an analysis into elementary steps. An examination of (i) - (iii) in Theorem 2.1 on the basis of Definition 2.2 shows that given $f \in L^2_g$, $M_\Theta^{-1} \widetilde{R}_\Theta M_\Theta f$ is in $(H^2(Q))^3$ for $\Theta \in \widetilde{\mathcal{M}}_\varepsilon$, and that

$$\|M_\Theta^{-1} \widetilde{R}_\Theta M_\Theta f\|_{H^2} \leq C \|f\|_{L^2} \tag{4.10}$$

for some positive Θ-independent constant C. The procedure in this connection is to look at the appearing Fourier series, e.g. $\sum A_{\alpha\beta j} e_{\alpha\beta\tau_j}$, not as an expansion $e_{\alpha\beta\tau_j}$, which for complex $\Theta \in \widetilde{\mathcal{M}}_\varepsilon$ is not an orthonormal system, but rather as an ordinary Fourier expansion in $\widetilde{e}_{\alpha\beta\tau_j}$, i.e. of the form $\frac{1}{m}\sum A_{\alpha\beta j}\widetilde{e}_{\alpha\beta\tau_j}$, and to apply to this series the arguments in Proposition 2.3 and the Corollaries to Theorems 2.1 and 2.1*, treating thereby m and $\frac{1}{m}$ as a smooth bounded multiplier. This remark also applies to the situations below.

Let I be the 3×3 unit matrix and $I\Delta$ the operator acting on its domain $(H^2(Q))^3$ componentwise like Δ. By termwise differentiation of the appearing Fourier series, one recognizes that $\{I\Delta M_\Theta^{-1}\widetilde{R}_\Theta M_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$ is a holomorphic family of bounded operators on L_g^2. Since a product of holomorphic factors is again holomorphic, we have that $\{P'_\Theta \Delta R_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$ is a holomorphic family.

Proposition 4.1. *The equalities*

(i) $(P'_\Theta)^2 = P'_\Theta$

(ii) $P'_\Theta\Delta R_\Theta = P'_\Theta$

are true.

Proof. We note that a scalar holomorphic function f on $\widetilde{\mathcal{M}}_\varepsilon$ which vanishes on $\dot{M}_\varepsilon$, vanishes on all of $\widetilde{\mathcal{M}}_\varepsilon$. This property is inherited by holomorphic families of bounded linear operators $\{B_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$. Now assume $\Theta \in \dot{M}_\varepsilon$. According to Lemma 4.1, $R_\Theta = A_s(\Theta)^{-1}P_\Theta^g$, while $P'_\Theta = P_\Theta^g$. The range of $A_s(\Theta)^{-1}P_\Theta^g$ is $\operatorname{dom} A_s(\Theta) \cap L_g^2$, i.e. $(H_\Theta^2 \cap H_{\Theta,0}^1)^3$, div $= 0$ intersected with L_g^2. But $P_\Theta^g\Delta$, restricted to $\operatorname{dom} A_s(\Theta)$, coincides with $A_s(\Theta)$, whence

$$P'_\Theta\Delta R_\Theta = A_s(\Theta)A_s(\Theta)^{-1}P_\Theta^g = P_\Theta^g = P'_\Theta \qquad (\Theta \in \dot{M}_\varepsilon). \tag{4.11}$$

By the preliminary remarks, this extends to all of $\widetilde{\mathcal{M}}_\varepsilon$, proving assertion (ii). Assertion (i) is treated likewise ∎

Next we consider the operator T_0, given by (1.3), supplied with $\operatorname{dom} T_0 = (H^2(Q))^3$. We recall assumption (3.21), according to which T_0 leaves L_g^2 invariant. By decomposing the action of T_0 into elementary steps, one is ultimately led to recognize $\{I\partial_x R_\Theta\}_\Theta$, $\{I\partial_y R_\Theta\}_\Theta$ and $\{I\partial_z R_\Theta\}_\Theta$ as holomorphic families of bounded linear operators on L_g^2, with values in L_g^2 in the first two cases, and in L_u^2 in the third one. The procedure is again by termwise differentiation of the Fourier series in Theorem 2.1. Since smooth, Θ-independent factors preserve holomorphy, we obtain

Proposition 4.2. $\{T_0 R_\Theta\}_{\Theta\in\mathcal{M}_\varepsilon}$ *and* $\{P'_\Theta T_0 R_\Theta\}_{\Theta\in\widetilde{\mathcal{M}}_\varepsilon}$ *are holomorphic families of bounded operators on* L_g^2.

4.2 Resolvents. In [12], the notion of a strongly holomorphic family of unbounded operators (see Rellich [9]) was used in the proofs of the basic theorems. This notion requires that all operators have the same domain of definition. This is definitely not so in the present case, what forces us to use the more general notion of a holomorphic family in [3: p. 366].

In order to study resolvents from this point of view we first note that by Lemma 4.1,

$$Q'_\Theta = (1 - P'_\Theta) \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon) \tag{4.12}$$

is a holomorphic extension of Q^g_Θ into $\widetilde{\mathcal{M}}_\varepsilon$. We now fix $0 \neq \tau \in \mathbb{R}$ arbitrarily for the moment and consider the holomorphic family

$$V_\Theta = P'_\Theta + P'_\Theta T_0 R_\Theta + \tau Q'_\Theta \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon). \tag{4.13}$$

The operators, in whose resolvents we are primarily interested are

$$\widetilde{H}_\Theta = P^g_\Theta(\mathcal{A}s(\Theta) + T_0)P^g_\Theta \qquad (\Theta \in \dot{M}_\varepsilon) \tag{4.14}$$

but it is advantageous to study instead the resolvents of

$$H_\Theta = \widetilde{H}_\Theta + \tau Q^g_\Theta \qquad (\Theta \in \dot{M}_\varepsilon). \tag{4.15}$$

For simplicity of notation we have suppressed the τ in $\widetilde{H}_\Theta, H_\Theta$ and V_Θ. Note also that

$$0 = P'_\Theta Q'_\Theta = Q'_\Theta P'_\Theta = R_\Theta Q'_\Theta = Q'_\Theta R_\Theta \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon) \tag{4.16}$$

holds by holomorphic extension from $\Theta \in \dot{M}_\varepsilon$ by virtue of Lemma 4.1. By Lemma 4.1, Proposition 4.1 and the relations (4.16) a simple computation shows that for $\Theta \in \dot{M}_\varepsilon$, V_Θ admits the factorization

$$V_\Theta = H_\Theta(R_\Theta + Q'_\Theta) \qquad (\Theta \in \dot{M}_\varepsilon). \tag{4.17}$$

Note that no holomorphic extension of H_Θ into $\widetilde{\mathcal{M}}_\varepsilon$ is defined, since no such extension is defined for $A_s(\Theta)$. We also note that for $\Theta \in \dot{M}_\varepsilon$ we have the implication

$$\lambda \neq \tau \quad \Longrightarrow \quad \left\{\lambda \in \rho_{E_\Theta}(\widetilde{H}_\Theta) \;\Leftrightarrow\; \lambda \in \rho_{L^2_g}(H_\Theta)\right\}. \tag{4.18}$$

Lemma 4.2. *Let $\Theta_0 \in \dot{M}_\varepsilon$, $\lambda_0 \in \rho_{E_\Theta}(\widetilde{H}_\Theta)$ and $\lambda_0 \neq \tau$. Then there are complex neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of Θ_0 and λ_0, respectively, with $\mathcal{U} \subseteq \widetilde{\mathcal{M}}_\varepsilon$ such that $\left(V_\Theta - \lambda(R_\Theta + Q'_\Theta)\right)^{-1}$ exists on L^2_g for $\Theta \in U$ and $\lambda \in \mathcal{V}$, and depends holomorphically on Θ and λ.*

Proof. We recall the relations (4.16) according to which

$$(P'_\Theta + Q'_\Theta)(R_\Theta + Q'_\Theta) = (R_\Theta + Q'_\Theta) \qquad (\Theta \in \widetilde{\mathcal{M}}_\varepsilon). \tag{4.19}$$

Using the factorization (4.17) and (4.19), we get the identity

$$V_\Theta - \lambda(R_\Theta + Q'_\Theta) = \left(H_\Theta - \lambda(P'_\Theta + Q'_\Theta)\right)(R_\Theta + Q'_\Theta) \tag{4.20}$$

for $\Theta \in \dot{M}_\varepsilon$. For $\Theta \in \dot{M}_\varepsilon$ we have $R_\Theta = A_s(\Theta)^{-1}P_\Theta^g$ by Lemma 4.1 what implies that R_Θ, restricted to E_Θ^g, maps E_Θ^g one-to-one onto dom $A_s(\Theta) \cap L_g^2$, i.e. dom $A_s(\Theta) \cap E_\Theta^g$. This in turn implies that $R_\Theta + Q'_\Theta$ maps L_g^2 one-to-one onto its range

$$\big(\mathrm{dom}\, A_s(\Theta) \cap E_\Theta^g\big) \times (L_g^2 \ominus E_\Theta^g). \tag{4.21}$$

On the other hand it follows from the definitions (4.14) and (4.15) that dom H_Θ and dom $(H_\Theta - \lambda)$ are given by (4.21). From our assumptions on Θ_0, λ_0 and τ and according to (4.18) we also have that $H_{\Theta_0} - \lambda_0$ maps its domain (i.e. (4.21)) one-to-one onto L_g^2. This, together with the remark previous to (4.21) and by clause (4.20) implies that $V_{\Theta_0} - \lambda_0(R_{\Theta_0} + Q'_{\Theta_0})$ maps L_g^2 one-to-one onto itself. We now invoke [3: Theorem 1.3/p. 367] according to which there are complex neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of Θ_0 and λ_0, respectively, with $\mathcal{U} \subseteq \widetilde{M}_\varepsilon$, such that the inverse of $V_\Theta - \lambda(R_\Theta + Q'_\Theta)$ exists for $\Theta \in \mathcal{U}$ and $\lambda \in \mathcal{V}$, and depends holomorphically on Θ and λ ∎

Corollary. *Let λ_0, Θ_0 and τ satisfy the assumptions of Lemma* 8. *Then there is a real neighbourhood $\mathcal{U}' \ni \Theta_0$ with $\mathcal{U}' \subseteq \dot{M}_\varepsilon$, and a complex neighbourhood $\mathcal{V} \ni \lambda_0$ such that $(H_\Theta - \lambda)^{-1}$ exists (i.e. $\lambda \in \rho_{L_g^2}(H_\Theta)$) on $\mathcal{U}' \times \mathcal{V}$ and is simultaneously real analytic in Θ and complex analytic in λ.*

Proof. By the assumptions, Lemma 4.1 is applicable, giving rise to complex neighbourhoods $\mathcal{U}_0 \ni \Theta_0$ and $\mathcal{V}_0 \ni \lambda_0$ having the properties of the lemma. Set $\mathcal{U}' = \mathcal{U}_0 \cap \dot{M}_\varepsilon$, and let $\Theta \in \mathcal{U}'$ and $\lambda \in \mathcal{V}_0$. From the lemma, the factorization (4.20) and the remark prior to (4.21) we infer that $(H_\Theta - \lambda)^{-1}$ exists as a bounded linear operator on L_g^2 and is given by

$$(H_\Theta - \lambda)^{-1} = (R_\Theta + Q'_\Theta)\big(V_\Theta - \lambda(R_\Theta + Q'_\Theta)\big)^{-1}. \tag{4.22}$$

The statement then follows from Lemma 4.1 ∎

Remark. The invoked Theorem 1.3 in [3: p. 367] is expressed for one complex variable only, but a technical check shows that it extends straightforwardly to several variables. We conclude with a statement on compactness.

Proposition 4.3. *Let $\Theta \in \dot{M}_\varepsilon$. Then $A_s(\Theta) + P_\Theta T_0$ has compact resolvents on E_Θ^g and E_Θ^u.*

Proof. We consider E_Θ^g. By Theorem 2.1 we have

$$\|A_s(\Theta)^{-1} f\|_{H^2} \le C \|f\|_{\mathcal{L}^2} \qquad (f \in E_\Theta^g)$$

for a positive Θ-independent constant C. Since Q, i.e. $(0, L)^2 \times (-\frac{1}{2}, +\frac{1}{2})$ has dimension $n = 3$, compact embeddings (see [1: p. 144]) imply that $A_s(\Theta)^{-1}$ is compact on E_Θ^g. It then follows from the resolvent formula that all resolvents of $A_s(\Theta)$ (on E_Θ^g) are compact. Next we recall the Corollary to Theorems 2.1 and 2.1* according to which $P_\Theta^g T_0$, restricted to E_Θ^g, is bounded relative to $A_s(\Theta)$ with relative bound zero. By [7: p. 80] we have that $\lambda \in \mathbb{R}$ sufficiently large is in the resolvent set of $A_s(\Theta) + P_\Theta^g T_0$ and that its resolvent has the form

$$\big(A_s(\Theta) + P_\Theta^g T_0 - \lambda\big)^{-1} = \big(A_s(\Theta) - \lambda\big)^{-1} B(\lambda, \Theta) \tag{4.23}$$

with $B(\lambda, \Theta)$ bounded on E_Θ^g. Since $A_s(\Theta)$ has compact resolvents on E_Θ^g, the compactness of the left-hand side of (4.23) follows. For arbitrary λ's in the resolvent set of $A_s(\Theta) + P_\Theta^g T_0$ (on E_Θ^g) the claim now follows from the resolvent formula. The proof for E_Θ^u via Theorem 2* is the same ∎

Remark. Formula (4.23), given explicitely by [7: Formula (2.3)], and the Θ-independence of the constants C and K_ϵ in Theorems 2.1 and 2.1* and their Corollaries imply the existence of constants γ and $C' > 0$ such that

$$\left\|(A_s(\Theta) + P_\Theta^g T_0 - \lambda)^{-1}\right\|_\infty \le C' \qquad (\Theta \in \dot{M}_\epsilon,\ \lambda \ge \gamma). \tag{4.24}$$

4.3 Local spectral relations. We now come to a first result relating the spectrum of $A_s(\Theta) + P_\Theta^g T_0$ (on E_Θ^g) with that of $A_s + P^g T_0$ (on E_g); we thereby use material from [12].

We recall that $\dot{M}$ is $M = [0, 2\pi]^2$ minus the corners, H_Θ and $\widetilde{H}_\Theta$ are as in (4.14) and (4.15). By $\int_M \cdot\, d\mu$ we always denote a direct integral with fibre space $\mathcal{H}_g''$ and values in $\mathcal{H}_g^*$ (see Remarks prior to Theorem 3.1*).

Theorem 4.1. *Let $0 \ne \tau \ne \lambda$. If λ is in the spectrum of H_{Θ_0} for some $\Theta_0 \in \dot{M}$, then λ is in the spectrum of $\int_M H_\Theta\, d\mu$.*

Corollary. *Let, for some $\Theta_0 \in \dot{M}$, λ be in the spectrum of $\widetilde{H}_{\Theta_0}$ (on $E_{\Theta_0}^g$). Then λ is in the spectrum of $As + P^g T_0$ (on E_g).*

Proof. Fix $0 \ne \tau \ne \lambda$. By our assumption and (4.18), λ is in the spectrum of H_{Θ_0} and that of

$$\int_M (P_\Theta^g (A_s(\Theta) + T_0) P_\Theta^g + \tau Q_\Theta^g)\, d\mu = \int_M H_\Theta\, d\mu \tag{4.25}$$

by Theorem 4.1. By Theorem 3.1* and (4.25), $\int_M H_\Theta\, d\mu$ is unitarily equivalent to

$$P^g(A_s + T_0)P^g + \tau Q^g \tag{4.26}$$

and hence λ is in the spectrum of the last operator. Since $\lambda \ne \tau$, we have by a remark similar to (4.18) that λ is in the spectrum of $P^g(A_s + T_0)P^g$ (on E_g), proving the corollary ∎

Theorem 4.1 is a consequence of two facts, the first of which follows directly from (4.24): Given $\tau \ne 0$ there are positive constants C and γ_0 with $\gamma_0 \ne \tau$ such that

$$\|(H_\Theta - \lambda)^{-1}\|_\infty \le C \qquad \text{for } \lambda \ge \gamma_0,\ \Theta \in \dot{M}. \tag{4.27}$$

Below, a set $\mathcal{U} \subseteq \dot{M}$ is relative open if $\mathcal{U} = \mathcal{U}' \cap \dot{M}$ for some open set $\mathcal{U}'$. The other fact is

Lemma 4.3. *Let $0 \ne \tau \ne \lambda_0$ and assume that, for some $\Theta_0 \in \dot{M}$, λ_0 is in the spectrum of H_{Θ_0}. Then there is a relatively open neighbourhood $\mathcal{U}_0$ of Θ_0, a mapping λ from $\mathcal{U}_0$ into $\mathbb{C}$ and a measurable mapping φ, mapping $\Theta \in \mathcal{U}_0$ into $\operatorname{dom} H_\Theta$ such that:*

(i) *$\lambda(\Theta_0) = \lambda_0$, and λ is continuous at Θ_0*

(ii) $H_\Theta \varphi(\Theta) = \lambda(\Theta)\varphi(\Theta)$ $(\Theta \in \mathcal{U}_0)$

(iii) *There are constants $a, b > 0$ such that $a \le \|\varphi(\Theta)\| \le b$ for all $\Theta \in \mathcal{U}_0$.*

The proof of Theorem 4.1 via (4.27) and Lemma 4.3 is by purely measure-theoretic reasoning and is given in [12: Proof of Theorem 1]. We now rephrase Lemma 4.3. First we note that by Lemma 4.2, its Corollary and (4.27) there is a complex neighbourhood $\mathcal{U}_1$ of Θ_0, ($\mathcal{U}_1 \subseteq \widetilde{\mathcal{M}}_\epsilon$) and a holomorphic family of bounded operators $\{F_\Theta\}_{\Theta\in\mathcal{U}_1}$ such that

$$F_\Theta = (H_\Theta - \gamma_0)^{-1} \qquad \text{for } \ \Theta \in \mathcal{U}_1 \cap \dot{M}_\epsilon \tag{4.28}$$

with γ_0 as in (4.27) and Θ_0 as in Lemma 4.3; we denote the complex extension F_Θ also by $(H_\Theta - \gamma_0)^{-1}$. Lemma 4.3 is easily seen to be equivalent to

Lemma 4.3*. *Let the assumptions of Lemma 4.3 hold and set $\mu_0 = (\lambda_0 - \gamma_0)^{-1}$. There is a relatively open neighbourhood $\mathcal{U}_0$ of Θ_0, a mapping δ from $\mathcal{U}_0$ into $\mathbb{C}$ and a measurable mapping φ from $\mathcal{U}_0$ into L^2_g, defined for all $\Theta \in \mathcal{U}_0$, such that:*

(i) $(H_\Theta - \gamma_0)^{-1}\varphi(\Theta) = (\mu_0 + \delta(\Theta))\varphi(\Theta) \ \ (\Theta \in \mathcal{U}_0)$

(ii) $\delta(\Theta_0) = 0$ *and* δ *is continuous at* Θ_0

(iii) $a \leq \|\varphi(\Theta)\| \leq b \ \ (\Theta \in \mathcal{U}_0)$ *for some constants* $a, b > 0$.

In order to investigate Lemma 4.3* we note that according to (4.13) and (4.14) we have

$$(H_{\Theta_0} - \gamma_0)^{-1} = (\widetilde{H}_{\Theta_0} - \gamma_0)^{-1} P^g_{\Theta_0} + (\tau - \gamma_0)^{-1} Q^g_{\Theta_0} \tag{4.29}$$

where $(\widetilde{H}_{\Theta_0} - \gamma_0)^{-1}$ is compact on $E^g_{\Theta_0}$. Now μ_0 is in the spectrum of $(\widetilde{H}_{\Theta_0} - \gamma_0)^{-1}$; since $\mu_0 \neq (\tau - \gamma_0)^{-1}$ and by (4.29), μ_0 is an eigenvalue of the compact operator $(\widetilde{H}_{\Theta_0} - \gamma_0)^{-1}$, which we denote temporarily by T. By the spectral theory for compact operators there are closed subspaces $\mathcal{L}$ and $\mathcal{N}$ of $E^g_{\Theta_0}$ with $\dim \mathcal{L} = N < \infty$, a basis $\{e_j\}_{j=1,\ldots,N}$ of $\mathcal{L}$ and functionals $e_j^* \in (E^g_{\Theta_0})^*$ $(j \leq N)$ such that:

(i) $\mathcal{L} \oplus \mathcal{N} = E^g_{\Theta_0}$

(ii) $\mathcal{L}$ and $\mathcal{N}$ are invariant under $T - \mu_0$

(iii) The restriction of $T - \mu_0$ to $\mathcal{L}$ is idempotent (i.e. $(T - \mu_0)^n = 0$ for some n)

(iv) The restriction of $T - \mu_0$ to $\mathcal{N}$ is boundedly invertible, i.e. maps $\mathcal{N}$ surjectively onto $\mathcal{N}$

(v) If $f \in E^g_{\Theta_0}$, then $f \in \mathcal{N}$ if and only if $\langle e_j^*, f\rangle = 0 \ \ (j = 1, \ldots, N)$.

Now $(H_{\Theta_0} - \gamma_0)^{-1}$ (denoted temporarily by $\widehat{T}$) is not compact but due to (4.29) inherits the above structure. That is, closed subspaces $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{N}}$ of L^2_g with $\dim \widehat{\mathcal{L}} = N$, a basis $\{\widehat{e}_j\}_{j\leq N} \subset \widehat{\mathcal{L}}$ and functionals $\widehat{e}_j^* \in (L^2_g)^*$ are defined by:

(α) $\widehat{\mathcal{L}} = \mathcal{L}$

(β) $\widehat{\mathcal{N}} = \mathcal{N} \oplus (L^2_g \ominus E^g_{\Theta_0})$

(γ) $e_j = \widehat{e}_j \ \ (j \leq N)$

(δ) If $z = x + y$ with $x \in E^g_{\Theta_0}$ and $y \in L^2_g \ominus E^g_{\Theta_0}$, then $\widehat{e}_j^*(z) = \widehat{e}_j^*(x) = e_j^*(x)$.

The objects $\widehat{\mathcal{L}}, \widehat{\mathcal{N}}$ and $\widehat{e}_j, \widehat{e}_j^*$ then satisfy properties (i) - (v) above with respect to $\widehat{T}_0 - \mu_0$ and L^2_g. An inspection of the proof of Lemma 8 (i.e. Proposition 4) in [12]) shows that this proof requires only the above structure plus the analytic extension property (4.28) in order to carry out the perturbation theoretic arguments. This proof therefore carries over to the present situation in a verbatim way and provides, as it stands, a proof of Lemma 4.3* and hence of Theorem 4.1 and its Corollary ∎

4.4 The corners. While Theorem 4.1 has to do with Θ's in $\dot{M}$, the corners of M need special consideration. It follows from Theorem 2.1 in case of $A_s(\Theta)^{-1}$ and from (4.5) in the case of pressure that if the integers α, β and j which label the Fourier coefficients, assume certain specific values, i.e. $j = 0$ and $\alpha, \beta \in \{0, -1\}$, then the corresponding Fourier coefficients are affected by singular factors, which in case of $\alpha = 0$ and $\beta = 0$ are of the form

$$\Theta_1^2\nu^{-1}, \qquad \Theta_2^2\nu^{-1}, \qquad \Theta_1\Theta_2\nu^{-1} \qquad \text{where } \nu = \Theta_1^2 + \Theta_2^2 \tag{4.30}$$

and with similar factors in the other cases. It suffices to investigate the case $\alpha = \beta = 0$ since $A_s(\Theta)$ and P_Θ, P_Θ^g are easily seen to be 2π-periodic in $\dot{M}_\varepsilon$, i.e. if $\Theta_0, \Theta_1 \in \dot{M}_\varepsilon$ and, e.g., $\Theta_1 = \Theta_0 + (2\pi, 0)$, then $A_s(\Theta_0) = A_s(\Theta_1)$, $P_{\Theta_0} = P_{\Theta_1}$, etc. In order to get rid of the singularities in (4.30), we make the substitution $\Theta_1 = r\cos\theta$ and $\Theta_2 = r\sin\theta$, where θ is in a small complex neighbourhood $\mathcal{W}$ of $[0, 2\pi]$ while the complex r satisfies $|r| < \varepsilon_0$ for some small ε_0. The real case is $r \in [0, \varepsilon_0)$ and $\theta \in [0, 2\pi]$, and we are in $\dot{M}_\varepsilon$ if also $r \neq 0$. An inspection of Definition 2.2, the representation of $A_s(\Theta)^{-1}$ in Theorem 2.1 and of P_Θ^g in (4.5) shows that by this substitution holomorphic families $\{R(r,\theta)\}_{r,\theta}$ and $\{P'_{r\theta}\}_{r,\theta}$ are defined for $|r| < \varepsilon_0$ and $\theta \in \mathcal{W}$ such that

$$R(r,\theta) = A_s(\Theta)^{-1}P_\Theta^g \qquad \text{and} \qquad P'_{r\theta} = P_\Theta^g \tag{4.31}$$

for $r \in (0, \varepsilon_0)$, $\theta \in [0, 2\pi]$ and $\Theta = (r\cos\theta, r\sin\theta)$. In order to study the limiting case $R(0,\theta)$ and $P'_{0\theta}$ we fix $\theta \in [0, 2\pi]$ and consider $\{R(r,\theta)\}$ and $\{P'_{r\theta}\}$ as a holomorphic families of bounded operators in one complex variable r, $|r| < \varepsilon_0$.

Notation. We write $\Theta = (r,\theta)$ if $\Theta = (\Theta_1, \Theta_2)$ and $\Theta_1 = r\cos\theta$, $\Theta_2 = r\sin\theta$. We also label operators and spaces by r and θ rather than by Θ; e.g. we write $P_{r\theta}^g$ instead of P_Θ^g and $R(r,\theta)$ instead of $R(\Theta)$.

Next we recall the two equivalent definitions of E_Θ and E_Θ^g, one given by (2.8), the other provided by Lemma 2.1. We rephrase the version in Lemma 2.1 slightly in terms of r and θ ($r \in [0, \varepsilon_0)$, $\theta \in [0, 2\pi]$). Let, to this end, $E_{r\theta}^0$ be the space of fields $(ae_{00}\rho_0, be_{00}\rho_0, 0)$ (recalling ρ_0 and π_j in (1.16)) such that

$$a\cos\theta + b\sin\theta = 0. \tag{4.32}$$

Let also $E_{r\theta}^1$ be the $\mathcal{L}^2$-closure of the linear hull of all fields

$$\begin{aligned} &\left(ae_{\alpha\beta}\rho_j, be_{\alpha\beta}\rho_j, ce_{\alpha\beta}\pi_j\right), \quad \alpha^2 + \beta^2 + j^2 > 0 \\ &\text{where } \gamma_{\alpha\beta j} = i\widehat{\alpha}a + i\widehat{\beta}b - c\sqrt{\mu_j} = 0. \end{aligned} \tag{4.33}$$

Then $E^0_{r\theta} \perp E^1_{r\theta}$ and $E^g_{r\theta} = E^1_{r\theta} \oplus E^0_{r\theta}$. From this representation one easily obtains a mild variant of definition (2.8) of $E^g_{r\theta}$, namely:

$E^g_{r\theta}$ is the $\mathcal{L}^2$-closure of all $f = (\mathcal{A},\mathcal{B},\mathcal{C})$ in L^2_g, which are in $(H^1_{\Theta,0})^3$ $(\Theta = (r,\theta))$, satisfy $\operatorname{div} f = 0$ and (4.32), where

$$a = (\mathcal{A}, e_{00}\rho_0) \qquad \text{and} \qquad b = (\mathcal{B}, e_{00}\rho_0). \tag{4.34}$$

In the periodic case $r = 0$, $e_{\alpha\beta}$ and $H^1_{\Theta,0}$ become $\widetilde{e}_{\alpha\beta}$ and $H^1_{\text{per},0}$, respectively. The spaces $E^g_{r\theta}, E^0_{r\theta}$ and $E^1_{r\theta}$ have orthogonal projections $P^g_{r\theta}, P^0_{r\theta}$ and $P^1_{r\theta}$ where $P^g_{r\theta}$ $(= P^g_{\Theta}, \Theta = (r,\theta))$ has the holomorphic extension $P'_{r\theta}$ into the complex $|r| < \varepsilon_0$, mentioned at the beginning of this subsection; evidently $P^g_{r\theta} = P^0_{r\theta} + P^1_{r\theta}$. For reference below we briefly describe $P^0_{r\theta}$. Given $f = (\mathcal{A},\mathcal{B},\mathcal{C})$ in L^2_g, $P^0_{r\theta}f$ has the form $(Ae_{00}\rho_0, Be_{00}\rho_0, 0)$ where

$$A = a\sin^2\theta - b\cos\theta\sin\theta \qquad \text{and} \qquad B = -a\cos\theta\sin\theta + b\cos^2\theta$$

with a and b given by (4.34).

For the use below we emphasize the dense set in the above definition of $E^g_{r\theta}$

$$D_{r\theta} = \left\{ f \in L^2_g \,\middle|\, \begin{array}{l} f \in (H^1_{\Theta,0})^3 \;\; (\Theta = (r,\theta)),\ \operatorname{div} f = 0 \\ f \text{ satisfy (4.32) with } a, b \text{ as in (4.34)} \end{array} \right\} \qquad \big(r \in [0,\varepsilon_0)\big). \tag{4.35}$$

We note in this connection $(\Theta = (r,\theta))$ the implication

$$r \in (0,\varepsilon_0) \quad\Longrightarrow\quad \operatorname{dom} A_s(r,\theta) = \operatorname{rg} R(r,\theta) = (H^2_{\Theta})^3 \cap D_{r\theta}. \tag{4.36}$$

Proposition 4.4. *The relation* $\operatorname{rg} R(0,\theta) \subseteq (H^2_{\text{per}})^3 \cap D_{0\theta}$ *holds.*

Proof. The substitution $\Theta_1 = r\cos\theta$, $\Theta_2 = r\sin\theta$ eliminates the singularities (4.30) which appear in the assertions (i) - (iii) of Theorem 2.1. An inspection based on Proposition 2.3 then shows that, after this substitution, the Fourier series given by (i) - (iii) define elements in $(H^2_{\Theta})^3$ $(\Theta = (r,\theta))$ for all $|r| < \varepsilon_0$. Next, it is easily established that an $f \in H^1_{\Theta}$ is in $H^1_{\Theta,0}$ if and only if

$$\int_Q (f\partial_z\varphi + \varphi\partial_z f)\, dz dx^2 = 0 \qquad \text{for all } \; \varphi \in H^1(Q).$$

By this characterization, $f \in (H^1_{\Theta})^3$ is in $(H^1_{\Theta,0})^3$ if and only if

$$\langle \partial_z g, f\rangle + \langle g, \partial_z f\rangle = 0 \qquad \text{for all } \; g \in (H^1(Q))^3$$

where $\partial_z(u,v,w) = (\partial_z u, \partial_z v, \partial_z w)$. By (4.35) and (4.36) we have that

$$\langle \partial_z g, R(r,\theta)f\rangle + \langle g, \partial_z R(r,\theta)f\rangle = 0 \qquad \big(r \in (0,\varepsilon_0)\big) \tag{4.37}$$

for $f \in L^2_g$ and $g \in (H^1(Q))^3$. By the arguments in Subsection 4.1 and the remarks at the beginning of this subsection, the family $\{\partial_z R(r,\theta)\}_{|r|<\varepsilon_0}$ is holomorphic with values in L^2_u, whence it follows that each term in (4.37) is holomorphic in $|r| < \varepsilon_0$. The left-hand side of (4.37) is thus a holomorphic function which vanishes on $r \in (0, \varepsilon_0)$ and hence on all of $|r| < \varepsilon_0$, in particular at $r = 0$. Since $f \in L^2_g$ and $g \in (H^1(Q))^3$ are arbitrary, this proves the first of three conditions.

The other two conditions involved in the definition of $D_{r\theta}$, i.e. $\operatorname{div} R(r,\theta)f = 0$ for all $f \in L^2_g$ and the validity of (4.32), with a and b defined by (4.34) in terms of $R(r,\theta)f$, can be characterized in a similar way by expressions which are holomorphic on $|r| < \varepsilon_0$ and vanish on $r \in (0, \varepsilon_0)$. The expressions then vanish at $r = 0$ implying that the conditions in question hold for $R(0,\theta)f$ for all $f \in L^2_g$ ∎

We now define the "limit" of $A_s(r,\theta)$ as $r \downarrow 0$:

$$A_s(0,\theta) = P^g_{0\theta}\Delta \qquad \text{on } \operatorname{dom} A_s(0,\theta) = (H^2_{\text{per}})^3 \cap D_{0\theta}, \tag{4.38}$$

i.e. $A_s(0,\theta)$ is the restriction of $P^g_{0\theta}\Delta$ to $(H^2_{\text{per}})^3 \cap D_{0\theta}$. Next recall that Proposition 4.1/(ii) rewritten in terms of r and θ yields

$$P'_{r\theta}\Delta R(r,\theta) = P'_{r\theta} \qquad (|r| < \varepsilon_0). \tag{4.39}$$

In fact, both sides of (4.39) are holomorphic in $|r| < \varepsilon_0$ and (4.39) holds for $r \in (0, \varepsilon_0)$ by virtue of Proposition 4.1, whence it holds for $|r| < \varepsilon_0$ by analytic continuation, and thus for $r = 0$. This fact, combined with (4.38) and Proposition 4.4 yields

$$A_s(0,\theta)R(0,\theta) = P^g_{0\theta}. \tag{4.40}$$

On the other hand, it is easily seen that $As(0,\theta)$ is symmetric, densely defined on $E^g_{0\theta}$ and $A_s(0,\theta) \le -\varepsilon$ for some $\varepsilon > 0$. By (4.40) we now also have that $\operatorname{rg} A_s(0,\theta) = E^g_{0\theta}$ whence

$$A_s(0,\theta) \text{ is selfadjoint} \qquad \text{and} \qquad A_s(0,\theta)^{-1}P^g_{0\theta} = R(0,\theta). \tag{4.41}$$

Now $R(r,\theta) = A_s(r,\theta)^{-1}P^g_{r\theta}$, $r \in (0, \varepsilon_0)$, is compact by Proposition 24 and $R(0,\theta)$ as the uniform limit $\lim_{r\downarrow 0} R(r,\theta)$ of compact operators is compact, whence by (4.41)

$$A_s(0,\theta)^{-1} \qquad \text{is compact.} \tag{4.42}$$

It remains to show that $A_s(0,\theta) + P^g_{0\theta}T_0$ has compact resolvents. First note that by the same arguments in Subsection 4.3 we have that

$$\{T_0R(r,\theta)\}_{|r|<\varepsilon_0} \qquad \text{is a holomorphic family.} \tag{4.43}$$

Next we recall that by Corollary 2.2 we have

$$\|T_0R(r,\theta)f\| \le \varepsilon\|P^g_{r\theta}f\| + K_\varepsilon\|R(r,\theta)f\| \tag{4.44}$$

for $r \in (0, \varepsilon_0)$, $f \in L^2_g$ and a positive r-independent constant K_ε. Since the expressions between the norm signs are holomorphic in $|r| < \varepsilon_0$, we may let $r \downarrow 0$ in order to infer

$$\|T_0R(0,\theta)f\| \le \varepsilon\|P^g_{0\theta}f\| + K_\varepsilon\|R(0,\theta)f\|. \tag{4.45}$$

Now pick $\mathcal{U} \in \operatorname{dom} A_s(0,\theta))$ and set $f = A_s(0,\theta)\mathcal{U}$ in (4.45). We then get

$$\|T_0\mathcal{U}\| \le \varepsilon\|A_s(0,\theta)\mathcal{U}\| + K_\varepsilon\|\mathcal{U}\| \qquad (\mathcal{U} \in \operatorname{dom} A_s(0,\theta)). \tag{4.46}$$

With this crucial inequality at hand, we can repeat the arguments in the proof of Proposition 4.3 in order to infer that

$$A_s(0,\theta) + P^g_{0\theta}T_0 \qquad \text{has compact resolvents.} \tag{4.47}$$

We now have reached a point where we are precisely in the same situation as in Subsections 4.2 and 4.3 with the only difference that here we have holomorphic functions $R(r,\theta)$ and $P'_{r\theta}, Q'_{r\theta}$ in one complex variable $|r| < \varepsilon_0$ ($\theta \in [0,2\pi]$ has been kept fixed) while in Subsections 4.2 and 4.3 one has holomorphic families $\{R(\Theta)\}$ and $\{P'_\Theta\}, \{Q'_\Theta\}$ in the complex variables $\Theta = (\Theta_1,\Theta_2)$, in the complex neighbourhood of some $\Theta_0 \in \dot{M}_\varepsilon$. In all other respects the situation is the same and so we are entitled to draw the same conclusions.

In particular, Lemma 4.3 holds in the present setting. We recall $\widetilde{H}_\Theta$ and H_Θ in (4.15) which in terms of $r \in [0,\varepsilon_0)$ and θ are

$$\widetilde{H}_{r\theta} = P^g_{r\theta}\big(A_s(r,\theta) + T_0\big)P^g_{r\theta} \qquad \text{and} \qquad H_{r\theta} = \widetilde{H}_{r\theta} + \tau Q^g_{r\theta} \tag{4.48}$$

with $\tau \in \mathbb{R}$ to be fixed suitably.

Lemma 4.4. *Assume $0 \ne \tau \ne \lambda_0$ and λ_0 in the spectrum of $H_{0\theta}$. Then there is a relative neighbourhood $\mathcal{U} \subseteq [0,\varepsilon_0)$ of $r = 0$, a mapping λ from $\mathcal{U}$ into $\mathbb{C}$ and a measurable mapping φ which maps $r \in \mathcal{U}_0$ into $\varphi(r) \in \operatorname{dom} H_{r\theta}$ such that:*

(i) $\lambda(0) = \lambda_0$, *and λ is continuous at $r = 0$*

(ii) $H_{r\theta}\varphi(r) = \lambda(r)\varphi(r) \quad (r \in \mathcal{U}_0)$

(iii) $a \le \|\varphi(r)\| \le b \quad (r \in \mathcal{U}_0)$ *for some positive constants a and b.*

The proof is again via a variant Lemma 4.4* in resolvent form, whose proof follows by precisely the same arguments that succeed Lemma 4.3*. The basic conclusion is

Theorem 4.2. *Let λ_0 be in the spectrum of $A_s(0,\theta) + P^g_{0\theta}T_0$ for some $\theta \in [0,2\pi]$. Then λ_0 is in the spectrum of $A_s + P^gT_0$.*

Proof. Let first $\theta \in [0,\frac{\pi}{2}]$. Fix $0 \ne \tau \in \mathbb{R}$ with $\tau \ne \lambda_0$. By assumption, λ_0 is in the spectrum of $\widetilde{H}_{0\theta}$ and thus in the spectrum of $H_{0\theta}$ since $\lambda_0 \ne \tau$. By Lemma 4.4 there is a neighbourhood $\mathcal{U} \in [0,\varepsilon_0)$ of $r = 0$ and a mapping λ from $\mathcal{U}$ into $\mathbb{C}$, continuous at $r = 0$ such that $\lambda(r)$ is in the spectrum of $H_{r\theta}$ for $r \in \mathcal{U}$, and $\lambda(0) = \lambda_0$. We may assume that $\lambda(r) \ne \tau$ for $r \in \mathcal{U}$.

If $0 \ne r \in \mathcal{U}$, then $\Theta = (r,\theta)$ is in $\dot{M}$ and $\lambda(r)$ is also in the spectrum of $\widetilde{H}_{r\theta}$, since $\lambda(r) \ne \tau$. By the Corollary to Theorem 4.1, $\lambda(r)$ is then also in the spectrum of $A_s + P^gT_0$ and since $\lambda_0 = \lim_{r\downarrow 0}\lambda(r)$, λ_0 too is in the spectrum of $A_s + P^gT_0$.

In case where, e.g., $\theta \in [\pi, \frac{3\pi}{2}]$, let $\Theta = (r,\theta)$ and $\Theta' = \Theta + (2\pi,2\pi)$; for $0 \ne r \in \mathcal{U}$, Θ' is a point in $\dot{M}$. With λ as above and the periodicity properties mentioned at the beginning of this subsection, $\lambda(r)$ is then also in the spectrum of $\widetilde{H}_{\Theta'}$, if $0 \neq r \in \mathcal{U}$. By the Corollary to Theorem 4, $\lambda(r)$ is then in the spectrum of $A_s + P^g T_0$, and so is $\lambda_0 = \lim_{r \downarrow 0} \lambda(r)$.

The cases $\theta \in [\frac{\pi}{2}, \pi]$ and $\theta \in [\frac{3\pi}{2}, \pi]$ are handled similarly via translations $(2\pi, 0)$ and $(0, 2\pi)$ ∎

Corollary. *Assume that λ_0 is in the spectrum of $A_s(\Theta) + P^g_\Theta T_0$ (on E^g_Θ) for some $\Theta \in \dot{M}$ or in the spectrum of $A_s(0,\theta) + P^g_{0\theta} T_0$ (on $E^g_{0\theta}$) for some $\theta \in [0, 2\pi]$. Then λ_0 is in the spectrum of $A_s + P^g T_0$ (on E_g).*

Proof. Via Theorem 4.2 and Corollary to Theorem 4.1 ∎

The question is if there is more in the spectrum of $A_s + P^g T_0$ than provided by the corollary. The answer is "no". In order to see this, we let now range θ over the whole complex neighbourhood $\mathcal{W}$ of $[0, 2\pi]$ introduced at the beginning of this subsection. $\{R(r,\theta)\}_{r,\theta}$ and $\{P'_{r\theta}\}_{r,\theta}, \{Q'_{r\theta}\}_{r,\theta}$ are now holomorphic families of bounded operators on L^2_g for $|r| < \varepsilon_0$ and $\theta \in \mathcal{W}$. Likewise, by termwise differentiation of the appearing Fourier series it is easily recognized that $\{I\Delta R(r,\theta)\}_{r,\theta}$ and $\{T_0 R(r,\theta)\}_{r,\theta}$ are holomorphic families on $|r| < \varepsilon_0$ and $\theta \in \mathcal{W}$. The relevant identities such as (4.39) above then hold for $\{P'_{r\theta}\Delta R(r,\theta)\}_{r,\theta}$ in the complex neighbourhood $|r| < \varepsilon_0$ and $\theta \in \mathcal{W}$ by analytic continuation since they hold for $r \in [0, \varepsilon_0)$ and $\theta \in [0, 2\pi]$ (Proposition 4.1 and (4.39) for $r = 0$). With $\{R(r,\theta)\}_{r,\theta}$ and $\{P'_{r\theta}, Q'_{r\theta}\}_{r,\theta}$ on $|r| < \varepsilon_0$ and $\theta \in \mathcal{W}$ we are precisely in the same situation as with $\{R(\Theta)\}_\Theta$ and $\{P'_\Theta\}_\Theta, \{Q'_\Theta\}_\Theta$ on $\Theta \in \widetilde{\mathcal{M}}_\varepsilon$, what allows us to handle them in the same way and to draw the same conclusions. In particular we may set

$$
\begin{aligned}
V_{r\theta} &= P'_{r\theta} + P'_{r\theta} T_0 R(r,\theta) + \tau Q'_{r\theta} && (|r| < \varepsilon_0,\ \theta \in \mathcal{W}) \\
\widetilde{H}_{r\theta} &= P^g_{r\theta}\big(A_s(r,\theta) + T_0\big) P^g_{r\theta} && (r \in [0, \varepsilon_0),\ \theta \in [0, 2\pi]) \qquad (4.49) \\
H_{r\theta} &= \widetilde{H}_{r\theta} + \tau Q^g_{r\theta} && (0 \neq \tau \in \mathbb{R})
\end{aligned}
$$

with τ a free parameter to be fixed later. We now repeat the proof of Lemma 4.2 as it stands, obtaining a variant of Lemma 4.2 in terms of $V_{r\theta}, \widetilde{H}_{r\theta}$ and $H_{r\theta}$. We content us to state a corollary of this variant, which is an immediate consequence of it.

Lemma 4.5. *Let $0 \neq \tau \neq \lambda_0$ and assume that λ_0 is in the resolvent set of $H_{r_0\theta_0}$ for some $r_0 \in [0, \varepsilon_0)$ and $\theta_0 \in [0, 2\pi]$. Then there are positive δ_0 and δ_1 such that:*

(i) *If $|r - r_0| < \delta_0$ for $r \in [0, \varepsilon_0$ and $|\theta - \theta_0| < \delta_1$ for $\theta \in [0, 2\pi]$, then λ_0 is in the resolvent set of $H_{r\theta}$.*

(ii) *$(H_{r\theta} - \lambda_0)^{-1}$ is continuous on $|r - r_0| < \delta_0$ and $|\theta - \theta_0| < \delta_1$ in the uniform topology.*

Theorem 4.3. *Let λ_0 be in the resolvent set of $A_s(0,\theta) + P^g_{0\theta} T_0$ (on $E^g_{0\theta}$) for all $\theta \in [0, 2\pi]$ and in the resolvent set of $A_s(\Theta) + P^g_\Theta T_0$ (on E^g_Θ) for all $\Theta \in \dot{M}$. Then λ_0 is in the resolvent set of $A_s + P^g T_0$ (on E_g).*

Proof. We fix $0 \neq \tau \in \mathbb{R}$ with $\tau \neq \lambda_0$ and proceed in two steps.

Step (S1). Since $\tau \neq \lambda_0$ and by our assumption we have by clause (4.18), which holds in the present setting, that $\lambda_0 \in \rho(H_{0\theta})$ for all $\theta \in [0, 2\pi]$. We now apply Lemma 4.5 with $r_0 = 0$ in order to find via continuity for each $\theta_0 \in [0, 2\pi]$ positive constants $\delta(\Theta_0)$ and $\mu(\Theta_0)$ with the property: if $r \in [0, \mu(\theta_0))$ and $|\theta - \theta_0| < \delta(\theta_0)$ for $\theta \in [0, 2\pi]$, then $\lambda_0 \in \rho(H_{r\theta})$ and

$$\|(H_{r\theta} - \lambda_0)^{-1}\|_\infty \leq \|(H_{0\theta_0} - \lambda_0)^{-1}\|_\infty + 1. \tag{4.50}$$

By the Heine-Borel theorem there are finitely many $\theta_1, \ldots, \theta_N \in [0, 2\pi]$ such that $\theta \in [0, 2\pi]$ implies $|\theta - \theta_j| < \delta_j$ for some j, where we set $\delta_j = \delta(\theta_j)$ and $\mu_j = \mu(\theta_j)$. Let also $\mu^* = \frac{1}{2} \min_j \mu_j$. By (4.50) we have

$$\|(H_{r\theta} - \lambda_0)^{-1}\|_\infty \leq \max_j \|(H_{0\theta_j} - \lambda_0)^{-1}\|_\infty + 1 = C^* \tag{4.51}$$

for $r \in [0, \mu^*]$ and $\theta \in [0, 2\pi]$. We now label the corners $(0,0)$, $(2\pi, 0)$, $(0, 2\pi)$ and $(2\pi, 2\pi)$ by e_1, e_2, e_3 and e_4, respectively, and let $S_j = \{\Theta : |\Theta - e_j| < \mu^*\}$, with $\overline{S}_j$ the closure. By (4.51) we have

$$\|(H_\Theta - \lambda_0)^{-1}\|_\infty \leq C^* \qquad \text{for } 0 \neq \Theta \in \overline{S}_1. \tag{4.52}$$

By the periodicity property stressed at the beginning of this subsection we have

$$\|(H_\Theta - \lambda_0)^{-1}\|_\infty \leq C^* \qquad \text{for } e_j \neq \Theta \in \overline{S}_j \ (j = 1, 2, 3, 4). \tag{4.53}$$

Step (S2). Next note that if $\Theta \in \dot{M} \setminus \cup_j S_j$, then $\lambda_0 \in \rho(H_\Theta)$ by our assumptions. Since $\dot{M} \setminus \cup_j S_j$ is closed, and by the Corollary to Lemma 4.2 we may use a covering argument similarly to that above, i.e. exactly the same as in the proof of Theorem 2 in [12], in order to find a positive constant C' such that

$$\|(H_\Theta - \lambda_0)^{-1}\|_\infty \leq C' \qquad \text{if } \Theta \in \dot{M} \setminus \cup_j S_j. \tag{4.54}$$

To sum up we have

$$\|(H_\Theta - \lambda_0)^{-1}\|_\infty \leq \max(C', C^*) = C \qquad \text{for } \Theta \in \dot{M}. \tag{4.55}$$

By exactly the same measure-theoretic arguments as in the proof of Theorem 2 in [12] we then infer

$$\lambda_0 \qquad \text{is in the resolvent set of } P^g(A_s + T_0)P^g + \tau Q^g. \tag{4.56}$$

Since $\tau \neq \lambda_0$ and again by (4.18) adopted to the present situation we conclude

$$\lambda_0 \qquad \text{is in the resolvent set of } A_s + P^g T_0 \text{ on } E_g \tag{4.57}$$

and the theorem is proved ∎

Corollary. *λ_0 is in the spectrum of $A_s + P^g T_0$ (on E_g) if and only if it is in the spectrum of $A_s(\Theta) + P^g_\Theta T_0$ (on E^g_Θ) for some $\Theta \in \dot{M}$ or in the spectrum of $A_s(0,\theta) + P^g_{0\theta} T_0$ (on $E^g_{0\theta}$) for some $\theta \in [0, 2\pi]$.*

Proof. Via corollaries to Theorems 4.2 and 4.3 ∎

4.5 Comments. Theorems 4.2 and 4.3 give a complete description of the spectrum of $A_s + PT_0$ as an unbounded operator on the invariant subspace E_g in terms of the spectra of the Θ-periodic constituents $A_s(\Theta) + P_\Theta T_0$ on the invariant subspaces E^g_Θ under the assumption that the L-periodic equilibrium solution $u_0 = (u_1, u_2, u_3)$ of Navier-Stokes which defines T_0 via (1.3) satisfies

$$u_1, u_2 \quad \text{are even in } z \qquad \text{and} \qquad u_3 \quad \text{is odd in } z. \tag{4.58}$$

This assumption simplifies the presentation, gives nicer results and admits a more precise analysis of what happens at the corners of M. However, it has to be stressed that only minor modifications are needed in order to extend the theory to the case of an arbitrary L-periodic equilibrium solution. The extension to the arbitrary rectangular case (L_1-periodicity in x and L_2-periodicity in y) is completely straightforward. This might be true to a lesser extent for lattice cells other than rectangular, since the corners might cause caution. Below, however, we base our discussion on (4.58). In Subsections (4.1) - (4.4) we have concentrated on the restriction of $A_s + PT_0$ to E_g, (i.e. $A_s + P^g T_0$) and neglected the other case $A_s + P^u T_0$ on E_u. This neglect is justified since the case of $A_s + P^u T_0$ on E_u is considerably simpler, in fact much closer to the reaction-diffusion case in [12]. The reason for this is that all difficulties related to the corners of $M = [0, 2\pi]^2$ are absent in this case. A glance at Theorem 2.1* and the formulas (2.24) which define Q_Θ and hence P_Θ, P^g_Θ and P^u_Θ shows that the denominators $\nu = \widehat{\alpha}^2 + \widehat{\beta}^2$ do not appear, what makes the considerations in Subsection 4.4 superfluous. For reasons of space we just state the relevant result

Theorem 4.4. *λ_0 is in the spectrum of $A_s + P^u T_0$ on E_u if and only if it is in the spectrum of $A_s(\Theta) + P^u_\Theta T_0$ on E^u_Θ for some $\Theta \in M = [0, 2\pi]^2$.*

Note that in Theorem 7 the corners of M appear on an even footing with all points of $\dot{M}$. We come back to this point after a brief digression into the periodic case which was the starting point but retired into the background in Subsections 4.1 - 4.4. In case of reaction-diffusion systems (see [12]) a simple relationship between the periodic and the $\mathcal{L}^2$-case emerges: a point in the periodic spectrum is a point in the $\mathcal{L}^2$-spectrum. Here, the situation is not so simple. In order to digress on this we briefly recall the periodic case. With $L^2 = (\mathcal{L}^2(Q))^3$ and L^2_g, L^2_u as before, we let E_{per} be the $\mathcal{L}^2$-closure of $f \in (H^1_{\text{per},0})^3$ such that $\operatorname{div} f = 0$. Then $E^g_{\text{per}} = E_{\text{per}} \cap L^2_g$ is the $\mathcal{L}^2$-closure of all f in $L^2_g \cap (H^1_{\text{per},0})^3$ such that $\operatorname{div} f = 0$; likewise with L^2_u and E^u_{per}. Then E_{per} is the orthogonal sum of E^g_{per} and E^u_{per}. The orthogonal projections onto $E_{\text{per}}, E^g_{\text{per}}$ and E^u_{per} are denoted by $P_{\text{per}}, P^g_{\text{per}}$ and P^u_{per}, respectively. The Stokes operator A_{per} now acts like $P_{\text{per}}\Delta$ on its domain $(H^2_{\text{per}} \cap H^1_{\text{per},0})^3$ with div $= 0$. It is well known that A_{per} is selfadjoint, $A_{\text{per}} \leq -\varepsilon$ for some ε and that A_{per} leaves E^g_{per} and E^u_{per} invariant, i.e. reduces to $P^g_{\text{per}}\Delta$ and $P^u_{\text{per}}\Delta$ on $\operatorname{dom} A_{\text{per}} \cap E^g_{\text{per}}$ and $\operatorname{dom} A_{\text{per}} \cap E^u_{\text{per}}$, respectively. The "perturbation" $A_{\text{per}} + P_{\text{per}}T_0$ is then recognized as a holomorphic semigroup generator on E_{per} which, by virtue of (4.58), leaves E^g_{per} invariant, i.e. coincides with $A_s + P^g_{\text{per}}T_0$ on E^g_{per}, and likewise with E^u_{per}. A straightforward analysis then shows that if Θ is one of the corners (e.g. $\Theta = (0,0)$), then the periodic case arises, i.e. $A_s(\Theta) + P^u_\Theta T_0$ on E^u_Θ becomes $A_{\text{per}} + P^u_{\text{per}}T_0$ on E^u_{per}.

Corollary. *If λ_0 is in the spectrum of $A_{\text{per}} + P^u_{\text{per}}T_0$ on E^u_{per}, then it is in the spectrum of $A_s + P^uT_0$ on E_u.*

The case of $A_{\text{per}} + P^g_{\text{per}}T_0$ on E^g_{per} is more delicate. Recall that $A_s(0,\theta)$ on $E^g_{0,\theta}$ is $P^g_{0\theta}\Delta$ restricted to $(H^2_{\text{per}})^3 \cap D_{0\theta}$ with $D_{0\theta}$ as in (4.35). Thus $A_s(0,\theta)$ on $E^g_{0,\theta}$ is simply the restriction of $P_{0\theta}A_{\text{per}}$ to $\operatorname{dom} A_s(0,\theta) \cap E^g_{0\theta}$. Moreover $E^g_{0\theta}$ is a closed subspace of E^g_{per}, whence $P^g_{0\theta}P^g_{\text{per}} = P^g_{0\theta}$. The relevant statement is

Lemma 4.6. *Let λ_0 be real and in the spectrum of $A_{\text{per}} + P^g_{\text{per}}T_0$ on E^g_{per}. Then λ_0 is in the spectrum of $A_s + P^gT_0$ on E_g.*

Proof. Since $A_{\text{per}} + P^g_{\text{per}}T_0$ has compact resolvents, λ_0 is necessarily an eigenvalue of it. Thus there is a real eigenfunction $0 \neq \varphi_0 \in \operatorname{dom} A_{\text{per}} \cap L^2_g$:

$$(A_{\text{per}} + P^g_{\text{per}}T_0)\varphi_0 = \lambda_0\varphi_0. \tag{4.59}$$

With φ_0 real, the two numbers a and b associated with φ_0 via (4.34) are real and hence there is a $\theta \in [0, 2\pi]$ such that (4.32) holds. Thus φ_0 is in $(H^2_{\text{per}})^3 \cap D_{0\theta}$, i.e. in $\operatorname{dom} A_s(0,\theta) \cap E^g_{0\theta}$, whence $P^g_{0\theta}\varphi_0 = \varphi_0$. Applying $P^g_{0\theta}$ to (4.59) yields

$$P^g_{0\theta}(P^g_{\text{per}}\Delta + P^g_{\text{per}}T_0)\varphi_0 = \lambda_0\varphi_0$$

that is

$$\big(A_s(0,\theta) + P^g_{0\theta}T_0\big)\varphi_0 = \lambda_0\varphi_0. \tag{4.60}$$

By (4.60) and Theorem 4.2 we have that λ_0 is in the spectrum of $A_s + P^gT_0$ on E_g ∎

Remarks. The basic open problem is if there may exist complex eigenvalues of $A_{\text{per}} + P^g_{\text{per}}T_0$ on E^g_{per} which are not in the spectrum of $A_s + P^gT_0$ on E_g. This would allow for the possibility of periodic equilibrium solutions which are periodically unstable but $\mathcal{L}^2$-stable, a situation that cannot arise in the diffusion case. Among the open problems there is of course the difficult task to determine quantitatively or qualitatively the spectrum of the operators $A_s(0,\theta) + P^g_{0\theta}T_0$ and $A_s(\Theta) + P^g_\Theta T_0$ $(\Theta \in \dot{M})$ and $A_s(\Theta) + P^u_\Theta T_0$ $(\Theta \in M)$, and to determine eventually classes of periodic equilibrium solutions for which this problem is solvable. This task is important for the stability analysis of the periodic equilibrium solution against $\mathcal{L}^2$-perturbations, a direction which might reveal new phenomena.

That the L^2_g-case is more difficult that the L^2_u-case can be seen as follows. E^g_{per} contains a two-dimensional subspace, the set of

$$(a\tilde{e}_{00}\rho_0, b\tilde{e}_{00}\rho_0, 0) \qquad (a, b \in \mathbb{C}),$$

call it $E^0_{\rm per}$, on which the divergence condition is trivially satisfied since $\widetilde{e}_{00}$ and ρ_0 are constant. The corresponding space in case of generic $\Theta \in \dot{M}$ is given by

$$(ae_{00}\rho_0, be_{00}\rho_0, 0) \qquad \text{with } a\widehat{\alpha} + b\widehat{\beta} = 0,$$

call it E^0_Θ. Since $\Theta \in \dot{M}$, $\widehat{\alpha}^2 + \widehat{\beta}^2 > 0$ whence $\dim E^0_\Theta = 1$. As $\Theta \to 0$, $e_{00} \to \widetilde{e}_{00}$, but the one-dimensional space E^0_Θ cannot converge toward the two-dimensional space $E^0_{\rm per}$. This is the major source of the difficulties in the L^2_g-case, absent in the L^2_u-case. All results and proofs obtained so far for a solution pair $u_0 = (u_1, u_2, u_3)$ and p_0 which is L-periodic in x and y carry over to the case where u_0, p_0 are L_1-periodic in x and L_2-periodic in y. The only exception is condition (4.32) which has to be replaced by the more general one

$$aL_1^{-1}\cos\theta + bL_2^{-1}\sin\theta = 0.$$

In view of its relation to stability it is a question of great interest to extend the above methods to domains such as half-planes or cylinders and to investigate whether the problem with the singularities persists. The infinite strip, a special case of the plate, has been studied in detail ($\Theta \in [0, 2\pi]$ and $u = (u_1, u_2)$). Here, the difficulties caused by the singularities disappear, but a little rest of the above paradox remains.

Acknowledgements. The author is indebted to Herbert Amann for reading and discussing the manuscript, and to W. von Wahl for valuable discussions on the topics of this paper.

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Received 08.11.1994